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Adaptive Mixed Finite Element Method for Reissner–Mindlin Plates

We use a modified mixed finite element method for the Reissner-Mindlin plate model to study its numerical properties in practical use. We derive an a posteriori error estimate to control adaptive mesh-refining algorithms and study the question of reliability. Numerical examples prove the new scheme efficient.

1. Mechanical Model and Finite Element Discretisation

Due to Reissner-Mindlin theory the deformation vector of a plate Ω with small thickness t and only transverse load f contains three independent components, the rotations $\vartheta = (\vartheta_x, \vartheta_y) \in H_0^1(\Omega)^2$ and the transverse displacement $w \in H_0^1(\Omega)$. The standard Reissner-Mindlin variation formulation is – because of the shear locking phenomena – not sufficient for effective finite element discretisation. So we reformulate the problem with an additional variable [1]

$$\gamma = (\frac{1}{t^2} - \alpha)(\nabla w - \vartheta) \qquad 0 < \alpha < t^{-2} \tag{1}$$

where α is a parameter to stabilize the discretisation and with the bilinear forms

$$a(w,\vartheta;v,\varphi) := \int_\Omega \varepsilon(\vartheta) : \mathbb{C}\varepsilon(\varphi)\,dx + \alpha\int_\Omega (\nabla w - \vartheta)\cdot(\nabla v - \varphi)\,dx \tag{2}$$

$$b(w,\vartheta;\eta) := \int_\Omega (\nabla w - \vartheta)\cdot\eta\,dx \tag{3}$$

$$c(\gamma;\eta) := \beta\int_\Omega \gamma\cdot\eta\,dx \qquad \beta = -t^2/(1-\alpha t^2) \tag{4}$$

The strain is $\varepsilon = \mathrm{sym}(\nabla\vartheta)$, the elasticity operator is defined by $\mathbb{C}\varepsilon = \frac{1}{12}\frac{\lambda}{\mu k}\mathrm{tr}\,\varepsilon I + \frac{1}{6k}\varepsilon$ (μ, λ Lamè–Constants). The **continuous problem** now reads: Find $(w,\vartheta,\gamma) \in H_0^1(\Omega) \times H_0^1(\Omega)^2 \times L^2(\Omega)^2$ such that

$$a(w,\vartheta;v,\varphi) + b(v,\varphi;\gamma) = \int_\Omega fv\,dx \tag{5}$$

$$b(w,\vartheta;\eta) + c(\gamma;\eta) = 0 \qquad \text{for all } (v,\varphi,\eta) \in H_0^1(\Omega) \times H_0^1(\Omega)^2 \times L^2(\Omega)^2. \tag{6}$$

We consider a regular triangulation $\mathcal{T}$ of Ω with discrete spaces of $\mathcal{T}$–piecewise polynomials of degree $\leq k (k \in \mathbb{N})$

$$\mathcal{P}_k(\mathcal{T}) := \{u \in L^2(\Omega)\ |\forall T \in \mathcal{T},\ u|_T \in \mathcal{P}_k(T)\} \qquad \mathcal{S}_k(\mathcal{T}) := \mathcal{P}_k(\mathcal{T}) \cap H_0^1(\Omega) \tag{7}$$

We choose here $H_w \times H_\vartheta \times L_\gamma = \mathcal{S}_2(\mathcal{T}) \times \mathcal{S}_2(\mathcal{T})^2 \times \mathcal{P}_0(\mathcal{T})^2$ as in [6]. The **discrete problem** now reads: Find $(W,\Theta,\Gamma) \in H_w \times H_\vartheta \times L_\gamma$ such that

$$a(W,\Theta;V,\Phi) + b(V,\Phi;\Gamma) = \int_\Omega fV\,dx \tag{8}$$

$$b(W,\Theta;H) + c(\Gamma;H) = 0 \qquad \text{for all } (V,\Phi,H) \in H_w \times H_\vartheta \times L_\gamma. \tag{9}$$

2. Stabilization Techniques

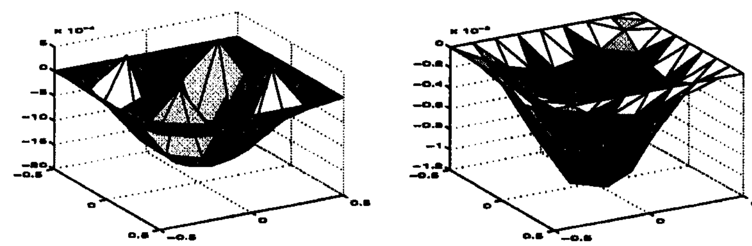

Fig. 1: Transversale displacement W of the plate with $\alpha = 1$ and $\alpha = 1/h_T^2$

In [1] is shown, that (5-6) lead asymtotically to a stable and locking free finite element discretisation. Here and in other theoretical contributions the stabilization parameter α is set equal 1 [3]. But in practical use – with coarse meshes – the choice of α has an essential influence on the solution quality. If α is too small, spurious modes appears. If it is too large, the system is too stiff. Trying to find an optimal α we made several numerical investigations. Here results are given for an all side clamped plate of $1 \cdot 1 \cdot 0.001$m under a unit load $f = 1000\mathrm{N/m}^3$, $\mu = 4.2\mathrm{N/m}^2$, $\lambda = 3.6\mathrm{N/m}^2$. It is meshed with $4 \cdot 4$ squares, each divided in 2 triangels (32 fe, 211 dof). If $\alpha = 1$ the fe-solution is unusable (Fig. 1). Best approximation we get with $\alpha = 5...50$, here the maximal transverse displacement is $w_{max} = 1.2$mm (analytical solution: $w_{max} = 1.26$mm). If e.g. $\alpha = 5000$ we get $w_{max} = 0.34$mm. Due to our numerical expierience we set α mesh dependent. With $h_T = \mathrm{diam}(T)$ we recommend

$$\alpha = h_T^{-2} \tag{10}$$

which is in the example $\alpha = 8$. (10) corresponds to similar results in [5] and [6].

3. A Posteriori Error Estimation

Let $(w, \vartheta, \gamma) \in H_0^1(\Omega) \times H_0^1(\Omega)^2 \times L^2(\Omega)^2$ solve (5-6) and suppose that (W, Θ, Γ) satisfies (8-9) for all $(V, \Phi, H) \in H_w \times H_\vartheta \times L_\gamma$. Then, there exists a positive constant C which is independent of t, h_T and α such that we have

$$\|\vartheta - \Theta\|_{H_0^1(\Omega)} + \|w - W\|_{H_0^1(\Omega)} + \|\gamma - \Gamma\|_{H^{-1}(\mathrm{div};\Omega)} + t\|\gamma - \Gamma\|_{L^2(\Omega)} \le C\big(\sum_{T\in\mathcal{T}} \eta_T^2\big)^{1/2}. \tag{11}$$

For each element $T \in \mathcal{T}$ we define our error indicator η_T ($\mathcal{E}$ edges of triangulation $\mathcal{E} \subset \partial\mathcal{T}$)

$$\begin{aligned}\eta_T^2 := h_T^2 \int_T |\mathrm{div}(\alpha(\nabla W - \Theta)) + \mathrm{div}\Gamma + f|^2\, dx + h_T^2 \int_T |\mathrm{div}\mathbb{C}\varepsilon(\Theta) + \alpha(\nabla W - \Theta)|^2\, dx + \\ \int_T |\nabla W|^2\, dx + \frac{h}{t}\int_T |\nabla\Theta|^2\, dx + \\ \sum_{E\in\mathcal{E}} \Big(h_E \int_E |[\alpha(\nabla W - \Theta) + \Gamma]\cdot n_E]|^2 ds + h_E \int_E |[\mathbb{C}\varepsilon(\Theta)]\cdot n_E|^2 ds\Big)\end{aligned} \tag{12}$$

The proof of (11) is given in [4], we will just mention, that – besides the standard a posteriori arguments – we need precise mapping properties of the weak formulation, t–dependent interpolation spaces is in [2] and an estimate of $\dot{B}_2^{1/2}(\mathcal{T})$ due to Tartar. Concluding we will show some results of our adaptive mesh refinement algorithm.

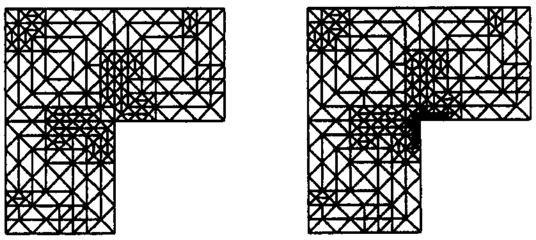

Fig. 2: All side clamped L–shaped domain (load and material as above) after 3 and 5 automatic refinement steps controlled by (12)

We see the expected refinement towards the corner. Here and in further examples (12) designs resonable adaptive refined meshes.

4. References

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