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On iteration methods without derivatives for the simultaneous determination of polynomial zeros

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Abstract

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Several algorithms for simultaneously approximating simple complex zeros of a polynomial are presented. These algorithms use Weierstrass' corrections and do not require any polynomial derivatives. It is shown that Nourein's method is, actually, regula falsi for Weierstrass' corrections. Convergence analysis and computational efficiency are given for the considered methods in complex and circular arithmetic. Special attention is paid to hybrid methods that combine the efficiency of floating-point arithmetic and the inclusion property of interval arithmetic.

Keywords: Polynomial zeros; simultaneous methods; interval arithmetic; computational efficiency.

1. Introduction

One of the most popular methods for simultaneous approximation of the zeros of a polynomial (cf. [14]) was indicated by Weierstrass [27, p.258] in 1891 and much later proposed independently by Durand [8], Dochev [7], Kerner [10] and others. Given n pairwise distinct approximants $(z_1, \ldots, z_n) \in \mathbb{C}^n$ for the n pairwise distinct zeros $(\zeta_1, \ldots, \zeta_n) \in \mathbb{C}^n$ of a monic polynomial f of degree $n \geqslant 3$, one iteration step of Durand–Kerner's methods reads

$$(z_1, \ldots, z_n) \mapsto (z_1, \ldots, z_n) - (W_1, \ldots, W_n), \tag{1}$$

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where for all $i \in \{1,\dots,n\}$,

$$W_i := W_i(z_1,\dots,z_n) := \frac{f(z_i)}{\prod_{k=1,k\neq i}^{n}(z_i - z_k)}. \tag{2}$$

We always assume that f has simple zeros. Then, it is well known that Durand–Kerner's method has local quadratic convergence.

Another iteration formula, which also does not use any polynomial derivatives, is of the form

$$(z_1,\dots,z_n) \mapsto (z_1,\dots,z_n) - (B_1, B_2,\dots,B_n), \tag{3}$$

where for all $i \in \{1,\dots,n\}$,

$$B_i := B_i(z_1,\dots,z_n) := \frac{W_i}{1 + \sum_{k=1,k\neq i}^{n}(W_k/(z_i - z_k))}.$$

This iteration formula was derived by Börsch-Supan [5], and later, by Nourein [16]. The iteration method (3) has local cubic convergence.

An improvement of the iteration method (3) was proposed by Nourein [17]. Again assuming simple approximations and zeros, one iteration step reads

$$(z_1,\dots,z_n) \mapsto (z_1,\dots,z_n) - (N_1,\dots,N_n),$$

where for all $i \in \{1,\dots,n\}$,

$$N_i := N_i(z_1,\dots,z_n) := \frac{W_i}{1 + \sum_{k=1,k\neq i}^{n}(W_k/(z_i - W_i - z_k))}. \tag{4}$$

In the sequel this method will be refered to as Nourein's method. Starting from reasonably good initial approximants $z_1^{(0)},\dots,z_n^{(0)}$, Nourein's method produces the sequences of approximants $\{z_i^{(m)}\}$:

$$z_i^{(m+1)} = z_i^{(m)} - \frac{W_i^{(m)}}{1 + \sum_{k=1,k\neq i}^{n}\left(W_k^{(m)}/\left(z_i^{(m)} - W_i^{(m)} - z_k^{(m)}\right)\right)}, \tag{5}$$

$$i = 1,\dots,n, \quad m = 0, 1,\dots,$$

which converge to the exact zeros, with the order of convergence equals four. $W_i^{(m)}$ in (5) is given by

$$W_i^{(m)} = \frac{f\left(z_i^{(m)}\right)}{\prod_{k=1,k\neq i}^{n}\left(z_i^{(m)} - z_k^{(m)}\right)}.$$

The aim of this paper is to present some algorithms without derivatives for the simultaneous refinement of sets of approximate zeros of a complex polynomial. We start with Theorem 2.1 proving that Nourein's method is, actually, regula falsi for Weierstrass' corrections (Section 2). Using this fact, an interesting and simple proof concerning the convergence rate of Nourein's method is given in Section 3 (Theorem 3.4). In addition, suitable initial conditions providing the safe convergence of Nourein's method are stated (Theorem 3.5).

In Section 4 we study the so-called inclusion methods that not only provide error bounds automatically, but also take into account rounding errors. First, a generalized interval method with the order of convergence $2q+1$, $q \geqslant 1$, is established, applying a repetition procedure.

Combining iterations in complex and circular arithmetic, a new hybrid algorithm of Nourein's type is constructed. Its main advantages are the great computational efficiency and the ability of inclusion of zeros. The presented hybrid algorithm suggests the construction of a new interval method of fourth order. The corresponding single-step method and its R-order of convergence are the subject of Theorem 4.3. Finally, in Section 5, we consider the computational efficiency of the proposed inclusion methods.

2. Derivation of Nourein's method

In this section we will present an interesting result which shows that Nourein's method is actually the regula falsi for Weierstrass' corrections.

Fix $i \in \{1,\dots,n\}$ and $z_1,\dots,z_{i-1}, z_{i+1},\dots,z_n \in \mathbb{C}$. Note that ζ_i is also a zero of

$$h_i : \begin{cases} \mathbb{C} \setminus \{z_1,\dots,z_{i-1}, z_{i+1},\dots,z_n\} \to \mathbb{C}, \\ z \mapsto W_i(z_1,\dots,z_{i-1}, z, z_{i+1},\dots,z_n). \end{cases}$$

Using two approximants z' and z'', the new approximant $\hat{z}$ obtained by the regula falsi is

$$\hat{z} := z' - \frac{z'' - z'}{h_i(z'') - h_i(z')} h_i(z').$$

Clearly, we assume that $h(z'') \neq h(z')$, which is implied by convergence.

Theorem 2.1. *If $z' = z_i$ and $z'' = z_i - W_i$, then $\hat{z} = z_i - N_i$.*

Proof. By (2) and Lagrange's interpolation formula applied to the polynomial

$$f(z) - \prod_{k=1}^{n} (z - z_k)$$

of degree $n-1$, we have for all $z \in \mathbb{C}$,

$$f(z) = \prod_{k=1}^{n} (z - z_k) + \sum_{j=1}^{n} W_j \prod_{k=1, k\neq j}^{n} (z - z_k). \tag{6}$$

Consequently,

$$\begin{aligned} h_i(z) = W_i(z_1,\dots,z_{i-1}, z, z_{i+1},\dots,z_n) &= \frac{f(z)}{\prod_{k=1,k\neq i}^{n}(z - z_k)} \\ &= W_i + (z - z_i)\left(1 + \frac{\sum_{k=1,k\neq i}^{n} W_k}{z - z_k}\right), \end{aligned} \tag{7}$$

where — as in the following — W_j denotes $W_j(z_1,\dots,z_n)$, while otherwise the arguments are specified explicitly. Using (7) we have

$$\frac{h_i(z')}{h_i(z'') - h_i(z')} = -\left(1 + \sum_{k=1,k\neq i}^{n} \frac{W_k}{z_i - W_i - z_k}\right)^{-1},$$

and by (4),

$$\hat{z} = z' - (z'' - z')\frac{h_i(z')}{h_i(z'') - h_i(z')} = z_i - N_i. \qquad \square$$

3. Convergence

The convergence rate of Nourein's method (5) can be determined in various ways, for instance as a corollary of [24, Theorem 1] where root-finding methods with recursive corrections were considered. In the beginning of this section we use the well-known results concerning regula falsi and Durand–Kerner's method as well as the relation (7) to prove the following assertion.

Theorem 3.1. *For Nourein's method, there exist a neighbourhood $V \subseteq \mathbb{C}^n$ of $(\zeta_1, \ldots, \zeta_n)$, $\zeta_1, \ldots, \zeta_n$ being the simple zeros of f, and a constant $K > 0$ such that for any $(z_1, \ldots, z_n) \in V$ and $i \in \{1, \ldots, n\}$ there holds*

$$|z_i - N_i - \zeta_i| \leqslant K |z_i - \zeta_i|^2 \max_{k=1,\ldots,n, k \neq i} |z_k - \zeta_k|^2. \tag{8}$$

Proof. Let $\hat{z}_i = z_i - N_i$, as in Theorem 2.1. It is well known from the convergence analysis of regula falsi that, taking the above notations, there holds for $i \in \{1, \ldots, n\}$,

$$|\hat{z}_i - \zeta_i| \leqslant \frac{\max\{|h_i''(z)|: z \in \mathrm{co}(z', z'', \zeta_i)\}}{\min\{|h_i'(z)|: z \in \mathrm{co}(z', z'', \zeta_i)\}} |z' - \zeta_i| \, |z'' - \zeta_i|,$$

where $\mathrm{co}(z', z'', \zeta_i)$ denotes the convex hull of the three complex numbers z', z'' and ζ_i. Using (7) we can easily compute the derivatives of h_i and obtain for any $z \in \mathbb{C} \setminus \{z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_n\}$,

$$h_i'(z) = 1 + \sum_{k=1,k\neq i}^{n} W_k \frac{z_i - z_k}{(z - z_k)^2}, \qquad h_i''(z) = -2 \sum_{k=1,k\neq i}^{n} W_k \frac{z_i - z_k}{(z - z_k)^3}.$$

Let $V_1, \ldots, V_n$ denote nonoverlapping compact disks in $\mathbb{C}$ with the centers $\zeta_1, \ldots, \zeta_n$ such that $V := V_1 \times \cdots \times V_n$ is a "sufficiently small" neighbourhood of $(\zeta_1, \ldots, \zeta_n)$ in $\mathbb{C}^n$. Then, define

$$\mathrm{dist}(V_i, V_k) := \min\{|z - \zeta|: z \in V_i, \zeta \in V_k\},$$

$$\mathrm{diam}(V_i, V_k) := \max\{|z - \zeta|: z \in V_i, \zeta \in V_k\},$$

$$k_1 := \max_{i=1,\ldots,n} \sum_{k=1,k\neq i}^{n} \frac{1}{\mathrm{dist}(V_i, V_k)},$$

$$k_2 := \max_{i=1,\ldots,n} \sum_{k=1,k\neq i}^{n} \frac{\mathrm{diam}(V_i, V_k)}{\mathrm{dist}(V_i, V_k)^2},$$

$$k_3 := \max_{i=1,\ldots,n} \sum_{k=1,k\neq i}^{n} \frac{\mathrm{diam}(V_i, V_k)}{\mathrm{dist}(V_i, V_k)^3}.$$

Since $h_i(\zeta_i)=0$ for any $i\in\{1,\dots,n\}$, we may assume that

$$\epsilon=\max_{k=1,\dots,n}\max_{(z_1,\dots,z_n)\in V}|W_k(z_1,\dots,z_n)|<\min\left\{\frac{1}{2k_1},\frac{1}{2k_2}\right\}.$$

Then, for $(z_1,\dots,z_n)\in V$ and $z\in V_i$,

$$|h_i'(z)|\geqslant 1-\epsilon k_2>\tfrac{1}{2},\qquad |h_i''(z)|\leqslant 2k_3\max_{k=1,\dots,n,k\neq i}|W_k|.$$

On the other hand, (7) implies

$$z_i-\zeta_i=\frac{W_i}{1+\sum_{k=1,k\neq i}^{n}W_k/(\zeta_i-z_k)},\tag{9}$$

$$z_i-W_i-\zeta_i=-W_i\frac{\sum_{k=1,k\neq i}^{n}W_k/(\zeta_i-z_k)}{1+\sum_{k=1,k\neq i}^{n}W_k/(\zeta_i-z_k)}=-(z_i-\zeta_i)\sum_{k=1,k\neq i}^{n}\frac{W_k}{\zeta_i-z_k},$$

so that

$$|W_i|<(1+\epsilon k_1)|z_i-\zeta_i|<\tfrac{3}{2}|z_i-\zeta_i|,\tag{10}$$

$$|z_i-W_i-\zeta_i|<k_1|z_i-\zeta_i|\max_{k=1,\dots,n,k\neq i}|W_k|<\tfrac{1}{2}|z_i-\zeta_i|.$$

Therefore, $z'=z_i\in V_i$ implies $z''=z_i-W_i\in V_i$ such that $\operatorname{co}(z',z'',\zeta_i)\subseteq V_i$.

Altogether,

$$\begin{aligned}|\hat{z}_i-\zeta_i|&<4k_3|z_i-\zeta_i|\,|z_i-W_i-\zeta_i|\max_{k=1,\dots,n,k\neq i}|W_k|\\&<4k_1k_3|z_i-\zeta_i|^2\max_{k=1,\dots,n,k\neq i}|W_k|^2\\&<9k_1k_3|z_i-\zeta_i|^2\max_{k=1,\dots,n,k\neq i}|z_k-\zeta_k|^2.\qquad\square\end{aligned}$$

Remark 3.2. Theorem 3.1 implies

$$|z_i-N_i-\zeta_i|\leqslant K\max_{k=1,\dots,n}|z_k-\zeta_k|^4,\quad i=1,\dots,n,$$

whence we obtain that the order of convergence of Nourein's method is *four* in the max-norm.

Remark 3.3. In [1] it is mentioned that (9) is Newton's method for Weierstrass' corrections. Indeed, using the above expression for h_i' there holds

$$B_i=\frac{h_i(z_i)}{h'(z_i)}.$$

In view of the last remark, from the convergence analysis of Newton's method, we obtain

$$|z_i-B_i-\zeta_i|\leqslant\frac{\max\{|h_i''(z)|:\ z\in\operatorname{co}(z_i,\zeta_i)\}}{\min\{|h_i'(z)|:\ z\in\operatorname{co}(z_i,\zeta_i)\}}|z_i-\zeta_i|^2.$$

Consequently, following the proof of Theorem 3.1, there holds the following assertion on method (3).

Theorem 3.4. *For method* (3), *there exist a neighbourhood* $V \subseteq \mathbb{C}^n$ *of* $(\zeta_1, \ldots, \zeta_n)$, $\zeta_1, \ldots, \zeta_n$ *being the simple zeros of* f, *and a constant* $K > 0$ *such that for any* $(z_1, \ldots, z_n) \in V$ *and* $i \in \{1, \ldots, n\}$ *there holds*

$$|z_i - B_i - \zeta_i| \leqslant K |z_i - \zeta_i|^2 \max_{k=1,\ldots,n, k \neq i} |z_k - \zeta_k|.$$

Theorem 3.1 was proved under certain conditions, including a "sufficiently small" neighbourhood of $(\zeta_1, \ldots, \zeta_n)$. Following the technique from [22] we are able to establish the convergence rate of Nourein's method under more precise initial conditions.

Let $m = 0, 1, \ldots$ be the iteration index and let

$$d = \min_{\substack{i,j \\ i \neq j}} |\zeta_i - \zeta_j|, \quad u_i^{(m)} = z_i^{(m)} - \zeta_i, \qquad i = 1, \ldots, n,$$

$$q = \frac{2(n+1)}{d}, \quad \alpha(n) = \frac{8}{n}\left(\frac{n-1}{2n+1}\right)^2, \quad \gamma(n) = q\alpha(n)^{1/3}, \qquad n \geqslant 3.$$

From (9) we find

$$W_i^{(m)} = \left(z_i^{(m)} - \zeta_i\right)\left(1 + \sum_{k=1, k \neq i}^{n} \frac{W_k^{(m)}}{\zeta_i - z_k^{(m)}}\right). \tag{11}$$

Theorem 3.5. *Assume that the initial conditions*

$$\left|u_i^{(0)}\right| < \frac{d}{2n+2} = \frac{1}{q} \tag{12}$$

are satisfied for all $i = 1, \ldots, n$. *Then Nourein's method* (5) *is convergent with the order of convergence equal to four.*

Proof. For $m = 0$ we estimate

$$\left|z_k^{(0)} - \zeta_i\right| \geqslant |\zeta_i - \zeta_k| - \left|z_k^{(0)} - \zeta_k\right| > d - \frac{d}{2n+2} = \frac{2n+1}{q},$$

$$\left|z_k^{(0)} - z_i^{(0)}\right| \geqslant \left|z_k^{(0)} - \zeta_i\right| - \left|z_i^{(0)} - \zeta_i\right| > \frac{2n+1}{q} - \frac{1}{q} = \frac{2n}{q} > \frac{1}{q},$$

$$\left|W_k^{(0)}\right| = \left|z_k^{(0)} - \zeta_k\right| \prod_{\substack{j=1 \\ j \neq k}}^{n} \left|\frac{z_k^{(0)} - \zeta_j}{z_k^{(0)} - z_j^{(0)}}\right| \leqslant \left|z_k^{(0)} - \zeta_k\right| \prod_{\substack{j=1 \\ j \neq k}}^{n} \left(1 + \frac{\left|z_j^{(0)} - \zeta_j\right|}{\left|z_k^{(0)} - z_j^{(0)}\right|}\right)$$

$$< \frac{1}{q}\left(1 + \frac{1}{2n}\right)^{n-1} < \frac{\mathrm{e}^{1/2}}{q},$$

$$\left|z_i^{(0)} - W_i^{(0)} - z_k^{(0)}\right| \geqslant \left|z_i^{(0)} - z_k^{(0)}\right| - \left|W_i^{(0)}\right| \geqslant \frac{2n}{q} - \frac{\mathrm{e}^{1/2}}{q} = \frac{2n - \mathrm{e}^{1/2}}{q},$$

$$\left|1+\sum_{k=1,k\neq i}^{n}\frac{W_k^{(0)}}{z_i^{(0)}-W_i^{(0)}-z_k^{(0)}}\right|\geqslant 1-\sum_{k=1,k\neq i}^{n}\frac{|W_k^{(0)}|}{|z_i^{(0)}-W_i^{(0)}-z_k^{(0)}|}$$

$$>1-\frac{(n-1)\,\mathrm{e}^{1/2}/q}{(2n-\mathrm{e}^{1/2})/q}=\frac{n(2-\mathrm{e}^{1/2})}{2n-\mathrm{e}^{1/2}}.$$

From (5) and (11) we obtain

$$z_i^{(m+1)}-\zeta_i=z_i^{(m)}-\zeta_i-\frac{\left(z_i^{(m)}-\zeta_i\right)\left(1+\sum_{k=1,k\neq i}^{n}\left(W_k^{(m)}/\left(\zeta_i-z_k^{(m)}\right)\right)\right)}{1+\sum_{k=1,k\neq i}^{n}\left(W_k^{(m)}/\left(z_i^{(m)}-W_i^{(m)}-z_k^{(m)}\right)\right)}$$

$$=\left(z_i^{(m)}-\zeta_i\right)^2\sum_{k=1,k\neq i}^{n}\frac{W_k^{(m)}}{z_k^{(m)}-\zeta_i}$$

$$\times\frac{\sum_{k=1,k\neq i}^{n}\left(W_k^{(m)}/\left(\left(z_k^{(m)}-\zeta_i\right)\left(z_i^{(m)}-W_i^{(m)}-z_k^{(m)}\right)\right)\right)}{1+\sum_{k=1,k\neq i}^{n}\left(W_k^{(m)}/\left(z_i^{(m)}-W_i^{(m)}-z_k^{(m)}\right)\right)}.$$

Then for the index $m=0$ and for all $i=1,\dots,n$ we get

$$|u_i^{(1)}|\leqslant|u_i^{(0)}|^2\sum_{k=1,k\neq i}^{n}\frac{|W_k^{(0)}|}{|z_k^{(0)}-\zeta_i|}$$

$$\times\frac{\sum_{k=1,k\neq i}^{n}\left(|W_k^{(0)}|/\left(|z_k^{(0)}-\zeta_i|\,|z_i^{(0)}-W_i^{(0)}-z_k^{(0)}|\right)\right)}{1-\sum_{k=1,k\neq i}^{n}\left|W_k^{(0)}/\left(z_i^{(0)}-W_i^{(0)}-z_k^{(0)}\right)\right|}.$$

Using the previous estimates, from the last inequality we obtain

$$|u_i^{(1)}|<|u_i^{(0)}|^2\left(\sum_{k=1,k\neq i}^{n}|u_k^{(0)}|\right)^2\frac{\mathrm{e}}{2-\mathrm{e}^{1/2}}\frac{q^3}{n(2n+1)^2},$$

that is,

$$|u_i^{(1)}|<\frac{\gamma(n)^3}{(n-1)^2}|u_i^{(0)}|^2\left(\sum_{k=1,k\neq i}^{n}|u_k^{(0)}|\right)^2,\quad i=1,\dots,n. \tag{13}$$

By virtue of the initial conditions (12) and the inequality $\alpha(n)<1$, from (13) it follows

$$|u_i^{(1)}|<\frac{q^3\alpha(n)}{(n-1)^2}\frac{1}{q^2}\left[(n-1)\frac{1}{q}\right]^2<\frac{1}{q},\quad i=1,\dots,n.$$

Applying mathematical induction, in a similar way as for $m=0$ we prove

$$|u_i^{(m+1)}|<\frac{\gamma(n)^2}{(n-1)^2}|u_i^{(m)}|^2\left(\sum_{k=1,k\neq i}^{n}|u_k^{(m)}|\right)^2<\frac{1}{q},\quad i=1,\dots,n, \tag{14}$$

for each iteration index $m=0,1,\dots$ if (12) holds.

Substituting $t_i^{(m)} = \gamma(n) | u_i^{(m)} |$, the inequalities (14) become

$$t_i^{(m+1)} < \frac{1}{(n-1)^2} t_i^{(m)^2} \left(\sum_{k=1,k\neq i}^{n} t_k^{(m)} \right)^2, \quad i=1,\dots,n. \tag{15}$$

With regard to (12) and the inequality $\alpha(n) < 1$ we find

$$t_i^{(0)} = \gamma(n) | u_i^{(0)} | = q\alpha(n)^{1/3} | u_i^{(0)} | < \alpha(n)^{1/3} < 1,$$

for each $i=1,\dots,n$. Let $t = \max_{1 \leqslant i \leqslant n} t_i^{(0)}$. Then

$$t_i^{(0)} \leqslant t < 1$$

holds for all $i=1,\dots,n$, wherefrom, taking into consideration the inequalities (15), we conclude that the sequences $\{t_i^{(m)}\}$ (and, consequently, $\{| u_i^{(m)} |\}$), $i=1,\dots,n$, tend to 0. Therefore, the iteration process (5) is convergent under the conditions (12). Further, putting

$$u^{(m)} = \max_{1 \leqslant i \leqslant n} | u_i^{(m)} |,$$

from (14) we obtain

$$u^{(m+1)} < \gamma(n)^3 u^{(m)^4},$$

which completes the proof of the theorem. □

Remark 3.6. Trigonometric and exponential polynomials have important applications in numerical analysis, in the theory of approximations as well as in many physical problems. The methods for finding the zeros of this kind of general polynomials have been considered in [4,12,13,26]. Nourein's method (5) can be also applied for solving trigonometric and exponential equations using simple transformations (see [26]), which is illustrated in the following example.

Example 3.7. An exponential polynomial

$$E_n(z) = a_0 + \sum_{k=1}^{n} \left(a_k e^{-kz} + b_k e^{kz} \right)$$

reduces to the algebraic polynomial

$$\hat{E}_n(w) = w^{2n} + c_{2n-1} w^{2n-1} + \cdots + c_1 w + c_0$$

by the substitution $e^z = w$. The coefficients c_j are determined by [26]

$$c_j = \frac{a_{n-1}}{b_n}, \quad j = 0, 1, \dots, n, \qquad c_{n+j} = \frac{b_j}{b_n}, \quad j=1,\dots,n.$$

The zeros ζ_k of $\hat{E}_n$ can now be found by Nourein's method; then the zeros of E_n are calculated as the principal values of $\xi_k = \log \zeta_k$, $k=1,\dots,2n$. In particular, we have considered the exponential polynomial

$$E_2(x) = a_0 + a_1 e^{-x} + b_1 e^{x} + a_2 e^{-2x} + b_2 e^{2x}$$

Table 1

	$m=0$	$m=1$	$m=2$
$w_1^{(m)}$	1	$0.367\underline{5}9$	$0.367879441171\underline{3}92$
$w_2^{(m)}$	10	$7.\underline{4}0101$	$7.389056098\underline{9}29027$
$w_3^{(m)}$	20	20.08554	20.085536923187668
$w_4^{(m)}$	40	$54.\underline{6}3882$	54.598150033146404

from [26] (see, also [13]), where

$$a_0 = e^3 + e^{-3} + pq, \qquad a_1 = -(e^{7/2}p + e^{1/2}q), \qquad a_2 = e^4,$$
$$b_1 = -(e^{-7/2}p + e^{-1/2}q), \qquad b_2 = e^{-4},$$
$$p = 2\cosh\tfrac{3}{2}, \qquad q = 2\cosh\tfrac{1}{2}.$$

The exact zeros of the polynomial E_2 are $\xi_1 = -1$, $\xi_2 = 2$, $\xi_3 = 3$ and $\xi_4 = 4$. The transformed polynomial (after normalization) is

$$\hat{E}_2(w) = w^4 - 82.44062249643399\, w^3 + 1678.667985874348\, w^2 - 8709.52403020384\, w + 2980.957987041728.$$

Using the initial approximations $w_1^{(0)} = 1$, $w_2^{(0)} = 10$, $w_3^{(0)} = 20$ and $w_4^{(0)} = 40$, Durand–Kerner's method produced the approximants which are exact to ten decimal places after five iterations [26]. On the other hand, the same accuracy was achieved by only two iterations of Nourein's method (see Table 1 where the underlined digit indicates the first incorrect digit).

We calculated the approximants to the zeros of the original function $E_2(x)$ as $x_x^{(2)} = \log w_k^{(2)}$, $k = 1, 2, 3, 4$, and obtained

$$x_1^{(2)} = -1.000000000000135, \qquad x_2^{(2)} = 1.999999999999780,$$
$$x_3^{(2)} = 3.000000000000001, \qquad x_4^{(2)} = 4.000000000000040.$$

4. Inclusion methods

During the last two decades many interval methods for the simultaneous inclusion of polynomial zeros have been established. These methods produce approximations (in the form of disks or rectangles) that not only contain the exact zeros providing error bounds automatically, but also take into account rounding errors without altering the fundamental structure of the interval formula. More about inclusion methods can be found in [20] and the references cited there. For the realization of interval methods the so-called (*rectangular* or *circular*) complex interval arithmetic can be usefully applied. We assume that this arithmetic is a well-established subject and we refer to [3] for more details.

In this section we will use circular interval arithmetic. A disk Z with the radius $r = \text{rad}(Z)$ and the center $c = \text{mid}(Z)$ will be denoted by the parametric notation $Z = \{c; r\}$. One of the most important properties of interval arithmetic is the *inclusion isotonicity*: if $z = g(z)$ and $z \in Z$, then $z \in G(Z)$, where $G(Z)$ is an interval extension of a function g. This property is the base for the construction of inclusion methods.

From (9) we obtain

$$\zeta_i = z_i - \frac{W_i}{1 + \sum_{k=1,k\neq i}^{n}(W_k/(z_i - z_k))}, \tag{16}$$

which is a fixed-point relation. Suppose that we have found disjoint disks $Z_i = \{z_i; r_i\}$ that contain simple real or complex zeros ζ_i of a monic polynomial f of degree n. Since $\zeta_i \in Z_i$, $i = 1, \ldots, n$, on the basis of the inclusion isotonicity from (16) there follows

$$\zeta_i \in z_i - \frac{W_i}{1 + \sum_{k=1,k\neq i}^{n}(W_k/(Z_i - z_k))} = \hat{Z}_i, \quad i = 1, \ldots, n,$$

where $\hat{Z}_i$ is the new circular approximant of the zero ζ_i. According to the last relation, the interval method of third order

$$Z_i^{(m+1)} = z_i^{(m)} - \frac{W_i^{(m)}}{1 + \sum_{k=1,k\neq i}^{n}\left(W_k^{(m)}/\left(Z_i^{(m)} - z_k^{(m)}\right)\right)}, \tag{17}$$

$$z_i^{(m)} = \operatorname{mid}\left(Z_i^{(m)}\right), \quad i = 1, \ldots, n, \quad m = 0, 1, \ldots,$$

has been established in [19]. If $\zeta_i \in Z_i^{(0)}$, $i = 1, \ldots, n$, then $\zeta_i \in Z_i^{(m)}$ for each $m = 1, 2, \ldots,$ if some suitable initial conditions are valid (see [19]).

Similarly as in [15,25] the interval method (17) can be generalized by applying a repetition procedure consisting of the use of the same values of $W_i^{(m)}$, $i = 1, \ldots, n$, several times. The generalized method is as follows:

$$Z_i^{(m+(\lambda+1)/q)} = z_i^{(m)} - \frac{W_i^{(m)}}{1 + \sum_{k=1,k\neq i}^{n}\left(W_k^{(m)}/\left(Z_i^{(m+\lambda/q)} - z_k^{(m)}\right)\right)}, \tag{18}$$

$$i = 1, \ldots, n, \quad \lambda = 0, 1, \ldots, q-1, \quad q \in \mathbb{N}, \quad m = 0, 1, \ldots .$$

Using Theorem 3.4, we easily obtain the following assertion which can also be proved as in [25].

Theorem 4.1. *Let* $\{Z_i^{(m)}\}$, $i = 1, \ldots, n$, *be the sequences of disks obtained by the interval method* (18). *If* $\zeta_i \in Z_i^{(0)}$ *and* $\operatorname{rad}(Z_i^{(0)})$ *is small enough for all* $i = 1, \ldots, n$, *then*

(1) $\zeta_i \in Z_i^{(m)}$ *for all* $m = 1, 2, \ldots$;

(2) *the order of convergence of* (18) *is at least* $2q + 1$.

The main objection of interval methods is their great computational amount of work. Following the idea of [6], a few effective methods for the simultaneous inclusion of polynomial zeros have been proposed in [21]. These methods combine the efficiency of ordinary floating-point iterations with the accuracy control which can be provided by interval arithmetic iterations. Using the procedure for the construction of *combined* algorithms described in [21], we can combine Nourein's method (5) and the interval method (17) to obtain a combined method which (i) has an improved computational efficiency and (ii) provides the enclosure of zeros. Evidently, since computational costs of interval arithmetic are still great, it is reasonable to apply the interval method at the end of a combined procedure, insuring in this way the inclusion of zeros. Altogether, our combined method, which does not use any derivatives, consists of the following steps.

(1) Using some searching procedure, find initial disks $Z_1^{(0)},\ldots,Z_n^{(0)}$ containing the zeros $\zeta_1,\ldots,\zeta_n$ of a given polynomial.

(2) Applying Nourein's method (5) (in complex arithmetic), compute the complex approximants $z_1^{(M)},\ldots,z_n^{(M)}$ to any required accuracy (after M iterations), starting with the centers $z_i^{(0)}$ of the initial disks $Z_i^{(0)}$, $i=1,\ldots,n$.

(3) In the final step apply the interval method (17) only once to compute the circular approximants

$$Z_i^{(M,1)} = z_i^{(M)} - \frac{W_i^{(M)}}{1+\sum_{k=1,k\neq i}^{n}\left(W_k^{(M)}/\left(Z_i^{(0)}-z_k^{(M)}\right)\right)}, \quad i=1,\ldots,n. \tag{19}$$

The inclusion disks $Z_1^{(M,1)},\ldots,Z_n^{(M,1)}$ are produced by M "point" iterations and one interval iteration, which is indicated by the superscript $(M, 1)$. Obviously, $\zeta_i \in Z_i^{(M,1)}$, $i=1,\ldots,n$, according to the inclusion isotonicity. The improved approximations $z_i^{(M)}$ force not only the contraction of the disks $Z_i^{(M,1)}$, but prevent division by a zero-interval in (19) if the initial disks $Z_i^{(0)}$ are not small enough (because $W_k^{(M)}$ becomes small enough in magnitude if $z_i^{(M)}$ is sufficiently close to the zero ζ_i). But, applying the interval method (17), the possibility of division by a zero-interval exists in the mentioned case, as shown in the following example.

Example 4.2. To illustrate the advantage of the combined method (19), we consider the polynomial

$$f(z) = z^9 + 3z^8 - 3z^7 - 9z^6 + 3z^5 + 9z^4 + 99z^3 + 297z^2 - 100z - 300,$$

with the zeros -3, ± 1, $\pm 2\mathrm{i}$, $\pm 2 \pm \mathrm{i}$. As the initial inclusion approximations containing the exact zeros we have taken the disks

$$\begin{array}{lll}
Z_1^{(0)} = \{-3.3 + 0.3\,\mathrm{i};\ 0.6\}, & Z_2^{(0)} = \{-1.3 - 0.2\,\mathrm{i};\ 0.6\}, & Z_3^{(0)} = \{0.3 + 1.7\,\mathrm{i};\ 0.6\}, \\
Z_4^{(0)} = \{-1.8 + 1.4\,\mathrm{i};\ 0.6\}, & Z_5^{(0)} = \{-1.7 - 0.7\,\mathrm{i};\ 0.6\}, & Z_6^{(0)} = \{2.4 + 1.2\,\mathrm{i};\ 0.6\}, \\
Z_7^{(0)} = \{1.8 - 0.6\,\mathrm{i};\ 0.6\}, & Z_8^{(0)} = \{1.2 + 0.2\,\mathrm{i};\ 0.6\}, & Z_9^{(0)} = \{-0.3 - 2.5\,\mathrm{i};\ 0.6\}.
\end{array}$$

The inclusion disks $Z_i^{(2)}$ and $Z_i^{(3)}$ obtained by the interval method (17) as well as the disks $Z_i^{(1,1)}$ and $Z_i^{(2,1)}$, produced by the combined method (19) applying one and two iterations of Nourein's method (5), are displayed in Table 2. We observe that the radii of the disks $Z_i^{(1,1)}$ and $Z_i^{(2,1)}$ are usually several orders of magnitude smaller than the corresponding radii of the disks $Z_i^{(2)}$ and $Z_i^{(3)}$. Besides, the computational effort of the combined method (19) is smaller compared to the interval method (17), which is confirmed in Section 5, in which the computational efficiency is calculated.

A further advantage of the combined method, discussed previously, is illustrated in the example of the same polynomial taking the initial disks $Z_i^{(0)}$ with the (slightly larger) radius $r_i^{(0)} = 0.8$, $i=1,\ldots,9$, and the centers

$$\begin{array}{lll}
z_1^{(0)} = -3.6 + 0.5\,\mathrm{i}, & z_2^{(0)} = -1.3 - 0.3\,\mathrm{i}, & z_3^{(0)} = 0.5 + 2.6\,\mathrm{i}, \\
z_4^{(0)} = -2.6 + 1.5\,\mathrm{i}, & z_5^{(0)} = -2.6 - 1.5\,\mathrm{i}, & z_6^{(0)} = 2.6 + 1.5\,\mathrm{i}, \\
z_7^{(0)} = 2.6 - 1.5\,\mathrm{i}, & z_8^{(0)} = 1.4 + 0.4\,\mathrm{i}, & z_9^{(0)} = -0.5 - 2.6\,\mathrm{i}.
\end{array}$$

Table 2
The inclusion disks obtained by the interval method (17) and the combined method (19)

i	Two iteration steps	
	Combined method (19); $Z_i^{(1,1)}$	Interval method (17); $Z_i^{(2)}$
1	$\{-3.000\,024-8.66\cdot 10^{-7}\mathrm{i};\ 4.34\cdot 10^{-5}\}$	$\{-2.999\,877-2.15\cdot 10^{-4}\mathrm{i};\ 1.12\cdot 10^{-3}\}$
2	$\{-0.999\,845+2.17\cdot 10^{-4}\mathrm{i};\ 4.68\cdot 10^{-4}\}$	$\{-0.998\,101-3.09\cdot 10^{-4}\mathrm{i};\ 8.29\cdot 10^{-3}\}$
3	$\{3.48\cdot 10^{-6}+2.000\,006\,\mathrm{i};\ 2.78\cdot 10^{-5}\}$	$\{4.04\cdot 10^{-5}+2.000\,599\,\mathrm{i};\ 2.41\cdot 10^{-3}\}$
4	$\{-1.999\,992+0.999\,907\,\mathrm{i};\ 2.06\cdot 10^{-4}\}$	$\{-2.001\,735+1.000\,316\,\mathrm{i};\ 4.31\cdot 10^{-3}\}$
5	$\{-2.000\,108-1.000\,271\,\mathrm{i};\ 4.32\cdot 10^{-4}\}$	$\{-2.000\,206-1.000\,220\,\mathrm{i};\ 2.62\cdot 10^{-3}\}$
6	$\{1.999\,926+0.999\,885\,\mathrm{i};\ 2.67\cdot 10^{-4}\}$	$\{2.000\,316+0.999\,875\,\mathrm{i};\ 1.13\cdot 10^{-3}\}$
7	$\{2.000\,106-1.000\,287\,\mathrm{i};\ 7.06\cdot 10^{-4}\}$	$\{1.997\,651-1.005\,877\,\mathrm{i};\ 1.51\cdot 10^{-2}\}$
8	$\{0.999\,923-4.68\cdot 10^{-5}\mathrm{i};\ 2.51\cdot 10^{-5}\}$	$\{0.998\,409-1.56\cdot 10^{-3}\mathrm{i};\ 9.12\cdot 10^{-3}\}$
9	$\{-7.18\cdot 10^{-6}-1.999\,998\,\mathrm{i};\ 3.07\cdot 10^{-5}\}$	$\{-4.68\cdot 10^{-5}-1.999\,819\,\mathrm{i};\ 1.22\cdot 10^{-3}\}$

i	Three iteration steps	
	Combined method (19); $Z_i^{(2,1)}$	Interval method (17); $Z_i^{(3)}$
1	$\{-2.999\,999\,999\,999\,999\,8-1.60\cdot 10^{-16}\mathrm{i};\ 5.46\cdot 10^{-16}\}$	$\{-2.999\,999\,999\,997\,206\,6+4.73\cdot 10^{-11}\mathrm{i};\ 5.47\cdot 10^{-10}\}$
2	$\{-0.999\,999\,999\,999\,988\,7-1.01\cdot 10^{-15}\mathrm{i};\ 2.32\cdot 10^{-14}\}$	$\{-0.999\,999\,996\,411\,974\,8+6.42\cdot 10^{-9}\mathrm{i};\ 3.94\cdot 10^{-8}\}$
3	$\{1.72\cdot 10^{-17}+2.000\,000\,000\,000\,000\,0\,\mathrm{i};\ 3.74\cdot 10^{-17}\}$	$\{1.90\cdot 10^{-10}+1.999\,999\,999\,980\,581\,1\,\mathrm{i};\ 2.59\cdot 10^{-9}\}$
4	$\{-2.000\,000\,000\,000\,002\,1+0.999\,999\,999\,999\,994\,2\,\mathrm{i};\ 1.03\cdot 10^{-14}\}$	$\{-2.000\,000\,001\,514\,381\,4+0.999\,999\,998\,927\,582\,4\,\mathrm{i};\ 1.41\cdot 10^{-8}\}$
5	$\{-2.000\,000\,000\,000\,011\,6-0.999\,999\,999\,999\,997\,0\,\mathrm{i};\ 1.62\cdot 10^{-14}\}$	$\{-2.000\,000\,000\,124\,503\,0-1.000\,000\,000\,011\,087\,2\,\mathrm{i};\ 1.77\cdot 10^{-9}\}$
6	$\{1.999\,999\,999\,999\,998\,0+1.000\,000\,000\,000\,003\,6\,\mathrm{i};\ 1.21\cdot 10^{-14}\}$	$\{1.999\,999\,999\,847\,133\,7+0.999\,999\,999\,793\,919\,1\,\mathrm{i};\ 1.21\cdot 10^{-9}\}$
7	$\{1.999\,999\,999\,999\,992\,4-1.000\,000\,000\,000\,008\,0\,\mathrm{i};\ 2.03\cdot 10^{-14}\}$	$\{2.000\,000\,014\,963\,835-1.000\,000\,042\,503\,593\,8\,\mathrm{i};\ 1.52\cdot 10^{-7}\}$
8	$\{1.000\,000\,000\,000\,001\,4-9.71\cdot 10^{-16}\mathrm{i};\ 2.85\cdot 10^{-15}\}$	$\{0.999\,999\,992\,209\,720\,4-1.54\cdot 10^{-8}\mathrm{i};\ 8.51\cdot 10^{-8}\}$
9	$\{6.68\cdot 10^{-18}-2.000\,000\,000\,000\,000\,0\,\mathrm{i};\ 3.17\cdot 10^{-17}\}$	$\{-4.70\cdot 10^{-11}-2.000\,000\,000\,017\,727\,8\,\mathrm{i};\ 5.42\cdot 10^{-10}\}$

Applying two iterations of Nourein's method (5) in complex arithmetic and one iteration of the interval method (17) in circular arithmetic, we obtain the following inclusion disks:

$$Z_1^{(2,1)} = \{-3.000\,000\,004\,0 - 3.49 \cdot 10^{-9}\mathrm{i};\ 6.29 \cdot 10^{-9}\},$$

$$Z_2^{(2,1)} = \{-1.000\,000\,012\,1 + 2.12 \cdot 10^{-9}\mathrm{i};\ 1.63 \cdot 10^{-8}\},$$

$$Z_3^{(2,1)} = \{7.15 \cdot 10^{-9} + 2.000\,000\,004\,3\,\mathrm{i};\ 9.71 \cdot 10^{-9}\},$$

$$Z_4^{(2,1)} = \{-2.000\,000\,016\,9 + 0.999\,999\,999\,6\,\mathrm{i};\ 2.83 \cdot 10^{-8}\},$$

$$Z_5^{(2,1)} = \{-1.999\,999\,996 - 1.000\,000\,017\,2\,\mathrm{i};\ 2.02 \cdot 10^{-8}\},$$

$$Z_6^{(2,1)} = \{2.000\,000\,000\,5 + 1.000\,000\,000\,5\,\mathrm{i};\ 8.54 \cdot 10^{-10}\},$$

$$Z_7^{(2,1)} = \{2.000\,000\,004\,7 - 1.000\,000\,001\,5\,\mathrm{i};\ 6.29 \cdot 10^{-9}\},$$

$$Z_8^{(2,1)} = \{0.999\,999\,999\,8 - 2.12 \cdot 10^{-10}\mathrm{i};\ 7.93 \cdot 10^{-10}\},$$

$$Z_9^{(2,1)} = \{-1.56 \cdot 10^{-9} - 2.000\,000\,000\,010\,1\,\mathrm{i};\ 1.34 \cdot 10^{-8}\}.$$

We cannot expect better results because the initial circular approximations are rather crude. But, the interval method (17) (starting with the same initial disks) must be terminated in the first iteration, since, calculating the disks $Z_2^{(1)}$ and $Z_8^{(1)}$, the denominator in (17) appears to be a zero-interval (a disk containing the origin).

In connection with combined methods, we remark that we can apply some other iteration method (in real or complex arithmetic) instead of Nourein's method (5) (step (2) of a combined algorithm). For example, Durand–Kerner's method (1) is convenient for that purpose because it possesses a comparable computational efficiency in regard to Nourein's method. Even more, Durand–Kerner's method always converges in practice for almost any starting point $(z_1^{(0)}, \ldots, z_n^{(0)})$ (see [9]) so it belongs to the root-finding methods which are very often applied at the present time. In this case the stages (2) and (3) of the new combined algorithm are as follows.

(2) Starting with $z_i^{(0)} = \mathrm{mid}(Z_i^{(0)})$, $i = 1, \ldots, n$, compute the point approximations

$$z_i^{(m+1)} = z_i^{(m)} - \frac{f(z_i^{(m)})}{\prod_{k=1,k\neq i}^{n}(z_i^{(m)} - z_k^{(m)})}, \quad i = 1, \ldots, n, \ m = 0, 1, \ldots, M-1,$$

where M is determined by some stopping criterion (for instance, when

$$\max_{1 \leqslant i \leqslant n} |f(z_i^{(M)})| < \epsilon,$$

where ϵ is a given accuracy).

(3) Compute the inclusion disks by (17) dealing with the point improved approximations $z_1^{(M)}, \ldots, z_n^{(M)}$ and the initial disks $Z_1^{(0)}, \ldots, Z_n^{(0)}$ (19).

The presented hybrid algorithm suggests naturally the construction of the following interval method (omitting the iteration index for simplicity);

$$
\begin{aligned}
&D_i = \prod_{k=1,k\neq i}^{n} (z_i - Z_k), \qquad W_i = \frac{f(z_i)}{\operatorname{mid}(D_i)}, \qquad \tilde{Z}_i = z_i - \frac{f(z_i)}{D_i},\\
&\hat{Z}_i = z_i - \frac{W_i}{1 + \sum_{k=1,k\neq i}^{n}\left(W_k/\left(\tilde{Z}_i - z_k\right)\right)},
\end{aligned}
\tag{20}
$$

for all $i = 1,\ldots,n$. One iteration step of the above circular arithmetic method means $(Z_1,\ldots,Z_n) \mapsto (\hat{Z}_1,\ldots,\hat{Z}_n)$. Using the properties of circular arithmetic and some estimations from the proof of Theorem 3.1, it is easy to derive the relation

$$
\forall i \in \{1,\ldots,n\}, \quad \operatorname{rad}\left(\hat{Z}_i\right) \leqslant K \operatorname{rad}(Z_i)^2 \max_{k=1,\ldots,n,k\neq i} \operatorname{rad}(Z_k)^2, \tag{21}
$$

where $K > 0$ is some real constant. Similarly as in Remark 3.2, (21) implies that the order of convergence of the interval method (20) is *four*.

The interval method (20) requires relatively great computational effort. The increase of the computational efficiency of this method can be attained to a certain degree if the inclusion disks $\tilde{Z}_i$ are calculated serially, using the already calculated disks $\tilde{Z}_1,\ldots,\tilde{Z}_{i-1}$ as soon as they are available (Weierstrass' single-step method, see [2] and [20, pp. 47, 48]). Then one iteration step of the single-step version of (20) is as follows.

(1) **for** $i = 1,\ldots,n$ **calculate**

$$
D_i = \prod_{k=1}^{i-1}\left(z_i - \tilde{Z}_k\right)\prod_{k=i+1}^{n}(z_i - Z_k), \qquad W_i = \frac{f(z_i)}{\operatorname{mid}(D_i)}, \qquad \tilde{Z}_i = z_i - \frac{f(z_i)}{D_i}. \tag{22}
$$

(2) **for** $i = 1,\ldots,n$ **calculate**

$$
\hat{Z}_i = z_i - \frac{W_i}{1 + \sum_{k=1,k\neq i}^{n}\left(W_k/\left(\tilde{Z}_i - z_k\right)\right)}.
$$

Using the concept of the R-order of convergence (introduced in [18]) we present the following result.

Theorem 4.3. *Assume that the initial disks $Z_1^{(0)},\ldots,Z_n^{(0)}$ are sufficiently small. Then the R-order of convergence of the single-step method (22) is greater than $\rho(A_n) > 4$, where $\rho(A_n)$ is the spectral radius of the $n \times n$ matrix defined by*

$$
A_n = \begin{bmatrix}
2 & 2 & & & & \\
1 & 2 & 1 & & 0 & \\
1 & 0 & 2 & 1 & & \\
\vdots & & 0 & \ddots & \ddots & \\
1 & & & & 2 & 1\\
2 & 1 & & & & 2
\end{bmatrix}, \quad n \geqslant 3.
$$

The proof of Theorem 4.3 is similar to that presented in [23] (see, also [3, Chapter 8] and [20, Chapter 2]) and will be omitted. The spectral radius $\rho(A_n)$ can be easily calculated by the well-known *power method*.

5. Computational efficiency

An estimation of computational efficiency of root-finding methods is of great interest from a practical point of view. For an implementation of these methods it is convenient to know the total number of numerical operations in calculating the zeros with the requested accuracy, convergence rate, processor time of a computer (CPU-time), the number of processors available to the user, etc. As an estimation of the efficiency of iterative methods for the simultaneous determination of polynomial zeros we will use the *coefficient of efficiency* [21], see also [20, Chapter 6]. This coefficient takes into account (i) the R-order of convergence and (ii) the total number of basic arithmetic operations per iteration, taken with certain *operation weights* depending on processor time. For more details see [21].

Actually, the computational efficiency of most numerical methods (including zero-finding algorithms) can be determined only approximately. The reasons for a variation of the number of operations have been discussed in [21]. Furthermore, the execution time of arithmetic operations depends on many complex factors (for example, the stocking cost or the communication cost is, in some case, equivalent to the computation cost, the computation time strongly depends on the precision of the employed arithmetic, etc.). Therefore, the values of the operation weight should be regarded as approximate. For demonstration, we have considered the computational effort for the CRAY X-MP/2 computer on the basis of data given in [11].

For comparison purposes we have calculated the computational efficiency for the interval methods (17) and (22) and for the combined methods (1), (17) and (5), (17) (quadruple-precision arithmetic was assumed). The entries are given in Table 3 where the polynomial degree n is a parameter. From this table we observe that the proposed combined methods have considerably greater computational efficiency compared to the interval methods (22) and (17), especially for the polynomials of lower degrees.

Table 3
The values of the coefficients of efficiency

Methods	$n=4$	$n=5$	$n=6$	$n=7$	$n=8$	$n=9$	$n=10$	$n=15$
Interval method (22)	1.365	1.218	1.146	1.105	1.080	1.062	1.050	1.022
Interval method (17)	1.507	1.301	1.201	1.144	1.108	1.085	1.068	1.030
Combined method (1), (17)	1.891	1.512	1.336	1.239	1.179	1.140	1.112	1.049
Combined method (5), (17)	1.984	1.553	1.358	1.253	1.189	1.146	1.117	1.051

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