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Convergence of adaptive finite element methods in elastoplasticity

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We show that an adaptive conforming finite element method for the solution of a variational inequality of the second kind, as model of one implicit time step of the primal problem of plasticity with positive hardening, yields the energy reduction property and the $R-$linear convergence of the stresses.

1 Introduction

Convergence of adaptive finite element methods (hereafter referred to as AFEM) have only recently experienced significant development following the work of Dörfler [6]. The introduction of the bulk criterion as marking strategy in place of the max refinement rule [1], along with the discrete efficiency estimate, and the orthogonality of the discrete conforming finite element approximations, represent the key ingredients for proving the linear convergence of AFEMs for conforming finite element approximations of the Poisson problem [6, 7]. Extensions to nonlinear problems have then been considered in [9] for the nonlinear laplacian whereas [3] presents a thorough analysis for uniformly convex and degenerated convex minimization problems, with applications to relaxed formulations in computational microstructures. Along the same line as [3] this short note states the energy reduction and the $R-$linear convergence of the stresses for a conforming FE method of the primal problem of plasticity [8]. Proof and details are given elsewhere [5].

2 Primal formulation of plasticity and the discrete problem

Let Ω be a bounded Lipschitz domain in $\mathbb{R}^d$, for $d = 2, 3$, with Dirichlet Γ_D and Neumann Γ_N boundary with $u_D = 0$ on Γ_D and prescribed traction force $g \in H^{1/2}(\Gamma_N; \mathbb{R}^d)$ on Γ_N. We denote by $f \in L^2(\Omega; \mathbb{R}^d)$ the volume force. As usual, it is assumed $\Gamma_D \cap \Gamma_N = \varnothing$ and $\Gamma_D \cup \bar{\Gamma}_N = \partial\Omega$. We consider small strain theory of associative elastoplasticity with positive hardening for the quite general model of isotropic-kinematic hardening with Von Mises yield condition. For $u \in V := H^1_D(\Omega; \mathbb{R}^d)$, the linear strain $\varepsilon(u) = 1/2(\nabla u + \nabla u^T)$ is split into an elastic e and plastic part p, with $e = \mathbb{C}^{-1}\sigma$, $\mathbb{C}$ being the elasticity tensor used to define the norm $|||\tau|||_{\mathbb{C}^{-1};\Omega} := (\tau; \mathbb{C}^{-1}\tau)^{1/2}_{L^2(\Omega;\mathbb{R}^{d\times d})}$ in $L^2(\Omega; \mathbb{R}^{d\times d})$. The kinematic hardening is described by the back stress tensor σ_b related to the conjugate internal variable α by $\sigma_b = \mathbb{H}\alpha$, with $\mathbb{H}$ the positive definite hardening tensor, whereas the isotropic hardening is described by the additional scalar internal variable R related to the conjugate kinematical variable A by $R = kA$ with $k > 0$. Using the Von Mises yield condition, the closed and convex domain $\mathbb{E} := \{(\tau, \tau_b, S) \in \mathbb{R}^{d\times d}_{sym} \times \mathbb{R}^{d\times d}_{sym} \times \mathbb{R} : |\operatorname{dev}(\tau - \tau_b)| - (S + \sigma_y) \leq 0,\ S \geq 0\}$ with σ_y a material constant, defines the set of admissible generalized stresses (σ, σ_b, R). Without loss of generality, we consider initial conditions equal to zero in the time discrete form of the evolution law, expressed as $(p, \alpha, A) \in \partial I_{\mathbb{E}}(\sigma, \sigma_b, R)$ or in the dual form $(\sigma, \sigma_b, R) \in \partial I^*_{\mathbb{E}}(p, \alpha, A)$ with $I_{\mathbb{E}}$ indicator function of $\mathbb{E}$, $I^*_{\mathbb{E}}$ its Legendre-Fenchel dual, and ∂f subdifferential of the convex function f. The elastoplastic problem is then defined by the above constitutive equation together with the equilibrium condition for the stress $\sigma = \mathbb{C}(\varepsilon(u) - p)$. The primal formulation of plasticity assumes $w := (u, p, \alpha, A) \in \mathcal{H} := V \times Q \times Q \times L$ as primary variable, with $Q := \{q \in L^2(\Omega; \mathbb{R}^{d\times d}_{sym}) : \operatorname{tr}(q) = 0\}$ and $L := L^2(\Omega; \mathbb{R})$. Let $z := (v, q, \beta, B) \in \mathcal{H}$ denote the test function, the weak form reduces to the solution $w \in \mathcal{H}$ of the following variational inequality

$$b(z - w) \leq a(w;\, z - w) + \psi(z) - \psi(w) \quad \text{for all } z \in \mathcal{H}, \tag{1}$$

with $a(w; z) := (\sigma; \varepsilon(v) - q)_{L^2(\Omega;\mathbb{R}^{d\times d})} + (\mathbb{H}\alpha; \beta)_{L^2(\Omega;\mathbb{R}^{d\times d})} + (kA; B)_{L^2(\Omega;\mathbb{R})}$, $b(z) := (f; v)_{L^2(\Omega;\mathbb{R}^d)} + \int_{\Gamma_N} g \cdot v\, ds$, and $\psi(z) := \int_\Omega I^*_{\mathbb{E}}(q, \beta, B)\, dx$. For the model here considered, the bilinear form a is continuous and $\mathcal{H}-$elliptic, b linear and bounded, and ψ convex, lower semicontinuous (lsc), and non differentiable [2, 8]. The variational inequality (1) is also equivalent to the following minimization problem

$$w = \operatorname*{arg\,min}_{z\in\mathcal{H}} J(z) \quad \text{with } J := \frac{1}{2}a(z; z) - b(z) + \psi(z) \text{ convex, lsc, and uniformly convex.} \tag{2}$$

Let $\mathcal{T}$ denote a shape regular triangulation of Ω into triangles, set of edges $\mathcal{E}$ and free nodes $\mathcal{K}$. We consider the conforming finite element approximation to w solution of (1) (or equivalently of (2)) with P_1 elements for the displacement field u, and

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P_0 elements for the kinematic internal variables (p, α, A). With $V_\ell \subseteq V$, and Q_0 and L_0 subspaces of Q and L of piecewise constant functions, respectively, we define the finite element space $\mathcal{H}_\ell := V_\ell \times Q_0 \times Q_0 \times L_0 \subseteq \mathcal{H}$, and the discrete problem as (1) (or (2)) by replacing $\mathcal{H}$ with $\mathcal{H}_\ell$.

3 Data oscillations

On the mesh $\mathcal{T}_\ell$ at the level $\ell = 0, 1, \ldots$, for $f \in L^2(\Omega; \mathbb{R}^d)$ we define $\operatorname{osc}_\ell^2(f) := \sum_{z \in \mathcal{K}_\ell} \|h_z(f - \bar{f})\|^2_{L^2(\omega_z;\mathbb{R}^d)}$, with $\omega_z := \{T \in \mathcal{T}_\ell : z \in T\}$, $|\omega_z|$ and h_z area and diameter of ω_z, and $\bar{f} = 1/|\omega_z| \int_{\omega_z} f\, dx$. For $g \in H^{1/2}(\Gamma_N; \mathbb{R}^d)$ we let $\operatorname{osc}_\ell^2(g) := \sum_{E \subset \Gamma_N} \|h_E^{1/2}(g - g_E)\|_{L^2(E;\mathbb{R}^d)}$ with $g_E := 1/h_E \int g\, ds$.

4 Adaptive algorithm, energy reduction, and convergence of stresses

Given an initial coarse shape-regular triangulation $\mathcal{T}_0$ of Ω into triangles with $\mathbb{C}$, $\mathbb{H}$, and k constant over each $T \in \mathcal{T}_0$, we consider triangulations $\mathcal{T}_\ell$ built according to the following algorithm.

*Input the triangulation $\mathcal{T}_0$ with set of edges $\mathcal{E}_0$, $0 < \Theta < 1$, and repeat **(a)** – **(e)**.*
***(a)** Solve the nonlinear discrete problem $w_\ell = \underset{z_\ell \in \mathcal{H}_\ell}{\arg\min}\, J(z_\ell)$ and evaluate $\sigma_\ell := \mathbb{C}(\varepsilon(u_\ell) - p_\ell)$.*
***(b)** For each $E \in \mathcal{E}_\ell$ with measure h_E, compute $\eta_E^2 := h_E \int_E |[\sigma_\ell \nu_E]|^2\, ds$ and $\eta_\ell = (\sum_{E \in \mathcal{E}_\ell} \eta_E^2)^{1/2}$. The symbol $[\sigma_\ell \nu_E]$ defines the jump of the discrete stresses σ_ℓ across the interior edges E with standard modification for $E \subseteq \Gamma_N$.*
***(c)** Select $\mathcal{M}_\ell \subseteq \mathcal{E}_\ell$ in the current triangulation $\mathcal{T}_\ell$ with*

$$\Theta \eta_\ell^2 \le \sum_{E \in \mathcal{M}_\ell} \eta_E^2 \,. \tag{3}$$

***(d)** Control $\operatorname{osc}_\ell(f)$ and $\operatorname{osc}_\ell(g)$, and add (possibly) further edges to $\mathcal{M}_\ell$ to decrease $\operatorname{osc}_{\ell+1}(f)$ and $\operatorname{osc}_{\ell+1}(g)$.*
***(e)** Refine all the elements T with some edge in $\mathcal{M}_\ell$ with the inner node* [7, 3] *and run the closure algorithm with red-green-blue refinement* [10]. *Denote with $\mathcal{T}_{\ell+1}$ the resulting shape-regular triangulation. Set $\ell := \ell + 1$ and go to **(a)**.*
Output discrete stress fields $\sigma_0, \sigma_1, \ldots$ in $L^2(\Omega; \mathbb{R}^{d\times d}_{sym})$ as approximation to $\sigma = \mathbb{C}(\varepsilon(u) - p)$.

Proof of the reliability of η uses Jensen inequality as in [4], whereas in [5] using inverse estimates and convex analysis we prove a local discrete efficiency estimate in terms of the discrete energies, and the existence of positive constants ρ_E, ρ with $\rho_E < 1$, depending on the regularity of the initial triangulation $\mathcal{T}_0$ and on the material parameters, such that there holds

$$\delta_{\ell+1} \le \rho_E \delta_\ell + \rho(osc_\ell^2(f) + osc_\ell^2(g))$$

with $\delta_\ell := J(w_\ell) - J(w)$ and similar definition for $\delta_{\ell+1}$. The control of the data oscillations $\operatorname{osc}_\ell(f)$ and $\operatorname{osc}_\ell(g)$ with conditions similar to (3) as step ***(e)*** of the adaptive algorithm using the inner node property leads finally to the existence of a sequence $(\alpha_\ell)_{\ell \in \mathbb{N}}$ linearly convergent to zero such that there holds

$$|||\sigma - \sigma_\ell|||_{\mathbb{C}^{-1};\Omega} \le \alpha_\ell. \tag{4}$$

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