Introduction to Dependence Modelling

Carole Bernard

GRENOBLE ECOLE DE MANAGEMENT
TECHNOLOGY & INNOVATION

Berlin, May 2015.

Outline

- Part 1: Introduction
  1. General concepts on dependence.
  2. Extreme Dependence in 2 or $N \geq 3$ dimensions.
  3. Minimizing the expectation of a convex function of a sum.
- Part 2: Application of 2-dimensional results on extreme dependence to portfolio choice and behavioral finance.
- Part 3: Application of N-dimensional results on extreme dependence to risk management problems and model risk assessment under dependence uncertainty.

References for Part 1 (not exhaustive lists)

General references on the topic:

- Quantitative Risk Management, McNeil, Frey, Embrechtsq.
- Frees and Valdez, 1997, (role of copulas in insurance).
- Nelsen, 1999, (standard reference on bivariate copulas).
- Joe, 1997, (on dependence in general).
- Aas, Czado, Frigessi, Bakken “Pair-copula constructions of multiple dependence." IME, 2009.

Specific references

- C. Bernard, X. Jiang and R. Wang (2014). “Risk Aggregation with Dependence Uncertainty”, IME.
- C. Bernard and D. McLeish (2015). “Algorithms for Finding Copulas Minimizing Convex Functions of Sums.” ArXiv.
- P. Embrechts, Puccetti, G. and L. Rüschendorf (2013). “Model uncertainty and VaR aggregation”. JBF.
- B Wang, R Wang (2011). Complete mixability and convex minimization problems with monotone marginal densities, JMVA

1. Modeling Dependence
   - Multivariate Models
   - Copulas
2. Extreme Dependence
   - Theory
   - Rearrangement Algorithm (practice)

- The overall risk of the company/ portfolio can be described as

$$X = X_1 + X_2 + ... + X_N$$

(total risk can be decomposed into risk components)

- In general there are dependencies between risks:
  - Structural
  - Empirical

Structural Dependencies

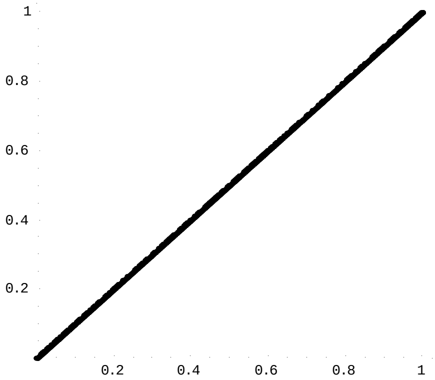
- Loss variables are driven by common variables:
  - Economic factors: inflation drives costs in various lines of insurance
  - Common shocks: an automobile accident can trigger several related claims
  - Uncertain risk variables: long term mortality changes affect all mortality-related insurance/annuities
  - Catastrophes: 9/11 ripple effect over many lines (life, business interruption, health, property, etc)
- Known relationships can be built into internal models

Empirical Dependencies

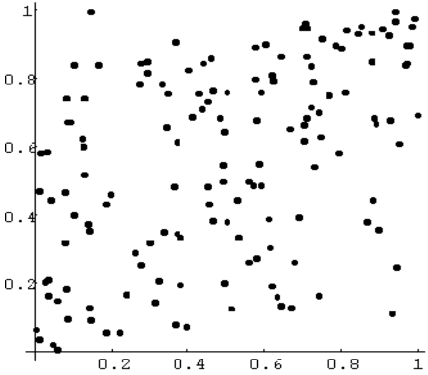
- Observed relationships between lines (usually) without necessarily well-defined cause-effect relationships.
  - Relationships may not be simple.
  - Relationships may not be over entire range of losses.
- In practice, observed relationships are at a macro level
  - Detailed data on relationships is often not available.
  - Detailed data on marginal distributions is available.

Dependence?

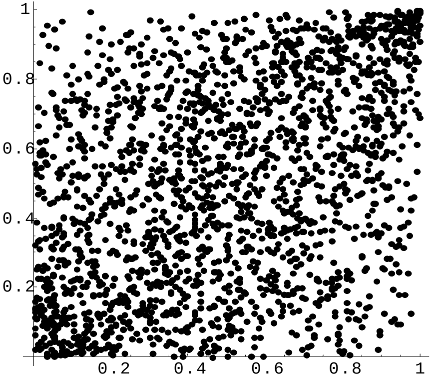
Dependence?



Dependence?



Dependence?



Two Approaches

- Financial and insurance risk models are multivariate
  - But variables are typically *not* independent
- Two common approaches to model multivariate (MV) risks
  1. Factor models, Standard MV models, e.g. MV Normal or MV Student
  2. Model the dependence structure and marginals separately (copula approach)

Multivariate Models

Multivariate Distribution

Let $X = (X_1, \ldots, X_N)'$ be a N-dimensional random vector from d.f.

$$F_X(X) = F_X(x_1, \ldots, x_N) = P(X_1 \leq x_1, \ldots, X_N \leq x_N)$$

Then

- $E(X) := (E(X_1), \ldots, E(X_N))'$ vector
- $Cov(X) := E\left[(X - E(X))(X - E(X))'\right]$ matrix

Further notations

- $Cov(X) = \Sigma$ with each element $\sigma_{ij} = Cov(X_i, X_j)$
- $\rho(X)$: correl. matrix with $\rho_{ij} = \sigma_{ij}/\sqrt{\sigma_{ii}\sigma_{jj}}$

If $\Sigma = Cov(X)$ is positive definite,

- Σ is invertible
- A Cholesky decomposition $\Sigma = AA'$ exists: A Cholesky factor A, is a lower triangular matrix with positive diagonals.
- A is often denoted by $\Sigma^{1/2}$

Multivariate Normal (MVN): Introduction

Definition: $X = (X_1, \ldots, X_N)'$ follows MVN if

$$X \overset{D}{=} \mu + AZ$$

where

- $Z = (Z_1, \ldots, Z_k)'$ is vector of iid univariate standard normal $N(0,1)$ (number of random factors)
- $A \in \mathbb{R}^{N \times k}$ and $\mu \in \mathbb{R}^d$
- Interested in non-singular case $rank(A) = N \leq k$ $\Rightarrow \Sigma$ is invertible

▶To generate a sample X from MVN (μ, Σ)

1. Perform a Cholesky decoposition of Σ to get $\Sigma^{1/2}$
2. Simulate $Z_i \overset{iid}{\sim} N(0,1)$, for $i = 1, 2, \ldots, N$
3. $X = \mu + \Sigma^{1/2} Z$

In Matlab simply use `mvnrnd`

MVN: parameters

MVN is completely characterized by μ and Σ.

- The sample estimates $\bar{X} = \hat{\mu}$ and $S = \hat{\Sigma}$ are the MLEs of μ and Σ, respectively
- MVN plays a central role in MV modeling

However, MVN itself is not the best model for financial and insurance data fitting

- Marginal distribution tails are symmetric and too short
- dependence structure too restrictive (see Fig 3.1 next slide)

▶Extension to normal mixture models, normal variance model...

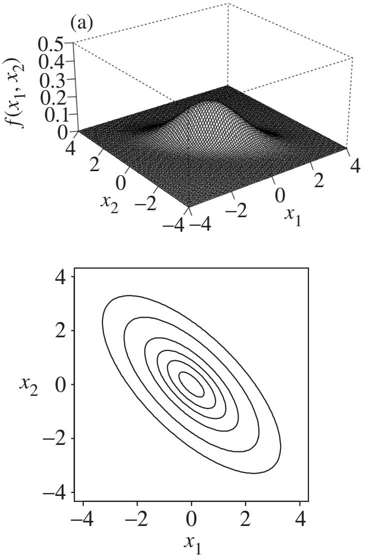


Figure 3.1. (a) Perspective and contour plots for the density of a bivariate normal distribution with standard normal margins and correlation -70%. (b) Corresponding plots for a bivariate *t* density with four degrees of freedom (see Example 3.7 for details) and the *same mean vector and covariance matrix* as the normal distribution. Contour lines are plotted at the same heights for both densities.

Copulas

Introduction to copulas...

Copulas will help us to separate the problem of choosing the dependence structure from the identification of the correct marginal behavior.

Example: Suppose that you want to model (X_1, X_2) so that $X_1, X_2 \sim N(0, 1)$ but you don't know how their dependence should be modeled. That is, you know the marginal distribution of each of X_1 and X_2 but don't know what the joint CDF $F(x_1, x_2) = P(X_1 \leq x_1, X_2 \leq x_2)$ should be.

Choice 1: X_1 and X_2 are independent.

In that case

$$F(x_1, x_2) = P(X_1 \leq x_1)P(X_2 \leq x_2) = \phi(x_1)\phi(x_2),$$

where $\phi(x) = P(N(0,1) \leq x)$. We can instead write

$$F(x_1, x_2) = C_{ind}(\phi(x_1), \phi(x_2)),$$

where $C_{ind} : [0,1]^2 \to [0,1]$ is defined by

$$C(u_1, u_2) =$$

and is called the *independence copula*.

Choice 2: X_1 and X_2 are defined so that $X_2 = -X_1$.

In that case

$$
\begin{aligned}
F(x_1, x_2) &= P(X_1 \leq x_1, X_1 \geq -x_2) \\
&= \begin{cases} \phi(x_1) - \phi(-x_2) & \text{if } x_1 \geq -x_2 \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} \phi(x_1) + \phi(x_2) - 1 & \text{if } \phi(x_1) \geq 1 - \phi(x_2) \\ 0 & \text{otherwise} \end{cases} \\
&= C_{neg}(\phi(x_1), \phi(x_2))
\end{aligned}
$$

where $C_{neg}(u_1, u_2) =$ is the *negative dependence copula* (antimonotonic copula).

E1. Modelling Dependence with Copulas

On Uniform Distributions

Lemma 1: probability transform

Let X be a random variable with continuous distribution function F. Then $F(X) \sim U(0,1)$ (standard uniform).

$$P(F(X) \le u) = P(X \le F^{-1}(u)) = F(F^{-1}(u)) = u, \quad \forall u \in (0,1).$$

Lemma 2: quantile transform

Let U be uniform and F the distribution function of any rv X. Then $F^{-1}(U) \stackrel{\text{d}}{=} X$ so that $P(F^{-1}(U) \le x) = F(x)$.

These facts are the key to all statistical simulation and essential in dealing with copulas.

Copula: definition

Let us give the general definition of a copula.

- A copula is a multivariate distribution on the unit N-dimensional cube with uniform $(0, 1)$ marginal distributions.

Definition

A copula C is a joint distribution function for a vector $(U_1, \dots, U_m)$ of random variables that each has a marginal $U(0, 1)$ distribution. I.e.,

$$C(u_1, \dots, u_m) = P(U_1 \leq u_1, \dots, U_m \leq u_m)$$

for a vector $(U_1, \dots, U_m)$ such that $P(U_i \leq u_i) = u_i$, for $0 \leq u_i \leq 1$.

Examples:

- **Independence copula**: $C_{ind}(u_1, u_2) = u_1 u_2$ is a copula for the case where U_1 and U_2 are independent (Their joint distribution is
$P(U_1 \leq u_1, U_2 \leq u_2) = P(U_1 \leq u_1)P(U_2 \leq u_2) = u_1 u_2$).
- If $U_1 \sim U(0,1)$ and $U_2 = 1 - U_1$, then $U_2 \sim U(0,1)$ and the joint CDF of (U_1, U_2) is

$$\begin{aligned} P(U_1 \leq u_1, U_2 \leq u_2) &= P(U_1 \leq u_1, 1 - U_1 \leq u_2) \\ &= \begin{cases} u_1 + u_2 - 1 & \text{if } 1 - u_2 \leq u_1 \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Therefore $C_{neg}(u_1, u_2) = \max(0, u_1 + u_2 - 1)$ is a copula (the **negative dependence copula** (antimonotonic copula) we described before).

Be careful!

At this point, you might think that any function $C(u_1, u_2)$ from $[0, 1]^2$ to $[0, 1]$ is a copula. Here is an example to show it's not the case.

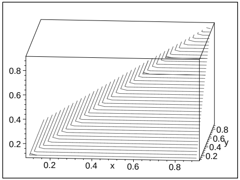
Example: Is $C(u_1, u_2) = u_1 + u_2$ a copula?

Be careful!

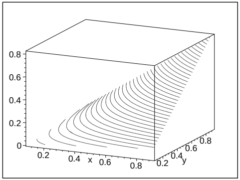
At this point, you might think that any function $C(u_1, u_2)$ from $[0, 1]^2$ to $[0, 1]$ is a copula. Here is an example to show it's not the case.
Example: Is $C(u_1, u_2) = u_1 + u_2$ a copula?

No, first of all we must have $0 \leq C(u_1, u_2) \leq 1$, which is not the case here. Second, we must have that $C(u_1, 1) = P(U_1 \leq u_1, U_2 \leq 1) = P(U_1 \leq u_1) = u_1$, but this is not case here since $C(u_1, 1) = u_1 + 1$.

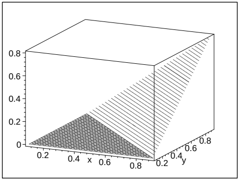
EXAMPLES



The Fréchet-Hoeffding upper bound, or the comonotonicity copula $\mathcal{M}$



The independence copula $\mathbf{\Pi}$



The Fréchet-Hoeffding lower bound, or the countermonotonicity copula $\mathcal{W}$

- Fréchet-Hoeffding inequality

$$\mathcal{W}(\boldsymbol{u}) \leq \mathcal{C}(\boldsymbol{u}) \leq \mathcal{M}(\boldsymbol{u}) \quad (3)$$

Simulation technique for these 3 dependencies...

How to get (X, Y) with the right marginal distribution F_X and F_Y and with copula C being the anti-monotonic copula? the comonotonic copula and the independence copula?

Properties of Copulas

- Invariance: The copula C is invariant under increasing transformations of the marginals: $f_1(X_1), ..., f_N(X_N)$ has the same copula as $(X_1, ..., X_N)$ if for all i, f_i are strictly increasing.
- Note that when C is a copula for (U, V), then for any v in $[0, 1]$, the partial derivative $\frac{\partial C(u,v)}{\partial u}$ exists for almost all u, and for such v and u,

$$0 \leq \frac{\partial C(u, v)}{\partial u} \leq 1.$$

  This is theorem 2.2.7 from Nelsen (2006), page 13.

- ▶ Interpretation of this derivative as a conditional distribution...

$$P(V \leq v | U = u)$$

General Simulation Techniques

To simulate (X, Y) with respective marginal cdf F_X and F_Y and joint copula C, one can proceed as follows.

1. Generate u and t, two independent uniform on $(0, 1)$.
2. Set $v = C_u^{-1}(t)$ where $C_u(v) = \frac{\partial C(u,v)}{\partial u}$. This derivative can be interpreted as $Q(V \leq v \mid U = u)$, the conditional distribution for V given $U = u$. Then u and v are uniformly distributed and linked with the copula C.
3. Set $x = F_X^{-1}(u)$ and $y = F_Y^{-1}(v)$. Then x and y are a random draw of the couple (X, Y).

The inverse functions are “pseudo-inverses” :

$$F^{-1}(t) = \inf\{x \mid F(x) \geq t\}$$

For more details see Nelsen (2006), page 41, section 2.9.
More material on copulas (appendix)

Extreme Dependence in 2 dimensions and in $N \geq 3$ dimensions

Dependence Uncertainty

Consider a joint portfolio $S = X_1 + \cdots + X_N$ of risky assets $X_1, \cdots, X_N$.

- The distribution of each X_i is modelled with statistical or financial tools.
- The dependence structure among $X_1, \cdots, X_N$ is unknown.
- *Marginal:* easier to statistically estimate/model/test.
  Dependence: difficult to estimate/model/test.

Fréchet Class

- Let $\mathbf{X} = (X_1, \cdots, X_N)$. Define the *Fréchet class*

$$\mathfrak{F}_N(F_1, \cdots, F_N) = \{\mathbf{X} : \; X_i \sim F_i, \; i = 1, \cdots, N\}.$$

  $\mathfrak{F}_N(F_1, \cdots, F_N)$ is the set of random vectors with a given marginal distributions $F_1, \cdots, F_N$.
- Extensively studied, sometimes using copulas.
- We want to know something about $S = X_1 + \cdots + X_N$ when $\mathbf{X} \in \mathfrak{F}_N(F_1, \cdots, F_N)$.

For example, let X_i be the price of stock i at the end of a period, then an European basket call option price is given by

$$\mathbb{E}_Q[(S - K)_+].$$

What can we tell about this price without knowing the dependence?

Q. To be more general, find bounds on

$$\mathbb{E}[g(S)], \quad \mathbf{X} \in \mathfrak{F}_N(F_1, \cdots, F_N)$$

for g being a convex function.
$\mathbb{E}[g(S)]$ is called a convex expectation. Why convex/concave functions?

- $\mathbb{E}[g(S)]$ includes important quantities such as
  - the variance, European option prices,
  - the stop-loss premium, the excess of loss,
  - a class of convex risk measures, and it is closely related to the risk measure TVaR,
  - Risk-avoiding/risk-seeking expected utility.
- Convex ordering/optimization.
- Nice mathematical properties.

definition: Convex order

X is smaller in convex order, $X \prec_{cx} Y$, if for all convex functions f

$$E[f(X)] \leq E[f(Y)]$$

Assume first that we trust the marginals $X_i \sim F_i$ but that we have no trust about the dependence structure between the X_i (copula).

definition: Convex order

X is smaller in convex order, $X \prec_{cx} Y$, if for all convex functions f

$$E[f(X)] \leq E[f(Y)]$$

Assume first that we trust the marginals $X_i \sim F_i$ but that we have no trust about the dependence structure between the X_i (copula).

Extreme Dependence with $N = 2$ Risks

In two dimensions, we have the following Fréchet-Hoeffding bounds or "extreme dependence".

$$F_1^{-1}(U) + F_2^{-1}(1-U) \prec_{cx} X_1 + X_2 \prec_{cx} F_1^{-1}(U) + F_2^{-1}(U)$$

Useful to build optimal portfolios (Part 2) and to assess dependence uncertainty (Part 3).

Dependence Uncertainty with $N = 2$ Risks

- For risk measures preserving convex order ($\rho(S) = var(S)$, $\rho(S) = E[g(S)]$ with convex g, $\rho(S) = TVaR(S)$), for $U \sim \mathcal{U}(0,1)$

$$\rho\left(F_1^{-1}(U) + F_2^{-1}(1-U)\right) \leq \rho(S) \leq \rho\left(F_1^{-1}(U) + F_2^{-1}(U)\right)$$

This does not apply to Value-at-Risk.

Dependence Uncertainty with $N = 2$ Risks

- For risk measures preserving convex order ($\rho(S) = var(S)$, $\rho(S) = E[g(S)]$ with convex g, $\rho(S) = TVaR(S)$), for $U \sim \mathcal{U}(0,1)$

$$\rho\left(F_1^{-1}(U) + F_2^{-1}(1-U)\right) \leq \rho(S) \leq \rho\left(F_1^{-1}(U) + F_2^{-1}(U)\right)$$

This does not apply to Value-at-Risk.

Example: $X_1 \sim \mathcal{N}(0,1)$ and $X_2 \sim \mathcal{N}(0,1)$ with distribution Φ

$$std\left(\Phi^{-1}(U) + \Phi^{-1}(1-U)\right) = 0$$
$$\leq std(S) \leq std(\Phi^{-1}(U) + \Phi^{-1}(U)) = 2$$

Issue: Wide bounds! Huge model risk...

Fréchet Hoeffding Bounds

In terms of copulas, the two extreme dependencies correspond to piecewise minimum and maximum over all possible copulas C:

$$\max(\mathbf{u}+\mathbf{v}-\mathbf{1},\mathbf{0}) \leq C(u,v) \leq \min(u,v)$$

(Fréchet-Hoeffding Bounds for copulas) (**anti-monotonic** copula as a lower bound)

Constrained Fréchet Hoeffding bounds in 2 dims

Applications in Part 2

Interesting to find bounds that are copulas as they are "best-possible" bounds... and they are attained.

- Portfolio selection problems
- Inferring the utility function of investors
- Designing strategies that are independent of the market when the market crashes...

Extensions in higher dimensions

Some extensions are possible:

- In general, it is difficult to find the “extreme copulas”: They may not even exist...
- Also, the optimization of a real-world problem such as minimizing a risk measure or the budget needed in a portfolio when it involves some constraints, may lead to a dependence structure that depends on the margins...
- It may thus be reasonable to consider *not to disentangle dependence and margins* but to work directly with the joint distribution.

Assessing Model Risk on Dependence with $N \geq 3$ Risks

- Fréchet upper bound : comonotonic scenario:
  $X_1 + X_2 + ... + X_N \prec_{cx} F_1^{-1}(U) + F_2^{-1}(U) + ... + F_N^{-1}(U)$
- In $N \geq 3$ dims, the Fréchet lower bound does not exist: It depends on F_1, F_2,..., F_N (Wang-Wang (2011, 2014))

(More details on CM in appendix)

Assessing Model Risk on Dependence with $N \geq 3$ Risks

- Fréchet upper bound : comonotonic scenario:
  $X_1 + X_2 + ... + X_N \prec_{cx} F_1^{-1}(U) + F_2^{-1}(U) + ... + F_N^{-1}(U)$
- In $N \geq 3$ dims, the Fréchet lower bound does not exist: It depends on F_1, F_2,..., F_N (Wang-Wang (2011, 2014))
  (More details on CM in appendix)
- In N dimensions
  - Puccetti and Rüschendorf (2012, JCAM): algorithm (RA) to approximate bounds on functionals.
  - Embrechts, Puccetti, Rüschendorf (2013, JBF): application of the RA to find bounds on VaR
  - Bernard, Jiang, Wang (2014, IME): explicit form of the lower bound for convex risk measures of an homogeneous sum.
- Issues
  - bounds are generally very wide
  - ignore all information on dependence.

Incorporating Partial Information on Dependence

- With $N = 2$:
  - subset of bivariate distribution with given measure of association Nelsen et al. (2001, 2004)
  - bounds for bivariate dfs when there are constraints on the values of its quartiles (Nelsen et al. (2004)).
  - 2-dim copula known on a subset of $[0, 1]^2 \Rightarrow$ find "improved Fréchet bounds", Tankov (2011), Bernard et al. (2012) and Sadooghi-Alvandi et al. (2013).
- With $N \geq 3$: Bounds on the VaR of the sum
  - with known bivariate distributions: Embrechts, Puccetti and Rüschendorf (2013)
  - with the variance of the sum (WP with Rüschendorf,Vanduffel)
  - with higher moments (WP with Denuit, Vanduffel)
  - with the joint distribution known on a subset (JBF with Vanduffel)

Rearrangement Algorithm

$N = 4$ observations of $N = 3$ variables: X_1, X_2, X_3

$$\mathrm{M} = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 6 & 3 \\ 4 & 0 & 0 \\ 6 & 3 & 4 \end{bmatrix}$$

Each column: **marginal** distribution
Interaction among columns: **dependence** among the risks

Same marginals, different dependence

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 6 & 3 \\ 4 & 0 & 0 \\ 6 & 3 & 4 \end{bmatrix} \qquad S_N = \overset{X_1 + X_2 + X_3}{\begin{bmatrix} 4 \\ 9 \\ 4 \\ 13 \end{bmatrix}}$$

$$\begin{bmatrix} 6 & 6 & 4 \\ 4 & 3 & 3 \\ 1 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \qquad S_N = \overset{X_1 + X_2 + X_3}{\begin{bmatrix} 16 \\ 10 \\ 3 \\ 0 \end{bmatrix}}$$

Aggregate Risk with Maximum Variance

comonotonic scenario

Rearrangement Algorithm: Sum with Minimum Variance

minimum variance with $N = 2$ risks X_1 and X_2

Antimonotonicity: $var(\mathbf{X_1^a} + X_2) \leq var(\mathbf{X_1} + X_2)$

How about in $N \geq 3$ dimensions?

Rearrangement Algorithm: Sum with Minimum Variance

minimum variance with $N = 2$ risks X_1 and X_2

Antimonotonicity: $var(\mathbf{X_1^a} + X_2) \leq var(\mathbf{X_1} + X_2)$

How about in $N \geq 3$ dimensions?
Use of the rearrangement algorithm on the original matrix M.

Aggregate Risk with Minimum Variance

- Columns of M are rearranged such that they become anti-monotonic with the sum of all other columns.

$$\forall k \in \{1, 2, ..., N\}, \mathbf{X_k^a} \textit{ antimonotonic with } \sum_{j \neq k} X_j$$

- After each step, $var\left(\mathbf{X_k^a} + \sum_{j \neq k} X_j\right) \leq var\left(\mathbf{X_k} + \sum_{j \neq k} X_j\right)$ where $\mathbf{X_k^a}$ is antimonotonic with $\sum_{j \neq k} X_j$

Aggregate risk with minimum variance
Step 1: First column

$$\begin{array}{c} \downarrow \\ \begin{bmatrix} 6 & 6 & 4 \\ 4 & 3 & 2 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \end{array} \quad \begin{array}{c} X_2 + X_3 \\ 10 \\ 5 \\ 2 \\ 0 \end{array} \quad \text{becomes} \quad \begin{bmatrix} 0 & 6 & 4 \\ 1 & 3 & 2 \\ 4 & 1 & 1 \\ 6 & 0 & 0 \end{bmatrix}$$

Aggregate risk with minimum variance

$$\begin{array}{c} \downarrow \\ \left[\begin{array}{ccc} \mathbf{6} & 6 & 4 \\ \mathbf{4} & 3 & 2 \\ \mathbf{1} & 1 & 1 \\ \mathbf{0} & 0 & 0 \end{array}\right] \end{array} \quad \begin{array}{c} X_2 + X_3 \\ 10 \\ 5 \\ 2 \\ 0 \end{array} \quad \text{becomes} \quad \left[\begin{array}{ccc} \mathbf{0} & 6 & 4 \\ \mathbf{1} & 3 & 2 \\ \mathbf{4} & 1 & 1 \\ \mathbf{6} & 0 & 0 \end{array}\right]$$

$$\begin{array}{c} \quad\downarrow \\ \left[\begin{array}{ccc} 0 & \mathbf{6} & 4 \\ 1 & \mathbf{3} & 2 \\ 4 & \mathbf{1} & 1 \\ 6 & \mathbf{0} & 0 \end{array}\right] \end{array} \quad \begin{array}{c} X_1 + X_3 \\ 4 \\ 3 \\ 5 \\ 6 \end{array} \quad \text{becomes} \quad \left[\begin{array}{ccc} 0 & \mathbf{3} & 4 \\ 1 & \mathbf{6} & 2 \\ 4 & \mathbf{1} & 1 \\ 6 & \mathbf{0} & 0 \end{array}\right]$$

$$\begin{array}{c} \qquad\quad\downarrow \\ \left[\begin{array}{ccc} 0 & 3 & \mathbf{4} \\ 1 & 6 & \mathbf{2} \\ 4 & 1 & \mathbf{1} \\ 6 & 0 & \mathbf{0} \end{array}\right] \end{array} \quad \begin{array}{c} X_1 + X_2 \\ 3 \\ 7 \\ 5 \\ 6 \end{array} \quad \text{becomes} \quad \left[\begin{array}{ccc} 0 & 3 & \mathbf{4} \\ 1 & 6 & \mathbf{0} \\ 4 & 1 & \mathbf{2} \\ 6 & 0 & \mathbf{1} \end{array}\right]$$

Aggregate risk with minimum variance

Each column is antimonotonic with the sum of the others:

$$\begin{array}{c} \downarrow \\ \begin{bmatrix} 0 & 3 & 4 \\ 1 & 6 & 0 \\ 4 & 1 & 2 \\ 6 & 0 & 1 \end{bmatrix} \end{array} \begin{array}{c} X_2+X_3 \\ 7 \\ 6 \\ 3 \\ 1 \end{array}, \quad \begin{array}{c} \downarrow \\ \begin{bmatrix} 0 & 3 & 4 \\ 1 & 6 & 0 \\ 4 & 1 & 2 \\ 6 & 0 & 1 \end{bmatrix} \end{array} \begin{array}{c} X_1+X_3 \\ 4 \\ 1 \\ 6 \\ 7 \end{array}, \quad \begin{array}{c} \downarrow \\ \begin{bmatrix} 0 & 3 & 4 \\ 1 & 6 & 0 \\ 4 & 1 & 2 \\ 6 & 0 & 1 \end{bmatrix} \end{array} \begin{array}{c} X_1+X_2 \\ 3 \\ 7 \\ 5 \\ 6 \end{array}$$

Aggregate risk with minimum variance

Each column is antimonotonic with the sum of the others:

$$\begin{array}{c} \downarrow \\ \begin{bmatrix} 0 & 3 & 4 \\ 1 & 6 & 0 \\ 4 & 1 & 2 \\ 6 & 0 & 1 \end{bmatrix} \end{array} \begin{array}{c} X_2+X_3 \\ 7 \\ 6 \\ 3 \\ 1 \end{array}, \quad \begin{array}{c} \downarrow \\ \begin{bmatrix} 0 & 3 & 4 \\ 1 & 6 & 0 \\ 4 & 1 & 2 \\ 6 & 0 & 1 \end{bmatrix} \end{array} \begin{array}{c} X_1+X_3 \\ 4 \\ 1 \\ 6 \\ 7 \end{array}, \quad \begin{array}{c} \downarrow \\ \begin{bmatrix} 0 & 3 & 4 \\ 1 & 6 & 0 \\ 4 & 1 & 2 \\ 6 & 0 & 1 \end{bmatrix} \end{array} \begin{array}{c} X_1+X_2 \\ 3 \\ 7 \\ 5 \\ 6 \end{array}$$

$$\begin{bmatrix} 0 & 3 & 4 \\ 1 & 6 & 0 \\ 4 & 1 & 2 \\ 6 & 0 & 1 \end{bmatrix} \qquad S_N = \overset{X_1+X_2+X_3}{\begin{bmatrix} 7 \\ 7 \\ 7 \\ 7 \end{bmatrix}}$$

The minimum variance of the sum is equal to 0! (ideal case of a constant sum (*complete mixability*, Wang - Wang (2011))

Improvement of the algorithm (with D. McLeish)

Necessary condition for a minimum in convex order

If $\sum_{i=1}^{N} X_i$ has minimum variance then corr $\left(\sum_{i\in\Pi} X_i, \sum_{i\in\bar{\Pi}} X_i\right)$ is minimized for every partition into two sets Π and $\bar{\Pi}$. However, the converse does not hold in general.

The RA can be implemented per “block” to design a Block RA so that

- at each step, $\sum_{j\in\Pi} X_j$ and $\sum_{j\in\bar{\Pi}} X_j$ are made countermonotonic

An index of convex order

Definition (Mutivariate dependence measure)

Let $\phi(X_1, X_2)$ be a measure of dependence between two columns of data X_1 and X_2 such as Spearman's rho, Kendall's tau, or Pearson correlation coefficient. For a matrix of data $X = [X_1, X_2, ..., X_{n-1}, X_N]$ with N columns, we define the multivariate measure of dependence

$$\varrho(X) := \frac{1}{2^{N-1}-1} \sum_{\Pi \in \mathcal{P}} \phi\left(\sum_{i \in \Pi} X_i, \sum_{i \in \bar{\Pi}} X_i \right) \qquad (1)$$

where the sum is over the set $\mathcal{P}$ consisting of $2^{N-1}-1$ distinct partitions of $\{1, 2, ..., N\}$ into **non-empty** subsets Π and its complement $\bar{\Pi}$.

interpretation as a measure of convex order in N dimensions.

Complexity and comments on the algorithm

- Easy to estimate it by computing the average over a subset of partitions.
- Use of this multivariate measure as a stopping rule for the Block RA
- Finding a copula achieving the minimum of the variance for instance is a NP complete problem.
- There exists no efficient algorithm in polynomial time.
- Our preliminary results with D. McLeish show that an algorithm that perform well in probability can be designed: probability to get to the global minimum may be small but error is typically very small...

Numerous applications in Part 3

- **Bounds on Value-at-Risk** (Embrechts et al. 2013, Journal of Banking and Finance)
- **Bounds on convex risk measures** (with X. Jiang and R. Wang IME 2014)
- Quantifying **model risk** (with M. Denuit, L. Rüschendorf, S. Vanduffel)
- **Infer the dependence structure among *N* variables** that is consistent with marginal distributions and the distribution of the sum (with S. Vanduffel)

Application: Toy example in B. and McLeish (2015)

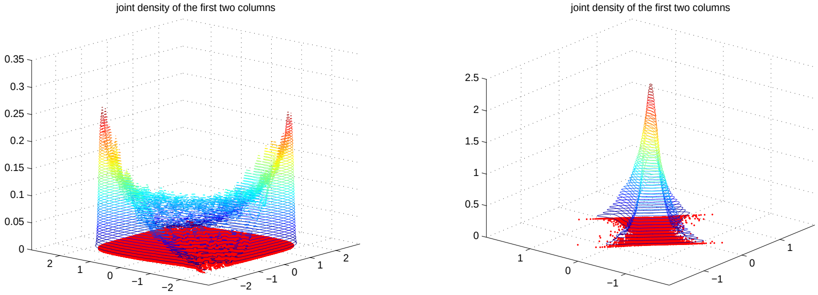


Figure: Left: Joint density of two $\mathcal{U}[-2.056, 2.056]$ random variables whose sum is $\mathcal{N}(0, 1)$. Right: Joint density of the two marginally normal random variables $\mathcal{N}(0, 0.3363^2)$ whose sum is $\mathcal{U}[-1, 1]$.

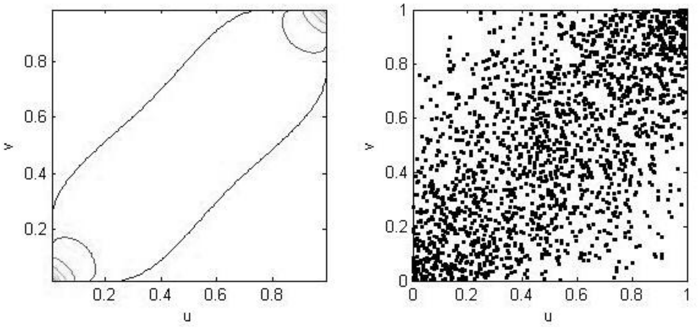
Appendix

- More on copulas here
- More on complete mixability here
- Constrained Fréchet Bounds here

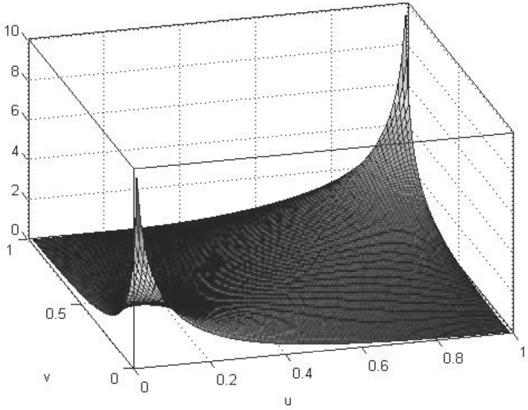
Copulas (additional comments)

(Back to presentation)

Normal (Gaussian) copula



Normal (Gaussian) copula



5

Example: Gauss copula

$$C(u, v, \rho) = N_2(N^{-1}(u), N^{-1}(v), \rho)$$

where N_2 is the bivariate cdf and N is the cdf of $N(0, 1)$.

$$\frac{\partial C}{\partial u}(u, v, \rho) = N\left[\frac{N^{-1}[v] - \rho N^{-1}[u]}{\sqrt{1-\rho^2}}\right]$$

$$\frac{\partial C}{\partial v}(u, v, \rho) = N\left[\frac{N^{-1}[u] - \rho N^{-1}[v]}{\sqrt{1-\rho^2}}\right]$$

Simulation of the Gaussian copula

(you can use the simulation of a multivariate Gaussian distribution)
How to get (X, Y) with the right marginal distribution F_X and F_Y and with copula C being the Gaussian copula with correlation coefficient ρ?

It is also easy to get the multivariate student copula, and more generally elliptical copulas.

Archimedean Copulas $d = 2$

These have simple closed forms and are useful for calculations. However, higher dimensional extensions are not rich in parameters.

- Gumbel Copula

$$C_\beta^{Gu}(u_1, u_2) = \exp\left(- \left((-\log u_1)^\beta + (-\log u_2)^\beta \right)^{1/\beta} \right).$$

$\beta \geq 1$: $\beta = 1$ gives independence; $\beta \to \infty$ gives comonotonicity.

- Clayton Copula

$$C_\beta^{Cl}(u_1, u_2) = \left(u_1^{-\beta} + u_2^{-\beta} - 1 \right)^{-1/\beta}.$$

$\beta > 0$: $\beta \to 0$ gives independence ; $\beta \to \infty$ gives comonotonicity.

Archimedean Copulas in Higher Dimensions

All our Archimedean copulas have the form

$$C(u_1, u_2) = \psi^{-1}(\psi(u_1) + \psi(u_2)),$$

where $\psi : [0, 1] \mapsto [0, \infty]$ is strictly decreasing and convex with $\psi(1) = 0$ and $\lim_{t \to 0} \psi(t) = \infty$.
The simplest higher dimensional extension is

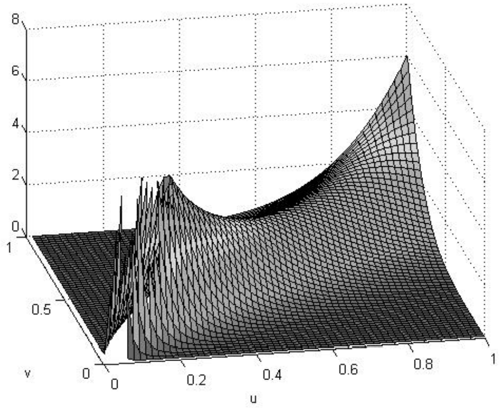
$$C(u_1, \ldots, u_d) = \psi^{-1}(\psi(u_1) + \cdots + \psi(u_d)).$$

Example: Gumbel copula: $\psi(t) = -(\log(t))^\beta$

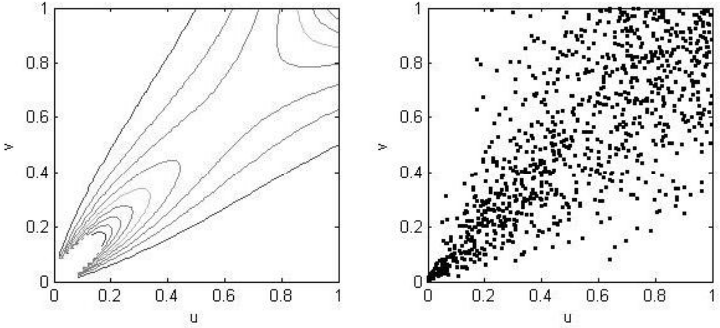
$$C_\beta^{\mathsf{Gu}}(u_1, \ldots, u_d) = \exp\left(-\left((-\log u_1)^\beta + \cdots + (-\log u_d)^\beta\right)^{1/\beta}\right).$$

These copulas are exchangeable (invariant under permutations).

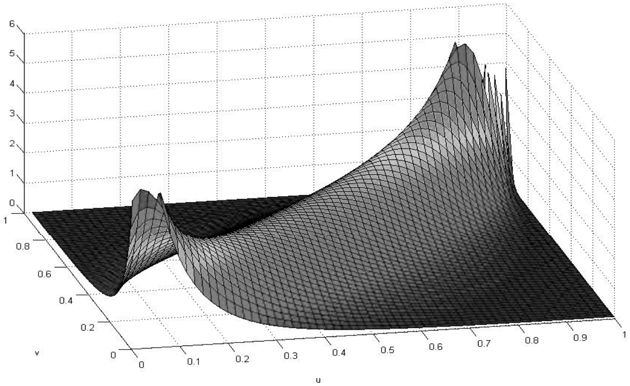
Clayton copula



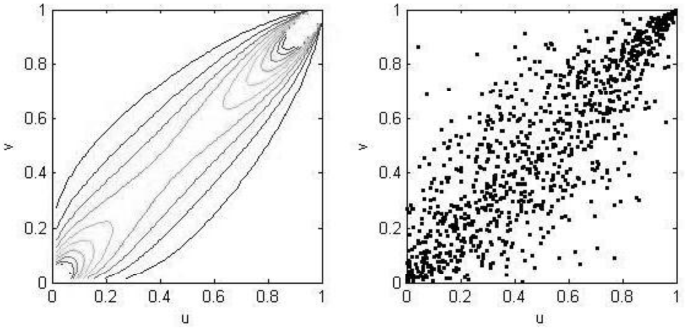
Clayton copula



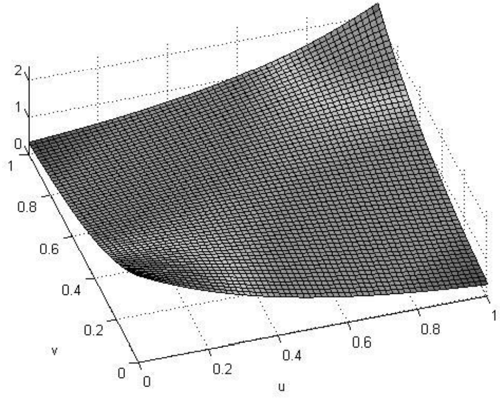
Gumbel copula



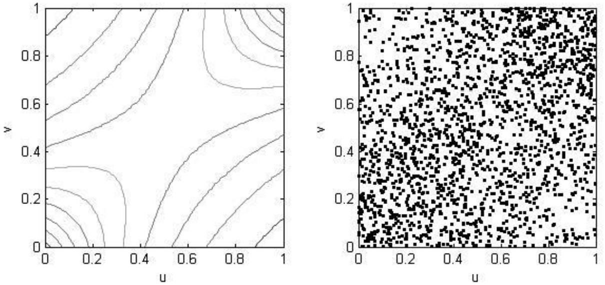
Gumbel copula



Frank copula



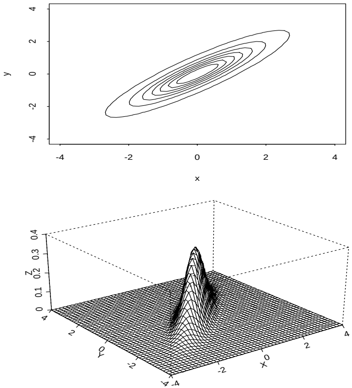
Frank copula



Copulas with R

- Use of the package copula.
- Representation of the copula on a bivariate normal distribution rather than on a uniform distribution.
- It could be more visual.
- See next slide for the bivariate normal case.

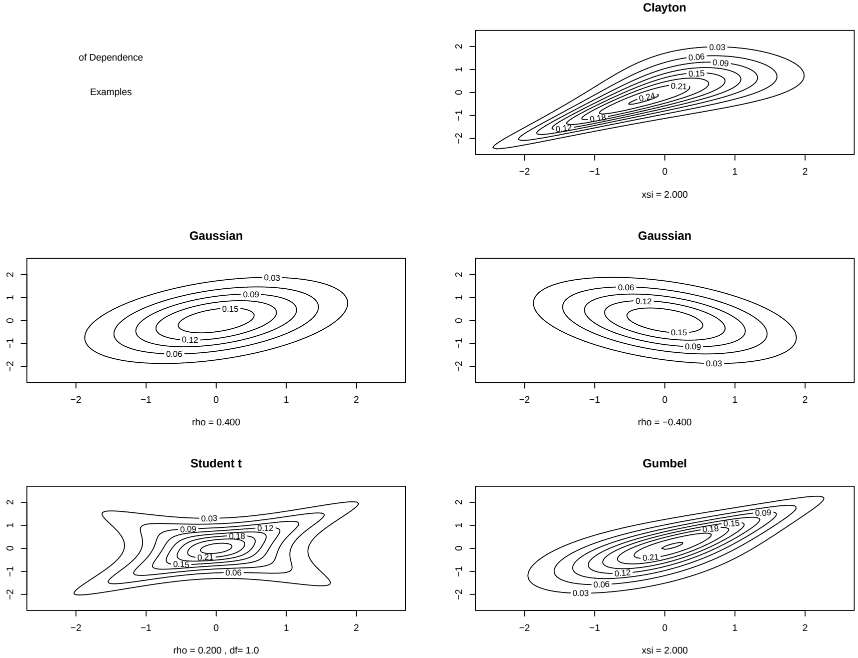
Bivariate Standard Normals



In left plots $\rho = 0.9$; in right plots$\rho = -0.7$.

of Dependence

Examples



Back to presentation

Back to appendix outline

Theory
- Complete Mixability

(Back to presentation)

Homogeneous Case

Convex expectation for positive risks

Now we consider the homogeneous case when $F_1 = \cdots = F_N = F$ is a distribution on $\mathbb{R}^+$ with a finite mean.

Q'. Find

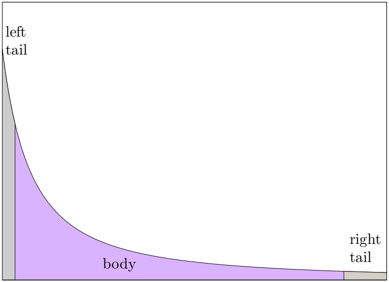
$$\inf_{\mathbf{X}\in\mathfrak{F}_N(F,\cdots,F)} \mathbb{E}[g(S)]$$

for g being a convex function and F on $\mathbb{R}^+$.

- Generally speaking, with the optimal structure the density of S should be concentrated as much as possible due to the convexity of g.

Observations.

- A decreasing density is CM (i.e. S could be a constant, proved by Wang and Wang, 2011) constrained in the middle part (body).
- To enhance concentration, when one of $\{X_i\}$ is very large (right tail), the others should be small (left tail).



Possible lower bound

Consider a dependence scenario that

- divides the probability space into two parts:

  (tails) when one of $\{X_i\}$ is large, all the other $\{X_i\}$ are small;

  (body) when one of $\{X_i\}$ is of medium size, treat S as a constant equal to its conditional expectation;

- make sure that the value of S is larger at the tails than at the body.

Define $H(x)$ and $D(a)$ for $x, a \in [0, \frac{1}{n}]$:

$$H(x) = (n-1)F^{-1}((n-1)x) + F^{-1}(1-x),$$

$$D(a) = \frac{n}{1-na}\int_a^{\frac{1}{n}} H(x)\underset{\cdot}{x} = \frac{n\int_{(n-1)a}^{1-a} F^{-1}(y)dy}{1-na}.$$

$H(x)$ gives the values of S at the tails and $D(a)$ is the value of S at the body.

Theorem (Lower bound for homogeneous risks)

For $a \in [0, \frac{1}{N}]$, suppose $H(x)$ is non-increasing on the interval $[0, a]$ and $\lim_{x \to a+} H(x) \geq D(a)$, then

$$\inf_{\mathbf{X} \in \mathfrak{F}_N(F, \cdots, F)} \mathbb{E}[g(S)] \geq N \int_0^a g(H(x)) \underset{\cdot}{x} + (1 - Na) g(D(a)). \quad (2)$$

Moreover, $g(k) := N \int_0^k f(H(x)) \underset{\cdot}{x} + (1 - Nk) f(D(k))$ is a non-decreasing function of k on $[0, a]$ so that the most accurate lower bound of $\mathbb{E}[f(S)]$ is obtained with the largest possible a.

Possible optimal structure

If possible, choose a dependence structure (copula Q_N^F) that

- divides the probability space into two parts: tails (with probability Na) and body (with probability $(1 - Na)$).
- makes a as large as possible. Define $c_N = \min\left\{c \in [0, \frac{1}{n}] : H(c) \leq D(c)\right\}$.
    - c_N is the largest possible a satisfying $\lim_{x \to a+} H(x) \geq D(a)$.
    - When F is a continuous distribution, $H(c_N) = D(c_N)$.
    - c_N is exactly the smallest possible a such that F on $I = [F^{-1}((N-1)a), F^{-1}(1-a)]$ satisfies the mean condition for CM (hence, a constant S is possible).
- When $N = 2$, this is automatically the Fréchet lower copula.

Theorem (Sharp lower bound for homogeneous risks)

Suppose
(A) $H(x)$ *is non-increasing on the interval* $[0, c_N]$*,*
then

$$\inf_{\mathbf{X} \in \mathfrak{F}_N(F,\cdots,F)} \mathbb{E}[g(S)] \geq N \int_0^{c_N} g(H(x))\underset{\cdot}{x} + (1 - Nc_N) g(D(c_N)). \quad (3)$$

Moreover, the equality in (3) *holds if*
(B) F *is* N*-CM on the interval*
$I = [F^{-1}((N-1)c_N), F^{-1}(1-c_N)]$.

I am sure you are wondering how conditions (A) and (B) are satisfied.

- For F with a decreasing density, we can show that (A) and (B) hold (Wang and Wang, 2011).
- Condition (A) is very easy to check. If $H(x)$ is convex, then (A) is satisfied.
- Knowledge of condition (B) for general distributions is very limited, need to use numerical techniques.

Back to appendix outline, Back to presentation

Constrained Fréchet Bounds in 2 dimensions

Back to presentation
Back to appendix outline

Constrained Fréchet Hoeffding Bounds

Let $\mathbb{S}$ be a set of constraints. The question is whether there exists a minimum copula B (or a maximum copula) satisfying $\mathbb{S}$ such that $\mathbf{B} \leq \mathbf{C}$ (pointwise) for all other copulas C satisfying $\mathbb{S}$.
Recall that $Q : [0,1]^2 \to [0,1]$ is a quasi-copula if it satisfies the following three properties.

1. For all $u \in [0,1]$, $Q(0,u) = Q(u,0) = 0$, and $Q(1,u) = Q(u,1) = u$ (boundary conditions).
2. Q is non-decreasing in each argument.
3. For all $u_1, v_1, u_2, v_2 \in [0,1]$, $|Q(u_2,v_2) - Q(u_1,v_1)| \leq |u_2 - u_1| + |v_2 - v_1|$ (Lipschitz property).

If, in addition, Q is 2$-$increasing (i.e. $V_Q(R) = Q(u_2,v_2) + Q(u_1,v_1) - Q(u_1,v_2) - Q(u_2,v_1) \geq 0$ for every rectangle $R = [u_1,u_2] \times [v_1,v_2] \subseteq [0,1]^2$) then it is a copula.

Constrained Fréchet Hoeffding Bounds

Let $\mathbb{S}$ denote a compact subset of the unit square $[0,1]^2$.
Tankov (2011) shows that $A^{\mathbb{S},Q}$ and $B^{\mathbb{S},Q}$ defined by

$$A^{\mathbb{S},Q}(u,v) = \min\left\{u, v, \min_{(a,b)\in\mathbb{S}}\{Q(a,b) + (u-a)^+ + (v-b)^+\}\right\},$$
$$B^{\mathbb{S},Q}(u,v) = \max\left\{0, u+v-1, \max_{(a,b)\in\mathbb{S}}\{Q(a,b) - (a-u)^+ - (b-v)^+\}\right\}$$

where $(u,v) \in [0,1]^2$, are the best possible upper (resp. lower) bounds for the set of all quasi-copulas Q' such that $Q'(a,b) = Q(a,b)$ for all $(a,b) \in \mathbb{S}$ (see Tankov (2011), Theorem 1). When $\mathbb{S}$ is the empty set, $B^{\mathbb{S},Q}(u,v) := \max(0, u+v-1)$ and $A^{\mathbb{S},Q}(u,v) := \min(u,v)$ are the Fréchet-Hoeffding bounds.

Sufficient condition of Tankov (2011) for $A^{\mathbb{S},Q}$ (resp. $B^{\mathbb{S},Q}$) to be a copula : $\mathbb{S}$ is non-increasing (resp. non-decreasing).
Weaker condition of Bernard, Jiang, Vanduffel (2012) : when Q is a copula, $A^{\mathbb{S},Q}$ (resp. $B^{\mathbb{S},Q}$) is a copula when $\mathbb{S}$ is a compact set with some "monotonicity" and "connectivity" conditions.

Theorem (Sufficient condition of BLMZ (DM2013))

If $\mathbb{S}$ is a compact set satisfying the following property:

$$\forall(a_0,b_0)\in\mathbb{S},\forall(a_1,b_1)\in\mathbb{S},\quad (a_0,b_1)\in\mathbb{S},(a_1,b_0)\in\mathbb{S}. \tag{4}$$

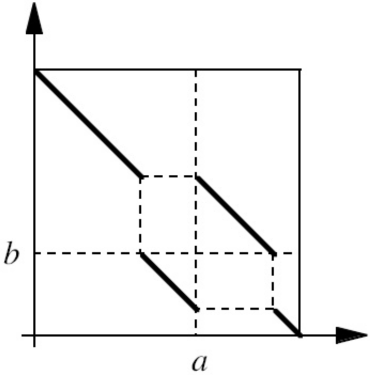
Furthermore, suppose Q is a quasi-copula such that $\forall(a_0,b_0),(a_1,b_1)\in\mathbb{S}$ with $a_0<a_1, b_0<b_1$, we have

$$Q(a_1,b_1)+Q(a_0,b_0)-Q(a_0,b_1)-Q(a_1,b_0)\geq 0, \tag{5}$$

then $A^{\mathbb{S},Q}$ and $B^{\mathbb{S},Q}$ are copulas. Note that condition (5) *is automatically satisfied when Q is a copula.*

Example 2: Illustration

Minimum copula with one constraint that $C(a, b) = \theta$.



Back to appendix outline

Back to presentation