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Coupling of Nonconforming Finite Elements and Boundary Elements I: A Priori Estimates

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Abstract

Nonconforming finite element methods are sometimes considered as a variational crime and so we may regard its coupling with boundary element methods. In this paper, the symmetric coupling of nonconforming finite elements and boundary elements is established and a priori error estimates are shown. The coupling involves a further continuous layer on the interface in order to separate the nonconformity in the domain from its boundary data which are required to be continuous. Numerical examples prove the new scheme useful in practice. A posteriori error control and adaptive algorithms will be studied in the forthcoming Part II.

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1. Introduction

One main motivation for applying nonconforming finite element methods such as the Crouzeix-Raviart elements is their divergence property which is very useful in the numerical modelling of incompressible media. Typical examples are the Navier-Stokes or Stokes equations where exterior problems arise naturally (cf. [11, Sect. 1.4]. If streaming fluids surround fixed structures, e.g. an aircraft's wing, we encounter inhomogeneities in the geometry (and possibly in the fluid as well) which are easily calculated with a nonconforming finite element method (nc-FEM) but also non-local effects in the infinite domain. The truncation of the finite element grid is not advisable because of increasing computer costs. Instead, infinite elements have to be employed and, along this class of schemes, we suggest to adapt the boundary element method (BEM).

This "mariage à la mode" was initiated by engineers. Its mathematical justification started in the later seventies with papers by Brezzi, Johnson, Nédélec, Bielak, MacCamy among others. Quasi-optimal a priori error estimates for the coupling of finite and boundary elements were then obtained for Lipschitz boundaries, systems of equations, and nonlinear problems (approximated by finite elements), e.g. in [5, 9, 10, 12, 18] (see also the literature quoted therein); the symmetric coupling, which is modified here, was introduced mathematically by Costabel in [5].

The coupling of boundary elements with nc-FE, treated here for the first time, may be regarded as a variational crime [1]. The energy space for the hypersingular operator is the trace space $H^{1/2}(\Gamma)$ which does *not* include discontinuous spline functions. As a consequence, we suggest the introduction of a continuous discrete variable to circumvent further variational crimes in the BE part. It is the aim of this paper to show in a simple model problem that the coupling is feasible and reliable and even competitive to the conforming case. In part II of this paper, we will study a posteriori error estimates and adaptive mesh-refinement algorithms [3]. Further investigations are necessary to handle Stokes or Navier-Stokes problems where we take real advantage of nonconforming finite elements.

The rest of Part I is organised as follows. In Section 2 we present the model problem which is a (nonlinear) interface problem for the Laplacian. The boundary integral operators and their mapping properties are recalled from the literature in Section 3 in order to rewrite equivalently the exterior part of the problem in Section 4. The weak form of the recast model problem is monotone, and we obtain unique solutions in Section 5. The discretisation is described in Section 6, and the discrete problem is analysed in Section 7, where we show quasi-optimal a priori error estimates. Numerical examples in Section 8 confirm our theoretical convergence results and illustrate the practical performance of the scheme.

2. Model Problem

In a bounded two–dimensional Lipschitz domain Ω with boundary $\Gamma = \partial\Omega$ and exterior domain $\Omega_c := \mathbb{R}^2 \setminus \overline{\Omega}$ we are given a possibly nonlinear mapping $A : L^2(\Omega)^2 \to L^2(\Omega)^2$, jumps $u_0 \in H^{1/2}(\Gamma)$, $t_0 \in H^{-1/2}(\Gamma)$ and a right-hand side $f \in L^2(\Omega)$ and look for functions $u \in H^1(\Omega)$, $v \in H^1_{loc}(\Omega_c)$ and real constants a and b satisfying

$$-\mathrm{div}\mathcal{A}(Du) = f \qquad \text{in } \Omega, \tag{2.1}$$

$$\Delta v = 0 \qquad \text{in } \Omega_c, \tag{2.2}$$

$$\lim_{|x|\to\infty} \{v(x) - b\log(x)\} = a, \tag{2.3}$$

$$u = v + u_0 \qquad \text{on } \Gamma, \tag{2.4}$$

$$\mathcal{A}(Du|_\Omega)\cdot n = \frac{\partial v}{\partial n} + t_0 \quad \text{on } \Gamma. \tag{2.5}$$

Here, D denotes the gradient, Δ denotes the Laplacian, and n is the unit normal on Γ pointing into Ω_c. The Lipschitz and uniform monotonicity properties of $\mathcal{A}$ are described in Section 5.

Remark 1. The model situation could be generalised to other operators, e.g. to linear elasticity, or other dimensions (with other radiation conditions (2.3)). Moreover we might add Dirichlet, Neumann or mixed boundary conditions and, furthermore, also could analyse the case that $\Omega_c \subset \mathbb{R}^2\backslash\overline{\Omega}$ is a (e.g. multiply connected) bounded domain. Finally, we could add a right-hand side in (2.2).

3. Preliminaries on Boundary Integral Operators

Let $H^s(\Omega)$ denote the usual Sobolev spaces [13] with the trace spaces $H^{s-1/2}(\Gamma)$ ($s \in \mathbb{R}$) for a bounded Lipschitz domain Ω with boundary Γ. Let $\|\cdot\|_{H^k(\omega)}$ and $|\cdot|_{H^k(\omega)}$ denote the norm and semi-norm in $H^k(\omega)$ for $\omega \subseteq \Omega$ and an integer k. The $L^2(\Omega)$-scalar product is denoted as $(\cdot,\cdot)$ while $\langle\cdot,\cdot\rangle$ denotes duality between $H^s(\Gamma)$ and $H^{-s}(\Gamma)$ (defined by extending the scalar product in $L^2(\Gamma)$).

In order to rewrite the exterior problem, we need some boundary integral operators. Given $v \in H^{1/2}(\Gamma)$ and $\phi \in H^{-1/2}(\Gamma)$ we define, for $z \in \Gamma$,

$$
\begin{aligned}
(\mathcal{V}\phi)(z) &:= -\frac{1}{\pi}\int_\Gamma \phi(\zeta)\log|z-\zeta|\,ds_\zeta,\\
(\mathcal{K}v)(z) &:= -\frac{1}{\pi}\int_\Gamma v(\zeta)\frac{\partial}{\partial n_\zeta}\log|z-\zeta|\,ds_\zeta,\\
(\mathcal{K}^*\phi)(z) &:= -\frac{1}{\pi}\int_\Gamma \phi(\zeta)\frac{\partial}{\partial n_z}\log|z-\zeta|\,ds_\zeta,\\
(\mathcal{W}v)(z) &:= \frac{1}{\pi}\frac{\partial}{\partial n_z}\int_\Gamma v(\zeta)\frac{\partial}{\partial n_\zeta}\log|z-\zeta|\,ds_\zeta.
\end{aligned}
$$

This defines linear and bounded boundary integral operators when mapping between the following Sobolev–spaces [6]

$$
\begin{aligned}
\mathcal{V} &: H^{s-1/2}(\Gamma) \to H^{s+1/2}(\Gamma),\\
\mathcal{K} &: H^{s+1/2}(\Gamma) \to H^{s+1/2}(\Gamma),\\
\mathcal{K}^* &: H^{s-1/2}(\Gamma) \to H^{s-1/2}(\Gamma),\\
\mathcal{W} &: H^{s+1/2}(\Gamma) \to H^{s-1/2}(\Gamma),
\end{aligned}
$$

where $s \in [-1/2, 1/2]$. The single layer potential $\mathcal{V}$ is symmetric, the double layer potential $\mathcal{K}$ has the dual $\mathcal{K}^*$ and the hypersingular operator $\mathcal{W}$ is symmetric. Both, $\mathcal{V}$ and $\mathcal{W}$ are strongly elliptic in the sense that they satisfy a Gårding inequality (in the above spaces with $s = 0$) [6].

Let $H^s(\Gamma)/\mathbb{R} := \{\phi \in H^s(\Gamma) : \langle 1, \phi\rangle = 0\}$. Then, it is known that $\mathcal{V} : H^{-1/2}(\Gamma)/\mathbb{R} \to H^{1/2}(\Gamma)$ and $\mathcal{W} : H^{1/2}(\Gamma)/\mathbb{R} \to H^{-1/2}(\Gamma)$ are positive definite. Assuming that the capacity of Γ is smaller than one, the single layer potential $\mathcal{V}$ is positive definite on $H^{-1/2}(\Gamma)$. (For a definition of the capacity of Γ, we refer, e.g. to [15] and mention here the sufficient condition that Ω lies in a ball with radius less than 1. Thus, this condition on Γ can always be achieved by scaling [8, 15].) We refer, e.g., to [6–8, 15–17] for proofs and more details.

4. Rewriting the Exterior Problem

There is an infinite set of formulae which characterise the Cauchy data $(v, \partial v/\partial n)|_\Gamma$ of a function v with (2.2)–(2.3). We quote only two of them from the literature.

Lemma 1 [7]. *Let $v \in H^1_{loc}(\Omega_c)$ satisfy (2.2) and (2.3), then $(\xi, \phi) := (v, \partial v/\partial n)|_\Gamma \in H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$ satisfies*

$$2\begin{pmatrix}\xi\\ \phi\end{pmatrix} = \begin{pmatrix}1+\mathcal{K} & -\mathcal{V}\\ -\mathcal{W} & 1-\mathcal{K}^*\end{pmatrix}\begin{pmatrix}\xi\\ \phi\end{pmatrix} + \begin{pmatrix}2a\\ 0\end{pmatrix}. \tag{4.1}$$

Conversely, for each $(\xi, \phi) \in H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$ there exists a function $v \in H^1_{loc}(\Omega_c)$ with (2.2)–(2.3) if and only if (4.1) holds. The function v is given by the representation formula

$$v(x) = \frac{1}{2\pi}\int_\Gamma \phi(z)\log|x-z|\,ds_z - \frac{1}{2\pi}\int_\Gamma \xi(z)\frac{\partial}{\partial n_z}\log|x-z|\,ds_z + a \tag{4.2}$$

for $x \in \Omega_c$. □

Since $\mathcal{V}$ is positive definite it is invertible and we may consider the Poincaré–Steklov operator $\mathcal{S} : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$, defined as

$$\mathcal{S} := (\mathcal{W} + (\mathcal{K}^* - 1)\mathcal{V}^{-1}(\mathcal{K} - 1))/2.$$

This operator is linear, symmetric and positive definite [4], and is a Dirichlet–Neumann map as shown in the following known lemma (see, e.g. [2, 4]). For uniqueness of solutions we may prescribe one of the constants a and b in (2.3) and we prescribe $a = 0$ in the sequel.

Lemma 2. *Let $v \in H^1_{loc}(\Omega_c)$ satisfy (2.2)–(2.3) with $a = 0$, then*

$$Dv|_\Gamma \cdot n = -\mathcal{S}v|_\Gamma. \tag{4.3}$$

Conversely, for $\xi \in H^{1/2}(\Gamma)$ there exists a unique function $v \in H^1_{loc}(\Omega_c)$ satisfying (2.2)–(2.3) (with $a = 0$) and

$$v|_\Gamma = \xi \quad \textit{and} \quad (Dv\cdot n)|_\Gamma = -\mathcal{S}\xi. \qquad \square \tag{4.4}$$

Remark 2. The Poincaré–Steklov operator as its inverse have various representations and we mention only

$$(\mathcal{V} + (1 + \mathcal{K})\mathcal{W}^{-1}(1 + \mathcal{K}^*))/2 = \mathcal{W}^{-1}(1 + \mathcal{K}^*) = \mathcal{S}^{-1}. \tag{4.5}$$

(For a proof of (4.5), take the second identity in (4.1) to obtain $\xi = -\mathcal{W}^{-1}(1+\mathcal{K}^*)\phi$ and put this into the first identity in (4.1).)

5. Weak Formulation of the Model Problem

Using Lemma 2 (to replace the traction on the interface) and direct calculations, we gain the weak form of the interface problem (2.1)–(2.5): *Find* $u \in H^1(\Omega)$ *satisfying, for all* $\eta \in H^1(\Omega)$,

$$(\mathcal{A}(Du), D\eta) + \langle \mathcal{S}u|_\Gamma, \eta|_\Gamma\rangle = (f, \eta) + \langle t_0 + \mathcal{S}u_0, \eta|_\Gamma\rangle. \tag{5.1}$$

In addition, suppose that $\mathcal{A} : L^2(\Omega)^2 \to L^2(\Omega)^2$ be uniformly monotone and Lipschitz continuous, i.e., there exists positive constants $c_\mathcal{A}$ and $C_\mathcal{A}$ with

$$c_\mathcal{A}\| \sigma - \tau \|^2_{L^2(\Omega)^2} \leq (\mathcal{A}(\sigma) - \mathcal{A}(\tau), \sigma - \tau), \tag{5.2}$$

$$\| \mathcal{A}(\sigma) - \mathcal{A}(\tau) \|_{L^2(\Omega)^2} \leq C_\mathcal{A}\| \sigma - \tau \|_{L^2(\Omega)^2}, \tag{5.3}$$

for all $\sigma, \tau \in L^2(\Omega)^2$.

It is known that the problems (2.1)–(2.5) and (4.2) are equivalent in the following sense.

Theorem 1. *If* $u \in H^1_{loc}(\Omega \cup \Omega_c)$ *is a solution of* (2.1)–(2.5), *then* $u|_\Omega$ *solves (5.1). Conversely, if* $u \in H^1(\Omega)$ *is a solution of* (5.1), *then* u *can be extended by using the representation formula* (4.2) *to a function* $u \in H^1_{loc}(\Omega \cup \Omega_c)$ *which solves* (2.1)–(2.5).

Proof: The proof is based on standard arguments in the context of strong and weak solutions of partial differential equations, and the use of Lemma 2; cf. [2, 10, 12] for details and related results. □

The left-hand side in (5.1) defines an operator B as

$$B(u)(\eta) := \int_\Omega \mathcal{A}(Du) \cdot D\eta\, dx + \langle \mathcal{S}u|_\Gamma, \eta|_\Gamma\rangle,$$

which maps $H^1(\Omega)$ into its dual $H^1(\Omega)^*$. Then, Equation (5.1) reads

$$B(u) = f \in H^1(\Omega)^*. \tag{5.4}$$

Since S is bounded and positive definite, B inherits monotonicity and Lipschitz continuity from A. Hence, from standard arguments in the theory of monotone operators, we gain existence and uniqueness of solutions in our model problem.

Theorem 2 [2]**.** *The operator B is uniform monotone and Lipschitz continuous. The problems* (2.1)–(2.5) *and* (5.1) *have unique solutions.*

6. Discretisation

Let the triangulation $\mathcal{T}$ be regular in the sense of Ciarlet [1] and cover the bounded Lipschitz domain Ω exactly

$$\overline{\Omega} = \bigcup_{T \in \mathcal{T}} T$$

such that $T \in \mathcal{T}$ is a closed triangle with interior angles greater than the (universal) constant $c_\theta > 0$ and diameter $h_T > 0$. We assume that two non-identical triangles share at most a common edge or a common vertex.

With any partition $\mathcal{T}$ we associate some discrete spaces. The nc-FE space $S^{NC}(\Omega)$ consists of, in general, discontinuous functions

$$S^{NC}(\mathcal{T}) \subset H^1(\Omega; \mathcal{T}) := \{u \in L^2(\Omega) : \forall\, T \in \mathcal{T}, u|_T \in H^1(T)\}.$$

For functions in $H^1(\Omega; \mathcal{T})$ we consider the elementwise gradient $D_\mathcal{T}$ defined by

$$(D_\mathcal{T} u)|_T = D(u)|_T \qquad (T \in \mathcal{T}; u \in H^1(\Omega; \mathcal{T}))$$

and endow $H^1(\Omega; \mathcal{T})$ with the discrete scalar-product (recall that $(\cdot, \cdot)$ is the scalar product in $L^2(\Omega)$)

$$(D_\mathcal{T} u, D_\mathcal{T} v) + (u, v) \qquad (u, v \in H^1(\Omega; \mathcal{T}))$$

and the semi-norm

$$|u|_{H^1(\Omega;\mathcal{T})} := \|D_\mathcal{T} u\|_{L^2(\Omega)} \qquad (u \in H^1(\Omega; \mathcal{T})).$$

For further reference, let

$$S^1(\mathcal{T}) := \{u \in H^1(\Omega) : \forall\, T \in \mathcal{T}, u|_T \in \mathbb{P}_1\}.$$

denote the conforming piecewise affine functions. The set of edges in $\mathcal{T}$ is denoted by $\mathcal{E}$ and the set of all midpoints of some edge $E \in \mathcal{E}$ is denoted by $\mathcal{M}$. Then, the Crouzeix-Raviart finite element space is

$$S^{NC}(\mathcal{T}) := \{u \in L^2(\Omega) : \forall\, T \in \mathcal{T}, u|_T \text{ affine and } \forall\, z \in \mathcal{M}, u \text{ continuous at } z\}.$$

In order to discretise the boundary integral operators, we associate with $\mathcal{T}$ a partition $\mathcal{G} := \{E \in \mathcal{E} : E \subset \Gamma\}$ of the boundary and consider

$$S^1(\mathcal{G}) := \{w \in C(\Gamma) : \forall\, E \in \mathcal{G}, w|_E \text{ affine}\}, \tag{6.1}$$

$$S^0(\mathcal{G}) := \{w \in L^2(\Gamma) : \forall\, E \in \mathcal{G}, w|_E \text{ constant}\}, \tag{6.2}$$

namely the piecewise affine continuous and the (in general discontinuous) piecewise constants on Γ. Then, the discrete problem reads: *Find* $(U, \Xi, \Phi, \lambda) \in S^{NC}(\mathcal{T}) \times S^1(\mathcal{G}) \times S^0(\mathcal{G}) \times \mathbb{R}$ *satisfying, for all* $(V, \Theta, \Psi, \mu) \in S^{NC}(\mathcal{T}) \times S^1(\mathcal{G}) \times S^0(\mathcal{G}) \times \mathbb{R}$,

$$\begin{aligned}
(\mathcal{A}(D_{\mathcal{T}}U), D_{\mathcal{T}}V) - \langle \Phi, V\rangle &= (f, V) + \langle t_0, V\rangle,\\
-2\langle U, \Psi\rangle - \langle \mathcal{V}\Phi, \Psi\rangle + \langle(\mathcal{K}+1)\Xi, \Psi\rangle &= -2\langle u_0, \Psi\rangle,\\
\langle(\mathcal{K}^*+1)\Phi, \Theta\rangle + \langle \mathcal{W}\Xi, \Theta\rangle + \langle \lambda, \Theta\rangle &= 0,\\
\langle \Xi, \mu\rangle &= 0.
\end{aligned} \tag{6.3}$$

Remark 3. The system is symmetric but, because of the minus sign in front of $\langle \mathcal{V}\Phi, \Psi\rangle$, *not* positive definite. Changing signs, we can rewrite this discrete problem to obtain a non-symmetric but positive definite system.

Remark 4. Because of the last identity in (6.3), $\langle \Xi, 1\rangle = 0$, whence $\Xi \in H^{1/2}(\Gamma)/\mathbb{R}$. Since $\mathcal{W}1 = (\mathcal{K}+1)1 = 0$ (take $(\xi, \phi) = (1, 0)$, $a = 1$ in Lemma 1 for a proof) and so, owing to the third identity in (6.3), the real number $-\lambda$ is the integral mean of $(\mathcal{K}^* + 1)\Phi + \mathcal{W}\Xi$, and so $\lambda = 0$. Hence, the discrete problem could be equally rewritten as: *Find* $(U, \Xi, \Phi) \in S^{NC}(\mathcal{T}) \times S^1(\mathcal{G})/\mathbb{R} \times S^0(\mathcal{G})$ *satisfying, for all* $(V, \Theta, \Psi) \in S^{NC}(\mathcal{T}) \times S^1(\mathcal{G})/\mathbb{R} \times S^0(\mathcal{G})$,

$$\begin{aligned}
3(\mathcal{A}(D_{\mathcal{T}}U), D_{\mathcal{T}}V) - \langle \Phi, V\rangle &= (f, V) + \langle t_0, V\rangle,\\
-2\langle U, \Psi\rangle - \langle \mathcal{V}\Phi, \Psi\rangle + \langle(\mathcal{K}+1)\Xi, \Psi\rangle &= -2\langle u_0, \Psi\rangle,\\
\langle(\mathcal{K}^*+1)\Phi, \Theta\rangle + \langle \mathcal{W}\Xi, \Theta\rangle &= 0.
\end{aligned} \tag{6.4}$$

Remark 5. The unknowns (U, Ξ, Φ) are discrete analogues of (u, ξ, ϕ) where u solves (5.1) and (ξ, ϕ) are the Cauchy data of the exterior part v of the interface problem of Section 2. The point is that $\Xi \neq U|_\Gamma - u_0$ as the later function does *not* belong to the energy space $H^{1/2}(\Gamma)$ of the hypersingular operator.

7. A Priori Convergence Estimate

Let $h_{\mathcal{T}} : L^\infty(\Omega) \to \mathbb{R}$ be piecewise constant with $h_{\mathcal{T}}|_T = \mathrm{diam}(T)$ for each element $T \in \mathcal{T}$. Suppose that $u \in H^2(\Omega)$ and that $\mathcal{A}(Du) \in H^1(\Omega)$.

Theorem 3. *There exist positive constants* c_0 *and* h_0 *such that for all meshes* $\mathcal{T}$

with mesh-size $\max_{T\in\mathcal{T}} h_T < h_0$, *the discrete problem* (6.4) *has a unique solution* U *and there holds*

$$| u - U |_{H^1(\Omega;\mathcal{T})} \le c_0\Big(\| h_{\mathcal{T}} D^2 u \|_{L^2(\Omega)} + \| h_{\mathcal{T}} \operatorname{div}\mathcal{A}(Du)) \|_{L^2(\Omega)} + \operatorname{dist}_{H^{-1/2}(\Gamma)}(\mathcal{S}u|_\Gamma - t_0; S^0(\mathcal{G}))\Big).$$

($\operatorname{dist}_X(w; Y)$ *denotes the best approximation error in the norm of* X *when approximating* $w \in X$ *with functions in* Y*.)*

Proof: With the solution $u \in H^1(\Omega)$, we define $\phi := -\mathcal{S}u|_\Gamma + \mathcal{S}u_0$ and $\xi := u|_\Gamma - u_0$. Let $e := u - U$, $\epsilon := \phi - \Phi \in H^{-1/2}(\Gamma)$, and $\delta := \xi - \Xi \in H^{1/2}(\Gamma)$. The same (laborious) calculations which lead from (4.1) to (4.4) show that

$$\rho_0 := -2e - \mathcal{V}\epsilon + (\mathcal{K} + 1)\delta \perp S^0(\mathcal{G}), \tag{7.1}$$

$$\rho_1 := (\mathcal{K}^* + 1)\epsilon + \mathcal{W}\delta \perp S^1(\mathcal{G})/\mathbb{R}, \tag{7.2}$$

where $\perp$ denotes orthogonality in $L^2(\Gamma)$. Let $t := \mathcal{A}(Du) \cdot n_E$, where n_E is the normal on each edge E and let $\mathcal{E}$ be the set of all edges in $\mathcal{T}$. Furthermore, let $w \in S^1(\mathcal{T})$ and perform an elementwise integration by parts to employ (5.3) and (2.1), (2.5) with $\partial v/\partial n = \phi$, and utilise (6.4) directly to infer that

$$\begin{aligned}
&(\mathcal{A}(Du) - \mathcal{A}(D_{\mathcal{T}}U), D_{\mathcal{T}}e)\\
&= (\mathcal{A}(Du) - \mathcal{A}(D_{\mathcal{T}}U), D(u - w)) + (\mathcal{A}(Du) - \mathcal{A}(D_{\mathcal{T}}U), D(w - U))\\
&\le C_{\mathcal{A}}\| D_{\mathcal{T}}e \|_{L^2(\Omega)}\| D(u - w) \|_{L^2(\Omega)} + (\mathcal{A}(Du), D(w - U))\\
&\qquad - \langle \Phi, w - U\rangle - (f, w - U) - \langle t_0, w - U\rangle\\
&\le C_{\mathcal{A}}\| D_{\mathcal{T}}e \|_{L^2(\Omega)}\| D(u - w) \|_{L^2(\Omega)} + \langle \epsilon, w - U\rangle - \int_{\cup\mathcal{E}\setminus\Gamma} t[U]\, ds,
\end{aligned} \tag{7.3}$$

where $[U]$ denotes the jump of U across an inner element side $E \in \mathcal{E}$ and $\cup\mathcal{E}\setminus\Gamma := (\cup\mathcal{E}) \setminus \Gamma$ denotes the union of all inner edges; $\cup\mathcal{E} := \cup_{E\in\mathcal{E}} E$ is the skeleton of all edges in $\mathcal{T}$

From the definition of ρ_0 and ρ_1 and the orthogonal relations (7.1) and (7.2) it follows, with $\langle\epsilon, (\mathcal{K} + 1)\delta\rangle = \langle(\mathcal{K}^* + 1)\epsilon, \delta\rangle$,

$$\begin{aligned}
2\langle\epsilon, e\rangle &= \langle\epsilon, (\mathcal{K} + 1)\delta\rangle - \langle\mathcal{V}\epsilon, \epsilon\rangle - \langle\rho_0, \epsilon\rangle\\
&= -\langle\mathcal{W}\delta, \delta\rangle - \langle\mathcal{V}\epsilon, \epsilon\rangle + \langle\rho_1, \delta\rangle - \langle\rho_0, \epsilon\rangle\\
&= -\langle\mathcal{W}\delta, \delta\rangle - \langle\mathcal{V}\epsilon, \epsilon\rangle - \langle\rho_0, \phi - \tilde{\Phi}\rangle + \langle\rho_1, \xi - \tilde{\Xi}\rangle
\end{aligned} \tag{7.4}$$

for all $\tilde{\Xi} \in S^1(\mathcal{G})/\mathbb{R}$ and all $\tilde{\Phi} \in S^0(\mathcal{G})$. Combining (7.3) and (7.4) and utilising

(4.2), we obtain

$$
\begin{aligned}
& c_{\mathcal{A}} \| D_{\mathcal{T}} e \|^2_{L^2(\Omega)} + \frac{1}{2}\langle \mathcal{W}\delta, \delta\rangle + \frac{1}{2}\langle \mathcal{V}\epsilon, \epsilon\rangle \\
& \leq C_{\mathcal{A}} \| D_{\mathcal{T}} e \|_{L^2(\Omega)} \| D(u-w) \|_{L^2(\Omega)} + \langle \epsilon; w-u\rangle - \int_{\cup\mathcal{E}\setminus\Gamma} t[U]\, ds \\
& \quad - \frac{1}{2}\langle \rho_0, \phi - \tilde{\Phi}\rangle + \frac{1}{2}\langle \rho_1, \xi - \tilde{\Xi}\rangle \qquad (7.5) \\
& \leq C_{\mathcal{A}} \| D_{\mathcal{T}} e \|_{L^2(\Omega)} \| D(u-w) \|_{L^2(\Omega)} + \| \epsilon \|_{H^{-1/2}(\Gamma)} \| w-u \|_{H^{1/2}(\Gamma)} \\
& \quad - \int_{\cup\mathcal{E}\setminus\Gamma} t[U]\, ds \; + \langle e, \phi - \tilde{\Phi}\rangle + \| \mathcal{V}\epsilon + (\mathcal{K}+1)\delta \|_{H^{1/2}(\Gamma)} \\
& \quad \times \| \phi - \tilde{\Phi} \|_{H^{-1/2}(\Gamma)} + \| \rho_1 \|_{H^{-1/2}(\Gamma)} \| \xi - \tilde{\Xi} \|_{H^{1/2}(\Gamma)}.
\end{aligned}
$$

Define $[U]|_\Gamma := \tilde{e} - e$ where $\tilde{e} \in S^0(\mathcal{G})$ is the $\mathcal{G}$-piecewise integral mean of e and let $\tilde{\Phi}$ denote the $\mathcal{G}$-piecewise integral mean of ϕ. Then,

$$\langle e, \phi - \tilde{\Phi}\rangle = \langle e - \tilde{e}, \phi - \tilde{\Phi}\rangle = \langle e - \tilde{e}, \phi\rangle = -\int_\Gamma t[U]\, ds \qquad (7.6)$$

if we notice that $t = \phi = \mathcal{A}(Du)\cdot n$ by (2.5). Hence, for all $\tilde{t}$ which are constant on each $E \in \mathcal{E}$,

$$-\int_{\cup\mathcal{E}\setminus\Gamma} t[U]\, ds + \langle e, \phi - \tilde{\Phi}\rangle = -\int_{\cup\mathcal{E}} t[U]\, ds = \int_{\cup\mathcal{E}} (\tilde{t} - t)[U]\, ds, \qquad (7.7)$$

because the integral of $[U]$ over each edge E vanishes. Define the constant $\tilde{t}$ as the integral mean of t on each edge E. A standard technique in nc-FE, cf., e.g. [1], leads to the estimate

$$\int_{\cup\mathcal{E}} (t - \tilde{t})[e]\, ds \leq c_1 \| h_{\mathcal{T}} D\mathcal{A}(Du)) \|_{L^2(\Omega)} \| h_{\mathcal{T}} D_{\mathcal{T}} e \|_{L^2(\Omega)}. \qquad (7.8)$$

Gathering (7.5)–(7.8) together, we conclude the theorem with standard estimations absorbing the error terms on the right-hand side. □

Remark 6. The best-approximation error $\mathrm{dist}_{H^{-1/2}(\Gamma)}(\mathcal{S}u|_\Gamma - t_0; S^0(\mathcal{G}))$ in Theorem 3 is bounded by $O(h^{3/2})$ when h denotes the maximal mesh-size on the boundary and $\mathcal{S}u|_\Gamma - t_0$ belongs to $H^1(\Gamma)$.

Remark 7. In the notation of the proof we furthermore have

$$
\begin{aligned}
&|\, e\,|_{H^1(\Omega;\mathcal{T})} + \|\, \epsilon\, \|_{H^{1/2}(\Gamma)/\mathbb{R}} + \|\, \delta\, \|_{H^{-1/2}(\Gamma)} \\
&\le C\left(\|\, h_{\mathcal{T}} D^2 u\, \|_{L^2(\Omega)} + \|\, h_{\mathcal{T}} \mathrm{div} A(Du))\, \|_{L^2(\Omega)} + \mathrm{dist}_{H^{-1/2}(\Gamma)/\mathbb{R}}(\mathcal{S}u|_\Gamma ;\, S^0(\mathcal{G}))\right).
\end{aligned}
$$

8. Numerical Example

To illustrate our convergence result we consider a numerical example and start with some remarks on the numerical implementation in Matlab. The duality pairs on the left hand side, e.g. $(\mathcal{A}(D_{\mathcal{T}}U), (D_{\mathcal{T}}V), \langle V\Phi, \Psi\rangle$ and $\langle K\,\Xi, \Psi\rangle$ where U, V, Φ, Ψ, Ξ are piecewise constant or piecewise linear functions can be calculated almost analytically. (See [14] for terms with integral operators.)

In order to approximate the right hand side for given functions $f \in L^2(\Gamma)$, $u_0 \in H^{1/2}(\Gamma)$, and $t_0 \in L^2(\Gamma)$ we compute $\int_\Omega f\eta_j\, dx$ via a mid-point quadrature rule on any triangle T and the integrals $\langle \Psi, u_0\rangle$ and $\langle t_0, V\rangle$ are approximated by an 8-point Gaussian quadrature formula.

Since $\mathcal{A}$ is a linear operator in our examples we get a linear system of equations which is solved directly.

Example 8.1. Let us consider the interface problem (2.1)–(2.5) on the L-shaped domain in Fig. 1 with exact solution

$$u(r,\theta) = r^{2/3}\sin(2\theta/3) \quad \text{and} \quad v(x,y) = \log(|(x+1/2,\, y-1/2|)$$

in polar resp. Cartesian coordinates (r,θ) resp. (x,y). The solution has a typical corner singularity such that the convergence rate of the h–version with a uniform mesh does not lead to the optimal convergence rate. The right hand side f and the jumps u_0 and t_0 are computed by (2.1) and (2.5) from u and v above. With those data, we computed the discrete solution of (6.3).

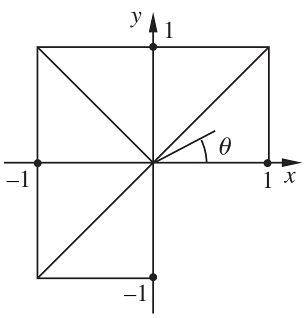

Figure 1. L-shape

Table 1. Error $|e|_1$ and convergence rate γ_h in Example 8.1

h	$\lvert e\rvert_1$	γ_h
1	0.23819	
1/2	0.23163	0.0403
1/4	0.14471	0.6786
1/8	0.09033	0.6799
1/16	0.05696	0.6636
1/32	0.03826	0.6701

In Table 1 we give the numerical results for the uniform meshes. For the sequence of uniform meshes we obtain experimentally a convergence which is approximately $h^{2/3}$ which coincides with the theoretically expected rate.

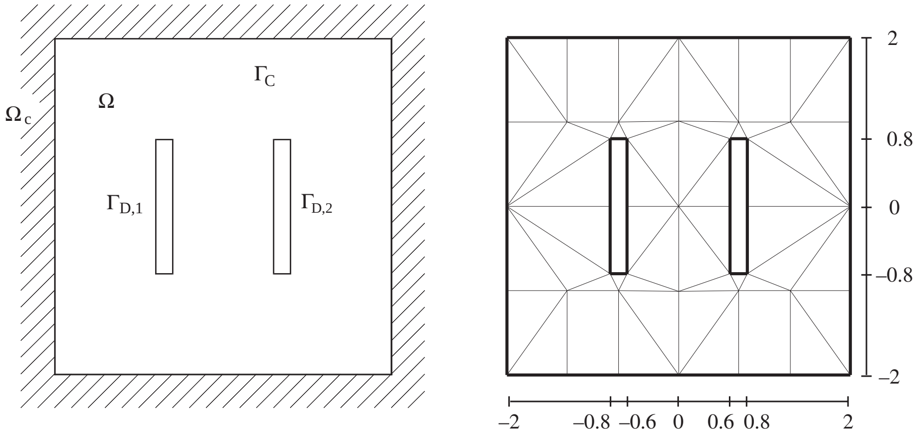

Figure 2. Configuration of Example 8.2

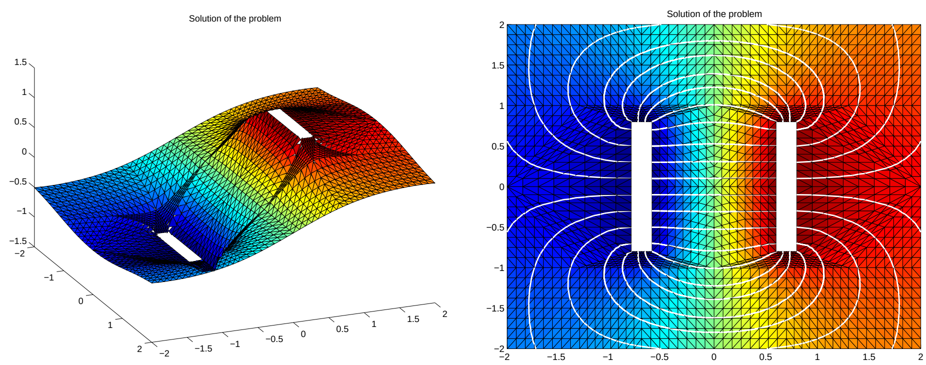

Figure 3. Solution of Example 8.2

Example 8.2. As a more practical example, we consider the following problem, where $A = 5\,id$ in (2.1), $u_0 = 0$ and $t_0 = 0$ on the coupling boundary $\Gamma_C := \partial\Omega \cup \partial\Omega_c$. This problem models the potential of a capacitor in an unbounded domain with different permeabilities in Ω_c and Ω. The charge at boundaries $\Gamma_{D,1}$ and $\Gamma_{D,2}$ are ± 1, respectively. Ω, Ω_c, Γ_C and $\Gamma_{D,\cdot}$ are given as depicted in Fig. 2. The exact geometry is given there, too. Three times uniform refinement, e.g. joining midpoints of each edge by straight lines, of the mesh shown in Fig. 2 gives the used mesh. The solution for this problem with $h = 0.12$ is shown in Fig. 3.

The calculation required 1225.27 sec CPU on a Ultra Sparc I and corresponds to 5825 degrees of freedom (5504 in the finite element part and 160 boundary elements, yielding 480 degrees of freedom on the boundary plus 1 for the Lagrange multiplier λ).

The streamlines in Fig. 3 give knowledge of gradients of the potential. Due to the higher permeability in Ω the streamlines look more depressed and flat than in the case $A = id$. Although we are using the nc-FEM in Ω the streamlines are smooth, also near the coupling boundary.

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