HU-EP-25/26 • HU-Mathematik 2025-02

MODULAR FAMILIES OF ELLIPTIC LONG-RANGE SPIN CHAINS FROM FREEZING

ROB KLABBERS[a], JULES LAMERS[b]

Abstract. We consider the construction of quantum-integrable spin chains with q-deformed long-range interactions by 'freezing' integrable quantum many-body systems with spins. The input is a spin-Ruijsenaars system along with an equilibrium configuration of the underlying spinless classical Ruijsenaars–Schneider system. For a distinguished choice of equilibrium, the resulting long-range spin chain has a real spectrum and admits a short-range limit, providing an integrable interpolation from nearest-neighbour to long-range interacting spins.

We focus on the elliptic case. We first define an action of the modular group on the spinless elliptic Ruijsenaars–Schneider system to show that, for a fixed elliptic parameter, it has a whole modular family of classical equilibrium configurations. These typically have constant but nonzero momenta. Then we use the setting of deformation quantisation to provide a uniform framework for freezing elliptic spin-Ruijsenaars systems at any classical equilibrium whilst preserving quantum integrability. As we showed in previous work, the results include the Heisenberg, Inozemtsev and Haldane–Shastry chains along with their XXZ-like q-deformations (face type), or the antiperiodic Haldane–Shastry chain of Fukui–Kawakami, its elliptic generalisation of Sechin–Zotov, and their completely anisotropic q-deformations due to Matushko–Zotov (vertex type).

Contents

1. Introduction

Integrable spin systems with long-range interactions provide theoretical laboratories to study physical phenomena such as fractional statistics and long-range order [Hal91, Tho69, HT13]. Unlocking this potential, however, requires understanding the exact underlying mathematical structures. The best-understood member of the long-range family is the Haldane–Shastry chain [Hal88, Sha88]. Its integrability descends from that of a related quantum many-body system, the trigonometric Calogero–Sutherland system with spins, through a procedure called *freezing* [Pol93]. This enables one to leverage the rich algebraic structure of that quantum many-body system [BGHP93], see also [LS24]. Freezing relies on a decoupling of the dynamical degrees of freedom (coordinates and momenta), which become classical, from the spins, which remain fully quantum mechanical. As suggested in [LRS24], this situation is analogous to the Born–Oppenheimer approximation, in which electrons moving around atomic nuclei are viewed as quantum-mechanical particles moving in a classical background given by the (much more massive) nuclei. Similarly, the Haldane–Shastry chain comprises quantum-mechanical spins interacting in a background governed by the classical trigonometric Calogero–Moser–Sutherland system. This situation extends to the q-deformed level, connecting the spin-Ruijsenaars–Macdonald system with the XXZ-type generalisation of the Haldane–Shastry chain [Ugl95, Lam18, LPS22].

For *elliptic* long-range spin chains, a recent surge of activity has started to fill in several long-standing gaps in the understanding of their integrability. For the Inozemtsev chain [Ino90], which interpolates between the Heisenberg XXX and Haldane–Shastry chains, it has long been known that its exact eigenfunctions are built from those of the scalar elliptic Calogero–Sutherland system [Ino95]. Our more recent reformulation in terms of physically motivated quantities [KL22] connects this solution directly to the exact (Bethe-ansatz/Jack) wave functions of the limiting spin chains, paving the way for a direct comparison of their integrable structures. A set of conserved quantities was proposed in [Ino96], but their mutual commutativity, and hence the integrability of Inozemtsev chain, remained an open problem for a long time. In [Cha24], Chalykh addressed it by constructing a hierarchy of commuting higher hamiltonians using elliptic Dunkl operators [BFV94] and freezing.

In this paper, we zoom in on freezing at the q-deformed elliptic level. Two long-range spin chains and underlying quantum many-body system with spins have recently been uncovered.

- **Vertex-type.** In [MZ23a, MZ23b], Matushko and Zotov (MZ) constructed a fully anisotropic elliptic spin-Ruijsenaars model based on the Baxter–Belavin R-matrix [Bax72, Bel81]. The corresponding spin chain q-deforms Sechin and Zotov's elliptic generalisation [SZ18] of the (antiperiodic) Fukui–Kawakami chain [FK96], cf. [KL25].
- **Face-type.** In [KL24], we defined a partially (an)isotropic, i.e. XXZ-like, elliptic spin-Ruijsenaars system based on Felder's dynamical R-matrix [Fel95b]. The associated spin chain q-deforms the Inozemtsev chain and moreover is an elliptic generalisation of the q-deformed Haldane–Shastry chain.

For a detailed analysis of the (almost entirely disjoint) landscapes in which these two families of integrable long-range spin chains live, see [KL25], summarised in Figures 2–3 therein.

In [MZ23a], the formalism for freezing from the trigonometric case [TH95, Ugl95, LPS22] was used to obtain a spin chain from their elliptic (vertex-type) spin-Ruijsenaars system. This involves expanding around a classical equilibrium configuration of the (elliptic) scalar Ruijsenaars system for which all momenta vanish, $p_i^\star = 0$, where we use '$\star$' for a classical equilibrium. This expansion coincides with a strong-coupling expansion of the shift operators (multiplicative momenta), $\Gamma_i = \mathrm{e}^{\epsilon \hat{p}_i} = 1 + \epsilon\, \hat{p}_i + O(\epsilon^2)$, in a parameter $\epsilon \propto 1/g$. However, compared to the trigonometric case, a new feature at the elliptic level is the existence of a large number of classical equilibria, almost all of which have non-vanishing momenta, as we shall see. It is precisely such equilibria that give rise to spin chains with a well-defined (i.e. convergent) short-range limit [KL24, KL25]. This requires a modification of the freezing procedure from [MZ23a] in order to account for the contribution from the nontrivial classical limit $\Gamma_i \to \mathrm{e}^{\epsilon p_i^\star}$. Surprisingly, the resulting 'MZ′ chain' [KL25] turns out to differ from the original MZ chain [MZ23a] simply by

a shift over a multiple of the identity [KL25]. In contrast, in the (face-type) dynamical case of [KL24] the two ways of freezing produce q-deformed Inozemtsev chains that differ more drastically — do not even commute — with only the version considered in [KL24, KL25] admitting a short-range limit. The upshot is that both the MZ′ chain and the q-deformed Inozemtsev chain enable an analytic comparison between long- and short-range regimes via interpolation. However, it remains to be proven that these interpolating spin chains are indeed integrable. In the vertex case, this provides a direct and more general alternative to the combination of the proof of integrability for the MZ chain [MZ23a] and the simple difference with the MZ′ chain established in [KL25]. In the face setting, it requires a separate proof. In this paper we provide a technical but crucial step to narrow this gap. Besides proving integrability, we expect this to be very useful for the study of the spectrum. Indeed, for the (q-deformed and ordinary) Haldane–Shastry chain, the eigenvalues, eigenvectors, and (nonabelian) symmetries are understood via the connection to quantum many-body systems provided by freezing.

Our main aim is to develop a framework for freezing elliptic spin-Ruijsenaars systems around *any* equilibrium of their spinless classical limit.

Outline. In §2 we review these systems at the quantum level in a framework allowing for a uniform treatment of the face and vertex versions. We *assume* the spin-QMBS is integrable (in the sense that it comes with as many commuting difference operators as there are particles); in the vertex case this was proven in [MZ23a], while in the face case the proof is forthcoming.

We study the classical level of the scalar model and its equilibria in §3, exhibiting an $\mathrm{SL}(2,\mathbb{Z})$-action that connects equilibria associated to the same (elliptic) lattice.

In §4 we employ deformation quantisation to freeze any of the QMBS with spins at any classical equilibrium configuration and *prove* that the resulting spin chain is integrable. This is closely related to the settings of [MV24, LRS24]. The result is an $\mathrm{SL}(2,\mathbb{Z})$-family of integrable long-range spin chains. We conclude in §5.

The appendix consists of two parts. §A contains all necessary definitions and properties of elliptic functions and R-matrices. §B has more details about the deformed spin permutations from which the spin-Ruijsenaars operators are built.

2. Elliptic quantum spin-Ruijsenaars systems

The (quantum, elliptic) Ruijsenaars system, see [Rui04] for an overview, describes N interacting particles moving on a circle with coordinates x_j. We will simply denote the space of states, which consists of suitable functions of $\boldsymbol{x} = (x_1, \dots, x_N)$, by $\mathrm{Fun}(\boldsymbol{x})$.[1] Fix an arbitrary elliptic parameter τ with $\mathrm{Im}\,\tau > 0$. We take the (odd) Jacobi theta function to be defined as

$$\theta(x\,|\,\tau) \coloneqq \frac{\sin(\pi\,x)}{\pi} \prod_{n=1}^{\infty} \frac{\sin\big(\pi(n\,\tau + x)\big)\sin\big(\pi(n\,\tau - x)\big)}{\sin^2(\pi\,n\,\tau)}\,. \tag{2.1}$$

More details on the elliptic functions that we use, which are all defined in terms of (2.1), can be found in §A. Given $\epsilon \in \mathbb{C}$, consider the shift operators $\Gamma_i \coloneqq \mathrm{e}^{-\mathrm{i}\,\hbar\,\epsilon\,\partial_{x_i}}$, acting on $\mathrm{Fun}(\boldsymbol{x})$ as

$$(\Gamma_i\,f)(\boldsymbol{x}) = f(x_1, \dots, x_i - \mathrm{i}\,\hbar\,\epsilon, \dots, x_N)\,. \tag{2.2}$$

We suppress the dependence of Γ_i on $\hbar$ and ϵ. Then, up to a conjugation ('gauge transformation') [Has97], the Ruijsenaars system is defined through the N difference operators [Rui87]

$$D_n(\boldsymbol{x};\hbar,\eta,\epsilon\,|\,\tau) \coloneqq \sum_{\substack{I\subset\{1,\dots,N\}\\ |I|=n}} A_I(\boldsymbol{x};\eta\,|\,\tau)\,\Gamma_I\,, \quad \Gamma_I \coloneqq \prod_{i\in I}\Gamma_i\,, \qquad 1 \leqslant n \leqslant N\,. \tag{2.3}$$

[1] We will be interested in formal and algebraic structures rather than (functional) analysis. For simplicity, we take $\mathrm{Fun}(\boldsymbol{x})$ to consist of meromorphic functions on which the action of finite products of Γ_i is well defined. One may wish to impose appropriate quasiperiodicity properties, cf. equation (26) of [Has94].

Here the sum is over n-element subsets of $\{1,\dots,N\}$ and the coefficients are given by

$$A_I(\boldsymbol{x};\eta\,|\,\tau) \coloneqq \prod_{i\in I\not\ni j} \frac{\theta(x_i-x_j+\eta\,|\,\tau)}{\theta(x_i-x_j\,|\,\tau)}\,, \tag{2.4}$$

where the product runs over $i\in I$ and $j\in\{1,\dots,N\}\setminus I$, and $\eta\in\mathbb{C}$ is a parameter. When there is no cause for confusion we will often suppress the range of the sum or the dependence of the coefficients on either or both parameters η and τ. Thus, (2.3) acquires the compact form $D_n=\sum_I A_I(\boldsymbol{x})\,\Gamma_I$.

The Ruijsenaars system is (quantum) integrable in the sense that the difference operators (2.3) commute,

$$[D_n,D_m]=0\,. \tag{2.5}$$

We are interested in *matrix-valued* generalisations of (2.3) for which this commutativity persists.

To this end, we generalise the state space to the space $\mathrm{Fun}(\boldsymbol{x})\otimes V^{\otimes N}$ of vector-valued functions, where each particle carries a 'spin' that lives in a complex vector space $V\cong\mathbb{C}^r$ for $r\in\mathbb{Z}_{\geqslant 1}$. The scalar case $r=1$ gives back the spinless Ruijsenaars system (2.3). The *integrable* matrix-valued difference operators that we will consider are built from R-matrices. In view of the coefficients (2.4) it is natural to consider *elliptic* R-matrices, which come in two well-known variants: the Baxter–Belavin ('vertex-type') R-matrix [Bax72, Bel81] and Felder's dynamical ('face-type') R-matrix [Bax73, Fel95b]. The resulting matrix-valued generalisations of (2.3) are the spin-Ruijsenaars systems of Matushko and Zotov [MZ23a, MZ24] and our dynamical spin-Ruijsenaars systems [KL24], respectively. For $r>1$ these two models are different, cf. §5.2 in [KL25], yet they look very similar. A uniform description goes as follows.

2.1. **Deformed spin permutations.** Consider a 'generalised R-matrix' in the sense of [Che92]. That is, suppose we have a family of linear operators $P_{i,i+1}(u)=P_{i,i+1}(u;\eta\,|\,\tau)$ on $V^{\otimes N}$ obeying the unitarity condition

$$P_{i,i+1}(u)\,P_{i,i+1}(-u)=\mathrm{id}\,, \tag{2.6}$$

the ('braided') Yang–Baxter equation [2]

$$P_{i,i+1}(u)\,P_{i+1,i+2}(u+v)\,P_{i,i+1}(v)=P_{i+1,i+2}(v)\,P_{i,i+1}(u+v)\,P_{i+1,i+2}(u)\,, \tag{2.7}$$

and the commutativity

$$[P_{i,i+1}(u),P_{j,j+1}(v)]=0\,,\qquad |i-j|>1\,. \tag{2.8}$$

We are particularly interested in the following two examples. Let $\mathbb{1}$ denote the identity on V, and P the flip on $V\otimes V$.

- **Vertex-type.** Let $R(u)=R(u;\eta\,|\,\tau)$ be the Baxter–Belavin R-matrix, which is Baxter's eight-vertex R-matrix if $r=2$, see §A.2.1. Set $\check R(u)=P\,R(u)$. Then (2.6)–(2.8) hold for

$$P^{\mathrm{V}}_{i,i+1}(u)=\mathbb{1}^{\otimes(i-1)}\otimes\check R(u;\eta\,|\,\tau)\otimes\mathbb{1}^{\otimes(N-i-1)}\,. \tag{2.9}$$

- **Face-type.** Let $R(u,\vec a)=R(u,\vec a;\eta\,|\,\tau)$ be Felder's dynamical R-matrix of type $\mathfrak{gl}_r$, involving 'dynamical' parameters $\vec a=(a_1,\dots,a_r)$, see §A.2.2. It obeys the *dynamical* Yang–Baxter equation, in which the dynamical parameters $\vec a$ are shifted depending on the weight of the factors of V to the left of the R-matrix. Namely, write $|\vec\mu\rangle\langle\vec\mu|$ for the projection onto the weight-$\vec\mu$ subspace of $V^{\otimes(i-1)}$. Again putting $\check R(u,\vec a)=P\,R(u,\vec a)$, we now take

$$P^{\mathrm{F}}_{i,i+1}(u)=\sum_{\vec\mu}|\vec\mu\rangle\langle\vec\mu|\otimes\check R(u,\vec a-\eta\,\vec\mu;\eta\,|\,\tau)\otimes\mathbb{1}^{\otimes(N-i-1)}\,. \tag{2.10}$$

  Since $\sum_{\vec\mu}|\vec\mu\rangle\langle\vec\mu|=\mathbb{1}^{\otimes(i-1)}$ is (a resolution of) the identity, (2.10) acts nontrivially only on the ith and $i+1$st factors of $V^{\otimes N}$. The shifts of $\vec a$ ensure that (2.7) is equivalent to the dynamical Yang–Baxter equation. The operators (2.10) also satisfy (2.6)–(2.8).

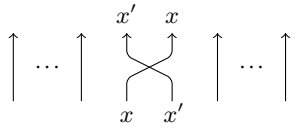
[2] The braided setting simplifies keeping track of the shifts of the dynamical parameters in the face-type setting.

Since both of these examples moreover obey the condition

$$P_{i,i+1}(u)\big|_{\eta=0} = P_{i,i+1} \coloneqq \mathbb{1}^{\otimes(i-1)} \otimes P \otimes \mathbb{1}^{\otimes(N-i-1)}\,, \tag{2.11}$$

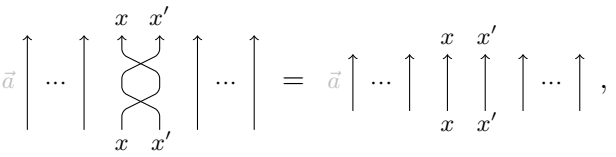
we think of $P_{i,i+1}(u)$ as a (q-)deformed nearest-neighbour spin Permutation operator.

We will use the standard graphical notation: each copy of the vector space V (containing a spin) is drawn as a vertical line, which carries an 'inhomogeneity parameter'. The 'deformed spin permutation' $P_{i,i+1}(u)$ is a crossing of the i and $i+1$st lines, which each carry along their inhomogeneities,

$P_{i,i+1}(u) =$ 

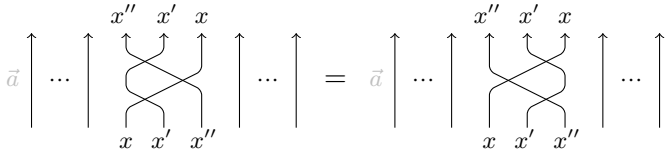
$, \quad u = x - x'\,.$ (2.12)

The only difference between the vertex- and face-type examples for $P(u)$ is that, in the latter case, the left-most face is decorated with the dynamical parameters $\vec{a}$ (indicated in gray), determining the dynamical parameters on all other faces, see §A.2.2. Then (2.6) with $u = x - x'$ becomes



(2.13)

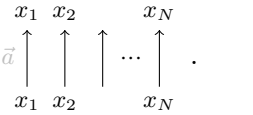
while (2.7) with in addition $v = x' - x''$ is



. (2.14)

Note that, while the inhomogeneities are carried around by the lines, (the subscripts labelling) the vector spaces do not move, since we work with $\check{R}(x) = P\,R(x)$. In the following, each diagram starts out (at the bottom) with inhomogeneity parameters given by the particle coordinates $x_1, \dots, x_N$.

We can now define an operator $P_w(\boldsymbol{x})$ for any permutation $w \in S_N$, cf. [Che92]. We start from the identity $e \in S_N$, for which

$P_e(\boldsymbol{x}) = \mathrm{id} =$ 

. (2.15)

All other operators are constructed from this by recursion via the *cocycle condition*

$$P_{w\,(i,i+1)}(\boldsymbol{x}) = P_w(x_1, \dots, x_{i+1}, x_i, \dots, x_N)\, P_{i,i+1}(x_i - x_{i+1})\,, \tag{2.16}$$

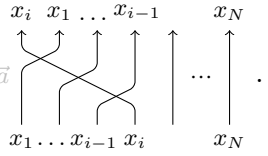
where the swap $x_i \leftrightarrow x_{i+1}$ for P_w takes into account how the parameters move around. The result is well defined thanks to (2.6)–(2.8). Note that $P_w(\boldsymbol{x})$ does not always actually depend on all parameters $x_1, \dots, x_N$, as (2.15) already illustrates.

Next, the permutations w that appear in the nth spin-Ruijsenaars operator can be concisely defined as follows. Given an n-element subset $I = \{i_1 < \dots < i_n\} \subseteq \{1, \dots, N\}$, consider the (Grassmannian) permutation $w_I \in S_N$ that sends $k \longmapsto i_k$ for all $1 \leqslant k \leqslant n$ while permuting neither any of $\{1, \dots, n\}$ amongst each other, nor any of $\{n+1, \dots, N\}$. From it, we define

$$P_I(\boldsymbol{x}) \coloneqq P_{w_I^{-1}}(\boldsymbol{x})\,, \qquad P_{-I}(\boldsymbol{x}) \coloneqq P_{\{1,\dots,N\}\setminus I}(\boldsymbol{x})\,. \tag{2.17}$$

For example, at $n = 1$ the cycle $w_{\{i\}} = (i\ i-1\ \dots\ 1)$ gives

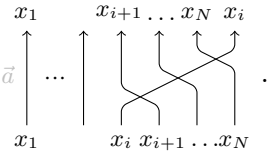
$$P_{\{i\}}(\boldsymbol{x}) = P_{(1\ \dots\ i-1\ i)}(\boldsymbol{x}) = P_{12}(x_1 - x_i)\cdots P_{i-1,i}(x_{i-1} - x_i) = \tag{2.18}$$



As n increases the $P_I(\boldsymbol{x})$ become more complicated, until $n = N/2$, when they simplify again, and it is more convenient to switch to the $P_{-I}(\boldsymbol{x})$. Since $w_{\{1,\dots,N\}} = e$, the first nontrivial example is

(2.19)

$$P_{-\{i\}}(\boldsymbol{x}) = P_{(N\ N-1\ \dots\ i)}(\boldsymbol{x}) = P_{N-1,N}(x_i - x_N)\cdots P_{i,i+1}(x_i - x_{i+1}) =$$



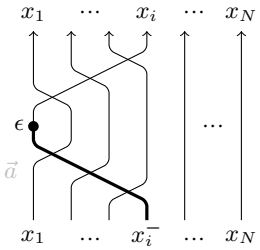
For more about these deformed spin permutations, including further examples, see §B.

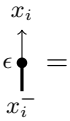
2.2. **Matrix-valued Ruijsenaars operators.** In terms of the deformed spin permutations, the spin-generalisations of the scalar Ruijsenaars operators (2.3) read

$$\widetilde{D}_n(\boldsymbol{x}; \hbar, \eta, \epsilon, \vec{a}\,|\,\tau) = \sum_{|I|=n} A_I(\boldsymbol{x})\, P_I(\boldsymbol{x})^{-1}\, \Gamma_I\, P_I(\boldsymbol{x})\,. \tag{2.20}$$

They depend on the parameters from the scalar case, along with any further parameters from the R-matrix (e.g. $\vec{a}$ in the vertex-type case); again, we will often suppress this from our notation. The first spin-Ruijsenaars operator thus takes the form

$$\begin{aligned}
\widetilde{D}_1 &= \sum_{i=1}^{N} A_i(\boldsymbol{x})\, P_{(1\ \dots\ i)}(\boldsymbol{x})^{-1}\, \Gamma_i\, P_{(1\ \dots\ i)}(\boldsymbol{x}) \\
&= \sum_{i=1}^{N} A_i(\boldsymbol{x}) \times
\end{aligned} \tag{2.21}$$





$$= \Gamma_i\,, \qquad x_i^- \equiv x_i - \mathrm{i}\,\hbar\,\epsilon$$

$$= \sum_{i=1}^{N} A_i(\boldsymbol{x})\, P_{(1\ \dots\ i)}(\boldsymbol{x})^{-1} P_{(1\ \dots\ i)}(x_1, \dots, x_i^-, \dots, x_N)\, \Gamma_i\,,$$

where in the last line we pushed the difference operator to the right. Higher spin-Ruijsenaars operators can be depicted and explicitly given similarly, see e.g. (4.45) below for $n = 2$.

Like for the spin permutations $P_I(\boldsymbol{x})$, there is a sort of symmetry between $\widetilde{D}_{N-n}$ and $\widetilde{D}_n$, as follows. The last difference operator $\widetilde{D}_N = \Gamma_1 \cdots \Gamma_N = D_N$ is just the total coordinate-shift operator, and acts as the identity on spins. Since all $\widetilde{D}_n$ only depend on *differences* of coordinates, they commute with $\widetilde{D}_N$ (for any choice of R-matrix). Using that $\widetilde{D}_N$ is clearly invertible, it will sometimes be convenient to simplify operators 'beyond the equator', i.e. with $n > N/2$, by defining

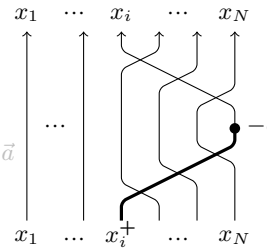
$$\begin{aligned}
\widetilde{D}_{-n} &\coloneqq D_N^{-1}\, \widetilde{D}_{N-n} \\
&= \sum_{|I|=n} A_{-I}(\boldsymbol{x})\, P_{-I}(\boldsymbol{x})^{-1}\, \Gamma_{-I}\, P_{-I}(\boldsymbol{x})\,,
\end{aligned} \tag{2.22}$$

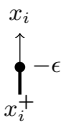
where $A_{-I}(\boldsymbol{x}) \coloneqq A_I(-\boldsymbol{x}) = A_I(\boldsymbol{x})|_{\eta\mapsto-\eta}$ and $\Gamma_{-I} \coloneqq \Gamma_I^{-1} = \Gamma_I|_{\epsilon\mapsto-\epsilon}$. The equality in the second line of (2.22) uses unitarity (2.6) and the Yang–Baxter equation (2.7). For example,

$$\widetilde{D}_{-1} = \sum_{i=1}^{N} A_{-i}(\boldsymbol{x})\, P_{(N\ N-1\ \dots\ i)}(\boldsymbol{x})^{-1}\, \Gamma_i^{-1}\, P_{(N\ N-1\ \dots\ i)}(\boldsymbol{x})$$

(2.23)

$$= \sum_{i=1}^{N} A_{-i}(\boldsymbol{x}) \times$$





$= \Gamma_i^{-1}\,, \qquad x_i^+ \equiv x_i + \mathrm{i}\,\hbar\,\epsilon$

$$= \sum_{i=1}^{N} A_{-i}(\boldsymbol{x})\, P_{(N\ N-1\ \dots\ i)}(\boldsymbol{x})^{-1} P_{(N\ N-1\ \dots\ i)}(x_1, \dots, x_i^+, \dots, x_N)\, \Gamma_i^{-1}\,.$$

For certain choices of R-matrices, the spin-Ruijsenaars operators all commute with each other,

$$[\widetilde{D}_n, \widetilde{D}_m] = 0\,, \tag{2.24}$$

for all n, m in $\{1, \dots, N\}$ (or $\{1-N, \dots, -1\}$). In the case of elliptic coefficients A_I given by (2.4), the R-matrices need to be elliptic too. In the (vertex) case with (2.9) the Baxter–Belavin R-matrix, (2.24) was proven by Matushko and Zotov [MZ23a]. For the (face) case with (2.10) Felder's dynamical R-matrix, we announced the analogous result in [KL24]; our proof goes well beyond this paper, and will appear elsewhere.

3. Classical level: elliptic Ruijsenaars–Schneider system

Now we turn to the classical level. The classical Ruijsenaars–Schneider system depends on the parameters $(\eta, \epsilon \,|\, \tau) \in \mathbb{C}^2 \times \mathbb{H}$, with $\mathbb{H} \coloneqq \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ the upper half plane.

Consider the classical phase space $M = T^*\,\mathbb{R}^N \cong \mathbb{R}^{2N}$ with coordinates x_i and conjugate momenta p_j obeying canonical Poisson brackets

$$\{x_i, x_j\} = \{p_i, p_j\} = 0\,, \quad \{x_i, p_j\} = \delta_{ij}\,. \tag{3.1a}$$

The multiplicative momenta $\gamma_i \coloneqq \mathrm{e}^{\epsilon\, p_i}$ provide coordinates satisfying

$$\{x_i, x_j\} = \{\gamma_i, \gamma_j\} = 0\,, \quad \{x_i, \gamma_j\} = \epsilon\, \delta_{ij}\, \gamma_j\,. \tag{3.1b}$$

We will work in the setting of algebraically integrable systems, with complexified phase space $M_{\mathbb{C}} = T^*\,\mathbb{C}^N \cong \mathbb{C}^{2N}$. The associated Poisson algebra of suitable[1] (p. 3) functions on $M_{\mathbb{C}}$ contains

$$\mathcal{A}_0 \coloneqq \operatorname{Fun}(\boldsymbol{x})[\gamma_1^{\pm 1}, \dots, \gamma_N^{\pm 1}] \tag{3.2}$$

as a Poisson subalgebra. The *Ruijsenaars–Schneider* (RS) system is defined by the functions

$$D_n^{\mathrm{cl}}(\boldsymbol{x}, \boldsymbol{p}; \eta \,|\, \tau) \coloneqq \sum_{\substack{I \subset \{1,\dots,N\} \\ \#I = n}} A_I(\boldsymbol{x}; \eta \,|\, \tau)\, \gamma_I \quad \in \mathcal{A}_0\,, \quad 1 \leqslant n \leqslant N\,, \qquad \gamma_I \coloneqq \prod_{i \in I} \gamma_i\,, \tag{3.3}$$

with the same coefficients (2.4). This (spinless, classical) many-body system is Liouville integrable [RS86]: the N functions (3.3) Poisson commute,

$$\{D_n^{\mathrm{cl}}, D_m^{\mathrm{cl}}\} = 0\,. \tag{3.4}$$

For the physical interpretation, first observe that the coefficients obey the symmetry properties $A_{I^{\mathrm{c}}}(\boldsymbol{x}) = A_{-I}(\boldsymbol{x})$ where $I^{\mathrm{c}} \coloneqq \{1, \dots, N\} \setminus I$ denotes the complement of I. Since $D_N^{\mathrm{cl}} = \gamma_1 \cdots \gamma_N = \exp(\epsilon \sum_j p_j)$, additional functions resembling (3.3) arise by setting

$$D_{-n}^{\mathrm{cl}} \coloneqq \frac{D_{N-n}^{\mathrm{cl}}}{D_N^{\mathrm{cl}}} = \sum_{\substack{I \subset \{1,\dots,N\} \\ \#I = n}} A_I(\boldsymbol{x}; -\eta)\, \gamma_I^{-1} = D_n^{\mathrm{cl}}\Big|_{\substack{\eta \mapsto -\eta \\ \epsilon \mapsto -\epsilon}} \quad \in \mathcal{A}_0\,, \quad 1 \leqslant n \leqslant n\,, \tag{3.5}$$

in complete analogy with (2.22). For the reader who likes physical units, let m_0 denote the rest mass, and c the speed of light. Now form the combinations

$$P^{\mathrm{cl}} := m_0\, c\, \frac{D_1^{\mathrm{cl}} - D_{-1}^{\mathrm{cl}}}{2}\,, \qquad H^{\mathrm{cl}} := m_0\, c^2\, \frac{D_1^{\mathrm{cl}} + D_{-1}^{\mathrm{cl}}}{2}\,. \tag{3.6}$$

For a single particle we recognise

$$N = 1: \qquad \begin{aligned} P^{\mathrm{cl}} &= m_0\, c\ \sinh(\epsilon\, p)\,, \\ H^{\mathrm{cl}} &= m_0\, c^2 \cosh(\epsilon\, p) = \sqrt{(m_0\, c^2\, 1)^2 + (c\, P^{\mathrm{cl}})^2}\,, \end{aligned} \qquad \epsilon = \frac{1}{m_0\, c}\,, \tag{3.7}$$

as the momentum and energy of a relativistic particle with p/m_0 playing the role of rapidity. The functions (3.6) are N-particle generalisations of (3.7) belonging to the family (3.3) that is Liouville integrable.

3.1. **Modularity.** We will be interested in families of hamiltonians associated to a *fixed* (quasi-) period lattice $\Lambda_\tau := \mathbb{Z} + \tau\,\mathbb{Z} \subset \mathbb{C}$, with modular parameter $\tau \in \mathbb{H}$. Such a lattice defines an elliptic curve $\mathbb{C}/\Lambda_\tau$, and homothetic lattices yield isomorphic curves. It is well known that all homothetic lattices are related by an action of the modular group

$$\mathrm{PSL}(2,\mathbb{Z}) = \langle S, T \,|\, S^2 = 1 = (S\,T)^3 \rangle\,, \tag{3.8}$$

whose action on $\mathbb{H}$ is generated by the signed inversion $S\colon \tau \longmapsto -1/\tau$ and the translation $T\colon \tau \longmapsto \tau + 1$, see e.g. [Sil94]. To keep the lattice *invariant*, this action has to be combined with a rescaling of the ambient copy of $\mathbb{C} \supset \Lambda_\tau$, see Fig. 1. Indeed, $\Lambda_\tau = c\,\Lambda_{\tau'}$ for some $\tau' \in \mathbb{H}$ and $c \in \mathbb{C}^\times$ precisely when $\tau = B\cdot\tau'$ for some $B \in \mathrm{PSL}(2,\mathbb{Z})$. Note that, given z in this ambient copy of $\mathbb{C}$, applying the S-transformation twice gives

$$(z\,|\,\tau) \ \longmapsto\ (-z/\tau\,|\,-1/\tau) =: (z'\,|\,\tau') \ \longmapsto\ (-z'/\tau'\,|\,-1/\tau') = (-z\,|\,\tau)\,, \tag{3.9}$$

where the second application uses the updated modular parameter τ'. Due to the sign on the right-hand side of (3.9), we are led to an action of the double cover

$$\mathrm{SL}(2,\mathbb{Z}) = \langle S, T \,|\, S^4 = 1 = (S\,T)^6 \rangle \tag{3.10}$$

of the modular group $\mathrm{PSL}(2,\mathbb{Z}) = \mathrm{SL}(2,\mathbb{Z})/\langle\pm 1\rangle$.[3] See Fig. 1.

It is not directly obvious how the Ruijsenaars–Schneider functions (3.3) with modular parameter $-1/\tau$ are related to their cousins with parameter τ. We will now construct a variant of the modular transformations that allows for a simple relation between the transformed and original Ruijsenaars–Schneider functions.

In the setting of dynamical systems, the coordinates x_j live in the ambient $\mathbb{C}$, so we simultaneously rescale $x_i \longmapsto c\,x_i$ with $c \in \mathbb{C}^\times$. To preserve the canonical Poisson brackets (3.1a) we also rescale all $p_j \longmapsto p_j/c$, while the Poisson brackets (3.1b) require rescaling the parameter ϵ as $\epsilon \longmapsto c\,\epsilon$.[4] It is then natural to rescale the remaining parameter $\eta \longmapsto c\,\eta$ as well, in order to preserve the form of the coefficients (2.4) appearing in (3.3). Altogether, we are thus led to an action of $\mathrm{SL}(2,\mathbb{Z})$ on $M_{\mathbb{C}} \times \mathbb{C}^2 \times \mathbb{H}$ given by

$$(\boldsymbol{x}, \boldsymbol{p}; \eta, \epsilon\,|\,\tau) \longmapsto (c\,\boldsymbol{x}, \boldsymbol{p}/c; c\,\eta, c\,\epsilon\,|\,B\cdot\tau)\,, \qquad B \in \mathrm{SL}(2,\mathbb{Z})\,, \quad c \in \mathbb{C}^\times\,, \tag{3.11}$$

which restricts to a symplectomorphism of the complexified phase space $M_{\mathbb{C}} = T^*\,\mathbb{C}^N$. Explicitly, this action is generated by

$$\begin{aligned} S &: (\boldsymbol{x}, \boldsymbol{p}; \eta, \epsilon\,|\,\tau) \longmapsto \big(-\boldsymbol{x}/\tau, -\tau\,\boldsymbol{p}; -\eta/\tau, -\epsilon/\tau\,\big|\,-1/\tau\big)\,, \\ T &: (\boldsymbol{x}, \boldsymbol{p}; \eta, \epsilon\,|\,\tau) \longmapsto (\boldsymbol{x}, \boldsymbol{p}; \eta, \epsilon\,|\,\tau + 1)\,. \end{aligned} \tag{3.12}$$

[3] One can also see that S now has order 4 by keeping track of both lattice vectors (periods). To preserve the orientation, the order of the two vectors is swapped under the action $S\colon \Lambda_{1,\tau} \longmapsto -\tau\,\Lambda_{1,-1/\tau} = \Lambda_{-\tau,1}$ (cf. Fig. 1). Applying S again thus gives $\Lambda_{-\tau,1} \longmapsto \Lambda_{-1,-\tau} = -\Lambda_{1,\tau}$, so $S^2 = -1$.

[4] Note that this rescaling preserves the multiplicative momenta $\gamma_j = \mathrm{e}^{\epsilon\, p_j}$ at the classical level, and also the operators (2.2) at the quantum level.

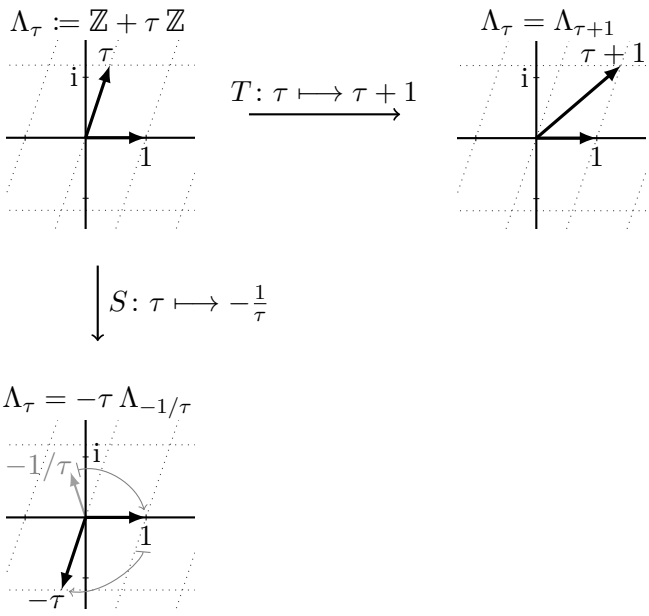


FIGURE 1. The generator T of $\mathrm{SL}(2,\mathbb{Z})$ simply shifts the modular parameter of the lattice $\Lambda_\tau \subset \mathbb{C}$ as $\tau \longmapsto \tau + 1$. The generator S acts by a signed inversion, $\tau \longmapsto -1/\tau$, which requires a simultaneous coordinate rescaling $x \longmapsto -\tau\,x$ in order to fix the lattice, as indicated.

On a theta function $\theta(x\,|\,\tau)$ where x is any linear combination of the x_i and p_j, these two generators yield nothing but the Jacobi imaginary transformation and a simple shift of τ:

$$\theta(-x/\tau\,|\,-1/\tau) = \mathrm{i}\,(-\mathrm{i}\,\tau)^{1/2}\,\mathrm{e}^{\mathrm{i}\,\pi\,x^2/\tau}\,\theta(x\,|\,\tau)\,, \qquad \theta(x\,|\,\tau+1) = \mathrm{e}^{\mathrm{i}\pi/4}\,\theta(x\,|\,\tau)\,. \tag{3.13}$$

Because of the prefactors in (3.13) it is not immediately obvious what the effect is of the modular action on the dynamics of the Ruijsenaars–Schneider system. We push forward the $\mathrm{SL}(2,\mathbb{Z})$-action on $M_{\mathbb{C}} \times \mathbb{C}^2 \times \mathbb{H}$ to functions $f\colon M_{\mathbb{C}} \times \mathbb{C}^2 \times \mathbb{H} \longrightarrow \mathbb{C}$, and consider the D_n^{cl}. The action of T then results in

$$T \cdot D_n^{\mathrm{cl}}(\boldsymbol{x},\boldsymbol{p};\eta,\epsilon\,|\,\tau) = D_n^{\mathrm{cl}}(\boldsymbol{x},\boldsymbol{p};\eta,\epsilon\,|\,\tau+1) = D_n^{\mathrm{cl}}(\boldsymbol{x},\boldsymbol{p};\eta,\epsilon\,|\,\tau)\,, \tag{3.14}$$

as follows from (3.13) and the fact that the coefficients A_I are homogeneous of degree 0 in theta functions. Since the Poisson brackets are preserved under this action, so are the dynamics.

The effect of the S-transformation is more complicated. Observe that the Jacobi imaginary transformation implies

$$\frac{\theta(-(x+\eta)/\tau\,|\,-1/\tau)}{\theta(-x/\tau\,|\,-1/\tau)} = \mathrm{e}^{\mathrm{i}\,\pi\,(\eta^2+2\,\eta\,x)/\tau}\,\frac{\theta(x+\eta\,|\,\tau)}{\theta(x\,|\,\tau)}\,. \tag{3.15}$$

Thus

$$\begin{aligned} S \cdot D_n^{\mathrm{cl}}(\boldsymbol{x},\boldsymbol{p};\eta,\epsilon\,|\,\tau) &= D_n^{\mathrm{cl}}(-\boldsymbol{x}/\tau, -\tau\,\boldsymbol{p}; -\eta/\tau, -\epsilon/\tau\,|\,-1/\tau) \\ &= \sum_{|I|=n} A_I(-\boldsymbol{x}/\tau; -\eta/\tau\,|\,-1/\tau)\,\gamma_I \\ &= \sum_{|I|=n} A_I(\boldsymbol{x};\eta\,|\,\tau)\,\exp\!\left(\frac{\mathrm{i}\,\pi}{\tau}\left(n\,(N-n)\,\eta^2 + 2\,\eta\sum_{j\in I\not\ni k}(x_j - x_k)\right)\right)\gamma_I \qquad (3.16) \\ &= \mathrm{e}^{\mathrm{i}\,\pi\,n\,(N-n)\,\eta^2/\tau}\sum_{|I|=n} A_I(\boldsymbol{x};\eta\,|\,\tau)\,\exp\!\left(\frac{2\pi\,\mathrm{i}\,\eta}{\tau}\left(N\sum_{j\in I} x_j\; - n\,|\boldsymbol{x}|\right) + \epsilon\sum_{j\in I} p_j\right), \end{aligned}$$

where $|\boldsymbol{x}| \coloneqq \sum_{i=1}^{N} x_i$. The exponential factor in the sum can be made independent of I by a suitable shift of the momenta $\boldsymbol{p}$, namely

$$p_j \longmapsto p_j + \frac{2\pi\,\mathrm{i}\,\eta}{\epsilon\,\tau}\sum_{k=1}^{N}(x_k - x_j) = p_j + \frac{2\pi\,\mathrm{i}\,\eta}{\epsilon\,\tau}\,(|\boldsymbol{x}| - N\,x_j)\,. \tag{3.17}$$

This shift does not only preserve the value $\sum_j p_j^\star$ of the total momentum and thus the value of $D_N^{\mathrm{cl}\,\star} = \gamma_1^\star \cdots \gamma_N^\star$, but also the symplectic structure. Indeed, one readily verifies that $\{p_i, p_j\} = 0$ is respected by (3.17).

We therefore modify the action (3.12) to incorporate the momentum shift (3.17):

$$\begin{aligned} S\colon (\boldsymbol{x},\boldsymbol{p};\eta,\epsilon\,|\,\tau) &\longmapsto \Big(-\frac{\boldsymbol{x}}{\tau}, -\tau\,\boldsymbol{p} - \frac{2\pi\,\mathrm{i}\,\eta}{\epsilon}\sum_{j=1}^{N}(|\boldsymbol{x}| - N x_j)\boldsymbol{e}_j\,; -\frac{\eta}{\tau}, -\frac{\epsilon}{\tau}\,\Big|\,-\frac{1}{\tau}\Big)\,,\\ T\colon (\boldsymbol{x},\boldsymbol{p};\eta,\epsilon\,|\,\tau) &\longmapsto (\boldsymbol{x},\boldsymbol{p};\eta,\epsilon\,|\,\tau+1)\,, \end{aligned} \tag{3.18}$$

with $\boldsymbol{e}_j$ the unit vector with a 1 in the jth entry. A direct computation shows that this still defines an action of $\mathrm{SL}(2,\mathbb{Z})$. Indeed, the shifts (3.17) cancelling after every second S-transformation to leave just an overall minus sign for the x_j as well as η; since the theta function is odd, these signs cancel in $A_I(\boldsymbol{x})$. Hence, on the Ruijsenaars functions D_n^{cl}, and more generally on all functions invariant under $(\boldsymbol{x};\eta) \longmapsto (-\boldsymbol{x};-\eta)$, we get an action of $\mathrm{PSL}(2,\mathbb{Z})$ obtained by quotienting out $S^2 = -1$. Setting $c_n(\eta) \coloneqq \mathrm{e}^{\mathrm{i}\,\pi\,n\,(N-n)\,\eta^2/\tau}$ allows us to write

$$S\cdot D_n^{\mathrm{cl}}(\boldsymbol{x},\boldsymbol{p};\eta,\epsilon\,|\,\tau) = c_n(\eta)\,D_n^{\mathrm{cl}}(\boldsymbol{x},\boldsymbol{p};\eta,\epsilon\,|\,\tau)\,. \tag{3.19}$$

3.2. **Equilibrium configurations.** Due to the Poisson commutativity (3.4), each of the integrals of motion (3.3) generates a time flow $\partial/\partial t_n = \{\,\cdot\,, D_n^{\mathrm{cl}}\}$ under which all other D_m^{cl} are preserved. The corresponding velocities and jerks (jolts) are

$$\frac{\partial x_i}{\partial t_n} = \{x_i, D_n^{\mathrm{cl}}\} = +\frac{\partial D_n^{\mathrm{cl}}}{\partial p_i} = \epsilon\sum_{I\colon I\ni i} A_I(\boldsymbol{x})\,\gamma_I \tag{3.20a}$$

and

$$\frac{\partial p_j}{\partial t_n} = \{p_j, D_n^{\mathrm{cl}}\} = -\frac{\partial D_n^{\mathrm{cl}}}{\partial x_j} = -\sum_{I}\partial_{x_j}A_I(\boldsymbol{x})\,\gamma_I\,, \tag{3.20b}$$

where the sums run over all n-element subsets $I \subset \{1,\dots,N\}$, for (3.20a) subject to $i \in I$. We are interested in *classical equilibrium configurations*: points $(\boldsymbol{x}^\star,\boldsymbol{p}^\star) \in M_{\mathbb{C}}$ at which the functions (3.20) are fixed to values

$$\frac{\partial x_i^\star}{\partial t_n} = v_n^\star\,, \quad \frac{\partial p_j^\star}{\partial t_n} = 0\,, \qquad 1 \leqslant n \leqslant N\,, \tag{3.21}$$

for constants $v_n^\star \in \mathbb{C}$ depending on the point $(\boldsymbol{x}^\star,\boldsymbol{p}^\star)$ and the system's parameters η,ϵ,τ, but *not* on $1 \leqslant i \leqslant N$. In particular, by (3.20a), the partial sums

$$v_n^\star = \frac{\partial x_i^\star}{\partial t_n} = \epsilon\sum_{I\colon I\ni i} A_I(\boldsymbol{x}^\star)\,\gamma_I^\star\,, \qquad \gamma_I^\star \coloneqq \prod_{i\in I}\gamma_i^\star\,, \quad \gamma_i^\star \coloneqq \mathrm{e}^{\epsilon\,p_i^\star}\,, \tag{3.22}$$

must be independent of i. For the time flow with $n = 1$ this requires all summands $\epsilon\,A_i(\boldsymbol{x}^\star)\,\gamma_i^\star = v_1^\star$ to coincide. Classical equilibrium configurations are 'frozen' in the sense that they remain stationary in the linearly co-moving frame with (constant) velocity $v_n^\star$. Let us write $M_{\mathbb{C}}^\star \subset M_{\mathbb{C}}$ for the set of all such classical equilibrium configurations in the complexified phase space.

Given parameters $(\eta,\epsilon\,|\,\tau) \in \mathbb{C}^2\times\mathbb{H}$, suppose we have an equilibrium configuration $(\boldsymbol{x}^\star,\boldsymbol{p}^\star)$ for $D_n^{\mathrm{cl}}(\boldsymbol{x},\boldsymbol{p};\eta,\epsilon\,|\,\tau)$ *for which the associated constant velocities obey the symmetry properties*

$$v_{N-n}^\star(\eta) = v_n^\star(-\eta) = v_n^\star(\eta)\,. \tag{3.23}$$

In due course we will verify that this is indeed the case.

Then the dynamical system defined by $B \cdot D_n^{\mathrm{cl}}(\boldsymbol{x}, \boldsymbol{p}; \eta, \epsilon \,|\, \tau)$ also has an equilibrium configuration. To show this, it suffices to do so for the two generators. Since the action of T changes neither the symplectic structure nor the hamiltonians, cf. (3.14), it obviously preserves equilibrium configurations. So we concentrate on S. Let us denote by $\boldsymbol{x}' \coloneqq S \cdot \boldsymbol{x} = -\boldsymbol{x}/\tau$ and $\boldsymbol{p}' \coloneqq S \cdot \boldsymbol{p} = -\tau\, \boldsymbol{p} - 2\pi\,\mathrm{i}\,\eta\,\epsilon^{-1}(|\boldsymbol{x}| - N x_j)$ the coordinates on $M_{\mathbb{C}}$ obtained by applying the action of S. Since $(\boldsymbol{x}, \boldsymbol{p}) \longmapsto (\boldsymbol{x}', \boldsymbol{p}')$ is a canonical transformation, for any two functions $f, g \colon M_{\mathbb{C}} \longrightarrow \mathbb{C}$ the Poisson bracket in the transformed coordinates $\boldsymbol{x}', \boldsymbol{p}'$ can be expressed in terms of that in terms of the original coordinates as

$$\big\{f(\boldsymbol{x}', \boldsymbol{p}'), g(\boldsymbol{x}', \boldsymbol{p}')\big\}' = \big\{f(\boldsymbol{x}'(\boldsymbol{x}, \boldsymbol{p}), \boldsymbol{p}'(\boldsymbol{x}, \boldsymbol{p})), g(\boldsymbol{x}'(\boldsymbol{x}, \boldsymbol{p}), \boldsymbol{p}'(\boldsymbol{x}, \boldsymbol{p}))\big\}\,, \tag{3.24}$$

and likewise for functions on $M_{\mathbb{C}} \times \mathbb{C}^2 \times \mathbb{H}$. For the transformed velocities (3.20a) we compute

$$\begin{aligned}
\frac{\partial x_i'}{\partial t_n} &= \{x_i', D_n^{\mathrm{cl}}(\boldsymbol{x}', \boldsymbol{p}'; \eta', \epsilon' \,|\, \tau')\}' \\
&= \{x_i'(\boldsymbol{x}), D_n^{\mathrm{cl}}(\boldsymbol{x}'(\boldsymbol{x}), \boldsymbol{p}'(\boldsymbol{p}); \eta', \epsilon' \,|\, \tau')\} \quad \text{by (3.24)} \\
&= \frac{c_n(\eta)}{-1/\tau}\, \{x_i, D_n^{\mathrm{cl}}(\boldsymbol{x}, \boldsymbol{p}; \eta, \epsilon \,|\, \tau)\} \qquad \text{by definition of } x_i' \text{ and (3.19)} \\
&= \frac{c_n(\eta)}{-1/\tau}\, \frac{\partial D_n^{\mathrm{cl}}(\boldsymbol{x}, \boldsymbol{p}; \eta, \epsilon \,|\, \tau)}{\partial p_i}\,.
\end{aligned} \tag{3.25}$$

Evaluating the coordinates at the equilibrium $(\boldsymbol{x}^\star, \boldsymbol{p}^\star)$, we obtain new velocities $v_n'(\eta) = \frac{c_n(\eta)}{-1/\tau}\, v_n(\eta)$. Similarly, for the transformed jerks (3.20b) we calculate

$$\begin{aligned}
\frac{\partial p_i'}{\partial t_n} &= \{p_i', D_n^{\mathrm{cl}}(\boldsymbol{x}', \boldsymbol{p}'; \eta', \epsilon' \,|\, \tau')\}' \\
&= \{p_i'(\boldsymbol{x}, \boldsymbol{p}), D_n^{\mathrm{cl}}(\boldsymbol{x}'(\boldsymbol{x}, \boldsymbol{p}), \boldsymbol{p}'(\boldsymbol{x}, \boldsymbol{p}); \eta', \epsilon' \,|\, \tau')\} \\
&= -c_n(\eta)\, \Big\{\tau\, p_i + \frac{2\pi\mathrm{i}\eta}{\epsilon}(|\boldsymbol{x}| - N x_i), D_n^{\mathrm{cl}}(\boldsymbol{x}, \boldsymbol{p}; \eta, \epsilon \,|\, \tau)\Big\} \\
&= -c_n(\eta) \left(\frac{\partial D_n^{\mathrm{cl}}(\boldsymbol{x}, \boldsymbol{p}; \eta, \epsilon \,|\, \tau)}{\partial x_i} + \frac{2\pi\mathrm{i}\eta}{\epsilon} \sum_{j=1}^{N} (1 - \delta_{ji}\, N) \times - \frac{\partial D_n^{\mathrm{cl}}(\boldsymbol{x}, \boldsymbol{p}; \eta, \epsilon \,|\, \tau)}{\partial p_i} \right).
\end{aligned} \tag{3.26}$$

At equilibrium this gives

$$\frac{\partial p_i'}{\partial t_n}\bigg|_{(\boldsymbol{x}^\star, \boldsymbol{p}^\star)} = -c_n(\eta) \left(0 - \frac{2\pi\mathrm{i}\eta}{\epsilon}\, n(N-n)\, (v_n^\star(\eta) - v_{N-n}^\star(-\eta))\right) = 0\,, \tag{3.27}$$

as desired. We conclude that the transformed hamiltonians $S \cdot D_n^{\mathrm{cl}}(\boldsymbol{x}, \boldsymbol{p}; \eta, \epsilon \,|\, \tau)$ have an equilibrium configuration at $S \cdot (\boldsymbol{x}^\star, \boldsymbol{p}^\star)$. Moreover, the new velocities $c_n(\eta)/(-1/\tau)\, v_n(\eta)$ also obey the symmetry property (3.23). Hence it follows that for any $B \in \mathrm{PSL}(2, \mathbb{Z})$ the dynamical system $B \cdot D_n^{\mathrm{cl}}(\boldsymbol{x}, \boldsymbol{p}, \eta, \epsilon \,|\, \tau)$ also has an equilibrium configuration.

3.3. **A modular family of equilibrium configurations.** We would like to find all classical equilibrium configurations. Actually, we will settle for a little less: we will exploit the preceding modular action to generate a whole family of classical equilibrium configurations starting from a simple well-known 'seed' solution $(\boldsymbol{x}^\star, \boldsymbol{p}^\star)$. While it seems to be common lore at least in the 'non-relativistic' limit ($\epsilon \propto \eta/g$, $\eta \to 0$) of the elliptic Calogero–Sutherland system, cf. [Dor99, BT15, BT16], we do not know how to prove that no further (discrete) classical equilibria exist.

Let $\omega \in \mathbb{H}$. Consider the standard classical equilibrium configuration $x_i^{\star\,(1)} = i/N$, $p_j^{\star\,(1)} = 0$, with particles that are fixed equidistantly along the real cycle of the torus. To guarantee a more uniform notation it is prudent to choose an analogous dependence of the parameters on N as

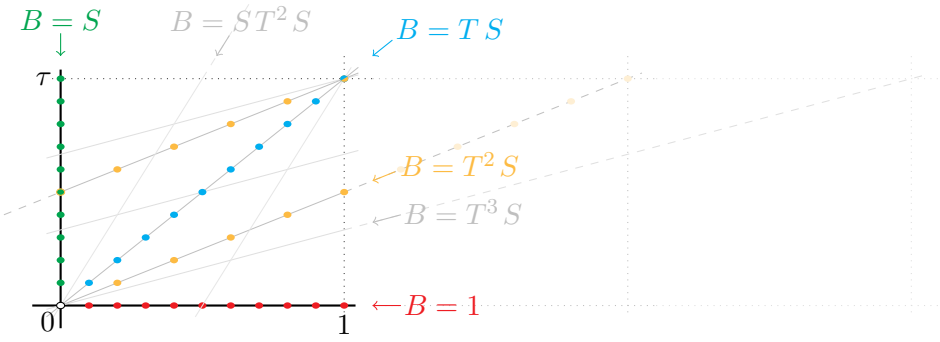


FIGURE 2. Examples of equilibrium positions $x_j^{\star\,(B)}$ for simple $B \in \mathrm{PSL}(2,\mathbb{Z})$, where we take $\tau \in \mathrm{i}\,\mathbb{R}_{>0}$. For freezing, the choices $B = 1$ ($x_j^{\star\,(1)} = j/N$ real) and especially $B = S$ ($x_j^{\star\,(S)} = \tau\, j/N$ imaginary) are particularly important. In general, equilibrium positions lie equispaced on any line $[0, t]$ (for $t \in \Lambda_\tau \setminus \{0\}$) that does not intersect any other point in Λ_τ.

well. Thus we start from

$$\begin{aligned} x_i^{\star\,(1)} &= \frac{i}{N}\,, \qquad p_j^{\star\,(1)} = 0\,, & & \\ \eta^{(1)} &= \frac{\eta}{N}\,, \qquad \epsilon^{(1)} = \frac{\epsilon}{N}\,, & \tau^{(1)} &= \frac{\omega}{N}\,, \end{aligned} \tag{3.28}$$

with the superscript indicating the neutral element $1 \in \mathrm{PSL}(2,\mathbb{Z})$. It is straightforward to check that the associated velocities satisfy the symmetry property (3.23). By applying the S-transformation we obtain the new solution

$$\begin{aligned} x_i^{\star\,(S)} &= -\frac{i}{\omega}\,, \qquad p_j^{\star\,(S)} = (N+1-2j)\,\frac{\pi\,\mathrm{i}\,\eta}{\omega}\,, & & \\ \eta^{(S)} &= -\frac{\eta}{\omega}\,, \qquad \epsilon^{(S)} = -\frac{\epsilon}{\omega}\,, & \tau^{(S)} &= -\frac{N}{\omega}\,. \end{aligned} \tag{3.29}$$

In this equilibrium configuration the particles are equally spaced along the complex (e.g. imaginary) cycle of the torus, see Fig. 2.

Applying the T-transformation to any solution only shifts the lattice parameter, which, as discussed above, does not change the dynamics. Nevertheless, any subsequent application of the S-transformation propagates the shifted lattice parameter to all other parameters. In particular, this tilts the line in the complex plane on which the equilibrium positions $x_i^\star$ lie. The associated spin-chain hamiltonians that we will construct below differ accordingly.

More generally, we obtain a new solution from the 'seed' configuration (3.28) for most products of S, T and their inverses; the only duplicate solutions arise from elements related as $B = B'\,T^n$ for some n, or due to the relations of $\mathrm{PSL}(2,\mathbb{Z})$. Hence the orbit under the modular group produces a whole family of equilibrium configurations,

$$\mathrm{PSL}(2,\mathbb{Z}) \cdot (\boldsymbol{x}^{\star\,(1)}, \boldsymbol{p}^{\star\,(1)}\,; \eta^{(1)}, \epsilon^{(1)} \,|\, \tau^{(1)}) \subseteq M_{\mathbb{C}}^\star\,. \tag{3.30}$$

Note that the equilibrium *positions* $\boldsymbol{x}^\star$ are independent of the deformation parameter η, i.e. the same as in the non-relativistic (Calogero–Sutherland–Moser) limit. In contrast, the associated momenta $\boldsymbol{p}^\star$ may depend on η due to the canonical transformation (3.17) needed to preserve the equilibrium condition. Notably, since the non-relativistic limit requires setting $\epsilon = \eta/g$, the shift seems to be required in the elliptic Calogero–Sutherland–Moser case as well.

In conclusion, we have shown how the extended modular action (3.12) on $(\boldsymbol{x}, \boldsymbol{p}; \eta, \epsilon \,|\, \tau)$ relates different members of the family of Ruijsenaars systems parametrised by $(\eta, \epsilon \,|\, \tau) \in \mathbb{C}^2 \times \mathbb{H}$. More precisely, we are free to rescale the coordinates ($\boldsymbol{x}, \boldsymbol{p}$ classically; $\boldsymbol{x}$ quantum-mechanically) and other parameters in such a way that the resulting system is defined on the same lattice. Hence physically *in*equivalent Ruijsenaars systems are parametrised by the quotient $(\mathbb{C}^2 \times$

$\mathbb{H})/\mathrm{PSL}(2,\mathbb{Z})$. Put differently, for any fixed $\tau \in \mathbb{H}/\mathrm{PSL}(2,\mathbb{Z})$ in the fundamental domain, the modular action generates a family of physically *equivalent* Ruijsenaars systems. At the classical level, this action allows one to construct a modular family of (discrete) equilibrium configurations.

4. Freezing

A suitable expansion of our quantum many-body system with spins yields a family of commuting operators that act (nontrivially) on spins only via a method called 'freezing' [Pol93]. This is a two-step process,

$$\begin{array}{c}\text{quantum}\\ \text{many-body system}\\ \text{with spins}\end{array} \longmapsto \begin{array}{c}\text{hybrid}\\ \text{many-body system}\end{array} \longmapsto \begin{array}{c}\text{long-range}\\ \text{(quantum) spin chain}\,.\end{array} \tag{4.1}$$

Starting from a (quantum) spin-Ruijsenaars system, one takes a limit in which the potential energy dominates the kinetic energy. It can be viewed as a strong-coupling or partial (semi)classical limit. In this limit, the system decouples into a 'hybrid system', consisting of a classical Ruijsenaars–Schneider system that governs the dynamics (times the identity on spins) plus a quantum part with spin interactions (but no difference/differential operators). The dynamics completely disappears at any classical equilibrium configuration, yielding a system of quantum-mechanical spins at fixed positions: this is the spin chain.

The aim of this section is to make this process precise. We follow [Ugl95, LPS22], supplemented by further insights from [MZ23a] for the elliptic case, and [Cha24, MV24, LRS24] for the setting of deformation quantisation.

4.1. **Formalism: deformation quantisation.** In order to relate the quantum Ruijsenaars system (2.3) to its classical counterpart (3.3), we need to have a way to take the classical limit of Ruijsenaars systems. The appropriate setting is provided by deformation quantisation [BFF+78a, BFF+78b], cf. [Kon03] and [Eti07]. For the setting with spins we follow [Cha24, MV24, LRS24].

To warm up, we start with the scalar case. We interpret the difference operators (2.3) as elements $D_n \in \mathcal{A}_\hbar$ of the associative algebra

$$\mathcal{A}_\hbar \coloneqq \mathrm{Fun}(\boldsymbol{x})[\Gamma_1^{\pm 1},\dots,\Gamma_N^{\pm 1}] \otimes \mathbb{C}[\![\hbar]\!]\,, \tag{4.2}$$

consisting of Laurent polynomials in the difference operators Γ_j with coefficients that are meromorphic functions in $x_1,\dots,x_N$ times a formal power series in $\hbar$.[5,6] From the formal power series

$$(\Gamma_j f)(\boldsymbol{x}) = f(x_1,\dots,x_j - \mathrm{i}\,\epsilon\,\hbar,\dots,x_N) = \sum_{k\geqslant 0} \frac{1}{k!}(-\mathrm{i}\,\epsilon)^k\,\partial_j^k f(\boldsymbol{x})\,\hbar^k \in \mathcal{A}_\hbar \tag{4.3}$$

one obtains the nontrivial commutation relations

$$-\mathrm{i}\,\hbar^{-1}\,[f(\boldsymbol{x}),\Gamma_j] = \epsilon\,\partial_j f(\boldsymbol{x}) + O(\hbar) \tag{4.4a}$$

for $\mathcal{A}_\hbar$. In particular, taking $f(\boldsymbol{x}) = x_i$ a coordinate function gives

$$[x_i,\Gamma_j] = \mathrm{i}\,\hbar\,\epsilon\,\delta_{ij}\,\Gamma_j\,, \tag{4.4b}$$

which is an intermediate version of the Heisenberg commutation relations $[x_i,\hat{p}_j] = \mathrm{i}\,\hbar\,\delta_{ij}$ (additive notation) and the Weyl-algebra relations $\mathrm{e}^{\mathrm{i}\,x_i}\,\Gamma_j = \mathrm{e}^{\epsilon\,\hbar}\,\Gamma_j\,\mathrm{e}^{\mathrm{i}\,x_i}$ (fully multiplicative notation).

[5] To be precise, by $\mathrm{Fun}(\boldsymbol{x})[\Gamma_1^{\pm 1},\dots,\Gamma_N^{\pm 1}]$ we mean the vector space $\mathrm{Fun}(\boldsymbol{x}) \otimes \mathbb{C}[\Gamma_1^{\pm 1},\dots,\Gamma_N^{\pm 1}]$ with product induced by $f(\boldsymbol{x})\,\Gamma_i\,g(\boldsymbol{x})\,\Gamma_j = f(\boldsymbol{x})\,g(\boldsymbol{x} - \mathrm{i}\,\hbar\,\epsilon\,e_i)\,\Gamma_i\,\Gamma_j$, cf. (4.3)–(4.4). This is extended $\mathbb{C}[\![\hbar]\!]$-bilinearly to $\mathcal{A}_\hbar$.

[6] The difference operators can (formally) be expressed as $\Gamma_j = \exp(\epsilon\,\hat{p}_j)$. While quantum-mechanically $\hat{p}_j = -\mathrm{i}\,\hbar\,\partial_{x_j}$, we emphasise that the latter $\hbar$ is viewed as a part of the operator $\hat{p}_j$ rather than $\mathbb{C}[\![\hbar]\!]$. However, when $\hat{p}_j$ acts on a function f, the resulting $\partial_j f(\boldsymbol{x}) \in \mathrm{Fun}(\boldsymbol{x})$ has coefficient $-\mathrm{i}\,\hbar \in \mathbb{C}[\![\hbar]\!]$, see e.g. (4.4b).

The commutative algebra $\mathcal{A}_0 = \mathrm{Fun}(\boldsymbol{x})[\gamma_1^{\pm 1}, \dots, \gamma_N^{\pm 1}]$ from §3, with $\gamma_j = \mathrm{e}^{\epsilon\, p_j}$, is the classical limit of $\mathcal{A}_\hbar$ in the following precise sense. Since the commutator (4.4b) is of order $\hbar$, the quotient $\mathcal{A}_\hbar/\hbar\mathcal{A}_\hbar$ is also a commutative algebra. It is isomorphic to $\mathcal{A}_0$ via the $\mathbb{C}$-linear map

$$\mathcal{A}_\hbar/\hbar\mathcal{A}_\hbar \xrightarrow{\sim} \mathcal{A}_0\,, \qquad f(\boldsymbol{x}) \longmapsto f(\boldsymbol{x})\,, \quad \Gamma_i \longmapsto \gamma_i \quad (\text{i.e. } \hat{p}_i \longmapsto p_i)\,. \tag{4.5}$$

This exhibits $\mathcal{A}_\hbar$ as a (flat, formal) deformation of $\mathcal{A}_0$. Now choose an identification of $\mathbb{C}[\![\hbar]\!]$-modules

$$c_\hbar \colon \mathcal{A}_\hbar \xrightarrow{\sim} \mathcal{A}_0[\![\hbar]\!] \tag{4.6}$$

that reduces mod $\hbar$ to (4.5). Concretely, this isomorphism boils down to a choice of a (normal) ordering of the quantum operators. In turn, $c_\hbar$ allows one to transport the (non-commutative) product on $\mathcal{A}_\hbar$ to $\mathcal{A}_0[\![\hbar]\!]$, equipping the latter with the (associative, $\mathbb{C}[\![\hbar]\!]$-bilinear) Moyal star-product

$$a \star b \coloneqq c_\hbar\big(c_\hbar^{-1}(a)\, c_\hbar^{-1}(b)\big) = \sum_{k\geqslant 0} m_k(a,b)\,\hbar^k\,, \qquad m_0(a,b) = a\,b\,, \tag{4.7}$$

Each coefficient function m_k, which in general are non-zero due to (4.3), in the formal power series (4.7) is itself a product on $\mathcal{A}_0$, with m_0 the original (commutative) product. At the next order in $\hbar$, compatibility with (4.4) requires the bracket on $\mathcal{A}_0$ given by

$$\{f,g\} \coloneqq -\mathrm{i}\,\big(m_1(f,g) - m_1(g,f)\big) = -\mathrm{i}\,\hbar^{-1}\, c_\hbar\big(\big[c_\hbar^{-1}(f), c_\hbar^{-1}(g)\big]\big) \mod \hbar \tag{4.8}$$

to obey the Poisson relations (3.1b) of the (scalar) Ruijsenaars–Schneider system. Note that (4.8) is independent of the choice of quantisation scheme $c_\hbar^{-1}$ since nonvanishing commutators (4.4) are already of order $\hbar$. The *classical limit* corresponds to the composition $\mathcal{A}_\hbar \twoheadrightarrow \mathcal{A}_\hbar/\hbar\mathcal{A}_\hbar \xrightarrow{\sim} \mathcal{A}_0$, which amounts to the map

$$c_0 = c_\hbar\big|_{\hbar=0} \colon \mathcal{A}_\hbar \twoheadrightarrow \mathcal{A}_0\,, \qquad f(\boldsymbol{x}) \longmapsto f(\boldsymbol{x})\,, \quad \Gamma_i \longmapsto \gamma_i\,, \quad \hbar \longmapsto 0\,. \tag{4.9}$$

Equipping the quantum space $\mathcal{A}_\hbar$ with the rescaled commutator $[\,\cdot\,,\cdot\,]_\hbar \coloneqq -\mathrm{i}\,\hbar^{-1}\,[\,\cdot\,,\cdot\,]$ would turn (4.9) into a Poisson-algebra homomorphism, but we prefer to work with the ordinary commutator to keep all factors of $\hbar$ explicit.

We are now ready to discuss the spin case. Let $r \geqslant 2$ and upgrade $\mathcal{A}_\hbar$ to its matrix-valued variant [7]

$$\widetilde{\mathcal{A}}_\hbar \coloneqq \mathrm{Fun}(\boldsymbol{x}) \otimes \mathrm{Mat}(r^N, \mathbb{C}) \otimes \mathbb{C}[\Gamma_1^{\pm 1}, \dots, \Gamma_N^{\pm 1}] \otimes \mathbb{C}[\![\hbar]\!]\,, \tag{4.10}$$

which again is non-commutative and thus has a nontrivial Poisson bracket given by the commutator. However, its classical version

$$\widetilde{\mathcal{A}}_0 \coloneqq \mathrm{Fun}(\boldsymbol{x}) \otimes \mathrm{Mat}(r^N, \mathbb{C})[\gamma_1^{\pm 1}, \dots, \gamma_N^{\pm 1}]\,, \tag{4.11}$$

is no longer a commutative algebra, and hence unsuitable as the phase space of an (ordinary) Hamiltonian system. Proceeding as before, we extend (4.9) to the surjective algebra homomorphism

$$\tilde{c}_0 \colon \widetilde{\mathcal{A}}_\hbar \twoheadrightarrow \widetilde{\mathcal{A}}_0\,, \qquad f(\boldsymbol{x}) \longmapsto f(\boldsymbol{x})\,, \quad M \longmapsto M\,, \qquad \Gamma_i \longmapsto \gamma_i\,, \quad \hbar \longmapsto 0\,, \tag{4.12}$$

with $M \in \mathrm{Mat}(r^N, \mathbb{C})$. Observe that this only a partial classical limit: the dependence on $\boldsymbol{x}$ and $\boldsymbol{p}$ becomes classical, but the spin part M remains fully quantum mechanical. Like before, its kernel is $\hbar\,\widetilde{\mathcal{A}}_\hbar$, but this time the corresponding map $\widetilde{\mathcal{A}}_\hbar/\hbar\,\widetilde{\mathcal{A}}_\hbar \xrightarrow{\sim} \widetilde{\mathcal{A}}_0$ is an isomorphism of *non*-commutative algebras. For our purposes, a suitable candidate for the classical phase space is the centre $Z(\widetilde{\mathcal{A}}_0)$, which can be identified with $\mathcal{A}_0 \cong Z(\widetilde{\mathcal{A}}_0)$ through the inclusion $a \longmapsto a\,\mathrm{id}$. It acquires a Poisson bracket as in (4.8): for any $a, b \in Z(\widetilde{\mathcal{A}}_0)$ we set

$$\{a,b\} \coloneqq -\mathrm{i}\,\hbar^{-1}\,\tilde{c}_\hbar\big(\big[\tilde{c}_\hbar^{-1}(a), \tilde{c}_\hbar^{-1}(b)\big]\big) \mod \hbar\,, \tag{4.13}$$

[7] One may replace $\mathrm{Mat}(r^N, \mathbb{C}) \cong \mathrm{End}(V^{\otimes N})$ for $V \cong \mathbb{C}^r$ by any representation U of S_N, or, if one prefers to work abstractly rather than in a representation, the group algebra $\mathbb{C}S_N$, cf. e.g. [Cha24].

retrieving the Poisson structure (4.8) (times the identity matrix). When $a \in Z(\widetilde{\mathcal{A}}_0)$ but $\tilde{b} \in \widetilde{\mathcal{A}}_0$, then a does not see the matrix structure of $\tilde{b}$, so the commutator $c_\hbar([\tilde{c}_\hbar^{-1}(a), \tilde{c}_\hbar^{-1}(\tilde{b})])$ remains of order $\hbar$. Hence the formula (4.13) can be used to define an action of $Z(\widetilde{\mathcal{A}}_0)$ on $\widetilde{\mathcal{A}}_0$ by derivations, given by $a \cdot \tilde{b} \coloneqq \{a, \tilde{b}\}$ for $a \in \mathcal{A}_0 \cong Z(\widetilde{\mathcal{A}}_0)$ and $\tilde{b} \in \widetilde{\mathcal{A}}_0$. This plays well with the Poisson structure (4.13) of $Z(\widetilde{\mathcal{A}}_0)$, i.e. $\{a, a'\} \cdot \tilde{b} = a \cdot (a' \cdot \tilde{b}) - a' \cdot (a \cdot \tilde{b})$, thus turning $\widetilde{\mathcal{A}}_0$ into a $Z(\widetilde{\mathcal{A}}_0)$-Poisson module.[8] Such an algebra, which is a (simple) Poisson module of finite rank over its own centre, was studied in [MV24, LRS24]. The authors of [LRS24] called it a *Poisson–Azuyama algebra*, and the dynamical system that it defines a *hybrid* dynamical system.

4.2. **Step one: hybrid expansion.**

4.2.1. *Quantum spinless case.* We first consider the quantum model without spins. Recall that

$$[D_n, D_m] = 0\,, \qquad D_n = \sum_{\substack{I \subset \{1,\dots,N\} \\ |I|=n}} A_I(\boldsymbol{x};\eta)\,\Gamma_I\,, \quad \Gamma_I = \prod_{i\in I} \Gamma_i\,. \tag{4.14}$$

As elements in $\mathcal{A}_\hbar$ we can directly compute

$$c_0(D_n) = D_n^{\mathrm{cl}} = \sum_{\substack{I \subset \{1,\dots,N\} \\ |I|=n}} A_I(\boldsymbol{x};\eta)\,\gamma_I\,, \quad \gamma_I = \prod_{i\in I} \mathrm{e}^{\epsilon\, p_i}\,, \tag{4.15}$$

which are the Ruijsenaars–Schneider functions (3.3). In order to analyse the lowest orders in $\hbar$ of the commutator we first expand the product $D_n\,D_m$ in $\hbar$. For brevity we suppress the details of the summation ranges $I, J \subseteq \{1,\dots,N\}$ with $|I| = n$ and $|J| = m$ in the following, as well as the argument η of the coefficients. Then

$$\begin{aligned} D_n\,D_m &= \sum_{I,J} A_J(\boldsymbol{x})\,\Gamma_I\; A_J(\boldsymbol{x})\,\Gamma_J \\ &= \sum_{I,J} A_I(\boldsymbol{x})\,A_J\Big(\boldsymbol{x} - \mathrm{i}\,\epsilon\,\hbar \sum_{i\in I} e_i\Big)\,\Gamma_I\,\Gamma_J \\ &= \sum_{I,J} A_I(\boldsymbol{x})\left(A_J(\boldsymbol{x}) - \mathrm{i}\,\epsilon\,\hbar \sum_{i\in I} \partial_i A_J(\boldsymbol{x}) - \frac{\epsilon^2\,\hbar^2}{2} \sum_{i,i'\in I} \partial_i\,\partial_{i'} A_J(\boldsymbol{x}) + O(\hbar^3)\right)\Gamma_I\,\Gamma_J\,. \end{aligned} \tag{4.16}$$

The sum at order $\hbar^2$ is $\sum_{i,i'\in I} \partial_i\,\partial_{i'} = \big(\sum_{i\in I} \partial_i\big)^2$ applied to $A_J(\boldsymbol{x})$.

Let us further suppress the arguments $\boldsymbol{x}$ of the coefficients A_I and A_J. Since these functions commute with each other, and the difference operators Γ_I, Γ_J do so too, we get

$$\begin{aligned} 0 = [D_n, D_m] = &-\mathrm{i}\,\epsilon\,\hbar \sum_{I,J}\Big(A_I \sum_{i\in I} \partial_i A_J - A_J \sum_{j\in J} \partial_j A_I\Big)\,\Gamma_I\,\Gamma_J \\ &- \frac{\epsilon^2\,\hbar^2}{2} \sum_{I,J}\Big(A_I \sum_{i,i'\in I} \partial_i\,\partial_{i'} A_J - A_J \sum_{j,j'\in J} \partial_j\,\partial_{j'} A_I\Big)\,\Gamma_I\,\Gamma_J \\ &+ O(\hbar^3)\,. \end{aligned} \tag{4.17}$$

[8] As shown in [MV24], in general the natural candidate for the generalisation of the classical phase space to the matrix-valued (spin)-setting is not $Z(\widetilde{\mathcal{A}}_0)$, but rather the slightly bigger $\Pi(\widetilde{\mathcal{A}}_0) \coloneqq Z(\widetilde{\mathcal{A}}_0) \times \big(\widetilde{\mathcal{A}}_0 / Z(\widetilde{\mathcal{A}}_0)\big)$, which inherits a Poisson structure and compatible commutative product from $\widetilde{\mathcal{A}}_\hbar$ by considering it as the quotient $(Z(\widetilde{\mathcal{A}}_0) + \hbar\,\widetilde{\mathcal{A}}_0)/\hbar\,\big(Z(\widetilde{\mathcal{A}}_0) + \hbar\,\widetilde{\mathcal{A}}_0\big)$. It moreover defines a $\Pi(\widetilde{\mathcal{A}}_0)$-Poisson module structure on $\widetilde{\mathcal{A}}_0$. In our case, the product on $\Pi(\widetilde{\mathcal{A}}_0)$ simplifies due to the fact that for all $a, b \in Z(\widetilde{\mathcal{A}}_0)$ the product $a\,b \in Z(\widetilde{\mathcal{A}}_0)$, receiving no contributions from $m_2(a,b)$ as that is central. In the language of [LRS24], $\widetilde{\mathcal{A}}_0$ is *flat*. The $\Pi(\widetilde{\mathcal{A}}_0)$-Poisson module can then be restricted to a $Z(\widetilde{\mathcal{A}}_0)$-Poisson module, resulting in the structure introduced above. Note that the dynamical system defined by the former is fully classical – including spins –, which is not the setting we need in order to obtain (quantum) spin-chain hamiltonians.

From the terms linear in $\hbar$,

$$-\mathrm{i}\,\hbar^{-1}\,[D_n, D_m] = -\epsilon \sum_{I,\,J} \Bigg(A_I \sum_{i\in I} \partial_i A_J - A_J \sum_{j\in J} \partial_j A_I \Bigg)\, \Gamma_I\, \Gamma_J \;+ O(\hbar)\,, \tag{4.18}$$

we get

$$c_0\big(-\mathrm{i}\,\hbar^{-1}\,[D_n, D_m]\big) = -\epsilon \sum_{I,\,J} \Bigg(A_I \sum_{i\in I} \partial_i A_J - A_J \sum_{j\in J} \partial_j A_I \Bigg)\, \gamma_I\, \gamma_J\,. \tag{4.19}$$

This matches direct computation of the Poisson bracket

$$\begin{aligned} \{D_n^{\mathrm{cl}}, D_m^{\mathrm{cl}}\} &= \sum_{i,j=1}^{N} \left(\frac{\partial D_n^{\mathrm{cl}}}{\partial x_i}\, \frac{\partial D_m^{\mathrm{cl}}}{\partial \gamma_j}\, \{x_i, \gamma_j\} + \frac{\partial D_n^{\mathrm{cl}}}{\partial \gamma_i}\, \frac{\partial D_m^{\mathrm{cl}}}{\partial x_j}\, \{\gamma_i, x_j\} \right) \\ &= \sum_{I,\,J} \Bigg(\sum_{i=1}^{N} \partial_i A_I\, \gamma_I \sum_{j\in J} A_J\, \gamma_{J\setminus\{j\}} \times \epsilon\, \gamma_j\, \delta_{ij} + \sum_{i\in I} A_I\, \gamma_{I\setminus\{i\}} \sum_{j=1}^{N} \partial_j A_J\, \gamma_J \times -\epsilon\, \gamma_i\, \delta_{ji} \Bigg)\,, \end{aligned} \tag{4.20}$$

where the second equality uses the Poisson bracket (3.1b). Thus we verified that

$$c_0\big(-\mathrm{i}\,\hbar^{-1}\,[D_n, D_m]\big) = \{D_n^{\mathrm{cl}}, D_m^{\mathrm{cl}}\} = 0\,, \tag{4.21}$$

in line with (4.8).

More generally, the commutator (4.17) vanishes to all orders in $\hbar$. At higher orders this yields non-trivial identities for the coefficients $A_I(\boldsymbol{x};\eta)$ that will be helpful in the following.

4.2.2. *Quantum case with spin.* Now consider a quantum-mechanical elliptic Ruijsenaars system with spins, described by some hierarchy of matrix-valued difference operators that commute. Let us consider the expansion of these operators in the *explicitly* appearing $\hbar$,

$$\widetilde{D}_{\pm n} = \sum_I \widetilde{A}_I(\boldsymbol{x})\, \Gamma_I^{\pm 1} = \sum_{d\geqslant 0} \widetilde{D}_{\pm n}^{(d)}\, \hbar^d\,, \quad \widetilde{D}_{\pm n}^{(d)} \coloneqq \sum_I \widetilde{A}_{\pm I}^{(d)}(\boldsymbol{x})\, \Gamma_I^{\pm 1}\,, \tag{4.22}$$

where we use tildes to indicate matrix valuedness. For the present discussion, the explicit form of the coefficients following from §2 is not relevant, and only distracts from the argument. We will compute some examples of the $\widetilde{D}_{\pm n}^{(1)}$ in §4.4. For now, we only borrow from §2 the assumption that the matrix-valued coefficients do not contain any further difference operators, i.e. $\widetilde{A}_I(\boldsymbol{x}) \in \mathrm{Fun}(\boldsymbol{x}) \otimes \mathrm{Mat}(r^N, \mathbb{C})$, and are such that

$$\widetilde{A}_I(\boldsymbol{x}) = A_I(\boldsymbol{x})\, \mathrm{id} + O(\hbar)\,, \quad \text{i.e.} \quad \widetilde{A}_I^{(0)}(\boldsymbol{x}) = A_I(\boldsymbol{x})\,. \tag{4.23}$$

Then $\widetilde{D}_n^{(0)} = D_n\, \mathrm{id}$, which is still quantum mechanical through the implicit $\hbar$ in the difference operators Γ_i. The replacement of Γ_I by γ_I will be achieved by applying the map $\widetilde{c}_0$. Although $\widetilde{c}_0(\widetilde{D}_n^{(0)}) = D_n^{\mathrm{cl}}\, \mathrm{id}$ could be viewed as a classical quantity, for $n \neq N$ the $\widetilde{D}_{\pm n}^{(d)}$ with $d \geqslant 1$ will contain deformed spin permutations (quantum R-matrices) in addition to the difference operators Γ_i. Even after applying the (partial) classical limit $\widetilde{c}_0$, they remain fully quantum mechanical in terms of spins. To emphasise that we do *not* expand the implicit $\hbar$ in Γ_I (cf. footnote 6), we call (4.22) a *hybrid* expansion.

Pick two such operators, $\widetilde{D}_n = \sum_I \widetilde{A}_I(\boldsymbol{x})\, \Gamma_I$ and $\widetilde{D}_m = \sum_J \widetilde{A}_J(\boldsymbol{x})\, \Gamma_J$, where, as before, sums over n- and m- element subsets I and J, respectively, are understood. From the hybrid

expansion of their commutator we obtain the following refinement of (4.17):

$$
\begin{aligned}
0 = [\widetilde{D}_n, \widetilde{D}_m] &= \sum_{I,J} \left(\widetilde{A}_I(\boldsymbol{x})\, \widetilde{A}_J\Big(\boldsymbol{x} - \mathrm{i}\,\epsilon\,\hbar \sum_{i\in I} e_i\Big) - \widetilde{A}_J(\boldsymbol{x})\, \widetilde{A}_I\Big(\boldsymbol{x} - \mathrm{i}\,\epsilon\,\hbar \sum_{j\in J} e_j\Big)\right) \Gamma_I\, \Gamma_J \\
&= \sum_{I,J} \big[\widetilde{A}_I, \widetilde{A}_J\big]\, \Gamma_I\, \Gamma_J \;\; - \mathrm{i}\,\epsilon\,\hbar \sum_{I,J} \left(\widetilde{A}_I \sum_{i\in I} \partial_i\, \widetilde{A}_J - \widetilde{A}_J \sum_{j\in J} \partial_j\, \widetilde{A}_I \right) \Gamma_I\, \Gamma_J \\
&\quad - \frac{\epsilon^2\,\hbar^2}{2} \sum_{I,J} \left(\widetilde{A}_I \sum_{i,i'\in I} \partial_i\, \partial_{i'}\, \widetilde{A}_J - \widetilde{A}_J \sum_{j,j'\in J} \partial_j\, \partial_{j'}\, \widetilde{A}_I \right) \Gamma_I\, \Gamma_J \;\; + O(\hbar^3)\,.
\end{aligned}
\tag{4.24}
$$

If (4.23) then $\widetilde{c}_0(\widetilde{D}_n) = D_n\,\mathrm{id} \in Z(\widetilde{\mathcal{A}}_0)$. Moreover, the terms of order $\hbar^0$ in (4.24) vanish, and the part of (4.24) linear in $\hbar$ becomes precisely (4.20) (times id), which implies that

$$
\widetilde{c}_0\big(-\mathrm{i}\,\hbar^{-1}[\widetilde{D}_n, \widetilde{D}_m]\big) = \big\{\widetilde{c}_0(D_n), \widetilde{c}_0(D_m)\big\}\,\mathrm{id} = 0\,.
\tag{4.25}
$$

We stress that the right-hand side features the Poisson bracket of the *scalar* Ruijsenaars hamiltonians, times the identity on spins, as in (4.13). In fact, by virtue of (4.17) the part linear in $\hbar$ of (4.24) with (4.23) vanishes even *before* applying $\widetilde{c}_0$. Hence the first non-trivial terms occur at $\hbar^2$. They are given by

$$
\begin{aligned}
0 = h^{-2}\,\big[\widetilde{D}_n, \widetilde{D}_m\big] &= \sum_{I,J} \Big(\big[\widetilde{A}^{(2)}_I, \widetilde{A}^{(0)}_J\big] + \big[\widetilde{A}^{(1)}_I, \widetilde{A}^{(1)}_J\big] + \big[\widetilde{A}^{(0)}_I, \widetilde{A}^{(2)}_J\big] \Big)\, \Gamma_I\, \Gamma_J \\
&\quad - \mathrm{i}\,\epsilon \sum_{I,J} \left(\widetilde{A}^{(0)}_I \sum_{i\in I} \partial_i\, \widetilde{A}^{(1)}_J - \widetilde{A}^{(1)}_J \sum_{j\in J} \partial_j\, \widetilde{A}^{(0)}_I \right) \Gamma_I\, \Gamma_J \\
&\quad - \mathrm{i}\,\epsilon \sum_{I,J} \left(\widetilde{A}^{(1)}_I \sum_{i\in I} \partial_i\, \widetilde{A}^{(0)}_J - \widetilde{A}^{(0)}_J \sum_{j\in J} \partial_j\, \widetilde{A}^{(1)}_I \right) \Gamma_I\, \Gamma_J \\
&\quad - \frac{1}{2}\,\epsilon^2 \sum_{I,J} \left(\widetilde{A}^{(0)}_I \sum_{i,i'\in I} \partial_i\, \partial_{i'}\, \widetilde{A}^{(0)}_J - \widetilde{A}^{(0)}_J \sum_{j,j'\in J} \partial_j\, \partial_{j'}\, \widetilde{A}^{(0)}_I \right) \Gamma_I\, \Gamma_J \\
&\quad + O(\hbar)\,.
\end{aligned}
\tag{4.26}
$$

The first and third matrix commutator in the first line of the right-hand side vanish due to (4.23). The terms linear in ϵ on the next two lines come from to the first-order bracket m_1, and will give rise to the Poisson bracket (4.13) upon applying $\widetilde{c}_0$. The coefficient of ϵ^2 on the fourth line, which is the contribution from the second-order bracket

$$
\big\{\widetilde{c}_0(\widetilde{D}_n), \widetilde{c}_0(\widetilde{D}_m)\big\}_2 = m_2\big(\widetilde{c}_0(\widetilde{D}_n), \widetilde{c}_0(\widetilde{D}_m)\big) - m_2\big(\widetilde{c}_0(\widetilde{D}_m), \widetilde{c}_0(\widetilde{D}_n)\big)\,,
\tag{4.27}
$$

vanishes thanks to the ϵ^2-term of the identity (4.17) from the spinless case. This is an example of the fact that $\widetilde{\mathcal{A}}_0$ is flat in the sense of [LRS24] (cf. footnote 8 on p. 15) in a strong sense: (4.27) does not just lie in $Z(\widetilde{A}_0)$, but it vanishes. Therefore, comparing with (4.20) we conclude that, after taking the classical limit,

$$
\begin{aligned}
0 &= \widetilde{c}_0\big(\hbar^{-2}\,[\widetilde{D}_n, \widetilde{D}_m]\big) \\
&= \big\{\widetilde{c}_0(\widetilde{D}^{(0)}_n), \widetilde{c}_0(\widetilde{D}^{(1)}_m)\big\} + \big\{\widetilde{c}_0(\widetilde{D}^{(1)}_n), \widetilde{c}_0(\widetilde{D}^{(0)}_m)\big\} + \big[\widetilde{c}_0(\widetilde{D}^{(1)}_n), c_0(\widetilde{D}^{(1)}_m)\big]\,. \\
&= \big\{D^{\mathrm{cl}}_n, \widetilde{c}_0(\widetilde{D}^{(1)}_m)\big\} + \big\{\widetilde{c}_0(\widetilde{D}^{(1)}_n), D^{\mathrm{cl}}_m\big\} + \big[\widetilde{c}_0(\widetilde{D}^{(1)}_n), \widetilde{c}_0(\widetilde{D}^{(1)}_m)\big]\,.
\end{aligned}
\tag{4.28}
$$

In particular, for $m = N$, noting $\widetilde{D}_N = D_N\,\mathrm{id}$ (so that $\widetilde{D}^{(d)}_N = 0$ for $d > 1$), we obtain the non-trivial identity

$$
0 = \big\{\widetilde{c}_0(\widetilde{D}^{(1)}_n), \widetilde{c}_0(D_N)\big\} = \widetilde{c}_0(D_N) \sum_{i=1}^{N} \frac{\partial\, \widetilde{c}_0(\widetilde{D}^{(1)}_n)}{\partial x_j} \quad \Longrightarrow \quad \sum_{i=1}^{N} \frac{\partial\, \widetilde{c}_0(\widetilde{D}^{(1)}_n)}{\partial x_j} = 0,
\tag{4.29}
$$

which will come in handy momentarily.

4.3. **Step two: evaluation.** With these hybrid expansions in place, we now are ready to perform the second part of freezing, by evaluating (4.28) at one of the classical equilibria we obtained in §3.3. We will want to include the case with spins. To this end, note that the dynamical parameter $\vec{a}$ is readily included in our treatment of the equilibria since it occurs in the same way as the x_i and η. Thus, $\vec{a}^{\star(1)} = \vec{a}/N$ for (3.28) and $\vec{a}^{\star(S)} = -\vec{a}/\omega$ for (3.29). Let us introduce 'evaluation' maps

$$\begin{aligned} \mathrm{ev}_B \colon (\boldsymbol{x}, \boldsymbol{p}; \eta, \epsilon, \vec{a} \,|\, \tau) &\longmapsto \big(\boldsymbol{x}^{\star(B)}, \boldsymbol{p}^{\star(B)}; \eta^{\star(B)}, \epsilon^{\star(B)}, \vec{a}^{\star(B)} \,\big|\, \tau^{(B)}\big) \\ &= B \cdot \big(\boldsymbol{x}^{\star(1)}, \boldsymbol{p}^{\star(1)}; \eta^{\star(1)}, \epsilon^{\star(1)}, \vec{a}^{\star(1)} \,\big|\, \tau^{(1)}\big), \quad B \in \mathrm{PSL}(2,\mathbb{Z})\,, \end{aligned} \tag{4.30}$$

that specialise the parameters to the equilibrium generated by $B \in \mathrm{PSL}(2,\mathbb{Z})$. These maps may act on operators $\widetilde{O}$ by $\mathrm{ev}_B \circ \widetilde{O}$. We will write $\widetilde{O}_1 \overset{\mathrm{ev}_B}{=} \widetilde{O}_2$ as a shorthand for $\mathrm{ev}_B \circ \widetilde{O}_1 = \mathrm{ev}_B \circ \widetilde{O}_2$.

We can now compute what happens after evaluation to the first two Poisson brackets in (4.28):

$$\begin{aligned} \big\{D_n^{\mathrm{cl}}, \widetilde{c}_0(\widetilde{D}_m^{(1)})\big\} &= \sum_{i,j=1}^{N} \left(\frac{\partial D_n^{\mathrm{cl}}}{\partial x_i} \frac{\partial\, \widetilde{c}_0(\widetilde{D}_m^{(1)})}{\partial \gamma_j} \{x_i, \gamma_j\} + \frac{\partial D_n^{\mathrm{cl}}}{\partial \gamma_i} \frac{\partial\, \widetilde{c}_0(\widetilde{D}_m^{(1)})}{\partial x_j} \{\gamma_i, x_j\} \right) \\ &\overset{\mathrm{ev}_B}{=} \sum_{i,j=1}^{N} \frac{\partial D_n^{\mathrm{cl}}}{\partial \gamma_i} \frac{\partial\, \widetilde{c}_0(\widetilde{D}_m^{(1)})}{\partial x_j} \{\gamma_i, x_j\} \\ &\overset{\mathrm{ev}_B}{=} -v_n^\star \sum_{i,j=1}^{N} \frac{\partial\, \widetilde{c}_0(\widetilde{D}_m^{(1)})}{\partial x_j} = 0\,, \end{aligned} \tag{4.31}$$

by first recognising the (i-independent) velocities (3.20a) and then the identity (4.29). Rearranging (4.28) and applying the evaluation map then yields

$$\big[\widetilde{c}_0(\widetilde{D}_n^{(1)}), \widetilde{c}_0(\widetilde{D}_m^{(1)})\big] = -\big\{D_n^{\mathrm{cl}}, \widetilde{c}_0(\widetilde{D}_m^{(1)})\big\} - \big\{\widetilde{c}_0(\widetilde{D}_n^{(1)}), D_m^{\mathrm{cl}}\big\} \overset{\mathrm{ev}_B}{=} 0\,. \tag{4.32}$$

Since after taking the classical limit there are no more difference operators, the matrix-valued operators

$$H_{n,B} \coloneqq \mathrm{ev}_B\big(\widetilde{c}_0(\widetilde{D}_n^{(1)})\big) \tag{4.33}$$

act trivially *on functions*. They can thus be restricted to act on $V^{\otimes N}$ only, and hence define hamiltonians of a long-range spin chain that is integrable in the sense that

$$[H_{n,B}, H_{m,B}] = 0\,. \tag{4.34}$$

4.4. **Explicit spin-chain hamiltonians.** Let us compute the first few spin chain hamiltonians from the spin Ruijsenaars operators in (2.20). Starting with $\widetilde{D}_1$ from (2.21), we perform the hybrid expansion (4.22) in the *explicitly* appearing $\hbar$ as

$$\begin{aligned} \widetilde{D}_1 &= \sum_{j=1}^{N} A_j(\boldsymbol{x})\, P_{(1\,\ldots\,j)}(\boldsymbol{x})^{-1}\, P_{(1\,\ldots\,j)}(x_1, \ldots, x_j - \mathrm{i}\,\hbar\,\epsilon, \ldots, x_N)\, \Gamma_j \\ &= \widetilde{D}_1^{(0)} + \hbar\, \widetilde{D}_1^{(1)} + O(\hbar^2)\,. \end{aligned} \tag{4.35}$$

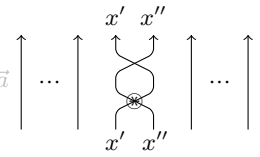
The first few coefficients are given by

$$\widetilde{D}_1^{(0)} \coloneqq \sum_{j=1}^{N} A_j(\boldsymbol{x})\, \Gamma_j = D_1\, \mathrm{id} \ \in Z(\widetilde{\mathcal{A}}_0)\,, \tag{4.36a}$$

which satisfies (4.23), and

$$\widetilde{D}_1^{(1)} \coloneqq \sum_{j=2}^{N} \widetilde{A}_j(\boldsymbol{x})\, \Gamma_j\,, \quad \widetilde{A}_j(\boldsymbol{x}) = \mathrm{i}\,\epsilon\, A_j(\boldsymbol{x}) \sum_{i=1}^{j-1} P_{(i+1\,\ldots\,j)}(\boldsymbol{x})^{-1}\, h_{i,i+1}(x_i - x_j)\, P_{(i+1\,\ldots\,j)}(\boldsymbol{x})\,, \tag{4.36b}$$

where $\widetilde{A}_1(\boldsymbol{x}) = 0$ and for $j \geqslant 2$ we introduced the nearest-neighbour spin interaction

(4.37) $$h_{i,i+1}(u) \coloneqq P_{i,i+1}(-u)\, P'_{i,i+1}(u) =$$ 

$$u = x' - x''\,,$$

with '$\circledast$' indicating the derivative of the R-matrix in our graphical notation. We emphasise again that these are still quantum-mechanical operators, through the implicit $\hbar$ in Γ_j as well as, for $d \geqslant 1$, the deformed spin interactions.

Similarly we compute the hybrid expansion of $\widetilde{D}_{-1}$ from (2.23),

(4.38) $$\widetilde{D}_{-1} = \widetilde{D}^{(0)}_{-1} + \hbar\, \widetilde{D}^{(1)}_{-1} + O(\hbar^2)\,,$$

with coefficients

(4.39)
$$\begin{aligned} \widetilde{D}^{(0)}_{-1} &\coloneqq \sum_{i=1}^{N} A_{-i}(\boldsymbol{x})\, \Gamma_i^{-1} = D_{-1}\,\mathrm{id} \ \in Z(\widetilde{\mathcal{A}}_0)\,, \\ \widetilde{D}^{(1)}_{-1} &\coloneqq \mathrm{i}\,\epsilon \sum_{i=1}^{N-1} A_{-i}(\boldsymbol{x}) \sum_{j=i+1}^{N} P_{(j+1\ \dots\ i)}(\boldsymbol{x})^{-1}\, h_{j-1,j}(x_i - x_j)\, P_{(j+1\ \dots\ i)}(\boldsymbol{x})\, \Gamma_i^{-1}\,. \end{aligned}$$

Again the coefficients take the form (4.23), so we can use the results from the previous section to conclude that, for any $B \in \mathrm{PSL}(2,\mathbb{Z})$,

(4.40) $$\big[\mathrm{ev}_B\big(\widetilde{c}_0(\widetilde{D}^{(1)}_1)\big), \mathrm{ev}_B\big(\widetilde{c}_0(\widetilde{D}^{(1)}_{-1})\big)\big] = 0\,.$$

Remember from §3.2 that the summands of the first Ruijsenaars operator D_1 all coincide in the classical limit at equilibrium: for any j,

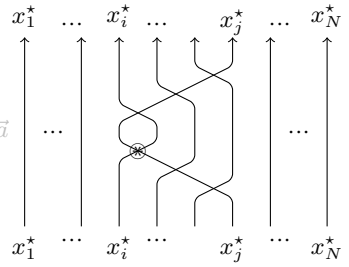
(4.41a) $$\widetilde{c}_0\big(\epsilon\, A_j(\boldsymbol{x})\, \Gamma_j / v_1\big) \overset{\mathrm{ev}_B}{=} \epsilon\, A_j(\boldsymbol{x}^\star)\, \gamma_j^\star / v_1^\star = 1\,.$$

Similarly, for $n = -1$ and any j

(4.41b) $$\widetilde{c}_0\big(\epsilon\, A_{-j}(\boldsymbol{x})\, \Gamma_j^{-1} / v_{-1}\big) \overset{\mathrm{ev}_B}{=} \epsilon\, A_{-j}(\boldsymbol{x}^\star)\, \gamma_j^{\star\,-1} / v_{-1}^\star = 1\,, \quad v_{-1}^\star = v_{N-1}^\star / v_N^\star\,.$$

We are thus led to define the normalised 'chiral' spin-chain hamiltonians

$$\bar{H}_{1,B} \coloneqq \mathrm{ev}_B\left(\frac{\widetilde{c}_0(\widetilde{D}^{(1)}_1)}{\mathrm{i}\, v_1^\star}\right) = \sum_{j=2}^{N} \sum_{i=1}^{j-1} P_{(i+1\ \dots\ j)}(\boldsymbol{x}^\star)^{-1}\, h_{i,i+1}(x_i^\star - x_j^\star)\, P_{(i+1\ \dots\ j)}(\boldsymbol{x}^\star)$$

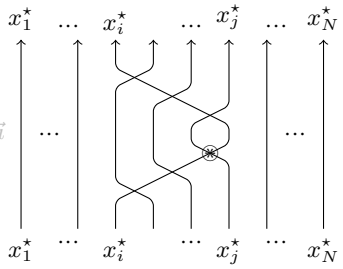
(4.42) $$= \sum_{i<j}^{N}$$ 

and

$$\bar{H}_{-1,B} \coloneqq \mathrm{ev}_B\left(\frac{\widetilde{c}_0(\widetilde{D}^{(1)}_{-1})}{\mathrm{i}\, v^\star_{-1}}\right) = \sum_{i=1}^{N-1}\sum_{j=i+1}^{N} P_{(j+1\ \dots\ i)}(\boldsymbol{x}^\star)^{-1}\, h_{j-1,j}(x^\star_i - x^\star_j)\, P_{(j+1\ \dots\ i)}(\boldsymbol{x}^\star)$$

(4.43)

$$= \sum_{i<j}^{N}$$



.

Since $\bar{H}_{\pm1,B} \propto H_{\pm1,B}$, by (4.40) they commute,

$$[\bar{H}_{1,B}, \bar{H}_{-1,B}] = 0\,. \tag{4.44}$$

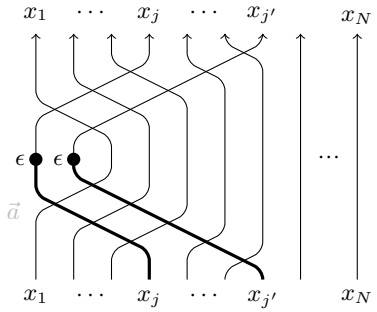
We stress that both the classical equilibrium positions $\boldsymbol{x}^\star = B\cdot\boldsymbol{x}^{\star\,(1)}$ and the parameters depend on $B \in \mathrm{PSL}(2,\mathbb{Z})$, while the classical equilibrium momenta $\boldsymbol{p}^\star$ dropped out thanks to (4.41). The form (4.42) for the hamiltonian of an integrable q-deformed long-range spin chain was first found in [Lam18], and (4.43) in [LPS22].

One proceeds similarly for higher hamiltonians, with the explicit result looking more and more involved. Let us illustrate this for $n = 2$. The operator $\widetilde{D}_2$ from (2.20) takes

$$\widetilde{D}_2 = \sum_{j<j'}^{N} A_{\{j,j'\}}(\boldsymbol{x})\, P_{\{j,j'\}}(\boldsymbol{x})^{-1}\, \Gamma_j\, \Gamma_{j'}\, P_{\{j,j'\}}(\boldsymbol{x})$$

(4.45)

$$= \sum_{j<j'}^{N} A_{\{j,j'\}}(\boldsymbol{x}) \times$$



$$= \sum_{j<j'}^{N} A_{\{j,j'\}}(\boldsymbol{x})\, P_{\{j,j'\}}(\boldsymbol{x})^{-1}\, P_{\{j,j'\}}(\boldsymbol{x} - \mathrm{i}\,\epsilon\,\hbar\,(\boldsymbol{e}_j + \boldsymbol{e}_{j'}))\, \Gamma_j\, \Gamma_{j'}\,,$$

where $\{\boldsymbol{e}_j\}_j$ is the standard orthonormal basis of $\mathbb{C}^N$, so that by (B.7) we have

(4.46)

$$P_{\{j,j'\}}(\boldsymbol{x} - \mathrm{i}\,\epsilon\,\hbar\,(\boldsymbol{e}_j + \boldsymbol{e}_{j'})) = P_{(2\ \dots\ j')}(x_j - \mathrm{i}\,\epsilon\,\hbar, x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_{j'} - \mathrm{i}\,\epsilon\,\hbar, \dots, x_N)$$
$$\times\, P_{(1\ \dots\ j)}(x_1, \dots, x_{j-1}, x_j - \mathrm{i}\,\epsilon\,\hbar, x_{j+1}, \dots, x_{j'} - \mathrm{i}\,\epsilon\,\hbar, \dots, x_N)\,,$$

where the arguments that do not actually appear in any R-matrix are printed in gray.

Working out the $\hbar$-expansion of (4.45) we see that $\widetilde{D}^{(0)}_2 = D_2\,\mathrm{id}$, like for $n = 1$. At first order in $\hbar$ there are two types of contributions. When expanding the argument with x_j, the other line can simply be straightened out using the unitarity relation (2.13), giving a contribution whose summands resemble the spin operators appearing in $\widetilde{D}^{(1)}_1$:

$$(\mathrm{i}\,\epsilon)^2 \sum_{j<j'}^{N} A_{\{j,j'\}}(\boldsymbol{x}) \sum_{i=1}^{j-1} P_{(i+1\ \dots\ j)}(\boldsymbol{x})^{-1}\, h_{i,i+1}(x_i - x_j)\, P_{(i+1\ \dots\ j)}(\boldsymbol{x})\, \Gamma_j\, \Gamma_{j'}\,. \tag{4.47}$$

When we expand the argument with $x_{j'}$ we obtain a more complicated nested product:

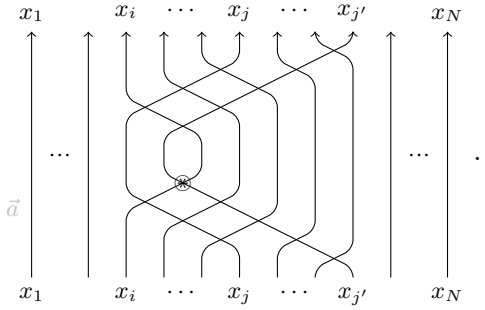
$$
\begin{aligned}
&(\mathrm{i}\,\epsilon)^2 \sum_{j<j'}^{N} A_{\{j,j'\}}(\boldsymbol{x}) \sum_{i=1}^{j-1} P_{(i\,\ldots\,j)}(\boldsymbol{x})^{-1}\, P_{(i+2\,\ldots\,j')}\big(s_{(i\,\ldots\,j)}\cdot\boldsymbol{x}\big)^{-1}\, h_{i+1,i+2}(x_i - x_{j'}) \\
&\qquad\qquad \times P_{(i+2\,\ldots\,j')}\big(s_{(i\,\ldots\,j)}\cdot\boldsymbol{x}\big)\, P_{(i\,\ldots\,j)}(\boldsymbol{x})\, \Gamma_j\, \Gamma_{j'} \\
&+ (\mathrm{i}\,\epsilon)^2 \sum_{j<j'}^{N} A_{\{j,j'\}}(\boldsymbol{x}) \sum_{i=j+1}^{j'-1} P_{(i+1\,\ldots\,j')}(\boldsymbol{x})^{-1}\, h_{i,i+1}(x_i - x_{j'})\, P_{(i+1\,\ldots\,j')}(\boldsymbol{x})\, \Gamma_j\, \Gamma_{j'}\,,
\end{aligned}
\tag{4.48}
$$

where

$$
s_{(i\,\ldots\,j)}\cdot\boldsymbol{x} = (x_1,\ldots,x_{i-1},x_j,x_i,\ldots x_{j-1},x_j,x_{j+1}\ldots,x_{j'},x_{j'+1},\ldots,x_N)\,,
\tag{4.49}
$$

again with in gray all coordinates that do not actually appear as the argument of any R-matrix contained in $P_{(i+2\,\ldots\,j')}$. The last line of (4.48) has the same structure as (4.47), while the spin operators in the first two lines of (4.48) correspond to

(4.50)



Now note that

$$
A_{j,j'} = A_j\, A_{j'}\, \frac{-\theta(x_j - x_{j'})^2}{\theta(x_j - x_{j'}+\eta)\,\theta(x_j - x_{j'}-\eta)}\,,
\tag{4.51}
$$

such that, upon taking the classical limit and evaluating at B,

(4.52)

$$
\begin{aligned}
\bar{H}_{2,B} &\coloneqq \mathrm{ev}_B\left(\frac{\widetilde{c}_0(\widetilde{D}_2^{(1)})}{(\mathrm{i}\,v_1)^2}\right) \\
&= \sum_{i<j}^{N}\left(\sum_{\substack{k=1\\ k\notin\{i,\ldots,j\}}}^{N} \frac{-\theta(x_j^\star - x_k^\star)^2}{\theta(x_j^\star - x_k^\star+\eta)\,\theta(x_j^\star - x_k^\star-\eta)}\right) P_{(i+1\,\ldots\,j)}(\boldsymbol{x}^\star)^{-1} \\
&\qquad\qquad \times h_{i,i+1}(x_i^\star - x_j^\star)\, P_{(i+1\,\ldots\,j)}(\boldsymbol{x}^\star) \\
&\quad + \sum_{i<j<k}^{N} \frac{-\theta(x_j^\star - x_k^\star)^2}{\theta(x_j^\star - x_k^\star+\eta)\,\theta(x_j^\star - x_k^\star-\eta)}\, P_{(i\,\ldots\,j)}(\boldsymbol{x}^\star)^{-1}\, P_{(i+2\,\ldots\,k)}\big((s_{(i\,\ldots\,j)}\cdot\boldsymbol{x})^\star\big)^{-1} \\
&\qquad\qquad \times h_{i+1,i+2}(x_i^\star - x_k^\star)\, P_{(i+2\,\ldots\,k)}\big((s_{(i\,\ldots\,j)}\cdot\boldsymbol{x})^\star\big)\, P_{(i\,\ldots\,j)}(\boldsymbol{x}^\star)
\end{aligned}
$$

The first line matches the $n=1$ hamiltonian (4.42), except that the summands are weighed differently. The second line is new to $n=2$. The form of this spin-chain hamiltonian is compatible with that of the second hamiltonians from [LPS22] and appeared more explicitly in [MZ23b].

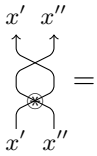
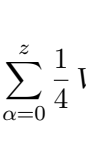
Just like $\bar{H}_{\pm1,B}$ from (4.42)–(4.43) are related by left-right flipping the diagram, $\bar{H}_{-2,B}$ is obtained from $\bar{H}_{2,B}$ by a vertical reflection of the diagrammatic representation. One can continue this procedure to generate explicit expressions for the higher hamiltonians $H_{n,B}$, until one reaches the 'equator' $n\sim N/2$.

In conclusion, at any fixed classical equilibrium configuration $(\boldsymbol{x}^\star,\boldsymbol{p}^\star) = B\cdot(\boldsymbol{x}^{\star\,(1)},\boldsymbol{p}^{\star\,(1)}) \in M_{\mathbb{C}}^\star$ (parametrised by $B\in PSL(2,\mathbb{Z})$), we thus obtain a hierarchy of commuting spin-chain

hamiltonians, namely one spin-chain operator $H_{n,B}$ for each spin-Ruijsenaars operator $\widetilde{D}_n$. The exception is $n = N$. Indeed, since the total shift operator $\widetilde{D}_N = \Gamma_1 \cdots \Gamma_N \,\mathrm{id}$ acts trivially on spins, its classical limit lies in the centre $Z(\mathcal{A}_\hbar) = \mathrm{Fun}(\boldsymbol{x})\,\mathrm{id}$, and commutes trivially with all other hamiltonians. Instead, it gives rise to the lattice translation operator, at least in the trigonometric case [LPS22]; we do not yet have the proof in the elliptic case.

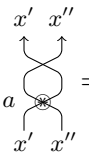
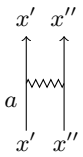
4.5. **Examples.** For specific choices of the input data the freezing procedure yields explicit spin-chain hamiltonians.

- **Vertex-type.** If, as in (2.9), $P_{i,i+1}(u) = P^{\mathrm{V}}_{i,i+1}(u)$ is the Baxter–Belavin R-matrix (A.7) of type $\mathfrak{gl}_r$, the elliptic spin-Ruijsenaars system in §2 is that of [MZ23a]. In the eight-vertex case, there exist (explicit) functions $V_\alpha(u;\eta\,|\,\tau)$ for $\alpha \in \{0,x,y,z\}$ such that the nearest-neighbour interaction from (4.37) can be recast via Pauli matrices as [KL25]

(4.53) $r=2:\quad h^{\mathrm{V}}(u) =$  $= \sum_{\alpha=0}^{z} \frac{1}{4}\, V_\alpha(u)$  $= \sum_{\alpha=0}^{z} \frac{1}{4}\, V_\alpha(u)\big(1 - P\,\sigma^\alpha \otimes \sigma^\alpha\big)\,, \qquad u = x' - x''\,.$

Freezing at $B = 1 \in \mathrm{PSL}(2,\mathbb{Z})$ gives the MZ chain [MZ23b]. Instead taking $B = S$ yields a variant that we call the MZ′ chain. For $r = 2$ the latter was introduced in [KL25], and various limits were evaluated; remarkably, the difference $H_{\pm 1,1} - H_{\pm 1,S}$ is a multiple of the identity [KL25], but in such a way that (only) the MZ′ chain admits a short-range limit.

- **Face-type.** When, like in (2.10), $P_{i,i+1}(u) = P^{\mathrm{F}}_{i,i+1}(u)$ is Felder's dynamical R-matrix of type $\mathfrak{gl}_r$, we instead obtain a dynamical elliptic spin-Ruijsenaars system which for $r = 2$ appeared in [KL24]. In that case, there is a physically meaningful factorisation

(4.54) $r=2:\quad h^{\mathrm{F}}(u,a) =$  $= \theta(\eta)\, V(u,a)$  $= \theta(\eta)\, V(u,a)\, E(u,a)\,, \qquad \begin{matrix} u = x' - x''\,, \\ a = a_1 - a_2\,, \end{matrix}$

for an (explicit) function $V(u,a;\eta\,|\,\tau)$, i.e. the potential, and operator $E(u,a;\eta\,|\,\tau)$, i.e. the nearest neighbour spin interaction [KL24]. Freezing at $B = S$ results in the q-deformed Inozemtsev chain, introduced for $r = 2$ in [KL24]. While its cousin with $B = 1$ also generalises the (long-range) q-deformed Haldane–Shastry chain [Ugl95, Lam18], it does not have a short-range limit.

The limits of both types of spin chains were evaluated in [KL25] for $r = 2$. In particular, the short-range limit only exists for $B = S$, which respectively include the antiperiodic XX chain (vertex) and the Heisenberg XXX chain (face). At higher rank ($r > 2$) the spin chains with $B = S$ have not yet been subject of detailed investigation.

5. Discussion

In this paper we demonstrated how elliptic spin-Ruijsenaars systems give rise to an elliptic long-range spin chain for each equilibrium configuration of the classical elliptic Ruijsenaars–Schneider system by freezing, and showed that this procedure preserves quantum integrability.

In more detail, in §2 we reviewed the structure of the matrix-valued difference operators $\widetilde{D}_n$ defining elliptic spin-Ruijsenaars systems, using a formulation that covers both the (vertex-type) systems of [MZ23a] and the (face-type) systems of [KL24]. In §3 we used an action of the modular group $\mathrm{PSL}(2,\mathbb{Z})$ on the (spinless) classical elliptic Ruijsenaars–Schneider system, and used it to exhibit a modular family of (discrete) equilibrium configurations, cf. Fig. 1. This generalises the known equilibria of the (spinless) classical elliptic Calogero–Sutherland–Moser system, which also appears in context of supersymmetric gauge theory [Dor99]. The construction requires a shift of the momenta, see (3.18), which survives in the Calogero–Sutherland–Moser limit. For any classical equilibrium, in §4 we used the framework of deformation quantisation,

building on [MV24, LRS24, Cha24], to derive a quantum spin chain with long-range interactions by freezing. We proved that quantum integrability is preserved in the sense that if the $\widetilde{D}_n$ commute among each other, then so do the spin-chain hamiltonians $H_{n,B}$ defined in (4.33), frozen at the equilibrium labelled by $B \in \mathrm{PSL}(2,\mathbb{Z})$.

The vertex- and face-type cases from [MZ23b] and [KL24] give rise to two separate landscapes of long-range spin chains that only overlap in a single point [KL25], see especially Figures 2–3 therein. In particular, our results allow one to freeze at $B = S$, which (unlike $B = 1$) provides spin chains admitting a short-range limit. Our results also carry over to the (trigonometric) long-range limit, and, in the face case, for $q \to 1$ agree with [Cha24], and §9 of [LRS24]. Besides proving integrability, we believe that freezing holds the key to a complete understanding of these q-deformed long-range spin chains — including exact descriptions of their spectra, like for the (face-type) Haldane–Shastry chain [BGHP93, LS24] and its q-deformation [Ugl95, LPS22]. Finally, it would be interesting to know whether any of the quantum many-body systems with spins discussed in this paper, or any of their limits, can be obtained from the double elliptic ('DELL') system with spins that was proposed in [KS20], or if they are related to the new spin-Calogero–Sutherland models recently introduced in [BM24, HLYZ24].

Acknowledgements

We thank O. Chalykh, M. Mikhailov, J. Pulmann and M. Volk for discussions, and A. Bourget for correspondence. Rob Klabbers's research is funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) - Projektnummer 417533893/GRK2575 "Rethinking Quantum Field Theory".

Appendix A. Elliptic preliminaries

A.1. **Elliptic functions.** We summarise our conventions for the elliptic functions we use and list their most important properties. Standard references are [OOL+, AS48, WW02].

The odd Jacobi theta function with lattice parameter $\tau \in \mathbb{C}$ ($\mathrm{Im}\,\tau > 0$) is defined as

$$\begin{aligned}\vartheta(x|\tau) &= 2\sum_{n=0}^{\infty}(-1)^n\, p^{(n+1/2)^2}\sin[(2n+1)x] \\ &= 2\,p^{1/4}\sin(x)\prod_{n=1}^{\infty}(1-p^{2n})(1-\mathrm{e}^{2\mathrm{i}x}p^{2n})(1-\mathrm{e}^{-2\mathrm{i}x}\,p^{2n})\,,\end{aligned} \qquad p = \mathrm{e}^{\mathrm{i}\pi\tau}\,, \tag{A.1}$$

where p is called the 'nome'. The normalised theta function $\theta(x) = \vartheta(\pi x)/(\pi\vartheta'(x))$ is odd and entire as well as is doubly quasiperiodic, $\theta(x+1) = -\theta(x)$ and $\theta(x+\tau) = -p^{-1}\,\mathrm{e}^{-2\pi\mathrm{i}x}\,\theta(x)$, with a simple zero at the origin, and satisfies $\theta'(0) = 1$. Its trigonometric and rational degenerations are $\theta(x) = \sin(\pi x)/\pi + O(p)$ and $\sin(\pi x)/\pi = x + O(N^{-2})$. It obeys the addition formula

$$\begin{aligned}\theta(x+y)\,\theta(x-y)\,\theta(z+w)\,\theta(z-w) &= \theta(x+z)\,\theta(x-z)\,\theta(y+w)\,\theta(y-w) \\ &\quad+\ \theta(x+w)\,\theta(x-w)\,\theta(z+y)\,\theta(z-y)\,.\end{aligned} \tag{A.2}$$

Other standard Jacobi theta functions are defined in terms of (A.1) by

$$\begin{aligned}\vartheta_1(z\,|\,\tau) &= \vartheta(z)\,, & \vartheta_2(z\,|\,\tau) &= \vartheta(z+1/2\,|\,\tau)\,, \\ \vartheta_3(z\,|\,\tau) &= \mathrm{e}^{\mathrm{i}\pi\tau/4}\,\mathrm{e}^{\mathrm{i}\pi x}\,\vartheta(z+(1+\tau)/2\,|\,\tau)\,, & \vartheta_4(z\,|\,\tau) &= -\mathrm{i}\,\mathrm{e}^{\mathrm{i}\pi\tau/4}\,\mathrm{e}^{\mathrm{i}\pi x}\,\vartheta(z+\tau/2\,|\,\tau)\,,\end{aligned} \tag{A.3}$$

hence one can think of these as (rescaled) versions of the odd Jacobi theta function (A.1) shifted by half-periods of the lattice, generalising the well-known identity $\cos(x) = \sin(x+\pi/2)$ between the elementary trigonometric functions. Expanding in the nome p we find

$$\begin{aligned}p^{-1/4}\,\vartheta_1(z\,|\,\tau) &= 2\sin\pi x + O(p^2)\,, & p^{-1/4}\,\vartheta_2(z\,|\,\tau) &= 2\cos\pi x + O(p^2)\,, \\ \vartheta_3(z\,|\,\tau) &= 1 + 2\,p\,\cos\pi x + O(p^2)\,, & \vartheta_4(z\,|\,\tau) &= 1 - 2\,p\cos\pi x + O(p^2)\,.\end{aligned} \tag{A.4}$$

There are many more relations between the ϑ_a, for our purposes the Jacobi imaginary transformation will be particularly important. For a given τ let $\tau' = -1/\tau$, then the following relations hold

$$\begin{aligned} \mathrm{i}\,(-\mathrm{i}\tau)^{1/2}\,\vartheta_1(z\,|\,\tau) &= \mathrm{e}^{\mathrm{i}\tau' z^2/\pi}\,\vartheta_1(\tau' z\,|\,\tau')\,, \\ (-\mathrm{i}\tau)^{1/2}\,\vartheta_2(z\,|\,\tau) &= \mathrm{e}^{\mathrm{i}\tau' z^2/\pi}\,\vartheta_4(\tau' z\,|\,\tau')\,, \\ (-\mathrm{i}\tau)^{1/2}\,\vartheta_3(z\,|\,\tau) &= \mathrm{e}^{\mathrm{i}\tau' z^2/\pi}\,\vartheta_3(\tau' z\,|\,\tau')\,, \\ (-\mathrm{i}\tau)^{1/2}\,\vartheta_4(z\,|\,\tau) &= \mathrm{e}^{\mathrm{i}\tau' z^2/\pi}\,\vartheta_2(\tau' z\,|\,\tau')\,, \end{aligned} \tag{A.5}$$

effectively relating the two regimes in which $\mathrm{Im}(\tau)$ is either large or small. Note that ϑ_2 and ϑ_4 switch places under this transformation, whereas ϑ_1 and ϑ_3 transform into themselves.

The (normalised) Kronecker elliptic function is

$$\begin{aligned} \phi(u,v\,|\,\tau) \coloneqq \frac{\theta(u+v\,|\,\tau)}{\theta(u\,|\,\tau)\,\theta(v\,|\,\tau)} &= \frac{\pi\sin\bigl(\pi(u+v)\bigr)}{\sin(\pi u)\,\sin(\pi v)} + O(p) \\ &= \pi\bigl(\cot(\pi u)+\cot(\pi v)\bigr) + O(p)\,. \end{aligned} \tag{A.6}$$

If no confusion can arise we will suppress its dependence on τ, simply writing $\phi(u,v)$. This function is symmetric and doubly (quasi)periodic, $\phi(u+1,v) = \phi(u,v)$ and $\phi(u+\tau,v) = \mathrm{e}^{-2\pi\mathrm{i}v}\,\phi(u,v)$.

A.2. **Elliptic *R*-matrices.** In the following, for $1 \leqslant \alpha,\beta \leqslant r$ let $E_{\alpha\beta} \in \mathrm{Mat}(r\times r,\mathbb{C})$ denote the $r\times r$ matrix units, with entries $(E_{\alpha\beta})_{\gamma\delta} = \delta_{\alpha\gamma}\,\delta_{\beta\delta}$.

A.2.1. *Vertex-type.* For $r \geqslant 2$ the Baxter–Belavin R-matrix for $\mathfrak{gl}_r$ can be given as

$$R(x;\eta\,|\,\tau) \coloneqq {\sum_{\alpha,\beta,\gamma,\delta=1}^{r}}' R_{\alpha\gamma,\beta\delta}(x;\eta\,|\,\tau)\,E_{\alpha\beta}\otimes E_{\gamma\delta}\,, \tag{A.7a}$$

where the prime indicates that sum is restricted to the 'weakened ice rule' $\alpha+\gamma \equiv \beta+\delta \bmod r$, and the nonzero entries read

$$\begin{aligned} R_{\alpha\gamma,\beta\delta}(x;\eta\,|\,\tau) \coloneqq \frac{1}{\phi(x,\eta\,|\,\tau)}\exp\Bigl(\frac{2\pi\mathrm{i}}{r}\bigl((\beta-\alpha)\,x+(\gamma-\beta)\,\eta+(\gamma-\beta)(\gamma-\alpha)\,\tau\bigr)\Bigr) \\ \times\,\phi(x+\gamma-\beta\,\tau,\eta+(\beta-\alpha)\,\tau\,|\,r\,\tau) \end{aligned} \tag{A.7b}$$

Then $\check{R}(x) = P\,R(x)$ again satisfies the relations (2.6)–(2.8) as well as (2.11). The Baxter–Belavin R-matrix is also commonly written in terms of a representation of the Heisenberg group [Bel81]; for this and many further relations see e.g. the appendix of [ZZ22]. [9]

For $r = 2$ the weakened ice rule allows for $R_{11,22}, R_{22,11} \neq 0$ in addition to nonzero entries in the same six positions as for the dynamical R-matrix and this definition yields the eight-vertex R-matrix in the conventions of [KL25], see Section 2.1 therein. The trigonometric limit gives the usual (symmetric) six-vertex R-matrix in the 'principal grading' (possibly up to a global spin rotation), see §B.2 in [KL25].

A.2.2. *Face-type.* The elliptic dynamical R-matrix of type $\mathfrak{gl}_r$ with $r \geqslant 2$ reads [Fel95b, FV97]

$$\begin{aligned} R(x,a;\eta\,|\,\tau) \coloneqq {} & \sum_{\alpha=1}^{r} E_{\alpha\alpha}\otimes E_{\alpha\alpha} + \frac{1}{\phi(x,\eta\,|\,\tau)}\sum_{\alpha\neq\beta}^{r}\phi(a_\beta-a_\alpha,\eta\,|\,\tau)\,E_{\alpha\alpha}\otimes E_{\beta\beta} \\ & + \frac{1}{\phi(x,\eta\,|\,\tau)}\sum_{\alpha\neq\beta}^{r}\phi(x,a_\beta-a_\alpha\,|\,\tau)\,E_{\alpha\beta}\otimes E_{\beta\alpha}\,. \end{aligned} \tag{A.8}$$

Then $\check{R}(x,a) \coloneqq P\,R(x,a)$ satisfies the unitarity relation (2.6), the (dynamical) Yang–Baxter equation (2.7), 'commutativity at a distance' (2.8), as well as the initial condition (2.11). For

[9] Our η is related to that of (B.14) in [ZZ22] via $\eta_{\mathrm{ZZ}} \coloneqq \eta/r$.

a graphical interpretation, including the connection to the 'interaction (a)round the face' (IRF) picture, see §B in [KL24]. This R-matrix is associated to the elliptic quantum group $E_{\tau,\eta}(\mathfrak{gl}_r)$ [FV97].

For $r = 2$, (A.8) coincides with the dynamical R-matrix in [KL25] and [KL24] after identifying $a = a_1 - a_2$.[10] Taking the trigonometric, and then non-dynamical limit, on obtains the trigonometric R-matrix in the 'homogeneous grading', related to the Hecke algebra, see e.g. §5.1 of [KL24].

A.2.3. *Face-vertex transformation.* The face- and vertex-type elliptic R-matrices are related by a 'face-vertex transformation' [Bax73, FV96], which can be interpreted as a Drinfeld twist [JKOS99]. As we emphasised in §5.2 of [KL25], this transformation does *not* extend from the level of the R-matrix to a simple conjugation of the corresponding spin-Ruijsenaars models, yielding very different spectral and physical properties.

Appendix B. Deformed spin permutations

In this appendix we provide more details for the deformed spin permutations defined in §2.1.

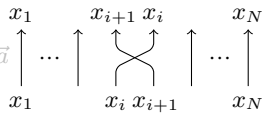
Let $w \longmapsto s_w$ denote the natural action of S_N on $\mathrm{Fun}(\boldsymbol{x})$ by permuting variables. According to (2.6)–(2.8), $\tilde{s}_i := s_{i,i+1}\, P_{i,i+1}(x_i - x_{i+1})$ gives an S_N-action on $\mathrm{Fun}(\boldsymbol{x}) \otimes V^{\otimes N}$ [Fel95a], see also [FWZJ16]. By (2.11) it deforms the diagonal action. Write $w \longmapsto \widetilde{w}$ for this representation. Define $P_w(\boldsymbol{x})$ by $\widetilde{w} = s_w\, P_w(\boldsymbol{x})$. Then considering $\widetilde{w}\widetilde{w'} = \widetilde{w}\,\widetilde{w'}$ yields the general cocycle condition[11]

$$P_{w\,w'}(\boldsymbol{x}) = s_{w'}^{-1}\, P_w(\boldsymbol{x})\, s_{w'}\, P_{w'}(\boldsymbol{x}) = P_w(s_{w'}^{-1} \cdot \boldsymbol{x})\, P_{w'}(\boldsymbol{x})\,. \tag{B.1}$$

This includes the cocycle condition (2.16) as the special case $w' = (i\ i+1)$, which allows one to recursively construct any $P_w(\boldsymbol{x})$ starting from the identity operator (2.15).

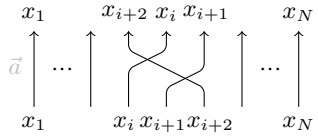
B.1. **Cycles.** The spin-Ruijsenaars operators $\widetilde{D}_n$ from (2.20) and (2.22) are all built from deformed cycles. Here are some examples. Clearly,

$$P_{(i,i+1)}(\boldsymbol{x}) = P_{i,i+1}(x_i - x_{i+1}) = \tag{B.2}$$



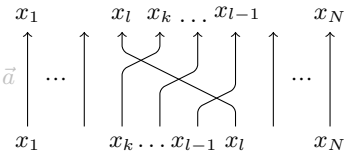
Next, for $(i\ i+1\ i+2) = (i\ i+1)\,(i+1\ i+2)$ we get

$$P_{(i\ i+1\ i+2)}(\boldsymbol{x}) = P_{i,i+1}(x_i - x_{i+2})\, P_{i+1,i+2}(x_{i+1} - x_{i+2}) = \tag{B.3}$$



More generally, for $k < l$ the (uninterrupted) cycle $(k\ k+1\ \ldots\ l) = (k\ k+1)\cdots(l-1\ l)$ yields

$$P_{(k\ k+1\ \ldots\ l)}(\boldsymbol{x}) = P_{k,k+1}(x_k - x_l)\cdots P_{l-1,l}(x_{l-1} - x_l) = \tag{B.4}$$



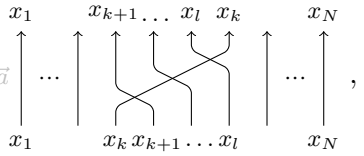
The case $k = 1$, $l = i$ gives (2.18).

In fact, as these examples illustrate, one can compute $P_w(\boldsymbol{x})$ graphically. First draw $w \in S_N$: start with a row of numbers $1\ 2\ \ldots\ N$, a little above it $w(1)\ w(2)\ \ldots\ w(N)$, and connect equal numbers, drawing so that no more than two lines cross at any point. Remove unnecessary double crossings, anticipating (2.13), to get a reduced decomposition of w. Now replace all i

[10] One obtains the dynamical R-matrix of [ZZ22] after a transposition and passing to $\eta_{\mathrm{ZZ}} := \eta/r$.

[11] Alternatively, one can set $\widetilde{w} = P'_w(\boldsymbol{x})\, s_w$ and work with $P'_w(\boldsymbol{x}) = s_w\, P_w(\boldsymbol{x})\, s_w^{-1} = P_w(s_w \cdot \boldsymbol{x})$; in particular, $P'_{(i\ i+1)}(\boldsymbol{x}) = P_{i,i+1}(x_{i+1} - x_i)$.

by x_i in both rows and reinterpret the crossings as deformed permutations via (2.12). For instance, if again $k < l$ we may compute

(B.5)
$$P_{(l\ l-1\ \dots\ k)}(\boldsymbol{x}) = P_{l-1,l}(x_k - x_l)\cdots P_{k,k+1}(x_k - x_{k+1}) =$$


by first drawing the diagram on the right and then reading off the expression in the middle. [12] The case $k = i$, $l = N$ yields (2.19).

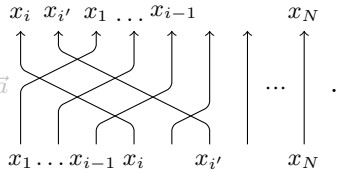
B.2. **Grassmannian permutations.** We are particularly interested in *Grassmannian permutations*, which are minimal-length representatives of the coset $S_N/(S_n \times S_{N-n})$ for some $1 \leqslant n \leqslant N$, i.e. permutations with (at most) one descent $w(n) > w(n+1)$. More explicitly, given an n-element subset $I = \{i_1 < \dots < i_n\} \subseteq \{1, \dots, N\}$, the corresponding Grassmannian permutation, which we denote by $w_I \in S_N$, sends $k \longmapsto i_k$ for all $1 \leqslant k \leqslant n$ without permuting either $\{1, \dots, n\}$ or $\{n+1, \dots, N\}$ amongst each other. It can be defined recursively, starting from $w_\varnothing = e$ the identity, through the recursion relation

(B.6)
$$w_{J\cup\{j\}} = w_J\,(j\ j-1\dots|J|+1) \quad \text{if} \quad j > \max(J)\,.$$

For instance, $n = 1$ gives the cycle $w_{\{i\}} = (i\ i-1\dots1)$, at $n = 2$ we get the product of cycles $w_{\{i,i'\}} = (i\,i{-}1\dots1)\,(i'\,i'{-}1\dots2)$, and so on. In general, $w_I = (i_1\,i_1{-}1\dots1)\cdots(i_n\,i_n{-}1\dots n)$. Note that at $n = N$ we simply retrieve $w_{\{1,\dots,N\}} = e$ the identity. This motivates further defining $w_{-I} \coloneqq w_{\{1,\dots,N\}\setminus I}$. For example, $w_{-\{i\}} = w_{\{1,\dots,i-1,i+1,\dots,N\}} = (i\ i+1\dots N)$ is again a cycle, $w_{-\{i,i'\}} = (i\ i+1\dots N-1)\,(i'\ i'+1\dots N)$ a product of two cycles, and in general $w_{-I} = (i_1\ i_1+1\dots N-n+1)\cdots(i_n\ i_n+1\dots N)$.

From these w_I we finally construct the operators $P_I(\boldsymbol{x}) \coloneqq P_{w_I^{-1}}(\boldsymbol{x})$ as in (2.17). Note the inverse! Let us again give a few examples. $P_\varnothing(\boldsymbol{x})$ is just (2.15). For $n = 1$ we get (2.18). Next,

$$P_{\{i,i'\}}(\boldsymbol{x}) = P_{(2\ \dots\ i'-1\ i')}(x_i, x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_N)\,P_{(1\ \dots\ i-1\ i)}(\boldsymbol{x})$$

(B.7)
$$=$$


Finally, since $w_{\{1,\dots,N\}} = e$ the first nontrivial example of $P_{-I}(\boldsymbol{x}) \coloneqq P_{\{1,\dots,N\}\setminus I}(\boldsymbol{x})$ is (2.19).

References

[AS48] M. Abramowitz and I. A. Stegun, *Handbook of mathematical functions with formulas, graphs, and mathematical tables*, Vol. 55, US Government printing office, 1948.

[Bax72] R. J Baxter, *One-dimensional anisotropic Heisenberg chain*, Ann. Phys. **70** (1972), no. 2, 323–337.

[Bax73] R. Baxter, *Eight-vertex model in lattice statistics and one-dimensional anisotropic Heisenberg chain. II. Equivalence to a generalized ice-type lattice model*, Ann. Phys. **76** (1973), no. 1, 25–47.

[Bel81] A. A. Belavin, *Dynamical symmetry of integrable quantum systems*, Nucl. Phys. B **180** (March 1981), no. 2, 189–200.

[BFF+78a] F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz, and D. Sternheimer, *Deformation theory and quantization. I. Deformations of symplectic structures*, Annals Phys. **111** (1978), no. 1, 61–110.

[BFF+78b] ———, *Deformation theory and quantization. II. Physical applications*, Annals Phys. **111** (1978), no. 1, 111–151.

[BFV94] V. M. Buchstaber, G. Felder, and A. P. Veselov, *Elliptic Dunkl operators, root systems, and functional equations*, Duke Math. J. **76** (1994), 885–911, available at arXiv:hep-th/9403178.

[12] Observe that (B.5) is not quite the inverse of (B.4) since the mismatch in inhomogeneities does not allow for composition. Rather, the inverse $P_w(\boldsymbol{x})^{-1} = P_{w^{-1}}(x_{w(1)}, \dots, x_{w(N)})$ is graphically obtained by flipping the diagram of $P_w(\boldsymbol{x})$ upside down and reversing the orientations of all lines back from bottom to top.

[BGHP93] D. Bernard, M. Gaudin, F. D. M. Haldane, and V. Pasquier, *Yang–Baxter equation in long-range interacting systems*, J. Phys. A: Math. Gen. **26** (1993), no. 20, 5219, available at arXiv:hep-th/9301084.

[BM24] J.-E. Bourgine and Y. Matsuo, *Calogero model for the non-Abelian quantum Hall effect*, Phys. Rev. B **109** (2024), 155158, available at arXiv:2401.0308.

[BT15] A. Bourget and J. Troost, *Duality and modularity in elliptic integrable systems and vacua of $\mathcal{N} = 1^*$ gauge theories*, JHEP **2015** (2015), 128, available at arXiv:1501.05074.

[BT16] ———, *On the $\mathcal{N} = 1^*$ gauge theory on a circle and elliptic integrable systems*, JHEP **2016** (2016), 97, available at arXiv:1511.03116.

[Cha24] O. Chalykh, *Integrability of the Inozemtsev spin chain* (2024), available at arXiv:2407.03276.

[Che92] I. Cherednik, *Quantum Knizhnik–Zamolodchikov equations and affine root systems*, Commun. Math. Phys. **150** (1992), no. 1, 109–136.

[OOL+] F. W. J. Olver, A. B. Olde Daalhuis, D. W. Lozier, B. I. Schneider, R. F. Boisvert, C. W. Clark, B. R. Miller, B. V. Saunders, H. S. Cohl, and M. A. McClain (eds.), *NIST Digital Library of Mathematical Functions*. `http://dlmf.nist.gov/`, release 1.2.4 of 2025-03-15.

[Dor99] N. Dorey, *An elliptic superpotential for softly broken $n = 4$ supersymmetric yang-mills theory*, JHEP **1999** (1999), 021, available at arXiv:hep-th/9906011.

[Eti07] P. Etingof, *Calogero–Moser systems and representation theory*, European Mathematical Society (EMS), Zürich, 2007. available at arXiv:math/0606233.

[Fel95a] G. Felder, *Conformal field theory and integrable systems associated to elliptic curves*, Proc. Int. Congr. Math., Vol. 1, 2 (Zürich, 1994), 1995, pp. 1247–1255. MR1404026

[Fel95b] ———, *Elliptic quantum groups*, Proc. ICMP Paris 1994, 1995, pp. 211.

[FK96] T. Fukui and N. Kawakami, *Exact solution and spectral flow for twisted Haldane–Shastry model*, Phys. Rev. Lett. **76** (1996), 4242, available at arXiv:cond-mat/9604143.

[FV96] G. Felder and A. Varchenko, *Algebraic Bethe ansatz for the elliptic quantum group $E_{\tau,\eta}(\mathfrak{sl}_2)$*, Nucl. Phys. B **480** (1996), no. 1, 485–503.

[FV97] ———, *Elliptic quantum groups and Ruijsenaars models*, J. Stat. Phys. **89** (1997), no. 5, 963–980, available at arXiv:q-alg/9704005.

[FWZJ16] P. E. Finch, R. Weston, and P. Zinn-Justin, *Theta function solutions of the quantum Knizhnik–Zamolodchikov–Bernard equation for a face model*, J. Phys. A: Math. Theor. **49** (2016), 064001, available at arXiv:1509.04594.

[Hal88] F. D. M. Haldane, *Exact Jastrow–Gutzwiller resonating-valence-bond ground state of the spin-1/2 antiferromagnetic Heisenberg chain with $1/r^2$ exchange*, Phys. Rev. Lett. **60** (1988), no. 7, 635.

[Hal91] ———, *"Fractional statistics" in arbitrary dimensions: A generalization of the Pauli principle*, Phys. Rev. Lett. **67** (1991), 937–940.

[Has94] K. Hasegawa, *L-operator for Belavin's R-matrix acting on the space of theta functions*, J. Math. Phys **35** (1994), 6158.

[Has97] ———, *Ruijsenaars' commuting difference operators as commuting transfer matrices*, Commun. Math. Phys. **187** (1997), 289, available at arXiv:q-alg/9512029.

[HLYZ24] S. Hu, S. Li, D. Ye, and Y. Zhou, *On the Hilbert space of the Chern–Simons matrix model, deformed double current algebra action, and the conformal limit* (2024), available at arXiv:2409.12486.

[HT13] P. Hauke and L. Tagliacozzo, *Spread of correlations in long-range interacting quantum systems*, Phys. Rev. Lett. **111** (2013Nov), 207202, available at arXiv:1304.7725.

[Ino90] V. I. Inozemtsev, *On the connection between the one-dimensional $s = 1/2$ Heisenberg chain and Haldane–Shastry model*, J. Stat. Phys. **59** (1990), no. 5-6, 1143–1155.

[Ino95] ———, *On the spectrum of $s = 1/2$ XXX Heisenberg chain with elliptic exchange*, J. Phys. A: Math. Gen. **28** (1995), no. 16, L439–L445, available at arXiv:cond-mat/9504096.

[Ino96] ———, *Invariants of linear combinations of transpositions*, Lett. Math. Phys. **36** (1996), no. 1, 55–63.

[JKOS99] M. Jimbo, H. Konno, S. Odake, and J. I. Shiraishi, *Quasi-Hopf twistors for elliptic quantum groups*, Transf. Groups **4** (1999), 303–327, available at arXiv:q-alg/9712029.

[KL22] R. Klabbers and J. Lamers, *How coordinate Bethe ansatz works for Inozemtsev model*, Commun. Math. Phys. **390** (2022), no. 2, 827–905, available at arXiv:2009.14513.

[KL24] ———, *The deformed Inozemtsev spin chain*, SciPost Phys. **17** (2024), 155, available at arXiv:2306.13066.

[KL25] ———, *Landscapes of integrable long-range spin chains*, The 16th MSJ-SI: Elliptic integrable systems, representation theory and hypergeometric functions (Tokyo, Japan, 31 July–4 August 2023), 2025. To appear, available at arXiv:2405.09718.

[Kon03] M. Kontsevich, *Deformation Quantization of Poisson Manifolds*, Lett. Math. Phys. **66** (2003), no. 3, 157–216.

[KS20] P. Koroteev and S. Shakirov, *The quantum DELL system*, Lett. Math. Phys. **110** (2020), no. 5, 969–999, available at arXiv:1906.10354.

[Lam18] J. Lamers, *Resurrecting the partially isotropic Haldane–Shastry model*, Phys. Rev. B. **97** (2018), 214416, available at arXiv:1801.05728.

[LPS22] J. Lamers, V. Pasquier, and D. Serban, *Spin-Ruijsenaars, q-deformed Haldane–Shastry and Macdonald polynomials*, Commun. Math. Phys. **393** (2022), 61–150, available at arXiv:2004.13210.

[LRS24] A. Liashyk, N. Reshetikhin, and I. Sechin, *Quantum integrable systems on a classical integrable background* (2024), available at arXiv:2405.17865.

[LS24] J. Lamers and D. Serban, *From fermionic spin-Calogero–Sutherland models to the Haldane–Shastry spin chain by freezing*, J. Phys. A: Math. Theor. **57** (2024), 235205, available at arXiv:2212.01373.

[MV24] A. V. Mikhailov and P. Vanhaecke, *Commutative Poisson algebras from deformations of noncommutative algebras*, Lett. Math. Phys. **114** (2024), no. 5, 108.

[MZ23a] M. Matushko and A. Zotov, *Anisotropic spin generalization of elliptic Macdonald–Ruijsenaars operators and R-matrix identities*, Ann. H. Poincaré **2023** (2023), available at arXiv:2201.05944.

[MZ23b] ———, *Elliptic generalization of integrable q-deformed Haldane–Shastry long-range spin chain*, Nonlin. **36** (2023), 319, available at arXiv:2202.01177.

[MZ24] ———, *Supersymmetric generalization of q-deformed long-range spin chains of Haldane–Shastry type and trigonometric $gl(n|m)$ solution of associative Yang–Baxter equation*, Nucl. Phys. B **1001** (2024), 116499, available at arXiv:2312.04525.

[Pol93] A. P. Polychronakos, *Lattice integrable systems of Haldane–Shastry type*, Phys. Rev. Lett. **70** (1993), 2329–2331, available at arXiv:hep-th/9210109.

[RS86] S. N. M. Ruijsenaars and H. Schneider, *A new class of integrable systems and its relation to solitons*, Annals Phys. **170** (1986), 370–405.

[Rui04] S. N. M. Ruijsenaars, *Elliptic integrable systems of Calogero–Moser type: A survey*, Proc. Workshop on elliptic integrable systems (2004, Kyoto) (2004).

[Rui87] ———, *Complete integrability of relativistic Calogero–Moser systems and elliptic function identities*, Commun. Math. Phys. **110** (1987), 191–213.

[Sha88] B. S. Shastry, *Exact solution of an $s = 1/2$ Heisenberg antiferromagnetic chain with long-ranged interactions*, Phys. Rev. Lett. **60** (1988), no. 7, 639.

[Sil94] J. H. Silverman, *Advanced Topics in the Arithmetic of Elliptic Curves*, Graduate Texts in Mathematics, vol. 151, Springer, New York, NY, 1994.

[SZ18] I. Sechin and A. Zotov, *R-matrix-valued Lax pairs and long-range spin chains*, Phys. Lett. B **781** (2018), 1–7, available at arXiv:1801.08908.

[TH95] J. C. Talstra and F. D. M. Haldane, *Integrals of motion of the Haldane–Shastry model*, J. Phys. A: Math. Gen. **28** (1995), 2369, available at arXiv:cond-mat/9411065.

[Tho69] D. J. Thouless, *Long-range order in one-dimensional Ising systems*, Phys. Rev. **187** (1969), no. 2, 732.

[Ugl95] D. Uglov, *The trigonometric counterpart of the Haldane–Shastry model* (1995), available at arXiv:hep-th/9508145. preprint.

[WW02] E. T. Whittaker and G. N. Watson, *A course of modern analysis: An introduction to the general theory of infinite series and of analytic functions, with an account of the principal transcendental functions*, Cambridge University Press, 1902.

[ZZ22] A. Zabrodin and A. Zotov, *Field analogue of the Ruijsenaars–Schneider model*, JHEP **2022** (2022), no. 7, 23, available at arXiv:2107.01697.

[a] Humboldt-Universität zu Berlin, Zum Grossen Windkanal 2, 12489 Berlin, Germany, [b] School of Mathematics and Statistics, University of Glasgow, University Place, Glasgow G12 8QQ, UK