Optimal trade execution in a stochastic order book model

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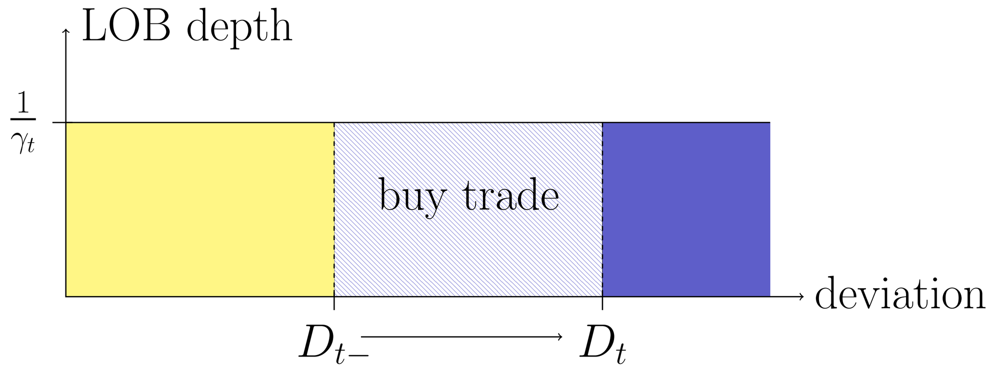
joint work with Thomas Kruse and Mikhail Urusov

Introduction

- task: close a large financial position of size $x \in \mathbb{R}$ up to a given time $T > 0$ with minimal execution costs
- trading according to a strategy X induces a price deviation D
- price = unaffected price (martingale) +price deviation
- model: symmetric block-shaped limit order book with order book depth $1/\gamma$ and resilience ρ both stochastic time-varying (generalizes [4])
- main findings (see [1]): necessity to adjust deviation dynamics and cost functional, minimal costs are characterized by a BSDE, existence result and formula for optimal strategies, optimal strategies with infinite variation
- deviation dynamics, cost functional and BSDE motivated by a related discrete-time problem ([2])



Stochastic control problem

$(\Omega, \mathcal{F}_T, (\mathcal{F}_s)_{s\in[0,T]}, P)$ with a continuous local martingale $M = (M_s)_{s\in[0,T]}$ s.t. for all $c \in (0,\infty)$, $E[\exp(c\,[M]_T)] < \infty$; $\mathcal{D}_M$ Doléans measure

ρ, μ, σ progressively measurable processes s.t. $2\rho + \mu - \sigma^2 > 0$ $\mathcal{D}_M$-a.e.

resilience process $\rho = (\rho_s)_{s\in[0,T]}$

price impact process $\gamma = (\gamma_s)_{s\in[0,T]}$:

$$d\gamma_s = \gamma_s\left(\mu_s\, d[M]_s + \sigma_s dM_s\right), \quad \gamma_0 > 0$$

$\rho_s < 0$ possible

given $t \in [0,T]$, $x, d \in \mathbb{R}$:

execution strategy $X = (X_s)_{s\in[t-,T]}$:

càdlàg semimartingale s.t. $X_{t-} = x$ and $X_T = 0$

typically, $X_t \neq X_{t-}$

associated deviation process $D = (D_s)_{s\in[t-,T]}$:

$$dD_s = -\rho_s D_s d[M]_s + \gamma_s dX_s + d[\gamma, X]_s, \quad D_{t-} = d$$

$\mathcal{A}_t(x,d)$ set of execution strategies $X = (X_s)_{s\in[t-,T]}$ with suitable integrability conditions

cost functional:

$$J_t(x,d,X) = E_t\left[\int_{[t,T]} D_{s-} dX_s + \int_{[t,T]} \frac{\gamma_s}{2} d[X]_s\right]$$

value function: $V_t(x,d) = \operatorname{ess\,inf}_{X\in\mathcal{A}_t(x,d)} J_t(x,d,X)$

optimal strategy: $X^* \in \mathcal{A}_t(x,d)$ s.t. $V_t(x,d) = J_t(x,d,X^*)$

new: $d[\gamma, X]_s$ in the deviation dynamics
relatively new: $d[X]_s$ in the cost functional (but already appeared in related settings, e.g., [3])

BSDE

$$Y_t = \frac{1}{2} + \int_t^T f(s, Y_s, Z_s) d[M]_s - \int_t^T Z_s dM_s - \left(M_T^\perp - M_t^\perp\right), \quad t \in [0,T],$$

where

$$f(s, Y_s, Z_s) = -\frac{\left((\rho_s + \mu_s)Y_s + \sigma_s Z_s\right)^2}{\sigma_s^2 Y_s + \frac{1}{2}(2\rho_s + \mu_s - \sigma_s^2)} + \mu_s Y_s + \sigma_s Z_s.$$

Assume that $\exists$ solution $(Y, Z, M^\perp)$ of BSDE s.t. Y is $[0,1/2]$-valued, $E\left[[M^\perp]_T\right] < \infty$ and $E\left[\int_0^T Z_s^2 d[M]_s\right] < \infty$, and fix any such solution.

Define

$$\widetilde{\beta}_s = \frac{(\rho_s + \mu_s)Y_s + \sigma_s Z_s}{\sigma_s^2 Y_s + \frac{1}{2}(2\rho_s + \mu_s - \sigma_s^2)}, \quad s \in [0,T].$$

Representation of the cost functional

For all $x, d \in \mathbb{R}$, $t \in [0,T]$ and $X \in \mathcal{A}_t(x,d)$ it holds

$$J_t(x,d,X) = \frac{Y_t}{\gamma_t}(d - \gamma_t x)^2 - \frac{d^2}{2\gamma_t} + E_t\left[\int_t^T \frac{\psi_s}{\gamma_s}\left(\widetilde{\beta}_s(\gamma_s X_s - D_s) + D_s\right)^2 d[M]_s\right]$$

where $\psi = \sigma^2 Y + \frac{1}{2}(2\rho + \mu - \sigma^2) > 0$ $\mathcal{D}_M$-a.e.

Optimal costs and optimal strategy

Further assumption: ρ, μ and $\widetilde{\beta}$ are $\mathcal{D}_M$-a.e. bounded.

It then holds for all $x, d \in \mathbb{R}$ and $t \in [0,T]$ that

$$V_t(x,d) = \frac{Y_t}{\gamma_t}(d - \gamma_t x)^2 - \frac{d^2}{2\gamma_t}.$$

If $x \neq d/\gamma_0$, then $\exists$ optimal strategy if and only if $\exists$ càdlàg semimartingale $\beta = (\beta_s)_{s\in[0,T]}$ s.t. $\widetilde{\beta} = \beta$ $\mathcal{D}_M$-a.e.

In this case, for $t \in [0,T]$ the (up to $\mathcal{D}_M|_{[t,T]}$-null sets unique) optimal strategy $(X_s^*)_{s\in[t-,T]}$ in $\mathcal{A}_t(x,d)$ is given by

$$X_{t-}^* = x, \quad X_T^* = 0,$$

$$X_s^* = \left(x - \frac{d}{\gamma_t}\right)(1 - \beta_s)\frac{\mathcal{E}(Q)_s}{\mathcal{E}(Q)_t}, \quad s \in [t,T),$$

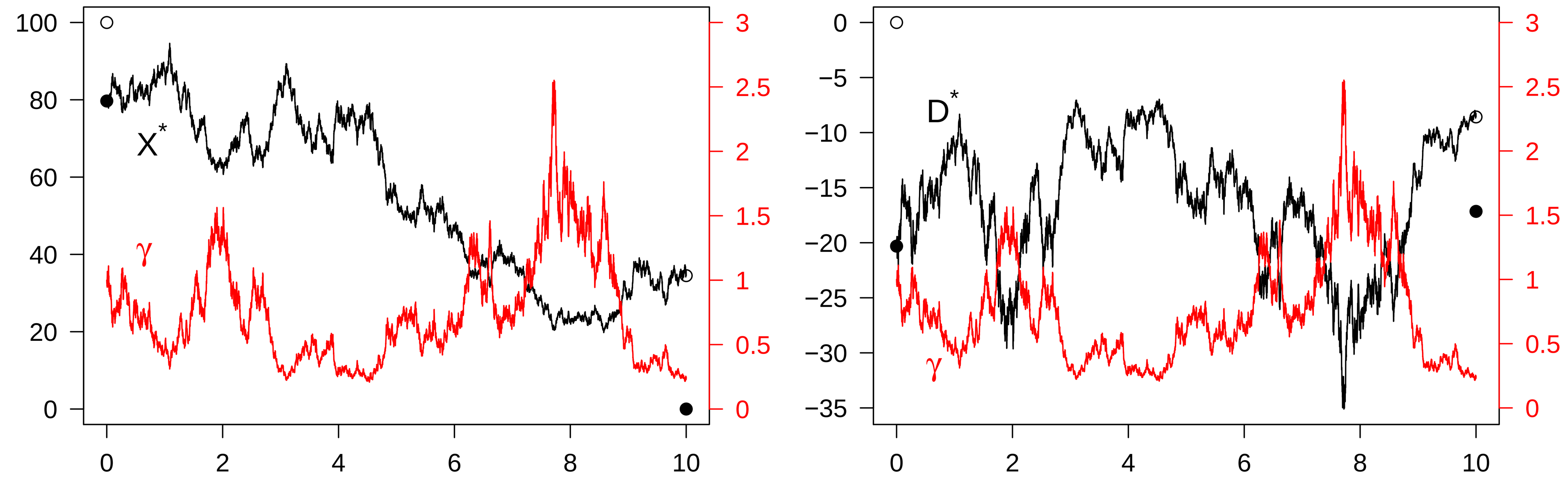
if $x = d/\gamma_t$, then immediate closure is optimal

where

$$Q_s = -\int_0^s \beta_r \sigma_r dM_r - \int_0^s \beta_r(\mu_r + \rho_r - \sigma_r^2) d[M]_r, \quad s \in [0,T].$$

Strategies with infinite variation

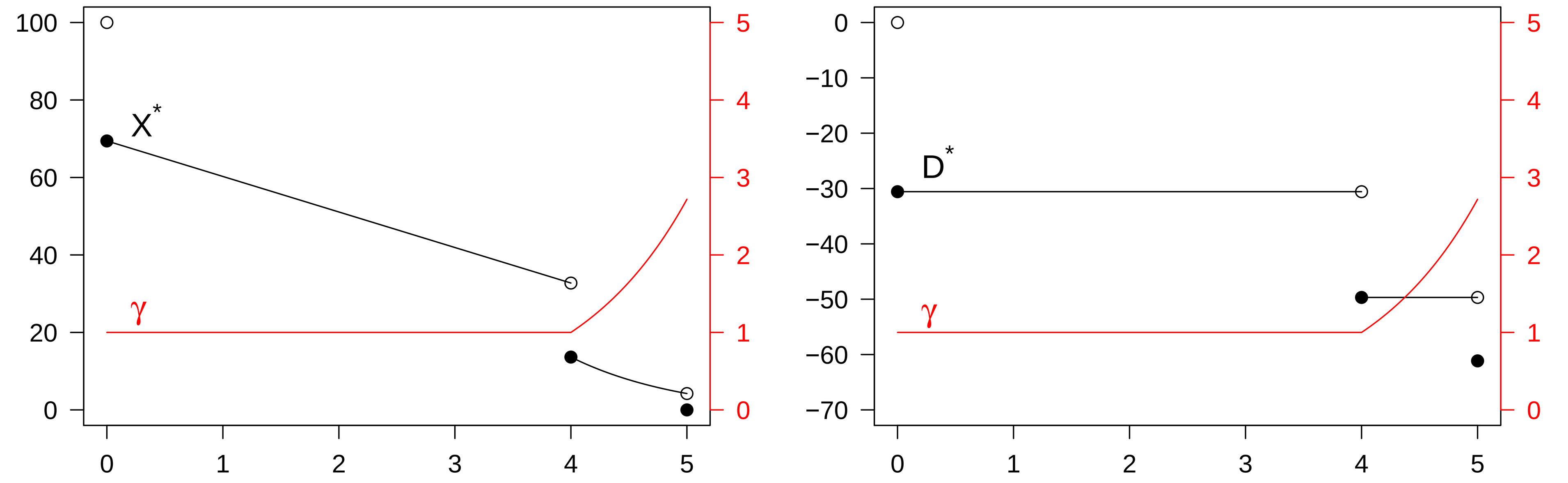
Example: $M = W$ $(\mathcal{F}_s)$-BM, $\mu \equiv 0$, σ, ρ constant with $2\rho - \sigma^2 > 0$.



But it is possible to obtain infinite variation in the optimal strategy also in situations where the price impact is smooth.

Block trades

Example: $M = W$ $(\mathcal{F}_s)$-BM, $\sigma \equiv 0$, $\rho > 0$ constant, $\mu_s = 1_{[t_0,T]}(s)$ for some $t_0 \in (0,T)$.



References

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