Cost-efficiency and Applications

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Part 2, Application to Portfolio Selection, Berlin, May 2015.

- This talk is joint work with **Phelim Boyle** (Wilfrid Laurier University, Waterloo, Canada), **Jit Seng Chen** (University of Waterloo) and with **Steven Vanduffel** (Vrije Universiteit Brussel (VUB), Belgium).

- **Outline (paper in *Finance* with Boyle and Vanduffel):**

  1. Traditional portfolio selection
  2. What is **cost-efficiency**? Illustration in the **binomial** model
  3. Characterization of optimal investment strategies for an investor with **law-invariant preferences** and a **fixed investment horizon**
  4. Illustration in the **Black and Scholes** model
  5. How to use cost-efficiency to **optimize** your investment strategies? Or your hedging strategies? To "choose" a utility? To model **state-dependent constraints**?

Traditional Approach to Portfolio Selection

Given an **investment horizon** T. Let X_T denote the **final** wealth at time T and x_0 the **initial** wealth. We define by $\mathcal{A}$ the set of admissible final wealths such that the cost of X_T is x_0 and they are "feasible" strategies.

- **Expected Utility Theory**.

$$\max_{X_T \in \mathcal{A}} E[U(X_T)]$$

where

- exponential utility $U(x) = -\exp(-\gamma x)$ with $\gamma > 0$.
- CRRA utility, $U(x) = \frac{x^{1-\eta}}{1-\eta}$ with $\eta > 0$ and $\eta \neq 1$.
- Log utility, $U(x) = log(x)$.
- increasing + concave (risk averse investor).

Traditional Approach to Portfolio Selection

- **Goal reaching**

$$\max_{X_T \in \mathcal{A}} P(X_T > K)$$

- **Sharpe ratio optimization**

$$\max_{X_T \in \mathcal{A}} \frac{E[X_T] - x_0 e^{rT}}{std(X_T)}$$

where x_0 is the initial budget.

- Minimize **Value-at-Risk** of X_T.
- Yaari's theory, Cumulative Prospect Theory, Rank Dependent Utility...

Traditional Approach to Portfolio Selection

- Common properties, the objective function is **law invariant**!

  If $X_T \sim Y_T$ (that is X_T and Y_T have the same distribution) then they must have the same objective function.

- and the objective function is **increasing**. If $X_T < Y_T$ almost surely, the investor prefers Y_T to X_T.
- Each problem needs different techniques since some of them have convexity properties, some don't...
- How to find the "**utility function**" of the investor? How to find the "right" objective?

Traditional Approach to Portfolio Selection

Consider an investor with **increasing law-invariant** preferences and a **fixed** horizon. Denote by X_T the investor's final wealth.

- Optimize an increasing law-invariant objective function
- for a given **cost** (budget)

$$cost\ at\ 0 = E_Q[e^{-rT} X_T]$$

Find optimal strategy X_T^* $\Rightarrow$ Optimal cdf F of X_T^*

Our idea is to start from F...

What is "cost-efficiency"?

Cost-efficiency is a criteria for evaluating payoffs independent of the agents' preferences.

Cost-Efficiency

A strategy (or a payoff) is **cost-efficient** if any other strategy that generates the same distribution under P costs at least as much.

This concept was originally proposed by Dybvig.

- Dybvig, P., 1988a. "Distributional Analysis of Portfolio Choice," *Journal of Business*, **61**(3), 369-393.
- Dybvig, P., 1988b. "Inefficient Dynamic Portfolio Strategies or How to Throw Away a Million Dollars in the Stock Market," *Review of Financial Studies*, **1**(1), 67-88.

Important observation

Consider an investor with

- Law-invariant preferences
- Increasing preferences
- A fixed investment horizon

It is clear that the optimal strategy must be **cost-efficient**.

Therefore optimal portfolios in the traditional settings discussed before are cost-efficient.

The rest of this section is about characterizing cost-efficient strategies.

Main Assumptions

- Consider an arbitrage-free and complete market.
- Given a strategy with final payoff X_T at time T.
- There exists a unique probability measure Q, such that its price at 0 is

$$c(X_T) = \mathbb{E}_Q[e^{-rT} X_T]$$

Cost-efficient strategies

- Given a cdf F under the **physical measure** P.

The distributional price is defined as

$$PD(F) = \min_{\{Y \mid Y \sim F\}} c(Y) = \min_{\{Y \mid Y \sim F\}} \mathbb{E}_Q[e^{-rT} Y]$$

- The strategy with payoff X_T is **cost-efficient** if

$$PD(F) = c(X_T)$$

- Given a strategy with payoff X_T at time T. Its price at 0 is

$$P_X = E_Q[e^{-rT} X_T]$$

- F: distribution of the cash-flow at T of the strategy

The "loss of efficiency" or "efficiency cost" is equal to

$$P_X - PD(F)$$

A Simple Illustration

Let's illustrate what the "efficiency cost" is with a simple example. Consider :

- A market with 2 assets: a bond and a stock S.
- A discrete 2-period binomial model for the stock S.
- A strategy with payoff X_T at the end of the two periods.

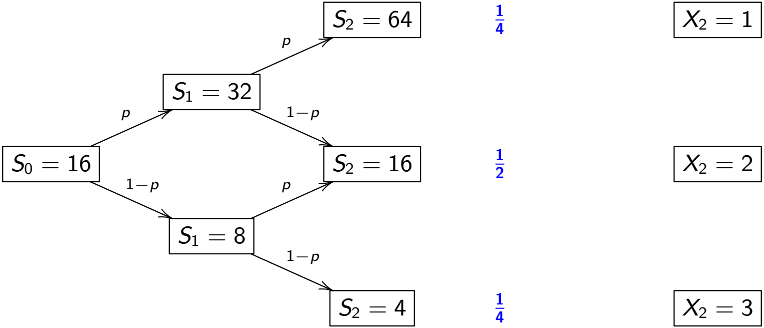
Example of

- $X_T \sim Y_T$ under P
- but with different prices

in a 2-period (arbitrage-free) binomial tree ($T = 2$).

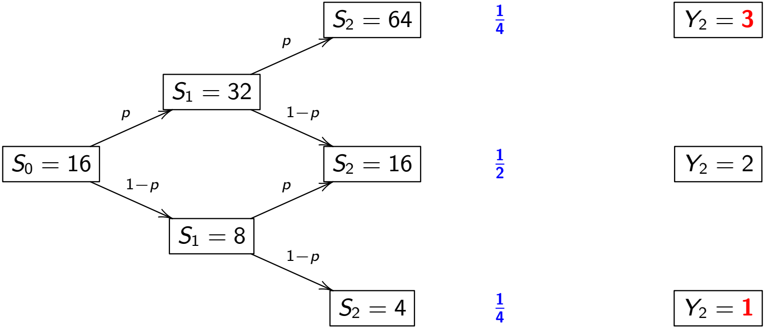
A simple illustration for X_2, a payoff at $T = 2$

Real-world probabilities: $p = \frac{1}{2}$



Y_2, a payoff at $T = 2$ distributed as X_2

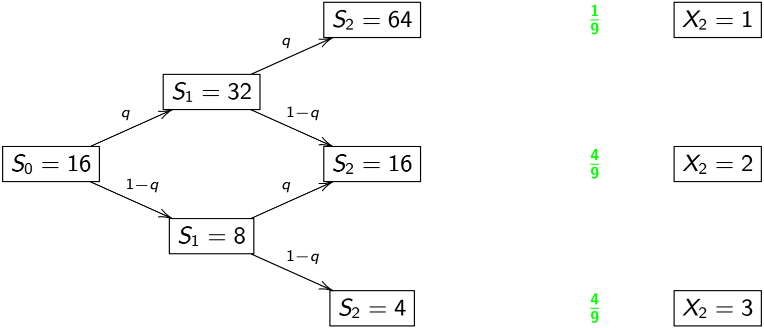
Real-world probabilities: $p = \frac{1}{2}$



X_2 and Y_2 have the same distribution under the physical measure

X_2, a payoff at $T = 2$

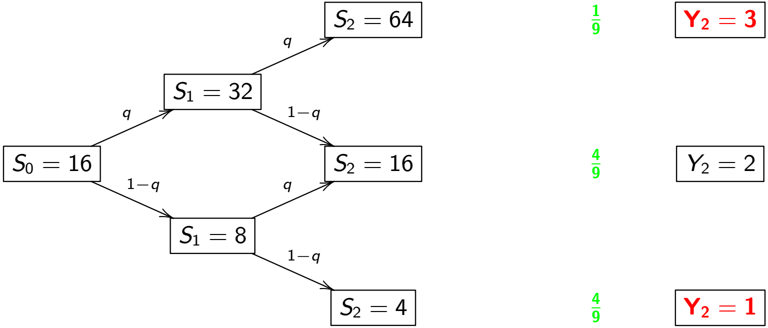
risk neutral probabilities: $q = \frac{1}{3}$.



$$c(X_2) \; = \; \text{Price of } X_2 = \left(\frac{1}{9} + \frac{4}{9}2 + \frac{4}{9}3\right) = \frac{21}{9}$$

Y_2, a payoff at $T = 2$

risk neutral probabilities: $q = \frac{1}{3}$.



$$c(Y_2) = \left(\frac{1}{9}3 + \frac{4}{9}2 + \frac{4}{9}1\right) = \frac{15}{9}$$

$$c(X_2) \;=\; \text{Price of } X_2 = \left(\frac{1}{9} + \frac{4}{9}2 + \frac{4}{9}3\right) = \frac{21}{9}$$

Characterization of Cost-Efficient Strategies

Assumptions: General setting

To characterize cost-efficiency, we need to introduce the "state-price process"

- Consider an arbitrage-free and complete market.
- Given a strategy with payoff X_T at time T. There exists a unique risk-neutral probability Q, such that its price at 0 is

$$c(X_T) = \mathbb{E}_Q[e^{-rT} X_T]$$

- P ("physical measure") and Q ("risk-neutral measure") are two equivalent probability measures:

$$\xi_T = e^{-rT} \left(\frac{dQ}{dP}\right)_T, \quad \mathbf{c(X_T)} = \mathbb{E}_Q[e^{-rT} X_T] = \mathbb{E}_{\mathbf{P}}[\xi_{\mathbf{T}} \mathbf{X_T}].$$

ξ_T is called "state-price process" and is also sometimes referred to as "deflator" or "pricing kernel".

Sufficient Condition for Cost-efficiency

A random pair (X, Y) is anti-monotonic if

there exists a non-increasing relationship between them.

Theorem (Sufficient condition for cost-efficiency)

Any random payoff X_T with the property that (X_T, ξ_T) is **anti-monotonic** *is* **cost-efficient**.

Note the absence of additional assumptions on ξ_T (it holds in discrete and continuous markets) and on X_T (no assumption on non-negativity).

Idea of the proof (1/2)

Minimizing the price $c(X_T) = E[\xi_T X_T]$ when $X_T \sim F$ amounts to finding the dependence structure that **minimizes the correlation** between the strategy and the state-price process

$$\min_{X_T} \ \mathbb{E}\left[\xi_T X_T\right]$$
$$\text{subject to} \quad \begin{cases} X_T \sim F \\ \xi_T \sim G \end{cases}$$

Recall that

$$\text{corr}(X_T, \xi_T) = \frac{\mathbb{E}[\xi_T X_T] - \mathbb{E}[\xi_T]\mathbb{E}[X_T]}{\text{std}(\xi_T)\,\text{std}(X_T)}.$$

Idea of the proof (2/2)

We can prove that when the distributions for both X_T and ξ_T are fixed, we have

$$(X_T, \xi_T) \text{ is anti-monotonic} \Rightarrow \text{corr}[X_T, \xi_T] \text{ is minimal.}$$

Minimizing the cost $E[\xi_T X_T] = c(X_T)$ of a strategy therefore amounts to minimizing the correlation between the strategy and the state-price process

Explicit Representation for Cost-efficiency

Assume ξ_T is continuously distributed (for example a Black-Scholes market)

Theorem

The cheapest strategy that has cdf F is given explicitly by

$$X_T^\star = F^{-1}\left(1 - F_\xi\left(\xi_T\right)\right).$$

Note that $X_T^\star \sim F$ and $X_T^\star$ is a.s. **unique** *such that*

$$PD(F) = c(X_T^\star) = \mathbb{E}[\xi_T X_T^\star]$$

where $PD(F)$ is the distributional price

$$PD(F) = \min_{\{X_T \mid X_T \sim F\}} e^{-rT}\mathbb{E}_Q[X_T] = \min_{\{X_T \mid X_T \sim F\}} \mathbb{E}[\xi_T X_T]$$

and F^{-1} is defined as follows:

$$F^{-1}(y) = \min\left\{x \ / \ F(x) \geqslant y\right\}.$$

Copulas and Sklar's theorem

The joint cdf of a couple (ξ_T, X) can be decomposed into 3 elements

- The marginal cdf of ξ_T: G
- The marginal cdf of X_T: F
- A copula C

such that

$$P(\xi_T < \xi, X_T < x) = C(G(\xi), F(x))$$

Idea of the proof (1/3)

Solving this problem amounts to finding bounds on copulas!

$$\min_{X_T} \mathbb{E}\left[\xi_T X_T\right]$$
$$\text{subject to} \quad \begin{cases} X_T \sim F \\ \xi_T \sim G \end{cases}$$

The distribution G is known and depends on the financial market.
Let C denote a copula for (ξ_T, X).

$$\mathbb{E}[\xi_T X] = \int\int (1 - G(\xi) - F(x) + C(G(\xi), F(x)))dxd\xi, \quad (1)$$

Bounds for $\mathbb{E}[\xi_T X]$ are derived from bounds on the copula C.

Idea of the proof (2/3)

It is well-known that any copula verify

$$\max(u+v-1,0) \leqslant C(u,v) \leqslant \min(u,v)$$

(Fréchet-Hoeffding Bounds for copulas) where the lower bound is the "anti-monotonic copula" and the upper bound is the "monotonic copula".

Let U be uniformly distributed on $[0,1]$.

- The cdf of $(U, 1-U)$ is $P(U \leqslant u, 1-U \leqslant v) = \max(u+v-1,0)$ (**anti-monotonic** copula)
- the cdf of (U, U) is $P(u,v) = \min(u,v)$ (**monotonic** copula).

Idea of the proof (3/3)

Consider a strategy with payoff X_T distributed as F. Note that $U = F_\xi(\xi_T)$ is uniformly distributed over $(0,1)$.

Note that ξ_T and $X_T^* := F^{-1}(1 - G(\xi_T))$ are anti-monotonic and that $X_T^* \sim F$.

Note that ξ_T and $Z_T^* := F^{-1}(G(\xi_T))$ are comonotonic and that $Z_T^* \sim F$.
The cost of the strategy with payoff X_T is $c(X_T) = E[\xi_T X_T]$.

$$E[\xi_T F^{-1}(1 - G(\xi_T))] \leqslant c(X_T) \leqslant E[\xi_T F^{-1}(G(\xi_T))]$$

that is

$$E[\xi_T X_T^*] \leqslant c(X_T) \leqslant E[\xi_T Z_T^*].$$

Path-dependent payoffs are inefficient

Corollary

To be cost-efficient, the payoff of the derivative has to be of the following form:

$$X_T^{\star} = F^{-1}\left(1 - F_{\xi}\left(\xi_T\right)\right)$$

It becomes a European derivative written on S_T when the state-price process ξ_T can be expressed as a function of S_T. Thus path-dependent derivatives are in general not cost-efficient.

Corollary

Consider a derivative with a payoff X_T which could be written as

$$X_T = h(\xi_T)$$

Then X_T is cost efficient if and only if h is non-increasing.

Examples

in the Black-Scholes setting

to improve strategies

Black-Scholes Model

Under the physical measure P,

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t^P$$

Then

$$\xi_T = e^{-rT}\left(\frac{dQ}{dP}\right) = a\left(\frac{S_T}{S_0}\right)^{-b}$$

where $a = e^{\frac{\theta}{\sigma}(\mu - \frac{\sigma^2}{2})t - (r + \frac{\theta^2}{2})t}$ and $b = \frac{\mu - r}{\sigma^2}$.

Theorem (Cost-efficiency in Black-Scholes model)

To be cost-efficient, the contract has to be a European derivative written on S_T and non-decreasing w.r.t. S_T (when $\mu > r$). In this case,

$$\mathbf{X_T^\star = F^{-1}\left(F_{S_T}\left(S_T\right)\right)}$$

Implications

In a Black Scholes model (with 1 risky asset), optimal strategies for an investor with a **fixed horizon investment** and **law-invariant preferences** are always of the form

$$g(S_T)$$

with g non-decreasing.

Maximum price = Least efficient payoff

Theorem

Consider the following optimization problem:

$$\max_{\{X_T \mid X_T \sim F\}} c(X_T) = \max_{\{X_T \mid X_T \sim F\}} \mathbb{E}[\xi_T X_T]$$

Assume ξ_T is continuously distributed. *The unique strategy $Z_T^\star$ that generates the same distribution as F with the highest cost can be described as follows:*

$$Z_T^\star = F^{-1}\left(F_\xi\left(\xi_T\right)\right) = F^{-1}\left(1 - F_{S_T}\left(S_T\right)\right)$$

Geometric Asian contract in Black-Scholes model

Assume a strike K. The payoff of the Geometric Asian call is given by

$$X_T = \left(e^{\frac{1}{T}\int_0^T \ln(S_t)dt} - K\right)^+$$

which corresponds in the discrete case to $\left(\left(\prod_{k=1}^n S_{\frac{kT}{n}}\right)^{\frac{1}{n}} - K\right)^+$.

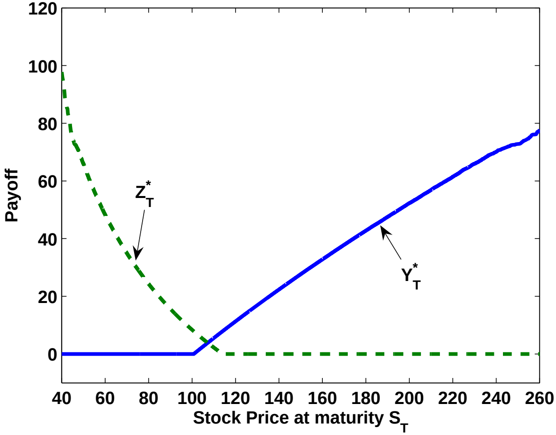
The efficient payoff that is distributed as the payoff X_T is a power call option

$$X_T^\star = d\left(S_T^{1/\sqrt{3}} - \frac{K}{d}\right)^+$$

where $d := S_0^{1-\frac{1}{\sqrt{3}}} e^{\left(\frac{1}{2}-\sqrt{\frac{1}{3}}\right)\left(\mu-\frac{\sigma^2}{2}\right)T}$.

Similar result in the discrete case.

Example: Discrete Geometric Option



With $\sigma = 20\%, \mu = 9\%, r = 5\%, S_0 = 100,\ T = 1$ year, $K = 100$.

$$C(X_T^\star) = 5.3 < Price(geometric\ Asian) = 5.5 < C(Z_T^\star) = 8.4.$$

Put option in Black-Scholes model

Assume a strike K. The payoff of the put is given by

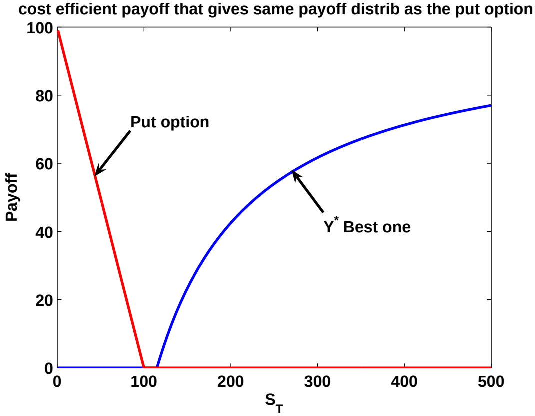
$$L_T = (K - S_T)^+ .$$

The payout that has the **lowest** cost and that has the same distribution as the put option payoff is given by

$$X_T^\star = F_L^{-1}\left(F_{S_T}\left(S_T\right)\right) = \left(K - \frac{S_0^2 e^{2\left(\mu - \frac{\sigma^2}{2}\right)T}}{S_T}\right)^+ .$$

This type of power option "dominates" the put option.

Cost-efficient payoff of a put



With $\sigma = 20\%, \mu = 9\%, r = 5\%, S_0 = 100,\ T = 1$ year, $K = 100$.

Distributional price of the put = 3.14

Price of the put = 5.57

Efficiency loss for the put = 5.57-3.14= 2.43

Up and Out Call option in Black and Scholes model

Assume a strike K and a barrier threshold $H > K$. Its payoff is given by

$$L_T = (S_T - K)^+ \mathbb{1}_{\max_{0 \leqslant t \leqslant T}\{S_t\} \leqslant H}$$

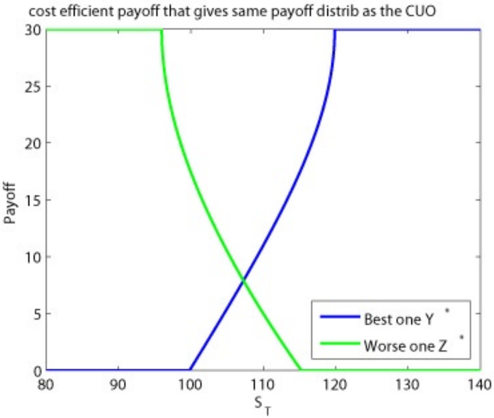
The payoff that has the **lowest** cost and is distributed such as the barrier up and out call option is given by

$$X_T^\star = F_L^{-1}\left(1 - F_\xi\left(\xi_T\right)\right)$$

The payoff that has the **highest** cost and is distributed such as the barrier up and out call option is given by

$$Z_T^\star = F_L^{-1}\left(F_\xi\left(\xi_T\right)\right)$$

Cost-efficient payoff of a Call up and out



With $\sigma = 20\%, \mu = 9\%, S_0 = 100,\ T = 1$ year, strike $K = 100,\ H = 130$

Distributional Price of the CUO $= 9.7374$

Price of CUO $= P_{cuo}$

Worse case $= 13.8204$

Efficiency loss for the CUO $= P_{cuo}$-9.7374

Some Applications of Cost-Efficiency

Applications

1. Solving well-known problems in a simpler way (mean variance or quantile hedging)
2. Equivalence between the Expected Utility Maximization setting and the Cost-Efficient strategy (Part 2, application to behavioral finance).
3. Extension to State Dependent preferences (Part 2, application to state dependent constraints).

Rationalizing Investors Choices

Carole Bernard (Grenoble Ecole de Management),

joint work with Jit Seng Chen (GGY)
and Steven Vanduffel (Vrije Universiteit Brussel)

GRENOBLE ECOLE DE MANAGEMENT
TECHNOLOGY & INNOVATION

Part 2, Application to Behavioral Finance, Berlin, May 2015.

Terminology

$V(\cdot)$ **denotes the objective function** of the agent to maximize (Expected utility, Value-at-Risk, Cumulative Prospect Theory...)

- **Law-invariant** preferences

$$X_T =_d Y_T \Rightarrow V(X_T) = V(Y_T)$$

- **First-order stochastic dominance (FSD)**

$$X_T \sim F_X, Y_T \sim F_Y, Y_T \prec_{fsd} X_T$$

$$\forall x, F_X(x) \leqslant F_Y(x) \Rightarrow V(X_T) \geqslant V(Y_T)$$

equivalently, for all non-decreasing U, $E[U(X_T)] \geqslant E[U(Y_T)]$.

Contributions

1. In any behavioral setting respecting **First-order Stochastic Dominance**, investors only care about the distribution of final wealth (**law-invariant** preferences).
2. Then the optimal portfolio is also the optimum for an **expected utility maximizer** (concave, non-decreasing utility).
3. Given a distribution F of terminal wealth, we **construct a utility function** (concave, non-decreasing, no differentiability conditions) such that the optimal solution to

$$\max_{X_T \mid \text{budget}=\omega_0} E[U(X_T)]$$

has the cdf F.
4. Use this utility to **infer risk aversion**.
5. **Decreasing Absolute Risk Aversion** (DARA) can be directly related to properties of the distribution of final wealth and of the financial market in which the agent invests.

FSD implies Law-invariance

Consider an investor with **fixed horizon** and objective $V(\cdot)$.

Theorem

Preferences $V(\cdot)$ are non-decreasing and law-invariant if and only if $V(\cdot)$ satisfies first-order stochastic dominance.

- **Law-invariant** preferences

$$X_T =_d Y_T \Rightarrow V(X_T) = V(Y_T)$$

- **Non-decreasing** preferences

$$X_T \geqslant Y_T a.s. \Rightarrow V(X_T) \geqslant V(Y_T)$$

- **First-order stochastic dominance (FSD)**

$$X_T \sim F_X, Y_T \sim F_Y, Y_T \prec_{fsd} X_T$$

$$\forall x, F_X(x) \leqslant F_Y(x) \Rightarrow V(X_T) \geqslant V(Y_T)$$

Main Assumptions

- Given a portfolio with final payoff X_T **(consumption only at time T)**.
- The market is complete and the initial value of X_T is given by

$$\mathbf{c(X_T)} = \mathbb{E}[\xi_\mathbf{T}\mathbf{X_T}].$$

  where ξ_T is called the pricing kernel or stochastic discount factor.
- ξ_T is **continuously distributed**.
- **Preferences satisfy FSD**.

Optimal Portfolio and Cost-efficiency

Optimal portfolio problem for an investor with preferences $V(\cdot)$ respecting FSD and final wealth X_T:

$$\max_{X_T \mid E[\xi_T X_T]=\omega_0} V(X_T). \tag{2}$$

Theorem: Cost-efficient strategies

If an optimum X_T^* of (2) exists, let F be its cdf. Then, X_T^* is the cheapest way (cost-efficient) to achieve F at T, i.e. X_T^* also solves

$$\min_{X_T | X_T \sim F} E[\xi_T X_T]. \tag{3}$$

Furthermore, for any cdf F, the solution $X_T^\star$ to (3) is unique (a.s.) and writes as $X_T^\star = F^{-1}(1 - F_\xi(\xi_T))$ where F_ξ is the cdf of ξ_T.

Optimal Portfolio and Cost-efficiency

Theorem

A cost-efficient payoff X_T with a **continuous** increasing distribution F corresponds to the optimum of an expected utility investor for

$$U(x) = \int_c^x F_\xi^{-1}(1 - F(y))dy$$

where F_ξ is the cdf of ξ_T, $F(c) > 0$, $\omega_0 = E[\xi_T F^{-1}(1 - F_\xi(\xi_T))]$. The utility function U is C^1, **strictly concave** and **increasing**.

- U is **unique** up to a linear transformation in a certain class.
- When the optimal portfolio in a behavioral setting respecting FSD is continuously distributed, then it can be obtained by **maximum expected (concave) utility.**
- All distributions can be approximated by continuous distributions. $\Rightarrow$ all investors are **approximately risk averse**...

Rationalizable consumption by EUT

Definition (Rationalization by Expected Utility Theory)

The optimal portfolio choice X_T with a finite budget ω_0 is rationalizable by the expected utility theory if there exists a utility function U such that X_T is also the optimal solution to

$$\max_{X \mid E[\xi X]=\omega_0} E[U(X)]. \tag{4}$$

Theorem (Rationalizable consumption by Standard EUT)

Consider a terminal consumption X_T at time T purchased with an initial budget ω_0 and distributed with a **continuous** *cdf F:*
The 8 following conditions are equivalent.

(i) *X_T is rationalizable by the standard Expected Utility Theory (**concave, increasing, and differentiable utility**).*

(ii) *X_T is cost-efficient with cdf F.*

(iii) $\omega_0 = E[\xi_T F^{-1}(1 - F_{\xi_T}(\xi_T))]$.

(iv) $X_T = F^{-1}(1 - F_{\xi_T}(\xi_T))$ *a.s.*

(v) *X_T is non-increasing in ξ_T a.s.*

(vi) *X_T is the solution to a maximum portfolio problem for some objective $V(\cdot)$ that satisfies FSD.*

(vii) *X_T is the solution to a maximum portfolio problem for some law-invariant and non-decreasing objective function $V(\cdot)$.*

(viii) *X_T is the solution to a maximum portfolio problem for some objective $V(\cdot)$ that satisfies SSD.*

Generalization

We can show that all distributions can be the optimum of an expected utility optimization with a “generalized concave utility”.

Definition: Generalized concave utility function

A generalized concave utility function $\widetilde{U}:\mathbb{R}\to\overline{\mathbb{R}}$ is defined as

$$\widetilde{U}(x) := \begin{cases} U(x) & \text{for } x\in(a,b),\\ -\infty & \text{for } x<a,\\ U(a^+) & \text{for } x=a,\\ U(b^-) & \text{for } x\geqslant b,\end{cases}$$

where $U(x)$ is concave and strictly increasing and $(a,b)\subset\mathbb{R}$.

General Distribution

Let F be **any** distribution (with possibly atoms...).

Theorem

A cost-efficient payoff X_T with a cdf F is also an optimal solution to

$$\max_{X_T \mid E[\xi_T X_T]=\omega_0} E\left[\widetilde{U}(X_T)\right]$$

where $\widetilde{U} : \mathbb{R} \to \overline{\mathbb{R}}$ is a generalized utility function given explicitly by the same formula as before:

$$\widetilde{U}(x) = \int_c^x F_\xi^{-1}(1 - F(y))dy.$$

where F_ξ is the cdf of ξ_T, $F(c) > 0$, $\omega_0 = E[\xi_T F^{-1}(1 - F_\xi(\xi_T))]$.

- $\widetilde{U}$ is **unique** up to a linear transformation in a certain class.

Theorem (Rationalizable consumption by Generalized EUT)

Consider a terminal consumption X_T at time T purchased with an initial budget w_0 and distributed with F.

The 8 following conditions are equivalent.

(i) X_T *is rationalizable by Generalized Expected Utility Theory.*

(ii) X_T *is cost-efficient.*

(iii) $w_0 = E[\xi_T F^{-1}(1 - F_{\xi_T}(\xi_T))]$.

(iv) $X_T = F^{-1}(1 - F_{\xi_T}(\xi_T))$ *a.s.*

(v) X_T *is non-increasing in* ξ_T *a.s.*

(vi) X_T *is the solution to a maximum portfolio problem for some objective* $V(\cdot)$ *that satisfies FSD.*

(vii) X_T *is the solution to a maximum portfolio problem for some law-invariant and non-decreasing objective function* $V(\cdot)$.

(viii) X_T *is the solution to a maximum portfolio problem for some objective* $V(\cdot)$ *that satisfies SSD.*

A comment: This theorem does not hold in a discrete setting

- One-period model with horizon T
- Finite space $\Omega = \{\omega_1, \omega_2, ..., \omega_N\}$ with equiprobable states
- $\frac{\xi(\omega_i)}{N}$: initial cost of the Arrow-Debreu security that pays 1 in state ω_i at time T and 0 otherwise.
- $\xi := (\xi_1, \xi_2, ..., \xi_N)$ the pricing kernel where $\xi_i := \xi(\omega_i)$.
- Terminal consumption $X := (x_1, x_2, ..., x_N)$ (with $x_i := X(\omega_i)$)
- Initial budget $E[\xi X] = \frac{1}{N}\sum_{i=1}^{N} \xi_i x_i$

The optimal consumption X^* of the agent with budget ω_0 and preferences $V(\cdot)$ (FSD) solves

$$\max_{X \mid E[\xi X]=\omega_0} V(X), \tag{5}$$

Optimal Consumption with Equiprobable States

- X^* and ξ must be antimonotonic (Peleg and Yaari (1975))

$$x_1^* \leqslant x_2^* \leqslant ... \leqslant x_N^* \quad \text{and} \quad \xi_1 \geqslant \xi_2 \geqslant ... \geqslant \xi_N.$$

Rationalizing Investment in a Discrete Setting

The optimal solution X^* of (5) solves also

$$\max_{X \ | \ E[\xi X]=\omega_0} E[U(X)].$$

for any concave utility $U(\cdot)$ such that the left derivative denoted by U' exists in x_i^* for all i and satisfies

$$\forall i \in \{1, 2, ..., N\}, \quad U'(x_i^*) = \xi_i. \tag{6}$$

- utility inferred only at a discrete number of consumption.
- **no uniqueness**

Optimal Consumption with Non-Equiprobable States

- take $\Omega = \{\omega_1, \omega_2\}$ with $P(\omega_1) = \frac{1}{3}$ and $P(\omega_2) = \frac{2}{3}$,
- $\xi_1 = \frac{3}{4}$ and $\xi_2 = \frac{9}{8}$. Budget $\omega_0 = 1$
- Consider X with $X(\omega_1) = a_1$ and $X(\omega_2) = a_2$ satisfying the budget condition $\frac{a_1}{4} + \frac{3a_2}{4} = 1$.
- **Objective** $V(X) := VaR^+_{1/3}(X)\mathbb{1}_{P(X<0)=0}$ (where $VaR^+_\alpha(X)$ is defined as $VaR^+_\alpha(X) := \sup\{x \in \mathbb{R}, F_X(x) \leqslant \alpha\}$). Note that $V(\cdot)$ is clearly **law-invariant and non-decreasing (FSD)**.
- $V(\cdot)$ is maximised for $X^*(\omega_1) = 0$ and $X^*(\omega_2) = \frac{4}{3}$.
- X^* is **never optimal for an EU maximizer with increasing concave utility U** on $[0, \frac{4}{3}]$ (range of consumptions).
- Proof: wlog $U(0) = 0$ and $U(\frac{4}{3}) = 1$. Consider Y such that $Y(\omega_1) = \frac{4}{3}$ and $Y(\omega_2) = \frac{8}{9}$. Observe that $E[\xi Y] = E[\xi X^*] = 1$ and $E[U(Y)] > E[U(X^*)] = \frac{2}{3}$.

Utility & Distribution in the Black-Scholes Model

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t^P, \quad \frac{dB_t}{B_t} = r dt, \quad \theta = \frac{\mu - r}{\sigma}, \quad \xi_T \sim \mathcal{LN}(M, \theta^2 T)$$

- **Power utility (CRRA) & LogNormal distribution:** $\mathcal{LN}(A, B^2)$ corresponds to a CRRA utility function with relative risk aversion $\nu := \frac{\theta\sqrt{T}}{B} \neq 1$ (where $\theta = \frac{\mu - r}{\sigma}$):

$$U(x) = a\,\frac{x^{1-\nu}}{1-\nu}.$$

- **Exponential utility & Normal Distribution:** $N(A, B^2)$ corresponds to the exponential utility $U(x) = -\exp(-\gamma x)$, with constant absolute risk aversion γ.

Explaining the Demand for Capital Guarantee Products

$$Y_T = \max(G, S_T)$$

where S_T is the stock price, $S_T \sim \mathcal{LN}(\mu T - \frac{\sigma^2}{2}T, \sigma^2 T)$ and G the guarantee. The utility function is then given by

$$\widetilde{U}(x) = \begin{cases} -\infty & x < G, \\ a\,\frac{x^{1-\frac{\theta}{\sigma}} - G^{1-\frac{\theta}{\sigma}}}{1-\frac{\theta}{\sigma}} & x \geqslant G,\ \frac{\theta}{\sigma} \neq 1. \end{cases} \tag{7}$$

- The mass point is explained by a utility which is infinitely negative for any level of wealth below the guaranteed level.
- The CRRA utility above this guaranteed level ensures the optimality of a Lognormal distribution above the guarantee.

Yaari's Dual Theory of Choice Model

Final wealth X_T. Objective function to maximize

$$\mathbb{H}_w\left[X_T\right] = \int_0^\infty w\left(1 - F(x)\right) \mathrm{d}x,$$

where the (distortion) function $w : [0,1] \to [0,1]$ is non-decreasing with $w(0) = 0$ and $w(1) = 1$. Then, the optimal payoff is solution to an expected utility maximization with

$$U(x) = \begin{cases} -\infty & x < 0 \\ \alpha(x - c) & 0 \leqslant x \leqslant b \\ \alpha(b - c) & x > b \end{cases}$$

where $\alpha > 0$ is constant.

Inferring preferences and utility

- more natural for an investor to describe her target distribution than her utility (Goldstein, Johnson and Sharpe (2008) discuss how to estimate the distribution at retirement using a questionnaire).
- From the investment choice, get the distribution and find the corresponding utility U. $\Rightarrow$ Inferring preferences from the target final distribution
- $\Rightarrow$ Inferring risk-aversion. The Arrow-Pratt measure for absolute risk aversion can be computed from a twice differentiable utility function U as $\mathcal{A}(x) = -\frac{U''(x)}{U'(x)}$.
- Always possible to approximate by a twice differentiable utility function...

Theorem (Arrow-Pratt Coefficient)

Consider an investor who wants a cdf F (with density f). The Arrow-Pratt coefficient for absolute risk aversion is for $x = F^{-1}(p)$,

$$\mathcal{A}(x) = \frac{f(F^{-1}(p))}{g(G^{-1}(p))},$$

where g and G are resp. the density and cdf of $-\log(\xi_T)$.

Theorem (Distributional characterization of DARA)

DARA iff $x \mapsto F^{-1}(G(x))$ is strictly convex.
In the special case of Black-Scholes: $x \mapsto F^{-1}(\Phi(x))$ is strictly convex, where $\Phi(\cdot)$ is the cdf of N(0,1).

- In BS, DARA iff target distribution F is fatter than normal.
- DARA iff target distribution F is fatter than cdf of $-\log(\xi_T)$.
- Many cdf are DARA. ex: Gamma, LogNormal, Gumbel...

Conclusions, Current & Future Work (1/2)

- Limitation of FSD preferences: **FSD** or law-invariant behavioral settings **cannot explain** all decisions. Need state-dependent preferences to explain investment decisions such as buying protection, path-dependent options...
- **"State-dependent"** regulation (systemic risk) with the idea of assessing risk and performance of a portfolio not only by looking at its final distribution but also by looking at its interaction with the economic conditions. Acharya (2009) explains that regulators should "be regulating each bank as a function of both its joint (correlated) risk with other banks as well as its individual (bank-specific) risk".
- **State-dependent preferences** can be modelled using a law-invariant objective and an additional constraint on the dependence of the portfolio with the market. Example: a portfolio that maximizes utility and is independent of "S_T" when the market crashes (QF, 2014).

Conclusions, Current & Future Work (2/2)

- **Inferring** preferences and risk-aversion from investment choice.
- **Understanding** the interaction between changes in the financial market, wealth level and utility on optimal terminal consumption for an agent with given preferences.
- Implications in some specific **non-expected utility** settings: Cumulative Prospect Theory is a setting which respects FSD.
- Remove the assumption on the **continuity of** F_{ξ_T} by using "randomized payoffs" (JAP 2015 with Rüschendorf and Vanduffel).
- What happens in an **incomplete** market? We can solve the problem under the assumption that $\xi_T = f(S_T)$
- Implications on equilibrium problems, pricing kernel puzzle...

Do not hesitate to contact me to get updated working papers!

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Optimal Investment under State-dependent Preferences

Carole Bernard (Grenoble Ecole de Management),

joint work with Franck Moraux (University of Rennes 1)
Ludger Rüschendorf (University of Freiburg)
and Steven Vanduffel (Vrije Universiteit Brussel)

Part 2, Application to Constrained Portfolio Selection,
Berlin, May 2015.

Investment with State-Dependent Constraints

Problem considered so far

$$\min_{\{X_T \mid X_T \sim F\}} \mathbb{E}\left[\xi_T X_T\right].$$

A payoff that solves this problem is **cost-efficient**.

New Problem

$$\min_{\{Y_T \mid Y_T \sim F, \, \mathbb{S}\}} \mathbb{E}\left[\xi_T Y_T\right].$$

where $\mathbb{S}$ denotes a set of constraints. A payoff that solves this problem is called a $\mathbb{S}-$**constrained cost-efficient payoff**.

"State-dependent preferences": Examples

- Investors also care about the **states** of the world in which wealth is received: Money can have "more value" in a crisis (insurance, puts).
- Often, we are looking at an isolated contract: the theory for law-invariant preferences only applies to the full portfolio.
- States can be described using the value of a **benchmark** A_T. (e.g. $A_T = S_T$ and states where stock market is low/high)

Definition: State-dependent preferences

Investors choose the distribution of a payoff X_T *and* additionally aim at obtaining a desired dependence with a benchmark asset A_T.

- Examples
  - Partial hedging min $\rho(A_T - H_T)$ where hedge is H_T and target is A_T and ρ is a law invariant risk measure.
  - Outperform a given benchmark: solve $\max_{X_T} P(X_T \geqslant A_T)$
  - Portfolio choice subject to some background risk, e.g. find X_T to $E[u(X_T + A_T)]$

Summary

- Part 1: Optimal payoffs for law-invariant preferences
  - Are always simple
  - Are increasing in the "market asset"
- Part 2: Limitations of law-invariance
  - Strategies perform badly during crises
  - Equivalence between first-order stochastic dominance and law-invariance (equivalence)
- Part 3: Optimal payoffs with additional state-dependent constraints are:
  - Conditionally increasing in the "market asset"
  - Able to cope with crises, background risk and benchmarking
- Part 4: Applications to Security Design and Portfolio Management

Main Assumptions on Market Model

- Given a portfolio with final payoff X_T **(consumption only at time T)**.
- P ("physical measure"). The initial value of X_T is given by

$$\mathbf{c_0(X_T)} = \mathbb{E}_{\mathbf{P}}[\xi_{\mathbf{T}}\mathbf{X_T}].$$

  where ξ_T is called the pricing kernel, state-price process, deflator, stochastic discount factor...
- All market participants **agree** on ξ_T, ξ_T is **continuously distributed** and

$$\xi_t = f(S_t), \; t \geqslant 0,$$

  for some suitable decreasing f and market asset S.
- *Another approach:* ξ_T is a Radon-Nikodym derivative. Let Q be a **"risk-neutral measure"** such that

$$\xi_T = e^{-rT}\left(\frac{dQ}{dP}\right)_T, \quad c_0(X_T) = \mathbb{E}_Q[e^{-rT}X_T].$$

Law-invariance - Sufficiency of Path-independent Payoffs

Theorem

Let X_T be a payoff with price c and having a cdf F. Then, there exists at least one path-independent payoff $f(S_T)$ with price c and cdf F.

- This characterization allows us to restrict the set of payoffs that are candidate solutions to optimal portfolio problems with an optimization of a **law-invariant objective** $V(\cdot)$.

$$\max_{X_T | c_0(X_T)=c} V(X_T)$$

- No other assumptions are needed (no risk-aversion, no non-decreasing preferences).

Law-invariance & Non-decreasing Preferences: Strict Optimality of Path-independent Products

Definition: **Non-decreasing** preferences

$$X_T \geqslant Y_T \ a.s. \Rightarrow V(X_T) \geqslant V(Y_T)$$

Theorem

For any payoff X_T with cdf F and price c for an investor with **non-decreasing** and **law-invariant** preferences, there exists an improved payoff X_T^* (almost surely non-decreasing in S_T) at same price c of the form

$$X_T^* = F^{-1}(F_{S_T}(S_T)) + a,$$

where $a \geqslant 0$.

Precisely, let c_0^* be the price of $F^{-1}(F_{S_T}(S_T))$ and F the cdf of X_T.

$$V\left(F^{-1}(F_{S_T}(S_T)) + (c - c_0^*)e^{rT}\right) \geqslant V(X_T).$$

Summary

Optimal payoffs for an investor with non-decreasing **law-invariant preferences** and a **fixed investment horizon**

- Optimal payoffs are "cost-efficient".
- **Cost-efficiency** $\Leftrightarrow$ Minimum correlation with the state-price process $\Leftrightarrow$ Anti-monotonicity with ξ_T $\Leftrightarrow$ Comonotonicity with S_T
  - *Optimality* of path-independent payoffs non-decreasing in S_T.
  - *Suboptimality* of path-dependent contracts.

State-Dependent preferences
Sufficiency of Bivariate Derivatives

Theorem: Bivariate payoff with given cdf with A_T and price c

Let X_T be a payoff with price c having joint distribution G with some **benchmark** A_T, where (S_T, A_T) has joint **density**. Then, there exists at least one bivariate derivative $f(S_T, A_T)$ with price c having the same joint distribution G with A_T.

Theorem: Bivariate payoff with given cdf with S_T and price c

Let X_T be a payoff with price c having joint distribution G with the **benchmark** S_T. Then, for any $0 < t < T$ there exists at least one payoff $f(S_t, S_T)$ with price c having joint distribution G with S_T. For example, for some $t \in (0, T)$,

$$f(S_t, S_T) := F^{-1}_{X_T|S_T}(F_{S_t|S_T}(S_t)).$$

State-Dependent preferences & Non-decreasing Preferences Strict Optimality of Bivariate Derivatives

Theorem:

Assume that (S_T, A_T) has joint density. Let G be a bivariate cumulative distribution function. The following optimization problem

$$\min_{(X_T,A_T)\sim G} c_0\,(X_T)$$

has an almost surely unique solution X_T^* which is a bivariate derivative almost surely increasing in S_T, conditionally on A_T and given by

$$X_T^* := F^{-1}_{X_T|A_T}(F_{S_T|A_T}(S_T)).$$

Improving Security Design

1. For **law-invariant preferences**: if the contract is *not* increasing in S_T, then there exists a strictly cheaper derivative (cost-efficient contract) that is strictly better.
2. If the investor buys the contract because of the **interaction with the market asset** S_T, and the contract depends on a more complex asset, then we simplify its design while keeping it "at least as good". For example, the contract can depend only on S_T and S_t for some $t \in (0, T)$.
3. If the investor buys the contract because of **its interaction with a benchmark** A_T, which has a joint density with S_T, and if the contract does not only depend on A_T and S_T, then there is a simpler contract which is "at least as good"and which writes as a function of S_T and A_T. Finally, if the obtained contract is *not* increasing in S_T conditionally on A_T, then it is also possible to construct a strictly cheaper alternative.

Geometric Asian option in Black-Scholes model

Geometric average G_T such that $\ln(G_T) := \frac{1}{T}\int_0^T \ln(S_s)\, ds$.

$$Y_T := (G_T - K)^+ .$$

Cheapest payoff with same distribution: Y_T^*. For some explicit constant $d > 0$

$$Y_T^* = d\left(S_T^{1/\sqrt{3}} - \frac{K}{d}\right)^+$$

Payoff X_T^* such that $(S_T, X_T^*) \sim (S_T, (G_T - K)^+)$.
For t freely chosen in $(0, T)$, $X_T^* = (f(S_t, S_T) - K)^+$ with

$$f(S_t, S_T) = S_0^{\frac{1}{2}-\frac{1}{2\sqrt{3}}\sqrt{\frac{T-t}{t}}} S_t^{\frac{T}{t}\frac{1}{2\sqrt{3}}\sqrt{\frac{t}{T-t}}} S_T^{\frac{1}{2}-\frac{1}{2\sqrt{3}}\sqrt{\frac{t}{T-t}}}$$

Maximal correlation $\rho_{\max}$ (for $t^* = T/2$) between $\ln(f(S_t, S_T))$ and $\ln(G_T)$ is

$$\rho_{\max} = \frac{3}{4} + \frac{\sqrt{3}\sqrt{(T-t^*)\,t^*}}{4T} = \frac{3}{4} + \frac{\sqrt{3}}{8} \approx 0.9665.$$

Portfolio Management

1. Extension of the standard Merton optimal portfolio choice problem
2. Extension of Browne, Spivak and Cvitanìc Target Probability Maximization Problem
3. Applications to partial hedging

Merton Problem

Theorem

Consider a utility function $u(\cdot)$ with Inada conditions. The optimal solution to

$$\max_{\mathbb{E}[\xi_T X_T]=W_0} \mathbb{E}[u(X_T)]$$

is given by

$$X_T^* = \left[u'\right]^{-1}(\lambda\xi_T) \tag{8}$$

where λ verifies $\mathbb{E}\left[\xi_T[u']^{-1}(\lambda X_T^*)\right] = W_0$.

We then solve the solution to the same problem with a constraint on the dependence between X_T and a benchmark is $X_T^\star = f(S_T, A_T)$.

$$\max_{\substack{c_0(X_T)=W_0 \\ \mathcal{C}_{(X_T,A_T)}=C}} \mathbb{E}\left(u(X_T)\right). \tag{9}$$

Standard Target Probability Maximization Problem

Theorem: Browne's original problem

Let W_0 be the initial wealth and $b > W_0 e^{rT}$ be the desired target. The solution to the target probability maximization problem

$$\max_{X_T \geqslant 0, c_0(X_T)=W_0} \mathbb{P}[X_T \geqslant b]$$

is $X_T^* = b\,\mathbb{1}_{\{S_T>\lambda\}}$ where λ is given by $\mathbb{E}\left(\xi_T X_T^*\right) = W_0$.

We propose two stochastic extensions for which bivariate derivatives are solutions.

$$\max_{X_T \geqslant 0, c_0(X_T)=W_0} \mathbb{P}[X_T \geqslant A_T]$$

$$\max_{\substack{X_T \geqslant 0, c_0(X_T)=W_0, \\ \mathcal{C}_{(X_T,A_T)}=C}} \mathbb{P}[X_T \geqslant b]$$

More Applications...

The best "partial" hedge X_T consists in minimizing the distance between X_T and a payoff B_T in some appropriate sense (assuming $c_0(B_T) > W_0$). Consider the following optimal hedging problems.

1. in the expected utility setting

$$\max_{\left\{X_T \middle| \substack{X_T \geqslant 0, \\ c_0(X_T) = W_0,}\right\}} \mathbb{E}[U(X_T - B_T)]$$

   where $U(\cdot)$ is concave and increasing

2. to minimize the risk as

$$\min_{\left\{X_T \middle| \substack{X_T \geqslant 0 \\ c_0(X_T) = W_0,}\right\}} \rho(X_T - B_T)$$

   where $\rho(\cdot)$ is a convex law-invariant risk measure.

3. in the quantile hedging problem setting

$$\max_{\{X_T | X_T \geqslant 0, c_0(X_T) = W_0\}} \mathbb{P}[X_T \geqslant B_T]$$

Another Application: Refining fraud detection tools

- Detect fraud based on mean and variance using the maximum possible Sharpe Ratio (SR) of a payoff X_T (terminal wealth at T when investing W_0 at $t = 0$) over all possible admissible strategies
$$SR(X_T) = \frac{\mathrm{E}[X_T] - W_0 e^{rT}}{\mathrm{std}(X_T)},$$
- But this ignores additional information available in the market: dependence between the investment strategy and the financial market?
- Include correlations of the fund with market indices (benchmarks) to refine fraud detection.

Ex: the so-called "market-neutral" strategy is typically designed to have very low correlation with market indices $\Rightarrow$ it reduces the maximum possible Sharpe ratio! (EJOR, 2014)

More Applications...

- Constraining the distribution in certain areas instead of the entire joint distribution.
- (QF 2014) "Optimal Portfolios under Worst-case Scenarios" for designing optimal strategies that offer protection in a crisis.

  "an increasing number of investors now want protection for financial end times"...
  "As the stock markets fell, a tail risk or black swan fund would profit..."
  (See "New Investment Strategy: Preparing for End Times")

Basic example

Y_T and S_T have given distributions.

- The investor wants to ensure a **minimum** when the market falls

$$\mathbb{P}(Y_T > 100 \mid S_T < 95) = 0.8.$$

  This provides some additional information on the joint distribution between Y_T and $S_T \Rightarrow$ information on the joint distribution of (ξ_T, Y_T) in the Black-Scholes framework.
- Y_T is **decreasing** in S_T when the stock S_T falls below some level (to justify the demand of a put option).
- Y_T is **independent** of S_T when S_T falls below some level.

All these constraints impose the strategy Y_T to pay out in given states of the world.

Formally

Goal: Find the **cheapest** possible payoff Y_T with the distribution F and which **satisfies additional constraints** of the form

$$\mathbb{P}(\xi_T \leqslant x, Y_T \leqslant y) = Q(F_{\xi_T}(x), F(y)),$$

with $x > 0, y \in \mathbb{R}$ and Q a given feasible function (for example a copula).

Each constraint gives information on the dependence between the state-price ξ_T and Y_T and is, for a given function Q, determined by the pair $(F_{\xi_T}(x), F(y))$.

Denote the finite or infinite set of all such constraints by $\mathbb{S}$.

Theorem (Case of one constraint)

Assume that there is only one constraint (a, b) *in* $\mathbb{S}$ *and let* $\vartheta := Q(a, b)$. *The* $\mathbb{S}-$*constrained cost-efficient payoff* $Y_T^\star$ *exists and is unique. It can be expressed as*

$$Y_T^\star = F^{-1}\left(G(F_{\xi_T}(\xi_T))\right), \tag{10}$$

where $G : [0,1] \to [0,1]$ *is defined as* $G(u) = \ell_u^{-1}(1)$ *and can be written as*

$$G(u) = \begin{cases} 1-u & \text{if } 0 \leqslant u \leqslant a-\vartheta, \\ a+b-\vartheta-u & \text{if } a-\vartheta < u \leqslant a, \\ 1+\vartheta-u & \text{if } a < u \leqslant 1+\vartheta-b, \\ 1-u & \text{if } 1+\vartheta-b < u \leqslant 1. \end{cases} \tag{11}$$

Example 1: $\mathbb{S}$ contains 1 constraint

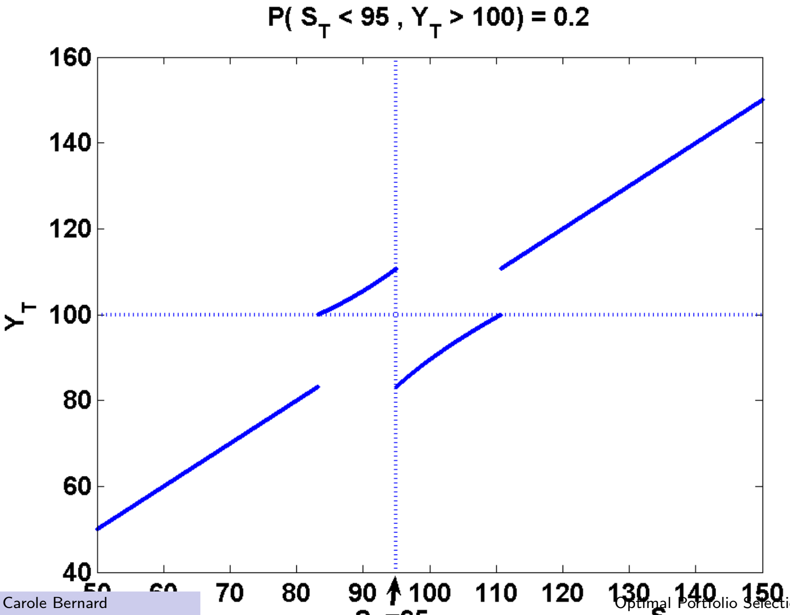
Assume a Black-Scholes market. We suppose that the investor is looking for the payoff Y_T such that $Y_T \sim F$ (where F is the cdf of S_T) and satisfies the following constraint

$$\mathbb{P}(S_T < 95,\ Y_T > 100) = 0.2.$$

The optimal strategy, where $a = 1 - F_{S_T}(95)$, $b = F_{S_T}(100)$ and $\vartheta = 0.2 - F_{S_T}(95) + F_{S_T}(100)$ is given by the previous theorem.

Its price is 100.2

Example: Illustration



Example 2: $\mathbb{S}$ is infinite

A cost-efficient strategy with the same distribution F as S_T but such that it is decreasing in S_T when $S_T \leqslant \ell$ is unique a.s. Its payoff is equal to

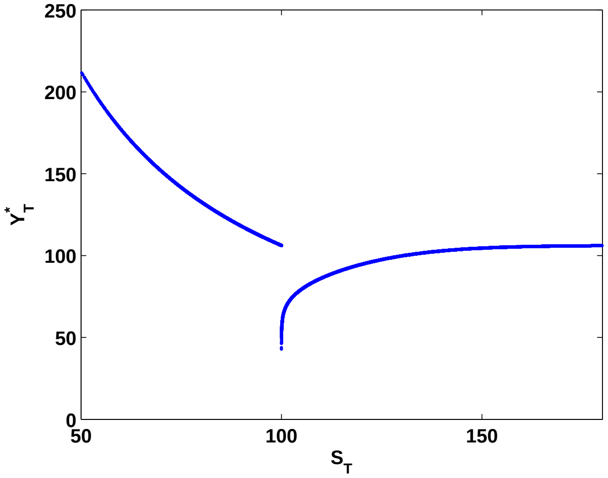
$$Y_T^\star = F^{-1}\left[G(F(S_T))\right],$$

where $G : [0,1] \to [0,1]$ is given by

$$G(u) = \begin{cases} 1-u & \text{if } 0 \leqslant u \leqslant F(\ell), \\ u - F(\ell) & \text{if } F(\ell) < u \leqslant 1. \end{cases}$$

The **constrained cost-efficient payoff** can be written as

$$Y_T^\star := F^{-1}\left[(1-F(S_T))\mathbb{1}_{S_T<\ell} + (F(S_T) - F(\ell))\,\mathbb{1}_{S_T \geqslant \ell}\right].$$



$Y_T^\star$ as a function of S_T. Parameters: $\ell = 100$, $S_0 = 100$, $\mu = 0.05$, $\sigma = 0.2$, $T = 1$ and $r = 0.03$. The price is 103.4.

"Tail Diversification" of Cost-Efficient Strategies

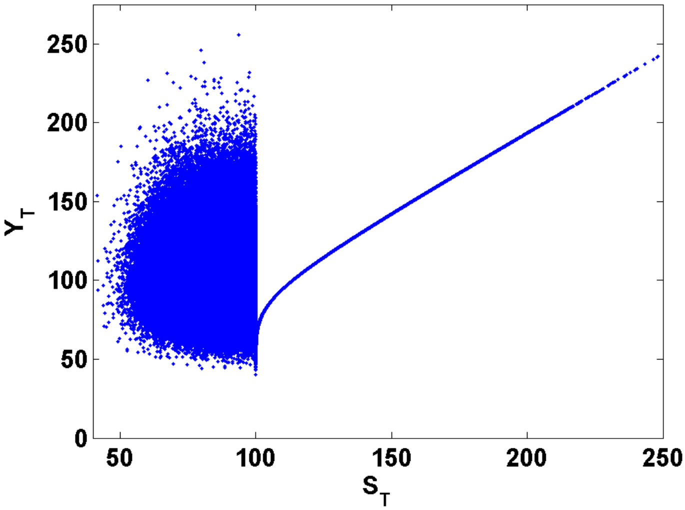
Theorem (Constraints on the tail)

In a one-dimensional Black-Scholes market, the cheapest path-dependent strategy with a cumulative distribution F but such that it is independent of $S_1(T)$ when $S_1(T) \leqslant q_\alpha$ can be constructed as

$$
\begin{cases}
F^{-1}\left(\frac{F_{S_1(T)}(S_1(T)) - F_{S_1(T)}(q_\alpha)}{1 - F_{S_1(T)}(q_\alpha)}\right) & \text{when} \quad S_1(T) > q_\alpha \\
F^{-1}\left(\Phi\left(\frac{\ln\left(\frac{S_1(t)}{(S_1(T))^{t/T}}\right) - (1 - \frac{t}{T})\ln(S_1(0))}{\sigma_1\sqrt{t - \frac{t^2}{T}}}\right)\right) & \text{when} \quad S_1(T) \leqslant q_\alpha
\end{cases}
$$

where $t \in (0, T)$ can be chosen freely.

(**No uniqueness and path-dependent optimum**).



10,000 realizations of $Y_T^\star$ as a function of S_T where $\ell = 100$, $S_0 = 100$, $\mu = 0.05$, $\sigma = 0.2$, $T = 1$, $r = 0.03$ and $t = T/2$. Its price is 101.1

Relaxing the assumptions on ξ_T

1. Use the Growth Optimal Portfolio (GOP) $\xi_T = \frac{1}{S^*_T}$. (Details in Platen & Heath (2006)) to replace the assumption $\xi_T = g(S_T)$. The **GOP**
   - maximizes expected logarithmic utility from terminal wealth.
   - is a diversified portfolio with the property that it almost surely accumulates more wealth than any other strictly positive portfolios after a sufficiently long time.
2. Model a multidimensional market: the state-price process (ξ_t) of the risk-neutral measure chosen for pricing is of the form $\xi_T = f(g(S^{(1)}_T, ..., S^{(n)}_T))$ with some real function g. All results in the paper apply by replacing $(S_t)_t$ by the one-dimensional market process $(g(\mathbf{S}_t))$.
3. Remove the assumption on the continuity of F_{ξ_T} by using "randomized payoffs" (JAP 2014).

Conclusions & Future Work

- **FSD** or law-invariant behavioral settings **cannot explain** all decisions. One needs to look at state-dependent preferences to explain investment decisions such as
  - Buying protection...
  - Investing in highly path-dependent derivatives...
- Our framework allows to take optimal decisions when there is a source of background risk and explains mildly path-dependent options.
- Applications for hedging, semi-static hedging...

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