Controlled paths and applications

Abstract. The aim of this series of talks is to give a personal perspective on the use of the idea of "controlled path" in various situations involving stochastic analysis and non-linear systems.

1. Introduction

A series of three lectures at Berlin:

- Tue 13:15–?
- Wed 15:00–17:00. TU RTG Lounge 748
- Thu either 12:00–14:00 or <17. TBA.

Topics I would like to touch (maybe briefly):

1. Regularization by noise of ODEs
2. Regularization by noise in stochastic dispersive equations
3. Controlled paths and rough paths (of course)
4. Stochastic partial differential equations driven by convolutional rough paths
5. Controlled distributions (work P. Imkeller and N. Perkowski)
6. Burgers like equations (Hairer's approach)
7. Energy (weak) solutions of KPZ equation (joint work with M. Jara)
8. Hairer's solutions for the KPZ equation
9. Tzvekvov and Burq super-critical solutions of non-linear Schrödinger equations (maybe)

The first and nowadays most important example of controlled path are Itô's processes:

$$\mathrm{d}X_t = a_t \mathrm{d}W_t + b_t \mathrm{d}t$$

Objects indexed by one parameter (time) are very special (they have a past and a future and the present is very simple).

2. Regularization by noise

I want like to look at the ODE (cfr Davie paper)

$$x_t = x_0 + \int_0^t b(x_s)\mathrm{d}s + w_t$$

where w is an irregular perturbation, $x \in C([0,1], \mathbb{R}^d)$ and $b: \mathbb{R}^d \to \mathbb{R}^d$ some "irregular" vector field. Take b a distribution (think about $b(x) = v\delta(x)$ with $v \in \mathbb{R}^d$) and you want to give a meaning to this equation.

KPZ (for me is Burgers) usual trajectories are "white noise" in space

$$u_t = u_0 + \int_0^t \Delta u_s \mathrm{d}s + \int_0^t \partial(u_s^2)\mathrm{d}s + \partial \int_0^t \xi_s \mathrm{d}s$$

The general idea is that maybe $b(x_s)$ does not make sense for any $s \in [0, 1]$ but the avereraged version provided by the integral in time is well defined

$$\int_0^t b(x_s)\mathrm{d}s.$$

We regularize b by convolution $b_n \to b$ in some distributional space and look at

$$\int_0^t b_n(x_s) \mathrm{d}s.$$

I'm thinking about having $b \in B^{\alpha}_{\infty,\infty}(\mathbb{R}^d)$ with $\alpha < 0$ and w a fBM of Hurst index sufficiently small $H \ll 1$.

$$\mathbb{E}[|w_t - w_s|] \sim |t - s|^{2H}$$

and w is a Gaussian process. And $f \in B^{\alpha}_{\infty,\infty}(\mathbb{R}^d)$ means that

$$\|f\|_{B^{\alpha}_{\infty,\infty}} = \sup_{i \geqslant -1} 2^{-i\alpha} \|\Delta_i f\|_{L^{\infty}(\mathbb{R}^d)} < +\infty$$

and

$$\widehat{\Delta_i f}(\xi) = \varphi(2^{-i}|\xi|) \hat{f}(\xi)$$

where $\varphi: \mathbb{R} \to \mathbb{R}_+$ is a smooth function with support in $[1/2, 2 + 1/2]$ and such that $\varphi(\xi) = 1$ for $\xi \in [1, 2]$.

$$f = \sum_{i \geqslant -1} \Delta_i f$$

in distribution and in $B^{\alpha}_{\infty,\infty}$.

For example $\delta \in B^{-d-\varepsilon}_{\infty,\infty}$ for any $\varepsilon > 0$. When $\alpha \in (0, 1)$ this are the usual Hölder spaces (more or less).

A fBm of Hurst index H is Hölder continuous for any index $<H$. Brownian motion $H = 1/2$.

$$\int_0^t b_n(x_s) \mathrm{d}s \to_{n \to +\infty} ?$$

$b_n = \sum_{i \leqslant n} \Delta_i b = Q_n * b$. The first observation is the following: take $x_s = w_s + x_0$ and look at

$$\sigma_t^w b_n(x) = \int_0^t b_n(x + w_s) \mathrm{d}s = L_t^w * b_n$$

this object has a limit as $n \to \infty$ for almost every sample of w if w is fbm of Hurst index H and $b \in C^{\alpha} = B^{\alpha}_{\infty,\infty}$ with $\alpha > -1/(2H)$. (joint work with R. Catellier, arXiv).

$$\int_0^t b(x + w_s) \mathrm{d}s = \lim_{n \to \infty} \sigma_t^w b_n(x).$$

People study the local time

$$L_t = \int_0^t \delta(x - w_s) \mathrm{d}s$$

$$Y_t(\xi) = \hat{L_t} = \int_0^t e^{i\langle \xi, w_s \rangle} \mathrm{d}s$$

and you can show the following estimation (almost surely)

$$|Y_t(\xi) - Y_s(\xi)| \lesssim C_w \frac{|t - s|^{\gamma}}{1 + |\xi|^{(1/2H) - \varepsilon}} \qquad \xi \in \mathbb{R}^d$$

for some small $\varepsilon > 0$ and some $\gamma > 1/2$.

$$\|\sigma_t b - \sigma_s b\|_{C^{\alpha+\rho}} \leqslant \|b\|_{C^{\alpha}} |t - s|^{\gamma}$$

for any $\rho < 1/2H$. Gain $1/2H$ derivatives. In particular you can have $\alpha < 0$ and $\alpha + \rho > 0$ (a function). The averaging remain Holder continuous. So you have some hope that the solution of the ODE looks like

$$x_t = x_0 + w_t + \theta_t$$

with θ a $C^{1/2}([0, 1]; \mathbb{R}^d)$ perturbation. This will be my notion of controlled path.

Related stuff

a) Regularization in kinetic theory $f(v, x)$ and $\partial_t + v \cdot \partial_x$

b) The work of Tao and W??? on integrability gain of averagin over paths

$$\sigma_t^{\gamma} f(x) = \int_0^t f(x + \gamma_s) \mathrm{d}s$$

and they show that $f \in L^q(\mathbb{R}^d) \rightarrow \sigma_t^{\gamma} f \in L^{q'}(\mathbb{R}^d)$ with $q' > q$.

$$\int_0^t b(w_s + \theta_s) \mathrm{d}s \simeq \sum_i \int_{t_i}^{t_{i+1}} b(w_s + \theta_{t_i}) \mathrm{d}s = \sum_i \sigma_{[t_i, t_{i+1}]}^w b(\theta_{t_i})$$

so the l.h.s. can be defined as the limit of the r.h.s when the size of the partition $\{t_0 = 0, \ldots, t_n = t\}$ goes to zero.

$$\lim_{|\Pi| \searrow 0} \sum_i \sigma_{[t_i, t_{i+1}]}^w b(\theta_{t_i}) = \int_0^t \sigma_{\mathrm{d}s}^w b(\theta_s)$$

The limit exists due to the fact that $\theta \in C^{1/2}$ and that

$$|\sigma_{[s,t]}^w b(x) - \sigma_{[s,t]}^w b(y)| \lesssim |t - s|^{\gamma} |x - y|$$

since in that case you have

$$|\sigma_{[s,s']}^w b(\theta_{s''}) - \sigma_{[s,s']}^w b(\theta_{s'})| \lesssim |s - s'|^{\gamma} |\theta_{s''} - \theta_{s'}| \lesssim |s - s'|^{\gamma} |s'' - s'|^{1/2}$$

and this allow to use the approach of Young integrals to take the limit over the partition. This needs Lipshitz regularity of $\sigma_t^w b$ which means that it works for $\alpha > 1 - 1/2H$.

$$\sigma_t^{w+\theta} b(x) = \int_0^t b(x + w_s + \theta_s) \mathrm{d}s = \int_0^t \sigma_{\mathrm{d}s}^w b(x + \theta_s)$$

and $\sigma^{w+\theta} b \in C^{\alpha + 1/2H - 1}$ if $b \in C^{\alpha}$.

$$x_t^n = x_0 + \int_0^t b_n(x_s^n) \mathrm{d}s + w_t$$

rewrite it as Young equation $x^n = w + \theta^n$

$$\theta_t^n = x_0 + \int_0^t \sigma_{\mathrm{d}s}^w b_n(\theta_s^n)$$

with $\theta^n \in C^{1/2}(\mathbb{R}^d)$. And when $b_n \rightarrow b$ we have that $\theta^n \rightarrow \theta$ in $C^{1/2}$ with θ unique solution to

$$\theta_t = x_0 + \int_0^t \sigma_{\mathrm{d}s}^w b(\theta_s).$$

Now define

$$x_t = x_0 + w_t + \theta_t$$

and you can show that x (since it is controlled by w and θ) satisfy

$$\int_0^t b_n(x_s) \mathrm{d}s = \int_0^t \sigma_{\mathrm{d}s}^w b_n(\theta_s) \rightarrow \int_0^t \sigma_{\mathrm{d}s}^w b(\theta_s) =: \int_0^t b(x_s) \mathrm{d}s$$

and x solve the original equation where the nonlinearity is interpreted in this sense

$$x_t = x_0 + \int_0^t b(x_s) \mathrm{d}s + w_t.$$

3. Regularization by noise in stochastic dispersive equations

Nonlinear dispersive equations with random modulation (on $\mathbb{T}$)

$$\mathrm{d}_t \varphi_t = A \varphi_t \circ \mathrm{d}W_t + \mathcal{N}(\varphi_t) \mathrm{d}t$$

where W is a Brownian motion and $\circ \mathrm{d}$ is Stratonovich.

- NLSE: $A = i\Delta$, $\mathcal{N}(\varphi) = i|\varphi|^2 \varphi$, φ: $[0, 1] \times \mathbb{T} \rightarrow \mathbb{C}$

- KdV: $A = \partial^3$ and $\mathcal{N}(\varphi) = \partial(\varphi^2)$, $\varphi: [0,1] \times \mathbb{T} \rightarrow \mathbb{R}$

Mild formulation

$$\varphi_t = U_t^W \varphi_0 + U_t^W \int_0^t (U_s^W)^{-1} \mathcal{N}(\varphi_s) \mathrm{d}s$$

where $U_t^W = \exp(A\, W_t)$. Difficulty: we want to start with initial conditions in $H^\alpha(\mathbb{T})$ (Sobolev spaces)

$$\|\varphi\|_\alpha^2 = \sum_{k \neq 0} |k|^{2\alpha} |\hat{\varphi}(k)|^2 < +\infty.$$

where $\alpha \geqslant 0$ in the case of NLSE or $\alpha < 0$ maybe in the KdV case. Change of variables

$$\varphi_t = U_t^W \theta_t$$

with $\theta \in C^{1/2}(H^\alpha)$. This is the controlled structure. Equation for θ:

$$\theta_t = \theta_0 + \int_0^t (U_s^W)^{-1} \mathcal{N}(U_s^W \theta_s) \mathrm{d}s$$

$$X_t^W(\psi) = \int_0^t (U_s^W)^{-1} \mathcal{N}(U_s^W \psi) \mathrm{d}s$$

and you can prove that

$$|X_t^W(\psi) - X_s^W(\psi)|_{H^\alpha} \lesssim |\psi|_{H^\alpha}^3 |t-s|^\gamma$$

where $\gamma > 1/2$. And the equation take the form of a Young equation

$$\theta_t = \theta_0 + \int_0^t X_{\mathrm{d}s}^W(\theta_s)$$

(joint work with K. Chouk, almost finished).

[end of first lecture]