Assessing Model Risk on Dependence in High Dimensions

Carole Bernard

based on joint work with Steven Vanduffel

Risk Aggregation and Diversification

- A key issue in capital adequacy and solvency is to **aggregate risks** (by summing capital requirements?) and potentially account for **diversification** (to reduce the total capital?)
- Using the standard deviation to measure the risk of aggregating X_1 and X_2 with standard deviation $std(X_i)$,

$$std(X_1 + X_2) = \sqrt{std(X_1)^2 + std(X_2)^2 + 2\rho std(X_1) std(X_2)}$$

If $\rho < 1$, there are "diversification benefits":

$$std(X_1 + X_2) < std(X_1) + std(X_2)$$

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If $\rho < 1$, there are "diversification benefits":

$$std(X_1 + X_2) < std(X_1) + std(X_2)$$

- This is not the case for instance for Value-at-Risk.

Objectives and Findings

- Model uncertainty on the risk assessment of an aggregate portfolio: the sum of d dependent risks.
  - Given all information available in the market, what can we say about the maximum and minimum possible values of a given risk measure of a portfolio?
- A non-parametric method based on the data at hand.
- Analytical expressions for the maximum and minimum

Objectives and Findings

- Model uncertainty on the risk assessment of an aggregate portfolio: the sum of d dependent risks.
  - Given all information available in the market, what can we say about the maximum and minimum possible values of a given risk measure of a portfolio?
- A non-parametric method based on the data at hand.
- Analytical expressions for the maximum and minimum
- Implications:
  - Current VaR based regulation is subject to high model risk, even if one knows the multivariate distribution "almost completely".
  - We can identify for which risk measures it is meaningful to develop accurate multivariate models.

Motivation on VaR aggregation

Full information on **marginal distributions**:
$X_j \sim F_j$ and represent risks as $X_j = F_j^{-1}(U_j)$ where U_j is $\mathcal{U}(0,1)$.

$$+$$

Full Information on **dependence**:
$(U_1, U_2, ..., U_n) \sim C$ (C is called the copula)

$$\Rightarrow$$

$$\mathrm{VaR}_q\left(X_1 + X_2 + ... + X_n\right) \text{ can be computed!}$$

Motivation on VaR aggregation

Full information on **marginal distributions**:
$X_j \sim F_j$ and represent risks as $X_j = F_j^{-1}(U_j)$
where U_j is $\mathcal{U}(0,1)$.

$+$

Partial or **no** Information on **dependence**:
$(U_1, U_2, ..., U_n) \sim$???

$\Rightarrow$

$\mathrm{VaR}_q(X_1 + X_2 + ... + X_n)$ **cannot** be computed!

Only a range of possible values for $\mathrm{VaR}_q(X_1 + X_2 + ... + X_n)$.

Model Risk

1. Goal: Assess the risk of a portfolio sum $S = \sum_{i=1}^{d} X_i$.
2. Choose a risk measure $\rho(\cdot)$: variance, Value-at-Risk...
3. "Fit" a multivariate distribution for $(X_1, X_2, ..., X_d)$ and compute $\rho(S)$
4. How about model risk? How wrong can we be?

Model Risk

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4. How about model risk? How wrong can we be?

Assume $\rho(S) = var(S)$,

$$\rho_{\mathcal{F}}^{+} := \sup\left\{ var\left(\sum_{i=1}^{d} X_i\right)\right\}, \quad \rho_{\mathcal{F}}^{-} := \inf\left\{ var\left(\sum_{i=1}^{d} X_i\right)\right\}$$

where the bounds are taken over all other (joint distributions of) random vectors $(X_1, X_2, ..., X_d)$ that "agree" with the available information $\mathcal{F}$

Assessing Model Risk on Dependence with d Risks

- Marginals known, Dependence fully unknown
- If $d = 2$, assessing model risk on variance is linked to the Fréchet-Hoeffding bounds or "extreme dependence".

$$var(F_1^{-1}(U)+F_2^{-1}(1-U)) \leqslant var(X_1+X_2) \leqslant var(F_1^{-1}(U)+F_2^{-1}(U))$$

- A challenging problem in $d \geqslant 3$ dimensions
  - Wang and Wang (2011, JMVA)
  - Puccetti and Rüschendorf (2012): algorithm (RA) useful to approximate the minimum variance.
  - Embrechts, Puccetti, Rüschendorf (2013): algorithm (RA) to find bounds on VaR
  - Bernard, Jiang, Wang (2014, IME): explicit form of a lower bound for the sum of homogeneous risks.
- **Issues**
  - bounds are generally very wide
  - ignore all information on dependence.
- **Our answer:** incorporating dependence information.

Rearrangement Algorithm

$N = 4$ observations of $d = 3$ variables: X_1, X_2, X_3

$$\mathbf{M} = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 6 & 3 \\ 4 & 0 & 0 \\ 6 & 3 & 4 \end{bmatrix}$$

Each column: **marginal** distribution

Interaction among columns: **dependence** among the risks

Same marginals, different dependence ⇒ Effect on the sum!

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 6 & 3 \\ 4 & 0 & 0 \\ 6 & 3 & 4 \end{bmatrix} \qquad S_N = \overset{X_1+X_2+X_3}{\begin{bmatrix} 4 \\ 9 \\ 4 \\ 13 \end{bmatrix}}$$

$$\begin{bmatrix} 6 & 6 & 4 \\ 4 & 3 & 3 \\ 1 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \qquad S_N = \overset{X_1+X_2+X_3}{\begin{bmatrix} 16 \\ 10 \\ 3 \\ 0 \end{bmatrix}}$$

Aggregate Risk with Maximum Variance

comonotonic scenario

Aggregate risk with minimum variance

Each column is antimonotonic with the sum of the others:

$$\begin{matrix}\downarrow & & \\ \end{matrix}\quad X_2+X_3 \qquad\qquad \begin{matrix} & \downarrow & \\ \end{matrix}\quad X_1+X_3 \qquad\qquad \begin{matrix} & & \downarrow \\ \end{matrix}\quad X_1+X_2$$

$$\begin{bmatrix} 0 & 3 & 4 \\ 1 & 6 & 0 \\ 4 & 1 & 2 \\ 6 & 0 & 1 \end{bmatrix} \begin{matrix} 7 \\ 6 \\ 3 \\ 1 \end{matrix}, \quad \begin{bmatrix} 0 & 3 & 4 \\ 1 & 6 & 0 \\ 4 & 1 & 2 \\ 6 & 0 & 1 \end{bmatrix} \begin{matrix} 4 \\ 1 \\ 6 \\ 7 \end{matrix}, \quad \begin{bmatrix} 0 & 3 & 4 \\ 1 & 6 & 0 \\ 4 & 1 & 2 \\ 6 & 0 & 1 \end{bmatrix} \begin{matrix} 3 \\ 7 \\ 5 \\ 6 \end{matrix}$$

$$\begin{bmatrix} \mathbf{0} & \mathbf{3} & \mathbf{4} \\ \mathbf{1} & \mathbf{6} & \mathbf{0} \\ \mathbf{4} & \mathbf{1} & \mathbf{2} \\ \mathbf{6} & \mathbf{0} & \mathbf{1} \end{bmatrix} \qquad S_N = \overset{X_1+X_2+X_3}{\begin{bmatrix} 7 \\ 7 \\ 7 \\ 7 \end{bmatrix}}$$

The minimum variance of the sum is equal to 0! (ideal case of a constant sum (*complete mixability*, see Wang and Wang (2011))

Bounds on variance

Analytical Bounds on Standard Deviation

Consider d risks X_i with standard deviation σ_i

$$0 \leqslant std(X_1 + X_2 + ... + X_d) \leqslant \sigma_1 + \sigma_2 + ... + \sigma_d$$

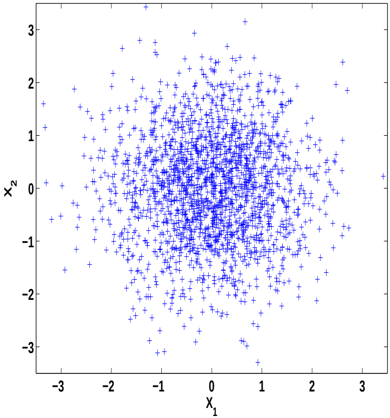
Example with 20 standard normal N(0,1)

$$0 \leqslant std(X_1 + X_2 + ... + X_{20}) \leqslant 20$$

and in this case, both bounds are sharp but too wide for practical use!

Our idea: Incorporate information on dependence.

Illustration with 2 risks with marginals N(0,1)



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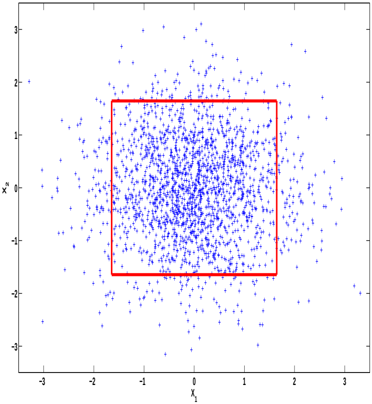
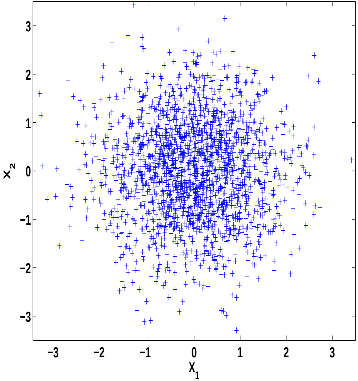
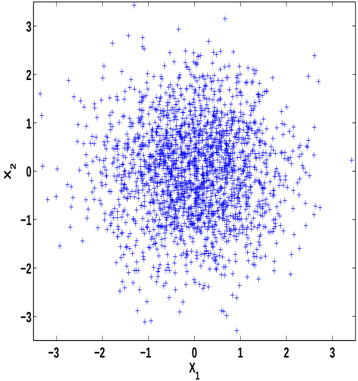


Assumption: Independence on $\mathcal{F} = \bigcap_{k=1}^{2} \{q_\beta \leqslant X_k \leqslant q_{1-\beta}\}$

Illustration with marginals N(0,1)





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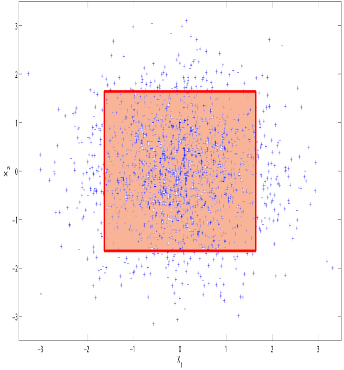
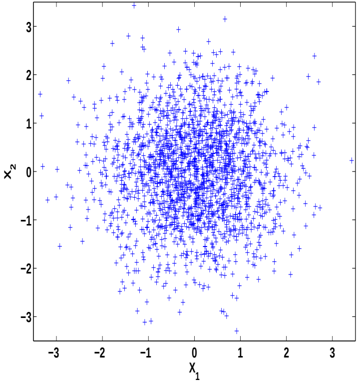
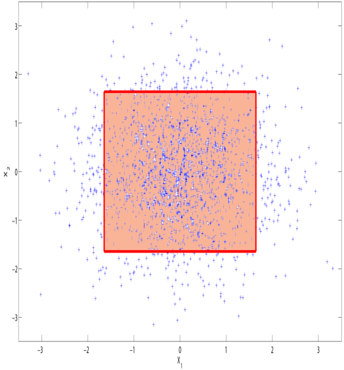
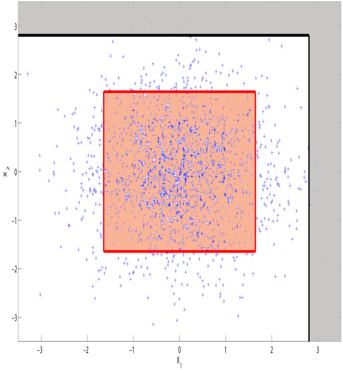




$$\mathcal{F}_1 = \bigcap_{k=1}^{2} \{q_\beta \leqslant X_k \leqslant q_{1-\beta}\}$$

Illustration with marginals N(0,1)

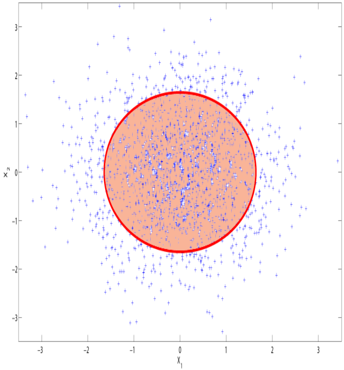


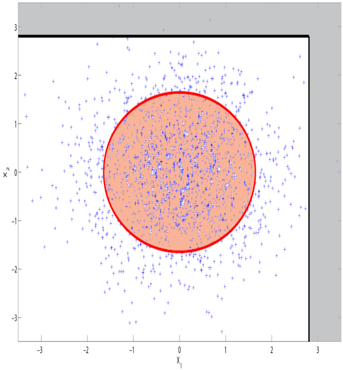


$$\mathcal{F}_1 = \bigcap_{k=1}^{2} \{q_\beta \leqslant X_k \leqslant q_{1-\beta}\}$$

$$\mathcal{F} = \bigcup_{k=1}^{2} \{X_k > q_p\} \bigcup \mathcal{F}_1$$

Illustration with marginals N(0,1)





$\mathcal{F}_1$ =contour of MVN at β

$$\mathcal{F} = \bigcup_{k=1}^{2} \{X_k > q_p\} \bigcup \mathcal{F}_1$$

Our assumptions on the cdf of $(X_1, X_2, ..., X_d)$

$\mathcal{F} \subset \mathbb{R}^d$ ("trusted" or "fixed" area)
$\mathcal{U} = \mathbb{R}^d \backslash \mathcal{F}$ ("untrusted").
We assume that we know:

(i) the marginal distribution F_i of X_i on $\mathbb{R}$ for $i = 1, 2, ..., d$,
(ii) the distribution of $(X_1, X_2, ..., X_d) \mid \{(X_1, X_2, ..., X_d) \in \mathcal{F}\}$.
(iii) $P((X_1, X_2, ..., X_d) \in \mathcal{F})$

- When only marginals are known: $\mathcal{U} = \mathbb{R}^d$ and $\mathcal{F} = \emptyset$.
- **Our Goal: Find bounds on** $var(S) := var(X_1 + ... + X_d)$ **when** $(X_1, ..., X_d)$ **satisfy (i), (ii) and (iii).**

Example:

$N = 8$ observations, $d = 3$ dimensions
and 3 observations trusted ($\ell_f = 3$, $p_f = 3/8$)

$$\begin{bmatrix} 3 & 4 & 1 \\ 1 & 1 & 1 \\ 0 & 3 & 2 \\ 0 & 2 & 1 \\ 2 & 4 & 2 \\ 3 & 0 & 1 \\ 1 & 1 & 2 \\ 4 & 2 & 3 \end{bmatrix} \qquad S_N = \begin{bmatrix} 8 \\ 3 \\ 5 \\ 3 \\ 8 \\ 4 \\ 4 \\ 9 \end{bmatrix}$$

Example: $N = 8$, $d = 3$ with 3 observations trusted ($\ell_f = 3$)
Maximum variance

$$M = \begin{bmatrix} 3 & 4 & 1 \\ 2 & 4 & 2 \\ 0 & 2 & 1 \\ 4 & 3 & 3 \\ 3 & 2 & 2 \\ 1 & 1 & 2 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad S_N^f = \begin{bmatrix} 8 \\ 8 \\ 3 \end{bmatrix}, \quad S_N^u = \begin{bmatrix} 10 \\ 7 \\ 4 \\ 3 \\ 1 \end{bmatrix}$$

The maximum variance is
$\frac{1}{8}\left(\sum_{i=1}^{3}(s_i - \bar{s})^2 + \sum_{i=1}^{5}(\tilde{s}_i^c - \bar{s})^2\right) \approx 8.75$ with $\bar{s} = 5.5$.

Example: $N = 8$, $d = 3$ with 3 observations trusted ($\ell_f = 3$)
Minimum variance

Minimum variance obtained when S_N^u has smallest variance (ideally constant, "mixability")

$$M = \begin{bmatrix} 3 & 4 & 1 \\ 2 & 4 & 2 \\ 0 & 2 & 1 \\ 1 & 1 & 3 \\ 0 & 3 & 2 \\ 1 & 2 & 2 \\ 3 & 1 & 1 \\ 4 & 0 & 1 \end{bmatrix}, \quad S_N^f = \begin{bmatrix} 8 \\ 8 \\ 3 \end{bmatrix}, \quad S_N^u = \begin{bmatrix} 5 \\ 5 \\ 5 \\ 5 \\ 5 \end{bmatrix}$$

The minimum variance is
$\frac{1}{8}\left(\sum_{i=1}^{3}(s_i - \bar{s})^2 + \sum_{i=1}^{5}(\tilde{s}_i^m - \bar{s})^2\right) \approx 2.5$ with $\bar{s} = 5.5$.

Example $d = 20$ risks N(0,1)

- $(X_1, ..., X_{20})$ independent N(0,1) on

$$\mathcal{F} := [q_\beta, q_{1-\beta}]^d \subset \mathbb{R}^d \qquad p_f = P\left((X_1, ..., X_{20}) \in \mathcal{F}\right)$$

(for some $\beta \leqslant 50\%$) where q_γ: γ-quantile of N(0,1)

- $\beta = 0\%$: no uncertainty (20 independent N(0,1))
- $\beta = 50\%$: full uncertainty

$\mathcal{F} = [q_\beta, q_{1-\beta}]^d$	$\mathcal{U} = \emptyset$ $\beta = 0\%$			$\mathcal{U} = \mathbb{R}^d$ $\beta = 50\%$
$\rho = 0$	4.47			(0 , 20)

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- $\beta = 0\%$: no uncertainty (20 independent N(0,1))
- $\beta = 50\%$: full uncertainty

$\mathcal{F} = [q_\beta, q_{1-\beta}]^d$	$\mathcal{U} = \emptyset$ $\beta = 0\%$	$p_f \approx 98\%$ $\beta = 0.05\%$	$p_f \approx 82\%$ $\beta = 0.5\%$	$\mathcal{U} = \mathbb{R}^d$ $\beta = 50\%$
$\rho = 0$	4.47	(4.4 , 5.65)	(3.89 , 10.6)	(0 , 20)

Model risk on the volatility of a portfolio is reduced a lot by incorporating information on dependence!

Bounds on Variance

Bounds on the variance of $\sum_{i=1}^{d} X_i$

Let $(X_1, X_2, ..., X_d)$ that satisfies properties (i), (ii) and (iii) and let

$$\mathbb{I} := \mathbb{1}_{(X_1, X_2, ..., X_d) \in \mathcal{F}},$$

$Z_i \sim F_{X_i | (X_1, X_2, ..., X_d) \in \mathcal{U}}$ are comonotonic and independent of $\mathbb{I}$ for $i = 1, 2, ..., d$. Then, with $S = \sum_{i=1}^{d} X_i$,

$$\text{var}\left(\mathbb{I}S + (1-\mathbb{I})\sum_{i=1}^{d} EZ_i\right) \leqslant \text{var}(S) \leqslant \text{var}\left(\mathbb{I}S + (1-\mathbb{I})\sum_{i=1}^{d} Z_i\right)$$

Other Risk Measures

- Assess model risk for variance of a portfolio of risks with given marginals but partially known dependence. Same method applies to TVaR (expected Shortfall) or any risk measure that satisfies convex order (but not for Value-at-Risk).

definition: Convex order

X is smaller in convex order, $X \prec_{cx} Y$, if for all convex functions f

$$E[f(X)] \leqslant E[f(Y)]$$

- **Next, let us study model risk on Value-at-Risk.**
  - Maximum Value-at-Risk is not caused by the comonotonic scenario.
  - Maximum Value-at-Risk is achieved when the variance is *minimum* in the tail. The RA is then used in the tails only.
  - Bounds on Value-at-Risk at high confidence level stay wide even when the trusted area covers 98% of the space!

Setting

- Model uncertainty on the VaR of an aggregate portfolio: the sum of d individual dependent risks.
  - Value-at-Risk at level q of $S = X_1 + X_2 + ... + X_d$
  - "Fit" a multivariate distribution for $(X_1, X_2, ..., X_d)$ and compute $VaR_q(S)$
  - How about model risk? How wrong can we be?

$$VaR^+_{q,\mathcal{F}} = \sup\left\{VaR_q\left(\sum_{i=1}^{d} X_i\right)\right\}, VaR^-_{q,\mathcal{F}} = \inf\left\{VaR_q\left(\sum_{i=1}^{d} X_i\right)\right\}$$

where bounds are taken over all other random vectors $(X_1, X_2, ..., X_d)$ that "agree" with the available information

Definitions

- **Value-at-Risk** of X at level $q \in (0,1)$

$$\mathrm{VaR}_q(X) = \inf\{x \in \mathbb{R} \mid F_X(x) \geqslant q\}$$

- **Tail Value-at-Risk** or **Expected Shortfall** of X

$$\mathrm{TVaR}_q(X) = \frac{1}{1-q}\int_q^1 \mathrm{VaR}_u(X)\mathrm{d}u \qquad q \in (0,1)$$

- **Left Tail Value-at-Risk** of X

$$\mathrm{LTVaR}_q(X) = \frac{1}{q}\int_0^q \mathrm{VaR}_u(X)\mathrm{d}u$$

Bounds on Value-at-Risk

First part works for all risk measures that satisfy convex order...
But not for Value-at-Risk.

- **Explicit sharp bounds**

· $n = 2$ Makarov (1981), Rüschendorf (1982)

· homogeneous portfolios: Rüschendorf & Uckelmann (1991), Denuit, Genest & Marceau (1999), Embrechts & Puccetti (2006), Wang & Wang (2011), Bernard, Jiang and Wang (2014)

· heterogeneous portfolios: Wang & Wang (2015)

- **Approximate sharp bounds**

· The Rearrangement Algorithm (Puccetti & Rüschendorf (2012), Embrechts, Puccetti & Rüschendorf (2013))

- VaR_q is **not** maximized for the comonotonic scenario:

$$S^c = X_1^c + X_2^c + ... + X_d^c$$

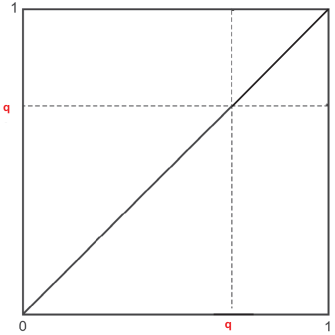
where all X_i^c are *comonotonic.*

Let us illustrate the problem with two risks:

"Riskiest" Dependence Structure maximum VaR at level q in 2 dimensions

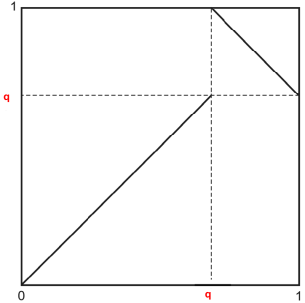
If L_1 and L_2 are U(0,1) comonotonic, then

$$VaR_q(S^c) = VaR_q(X_1) + VaR_q(X_2) = 2q.$$



"Riskiest" Dependence Structure maximum VaR at level q in 2 dimensions

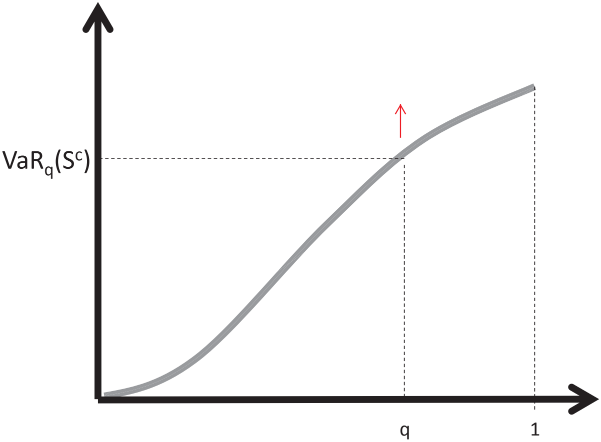
If L_1 and L_2 are U(0,1) and antimonotonic in the tail, then $VaR_q(S^*) = 1 + q$.



$$VaR_q(S^*) = 1 + q > VaR_q(S^c) = 2q$$

$\Rightarrow$ to maximize VaR_q, the idea is to change the comonotonic dependence such that the sum is constant in the tail

VaR at level q of the comonotonic sum w.r.t. q



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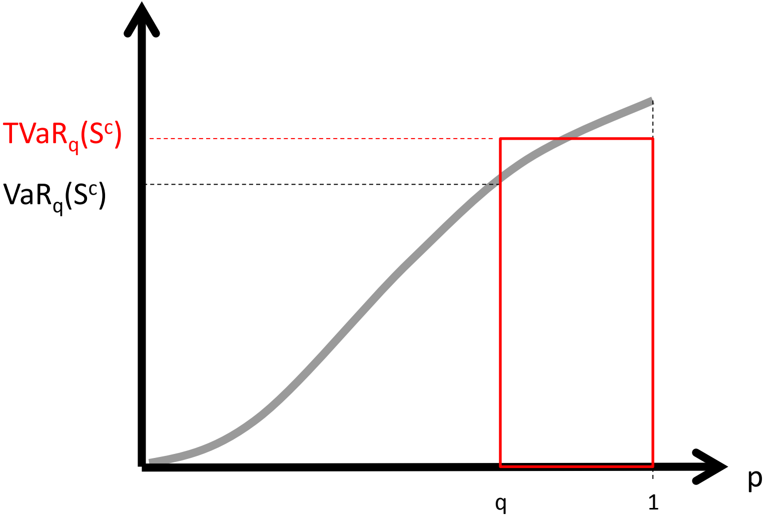
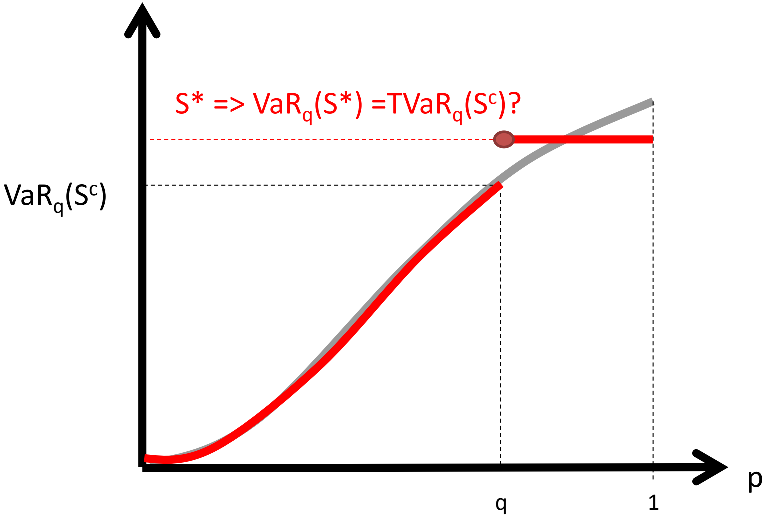


Riskiest Dependence Structure VaR at level q



Analytical Unconstrained Bounds with $X_j \sim F_j$

$$A = LTVaR_q(S^c) \leqslant \mathrm{VaR}_q\left[X_1 + X_2 + ... + X_n\right] \leqslant B = TVaR_q(S^c)$$

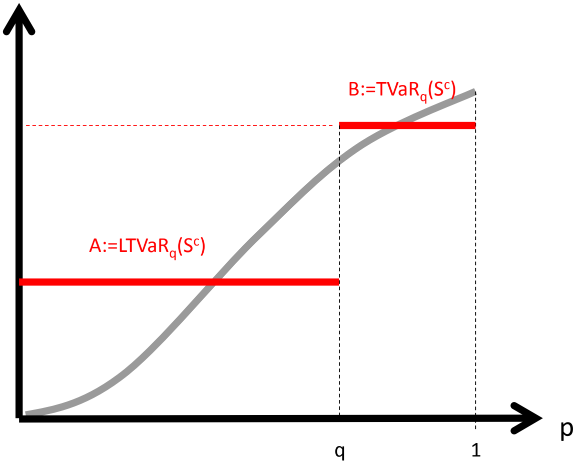


Illustration (1/3)

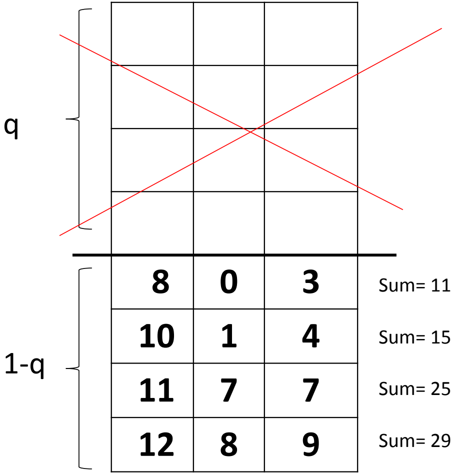


Illustration (2/3)

q				
1-q	8	0	3	Sum= 11
	10	1	4	Sum= 15
	11	7	7	Sum= 25
	12	8	9	Sum= 29

Rearrange **within** columns..to make the sums as constant as possible...
B=(11+15+25+29)/4=20

Illustration (3/3)

q				
1-q	**8**	**8**	**4**	Sum= 20
	10	**7**	**3**	Sum= 20
	12	**1**	**7**	Sum= 20
	11	**0**	**9**	Sum= 20

=B!

Numerical Results, 20 risks $N(0,1)$

When marginal distributions are given,

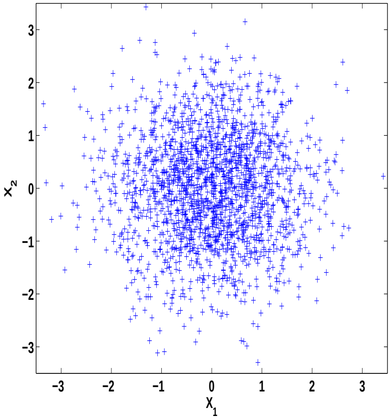
- What is the maximum Value-at-Risk?
- What is the minimum Value-at-Risk?
- A portfolio of 20 risks normally distributed N(0,1). Bounds on VaR_q (by the rearrangement algorithm applied on each tail)

q=95%	(-2.17 , 41.3)
q=99.95%	(-0.035 , 71.1)

- More examples in Embrechts, Puccetti, and Rüschendorf (2013): "Model uncertainty and VaR aggregation," *Journal of Banking and Finance*
- Very wide bounds
- All dependence information ignored

Our idea: add information on dependence from a fitted model where data is available...

Illustration with 2 risks with marginals N(0,1)



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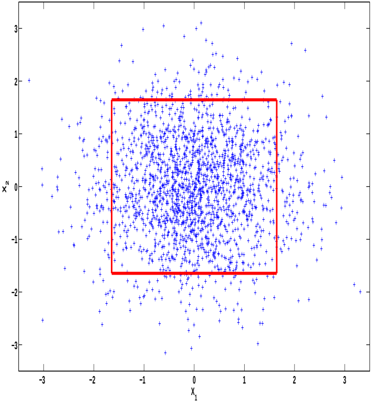

Assumption: Independence on $\mathcal{F} = \bigcap_{k=1}^{2} \{q_\beta \leqslant X_k \leqslant q_{1-\beta}\}$

Our assumptions on the cdf of $(X_1, X_2, ..., X_d)$

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(iii) $P((X_1, X_2, ..., X_d) \in \mathcal{F})$

- **Our Goal: Find bounds on $\mathbf{VaR}_q(S) := \mathbf{VaR}_q(X_1 + ... + X_d)$ when $(X_1, ..., X_d)$ satisfy (i), (ii) and (iii).**

In the paper entitled "*A New Approach to Assessing Model Risk in High Dimensions*" with S. Vanduffel,

- we adapt the rearrangement algorithm to solve for sharp bounds on VaR in the above case.
- we provide theoretical expressions as the VaR of a mixture for the lower and the upper bounds.

Numerical Results, 20 independent $N(0,1)$ on $\mathcal{F} = [q_\beta, q_{1-\beta}]^d$

	$\mathcal{U} = \emptyset$ $\beta = 0\%$			$\mathcal{U} = \mathbb{R}^d$ $\beta = 0.5$
q=95%	12.5			(-2.17 , 41.3)
q=99.95%	25.1			(-0.035 , 71.1)

- $\mathcal{U} = \emptyset$: 20 independent standard normal variables.

$$\text{VaR}_{95\%} = 12.5 \qquad \text{VaR}_{99.95\%} = 25.1$$

Numerical Results, 20 independent $N(0,1)$ on $\mathcal{F} = [q_\beta, q_{1-\beta}]^d$

	$\mathcal{U} = \emptyset$ $\beta = 0\%$	$p_f \approx 98\%$ $\beta = 0.05\%$	$p_f \approx 82\%$ $\beta = 0.5\%$	$\mathcal{U} = \mathbb{R}^d$ $\beta = 0.5$
q=95%	12.5	(12.2 , 13.3)	(10.7 , 27.7)	(-2.17 , 41.3)
q=99.95%	25.1	(24.2 , 71.1)	(21.5 , 71.1)	(-0.035 , 71.1)

- $\mathcal{U} = \emptyset$: 20 independent standard normal variables.

$$\text{VaR}_{95\%} = 12.5 \quad \text{VaR}_{99.95\%} = 25.1$$

- **The risk for an underestimation of VaR is increasing in the probability level used to assess the VaR.**
- **For VaR at high probability levels ($q = 99.95\%$), despite all the added information on dependence, the bounds are still wide!**

Bounds on VaR

Theorem (Constrained VaR Bounds for $\sum_{i=1}^d X_i$)

Assume $(X_1, X_2, ..., X_d)$ satisfies properties (i), (ii) and (iii), and let $(Z_1, Z_2, ..., Z_d)$, U and $\mathbb{I}$ as defined before. Define the variables L_i and H_i as

$$L_i = LTVaR_U(Z_i) \text{ and } H_i = TVaR_U(Z_i)$$

and let

$$m_p := VaR_p\left(\mathbb{I}\sum_{i=1}^d X_i + (1-\mathbb{I})\sum_{i=1}^d L_i\right)$$
$$M_p := VaR_p\left(\mathbb{I}\sum_{i=1}^d X_i + (1-\mathbb{I})\sum_{i=1}^d H_i\right)$$

Bounds on the Value-at-Risk are $m_p \leqslant VaR_p\left(\sum_{i=1}^d X_i\right) \leqslant M_p$.

Value-at-Risk of a Mixture

Lemma

Consider a sum $S = \mathbb{I}X + (1-\mathbb{I})Y$, *where* $\mathbb{I}$ *is a Bernoulli distributed random variable with parameter* p_f *and where the components* X *and* Y *are independent of* $\mathbb{I}$. *Define* $\alpha_* \in [0,1]$ *by*

$$\alpha_* := \inf\left\{\alpha \in (0,1) \mid \exists \beta \in (0,1) \left\{ \begin{array}{l} p_f\alpha + (1-p_f)\beta = p \\ VaR_\alpha(X) \geqslant VaR_\beta(Y) \end{array} \right.\right\}$$

and let $\beta_* = \frac{p - p_f\alpha_*}{1-p_f} \in [0,1]$. *Then, for* $p \in (0,1)$,

$$VaR_p(S) = \max\left\{VaR_{\alpha_*}(X), VaR_{\beta_*}(Y)\right\}$$

Applying this lemma, one can prove a more convenient expression to compute the VaR bounds.

Let us define $T := F^{-1}_{\sum_i X_i|(X_1,X_2,...,X_d)\in\mathcal{F}}(U)$.

Theorem (Alternative formulation of the upper bound for VaR)

Assume $(X_1, X_2, ..., X_d)$ *satisfies properties (i), (ii) and (iii), and let* $(Z_1, Z_2, ..., Z_d)$ *and* $\mathbb{I}$ *as defined before.*

With $\alpha_1 = \max\left\{0, \frac{p+p_f-1}{p_f}\right\}$ *and* $\alpha_2 = \min\left\{1, \frac{p}{p_f}\right\}$,

$\alpha_* := \inf\left\{\alpha \in (\alpha_1, \alpha_2) \mid VaR_\alpha(T) \geqslant TVaR_{\frac{p-p_f\alpha}{1-p_f}}\left(\sum_{i=1}^d Z_i\right)\right\}$

When $\frac{p+p_f-1}{p_f} < \alpha_* < \frac{p}{p_f}$,

$$M_p = TVaR_{\frac{p-p_f\alpha_*}{1-p_f}}\left(\sum_{i=1}^d Z_i\right)$$

The lower bound m_p is obtained by replacing "TVaR" by "LTVaR".

With Pareto risks

Consider $d = 20$ risks distributed as Pareto with parameter $\theta = 3$.

- Assume we trust the independence conditional on being in $\mathcal{F}_1$

$$\mathcal{F}_1 = \bigcap_{k=1}^{d} \{q_\beta \leqslant X_k \leqslant q_{1-\beta}\}$$

where $q_\beta = (1-\beta)^{-1/\theta} - 1$.

$\mathcal{F}_1$	$\mathcal{U} = \emptyset$ $\beta = 0\%$	$\beta = 0.05\%$	$\beta = 0.5\%$	$\mathcal{U} = \mathbb{R}^d$ $\beta = 0.5$
α=95%	16.6	(16 , 18.4)	(13.8 , 37.4)	(7.29 , 61.4)
α=99.95%	43.5	(26.5 , 359)	(20.5 , 359)	(9.83 , 359)

Incorporating Expert's Judgements

Consider $d = 20$ risks distributed as Pareto $\theta = 3$.

- Assume comonotonicity conditional on being in $\mathcal{F}_2$

$$\mathcal{F}_2 = \bigcup_{k=1}^{d} \{X_k > q_p\}$$

Comonotonic estimates of Value-at-Risk
$VaR_{95\%}(S^c) = 34.29$, $VaR_{99.95\%}(S^c) = 231.98$

$\mathcal{F}_2$	$\mathcal{U} = \emptyset$ (Model)	$p = 99.5\%$	$p = 99.9\%$	$p = 99.95\%$
α=95%	16.6	(8.35 , 50)	(7.47 , 56.7)	(7.38 , 58.3)
α=99.95%	43.5	(232 , 232)	(232 , 232)	(180 , 232)

Comparison

Independence within a rectangle $\mathcal{F}_1 = \bigcap_{k=1}^{d} \{q_\beta \leqslant X_k \leqslant q_{1-\beta}\}$

$\mathcal{F}_1$	$\mathcal{U} = \emptyset$ $\beta = 0\%$	$\beta = 0.05\%$	$\beta = 0.5\%$	$\mathcal{U} = \mathbb{R}^d$ $\beta = 0.5$
α=95%	16.6	(16 , 18.4)	(13.8 , 37.4)	(7.29 , 61.4)
α=99.95%	43.5	(26.5 , 359)	(20.5 , 359)	(9.83 , 359)

Comonotonicity when one of the risks is large $\mathcal{F}_2 = \bigcup_{k=1}^{d} \{X_k > q_p\}$

$\mathcal{F}_2$	$\mathcal{U} = \emptyset$ (Model)	$p = 99.5\%$	$p = 99.9\%$	$p = 99.95\%$
α=95%	16.6	(8.35 , 50)	(7.47 , 56.7)	(7.38 , 58.3)
α=99.95%	43.5	(232 , 232)	(232 , 232)	(180 , 232)

Adding Moment information

In addition to information about the distribution, we can add information about moments of the sum.

Example 1: variance constraint - Bernard, Rüchendorf and Vanduffel (2015)

$$M := \sup \mathrm{VaR}_q \left[L_1 + L_2 + ... + L_n\right],$$
$$\text{subject to } \; L_j \sim F_j, \mathrm{var}(L_1 + L_2 + ... + L_n) \leqslant s^2$$

Description

It appears that adding dependence information can sharpen the bounds considerably. Here,

- VaR bounds with higher order moments on the portfolio sum
  - Portfolio loss

$$L = \sum_{i=1}^{n} L_i \text{ where } L_i \sim F_i$$

  - We are interested in the problem:

$$M := \sup \ \mathrm{VaR}_q[L]$$
$$\text{subject to } L_i \sim F_i \ \text{ and } E[L^k] \leqslant c_k \ (k = 2, 3, ..., K).$$

VaR bounds with moment constraints

- Without moment constraints, VaR bounds are attained if there exists a dependence among risks L_i such that

$$L = \begin{cases} A & \text{probability } q \\ B & \text{probability } 1-q \end{cases} \quad \text{a.s.}$$

  - If the "distance" between A and B is too wide then improved bounds are obtained with

$$L^* = \begin{cases} a & \text{with probability } q \\ b & \text{with probability } 1-q \end{cases}$$

such that

$$\begin{cases} a^k q + b^k(1-q) \leqslant c_k \\ aq + b(1-q) = E[L] \end{cases}$$

in which a and b are "as distant as possible while satisfying the constraint"

Dealing with moment constraints

To find a and b, solve for each $k = 2, 3, .., K$ the system of equations ($A \leqslant B$)

$$\begin{cases} Aq + B(1-q) = E(L) \\ A^k q + B^k (1-q) = c_k \end{cases}$$

and obtain $K - 1$ pairs $\{A_j, B_j\}$. Then, take

$$\begin{aligned} b &= \min\{B_j | j = 2, 3, ..., K\} \\ a &= \frac{E[L] - b(1-q)}{q}. \end{aligned}$$

Approximating Sharp Bounds

- The bounds a and b are sharp if one can construct dependence among the risks L_i such that quantile function of their sum L becomes flat on $[0, q]$ and on $[q, 1]$. This holds true under certain conditions (see eg Wang and Wang, 2014).
- To approximate sharp VaR bounds: Extended Rearrangement Algorithm (RA).

Standard RA (Puccetti and Rüschendorf, 2012):

- Put the margins in a matrix
- Rearrange each column (adapt the dependence) such that L (row-sums) approximates a constant ($E[L]$)

Illustration

Extended RA

q	...	...	...	-a
	...	...	...	-a
	...	...	...	-a
	...	...	...	-a
1-q	8	8	4	-b
	10	7	3	-b
	12	1	7	-b
	11	0	9	-b

Rearrange now within all columns such that all sums becomes close to zero

Extended RA

- ERA: Apply RA on the new matrix and check:
  – If all constraints are satisfied, then L^* readily generates the approximate solutions to the problem
  – If not, decrease b by ε, and compute a such as the expectation of L is satisfied. Apply the extended RA again.

Conclusions (1/2)

We have shown that

- Maximum Value-at-Risk is not caused by the comonotonic scenario.
- Maximum Value-at-Risk is achieved when the variance is *minimum* in the tail. The RA is then used in the tails only.
- Bounds on Value-at-Risk at high confidence level stay wide even if the multivariate dependence is known in 98% of the space!

Conclusions (2/2)

- Assess model risk with partial information and given marginals (Monte Carlo from fitted model or non-parametrically)
- Design algorithms for bounds on variance, TVaR and VaR and many more risk measures.
- Challenges:
  - How to choose the trusted area $\mathcal{F}$ optimally?
  - Re-discretizing using the fitted marginal $\hat{f}_i$ to increase N
  - Amplify the tails of the marginals by re-discretizing with a probability distortion
- Additional information on dependence can be incorporated
  - expert opinions on the dependence under some scenarios (amounts to fix the dependence in some areas).
  - variance of the sum (work with Rüschendorf and Vanduffel).
  - higher moments (work with Denuit and Vanduffel)

Acknowledgments

- **BNP Paribas Fortis Chair in Banking**.
- Research project on "*Risk Aggregation and Diversification*" with Steven Vanduffel for the **Canadian Institute of Actuaries**.
- **Humboldt Research Foundation**.
- Project on *"Systemic Risk"* funded by the **Global Risk Institute** in Financial Services.
- **Natural Sciences and Engineering Research Council of Canada**
- **Society of Actuaries Center of Actuarial Excellence Research Grant**

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