

## A POSTERIORI FE ERROR CONTROL FOR P-LAPLACIAN BY GRADIENT RECOVERY IN QUASI-NORM

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ABSTRACT. A posteriori error estimators based on quasi-norm gradient recovery are established for the finite element approximation of the p-Laplacian on unstructured meshes. The new a posteriori error estimators provide both upper and lower bounds in the quasi-norm for the discretization error. The main tools for the proofs of reliability are approximation error estimates for a local approximation operator in the quasi-norm.

### 1. INTRODUCTION

In this work we introduce a class of a posteriori error estimators for the finite element approximation of the p-Laplacian with homogeneous Dirichlet data

$$(1.1) \quad -\operatorname{div}(|\nabla u|^{p-2}\nabla u) = f \text{ in } \Omega \quad \text{and} \quad u = 0 \text{ on } \partial\Omega.$$

Here,  $1 < p < \infty$ ,  $\Omega$  is a bounded open subset of  $\mathbb{R}^2$  with a Lipschitz boundary  $\partial\Omega$ , and  $f$  is a given right-hand side. This equation is viewed as one of the typical examples of a large class of nonlinear problems. Indeed it is believed that essential difficulties in studies of finite element approximations for nonlinear systems are contained in (1.1) where many techniques, e.g., the linearization or deformation procedure, do not seem to work well.

Finite element approximations of the p-Laplacian have been studied extensively in the literature [Ci, GM, Ch]. The quasi-norm approach for sharp a priori error bounds is summarized in [LY1, LY2]. An important aspect is the a posteriori error estimation of the p-Laplacian. In the contributions [ODSD, BA, BL2, P, V1] there are *gaps* in the power between the established upper and lower estimates. Recently [LY1, LY2], the quasi-norm techniques and improved a posteriori error estimates of residual type were derived for the p-Laplacian. Initial analysis and numerical tests indicate that the new estimators are sharper than the very different existing ones, and, indeed, lead to more efficient computational meshes [CK, LY1].

In engineering simulations, a posteriori error estimators based on gradient recovery are widely used; see [AO, V2] for an introduction and [Ca, CB, CF1, CF2] for their mathematical justification. There seems to be no difficulty in designing and

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using such estimators for the p-Laplacian using the standard Sobolev semi-norms, but, their mathematical justification was found difficult [CK].

The quasi-norm was used in [LY3] to construct a posteriori error estimators based on gradient recovery, but the construction approach therein is restricted to *uniform* meshes. This paper aims at a justification for unstructured and locally refined meshes with different mathematical ideas. We derive upper and lower error bounds for the estimators and establish several approximation error estimates in the quasi-norms for the operator  $\pi$  from [Ca] (see Definition 4.2 below).

Efficient and reliable averaging techniques are very popular tools in practice. But they are also of particular importance for the understanding of a posteriori error estimation because they show that pollution is excluded in the sense that the local approximation error dominates the global error.

The plan of this paper is as follows. In Section 2 we state some important inequalities and give the p-Laplacian its variational formulation. In Section 3 we set up the finite element approximation for the equation. We also introduce some quasi-norms and related results. In Section 4, we introduce the weighted Clement-type interpolator and we prove quasi-norm estimates for the interpolation error. In Section 5, we construct a posteriori error estimators based on quasi-norm gradient recovery on unstructured meshes, and we prove upper and lower error bounds for these estimators. Numerical experiments [CK] proved that the averaging estimators were surprisingly accurate in practice —quite in agreement with linear situations in [CB, CF1, CF2]— provided the error measure is the quasi-norm.

## 2. PRELIMINARIES

Throughout this paper we adopt the standard notation  $W^{m,q}(\Omega)$  for Sobolev spaces on  $\Omega$  with norm  $\|\cdot\|_{W^{m,q}(\Omega)}$  and semi-norm  $|\cdot|_{W^{m,q}(\Omega)}$ . We set  $W_0^{m,q}(\Omega) \equiv \{w \in W^{m,q}(\Omega) : w|_{\partial\Omega} = 0\}$ . We denote  $W^{m,2}(\Omega)$  by  $H^m(\Omega)$ . In addition  $c$  or  $C$  denotes a general positive constant independent of  $h$  and  $A \leq CB$  abbreviates  $A \lesssim B$ .

The generic constant  $C$  is only allowed to depend on  $p$ ,  $\Omega$ , and the aspect ratio of the finite elements or the polynomial degree of piecewise polynomials under consideration.

The trace theorem [A, BS] for  $v \in W^{1,q}(\Omega)$  and  $1 \leq q \leq \infty$  reads

$$\|v\|_{L^q(\partial\Omega)} \lesssim \|v\|_{L^q(\Omega)} + |v|_{W^{1,q}(\Omega)}.$$

For a triangle  $K \in \mathcal{T}^h$  and for all  $v \in W^{1,q}(K)$ , this is

$$\|v\|_{L^q(\partial K)} \lesssim h_K^{-\frac{1}{q}} \|v\|_{L^q(K)} + h_K^{1-\frac{1}{q}} |v|_{W^{1,q}(K)}.$$

We need a quasi-norm version of the trace theorem for polynomials.

**Lemma 2.1** (Lemma 3.6 in [LY2]). *Let  $K \in \mathcal{T}^h$  and let  $v$  be a polynomial of degree  $\leq k$ . Then*

$$(2.1) \quad h_K \int_{\partial K} (|\nabla u_h| + |\nabla v|)^{p-2} |\nabla v|^2 dx \lesssim \int_K (|\nabla u_h| + |\nabla v|)^{p-2} |\nabla v|^2 dx.$$

*The generic constant depends only on  $k$  and the aspect ratio of the finite elements.*  $\square$

A display of elementary (but sometimes tricky) estimates in  $\mathbb{R}^n$  that play an essential role in our error analysis concludes this section.

**Lemma 2.2** (Main tools for quasi-norm from [LY1, BL3]).

(A) For all  $a, \xi, \eta \geq 0, \theta > 0$ , it holds that

$$(a + \xi)^{p-2} \xi \eta \leq \theta^{-\gamma} (a + \xi)^{p-2} \xi^2 + \theta (a + \eta)^{p-2} \eta^2,$$

where

$$\gamma = \begin{cases} 1 & \text{if } 1 < p \leq 2, \theta \in [1, \infty) \text{ or if } 2 < p < \infty, \theta \in (0, 1), \\ \frac{1}{p-1} & \text{if } 1 < p \leq 2, \theta \in (0, 1) \text{ or if } 2 < p < \infty, \theta \in [1, \infty). \end{cases}$$

(B) For all  $a, \xi, \eta \geq 0$ , and  $\delta > 0$ , it holds that

$$\xi \eta \leq \delta^{-\beta} (a^{p-1} + \xi)^{p'-2} \xi^2 + \delta (a + \eta)^{p-2} \eta^2,$$

where  $\beta$  is such that  $\delta^{-\beta} = \max\{\delta^{-1}, \delta^{1/(1-p)}\}$  and  $1/p + 1/p' = 1$ .

(C) For all  $\xi, \eta \in \mathbb{R}^n$  and  $a \geq 0$ , it holds that

$$(a + |\xi + \eta|)^{p-2} |\xi + \eta|^2 \lesssim (a + |\xi|)^{p-2} |\xi|^2 + (a + |\eta|)^{p-2} |\eta|^2.$$

The generic constant depends only on  $1 < p < \infty$ .

### 3. FINITE ELEMENT APPROXIMATION OF THE P-LAPLACIAN

In this section we consider the finite element approximation of (WP) and introduce some quasi-norms. Given  $f \in L^2(\Omega)$ , the weak formulation of the p-Laplacian reads (WP) seek  $u \in W_0^{1,p}(\Omega)$  with

$$(3.1) \quad a(u, v) = (f, v) \quad \text{for all } v \in W_0^{1,p}(\Omega).$$

Here and throughout this paper,

$$a(u, v) = \int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla v \, dx \quad \text{and} \quad (w, v) = \int_{\Omega} wv \, dx.$$

It is well established that there exists a unique solution to (WP).

Let  $T^h$  be a regular triangulation [Ci, BS] of  $\Omega$  into disjoint open regular triangles  $K$ , so that  $\bar{\Omega} = \bigcup_{K \in T^h} \bar{K}$ . Each element has at most one edge on  $\partial\Omega$ , and  $\bar{K}$  and  $\bar{K}'$  have either only one common vertex or a whole edge if  $K$  and  $K' \in T^h$ . Let  $h_K$  denote the maximum diameter of the element  $K$  in  $T^h$  and let  $\rho_K$  denote the diameter of the largest ball contained in  $K$ . We assume that there is a regularity constant  $R$  of  $T^h$ , independent of  $h$ , such that  $1 \leq \max_{K \in T^h} (h_K/\rho_K) \leq R$ . Let  $h = \max_{K \in T^h} h_K$ .

Associated with  $T^h$  is a finite dimensional subspace  $V^h$  of  $C(\Omega^h)$ , such that  $\chi|_K$  are linear functions for all  $\chi \in V^h$  and  $K \in T^h$  and

$$V_0^h = \{v \in V^h : v = 0 \text{ on } \partial\Omega\}.$$

The weak formula of the finite element approximation for (3.1) reads (WP<sup>h</sup>) seek  $u_h \in V_0^h$  with

$$(3.2) \quad a(u_h, v_h) = (f, v_h) \quad \text{for all } v_h \in V_0^h.$$

One of the key ideas in our approach is to introduce some quasi-norms to handle the degeneracy of the p-Laplacian in order to obtain sharp error bounds. We briefly introduce a quasi-norm and some relations between it and the standard Sobolev norms. Given  $v, w \in W^{1,p}(\Omega)$ , set

$$(3.3) \quad |v|_{(w,p)}^2 \equiv \int_{\Omega} |\nabla v|^2 (|\nabla w| + |\nabla v|)^{(p-2)} \, dx.$$

**Proposition 3.1.** (i) It holds that  $|v|_{(w,p)} \geq 0$ , and, when  $v \in W_0^{1,p}(\Omega)$ ,  $|v|_{(w,p)} = 0$  if and only if  $v = 0$ .

(ii) It holds that  $|v_1 + v_2|_{(w,p)} \lesssim |v_1|_{(w,p)} + |v_2|_{(w,p)}$  for any  $v_1, v_2 \in W^{1,p}(\Omega)$ .

(iii) Furthermore, for  $1 < p \leq 2$ , it holds that

$$(3.4) \quad |v|_{W^{1,p}(\Omega)} \lesssim (|w|_{W^{1,p}(\Omega)}, |v|_{W^{1,p}(\Omega)})|v|_{(w,p)} \text{ and } |v|_{(w,p)}^2 \leq |v|_{W^{1,p}(\Omega)}^p.$$

(iv) For  $2 \leq p < \infty$ ,  $s \in [2, p]$ ,  $r = s(2-p)/(2-s)$ , it holds that

$$(3.5) \quad |v|_{W^{1,p}(\Omega)}^p \leq |v|_{(w,p)}^2 \leq C(|w|_{W^{1,r}(\Omega)}, |v|_{W^{1,r}(\Omega)})|v|_{W^{1,s}(\Omega)}^2.$$

*Proof.* The conclusion (ii) can be proved with Lemma 2.2(C). The rest of the proposition can be shown as in [BL3].  $\square$

*Remark 3.1.* Throughout an a priori error analysis,  $w$  is chosen to be  $u$ , the solution of (WP). To ensure the computability of the a posteriori error estimators it is replaced  $u_h$  (or some postprocessed approximation of  $u$ ) [LY1, LY2]. A triangle inequality yields

$$|u - u_h|_{(u,p)} \lesssim |u - u_h|_{(u_h,p)} \lesssim |u - u_h|_{(u,p)}.$$

We shall simply write  $|\cdot|_{(u,p)}$  as  $|\cdot|_{(p)}$  when no confusion is likely to be caused.

*Remark 3.2.* The essential relations between the quasi-norm and the equation are reflected in the following inequalities. If  $u$  solves (WP) and  $v \in W^{1,p}(\Omega)$ , then

$$(3.6) \quad |u - v|_{(u,p)}^2 \lesssim a(u, u - v) - a(v, u - v).$$

For any  $\theta > 0$ ,  $v, w \in W^{1,p}(\Omega)$ , and  $\gamma$  from Lemma 2.2(A),

$$(3.7) \quad |a(u, w) - a(v, w)| \lesssim \theta^\gamma |u - v|_{(u,p)}^2 + \theta |w|_{(u,p)}^2.$$

It follows from (3.6)–(3.7) that, for any  $u, v \in W^{1,p}(\Omega)$ ,

$$a(u, v - u) - a(v, u - v) \lesssim |u - v|_{(u,p)}^2 \lesssim a(u, u - v) - a(v, u - v).$$

Thus the quasi-norm is naturally related to the total energy difference.

*Remark 3.3.* The relations (3.6)–(3.7) are important to prove the optimal a priori error bound in the quasi-norm [BL1, LB5]

$$|u - u_h|_{(p)}^2 \lesssim \min_{v_h \in V_0^h} |u - v_h|_{(p)}^2 \lesssim h^2$$

when  $u$  is smooth enough, where

$$|u - v|_{(p)}^2 = |u - v|_{(u,p)}^2 = \int_{\Omega} (|\nabla u| + |\nabla(u - v)|)^{p-2} |\nabla(u - v)|^2 dx.$$

Thus when  $1 < p \leq 2$ , one has the optimal a priori error bound in  $W^{1,p}$ ,

$$\|u - u_h\|_{W^{1,p}(\Omega)} \lesssim |u - u_h|_{(p)} \lesssim h.$$

*Remark 3.4.* In [LY1, LY2], the quasi-norm has been used to derive improved a posteriori error estimates for the p-Laplacian. For instance, let  $u_h$  be the finite element approximation of (3.1) and let  $1/p + 1/p' = 1$ . Then

$$\eta_1^2 + \eta_2^2 + \epsilon_1 \lesssim |u - u_h|_{(p)}^2 \lesssim \eta_1^2 + \eta_2^2 + \epsilon_2,$$

with higher order terms  $\varepsilon_1$  and  $\varepsilon_2$  and

$$\begin{aligned}
 \eta_1^2 &= \sum_K \int_K (|\nabla u_h|^{p-1} + h_K |f|)^{p'-2} h_K^2 |f|^2 dx, \\
 \eta_2^2 &= \sum_l \int_{K_l} (|\nabla u_h|^{p-1} + |A_l|)^{p'-2} A_l^2 dx, \\
 A_l &= ((|\nabla u_h|^{p-2} \nabla u_h)_{K_l^1} - (|\nabla u_h|^{p-2} \nabla u_h)_{K_l^2})n,
 \end{aligned}
 \tag{3.8}$$

where  $n$  is the unit normal vector on  $l = \bar{K}_l^1 \cap \bar{K}_l^2$  outward  $K_l^1$ .

4. NEW RESULTS ON QUASI-NORM APPROXIMATION

This section establishes essential approximation error estimates in the quasi-norm. Here we take a general approach so that the arguments can be applied to a more general class of degenerate systems of Remark 4.3.

**Definition 4.1.** For  $x, y \geq 0$  and  $1 < p < \infty$ , let

$$G(x, y) := \begin{cases} y^2(x + y)^{p-2} & \text{if } x + y > 0, \\ 0 & \text{if } x = y = 0. \end{cases}$$

*Remark 4.1.* Without further (explicit) notice, we shall use that  $G(x, y)$  is monotone increasing and convex with respect to the variable  $y$ .

First, we prove a quasi-norm version of the quotient theorem.

**Lemma 4.1.** *Let  $\Omega$  be a nonempty bounded convex open set in  $\mathbb{R}^2$ . Let  $1 < p < \infty$  and  $f \in (W^{1,p}(\Omega))^*$  with  $\mathbb{R} \cap \text{Ker } f = \{0\}$ , where  $\mathbb{R}$  is the space of functions constant on the domain  $\Omega$ . Then there exists a constant  $c_1 = c(f, p, \Omega)$  such that, for all  $a \in \mathbb{R}$ ,  $a \geq 0$ , and  $v \in W^{1,p}(\Omega)$ ,*

$$\int_{\Omega} G(a, |v|) dx \leq c_1 G(a, |f(v)|) + c_1 \int_{\Omega} G(a, |\nabla v|) dx.$$

*Proof.* We argue by contradiction and suppose that the lemma is false. Then there would exist a sequence  $v_j$  in  $W^{1,p}(\Omega)$  with  $\delta_j := \|v_j\|_{W^{1,q}(\Omega)} > 0$ ,  $q = \min\{2, p\}$ , and a sequence  $a_j$  of nonnegative real numbers such that

$$G(a_j, |f(v_j)|) + \int_{\Omega} G(a_j, |\nabla v_j|) dx \leq 1/j \int_{\Omega} G(a_j, |v_j|) dx
 \tag{4.1}$$

for all  $j \in \mathbb{N}$ . We observe in any case there exists a  $u \in W^{1,q}(\Omega)$  with

$$u_j := v_j/\delta_j \text{ satisfies } \|u_j\|_{W^{1,q}(\Omega)} = 1, \quad u_j \rightharpoonup u \text{ in } W^{1,q}(\Omega).
 \tag{4.2}$$

Here we have chosen a weak convergent subsequence with Banach Alaoglu's theorem. In the first case we suppose that there exists a constant  $\gamma$ ,  $0 < \gamma < \infty$ , with

$$a_j \leq \gamma \delta_j \quad \text{for } j = 1, 2, 3, \dots
 \tag{4.3}$$

At least we suppose (4.3) for a subsequence we have not relabelled. If  $1 < p \leq 2$ , then  $G(a, x) \leq x^p$  for all  $x \geq 0$ . Therefore, even without (4.3),

$$\int_{\Omega} G(a_j/\delta_j, |u_j|) dx \leq \|u_j\|_{L^p(\Omega)}^p \leq 1.$$

If  $2 \leq p$ , then  $G(\cdot, |u_j|)$  is monotone increasing. Hence, (4.2)–(4.3) yields

$$(4.4) \quad \int_{\Omega} G(a_j/\delta_j, |u_j|) dx \leq \int_{\Omega} (\gamma + |u_j|)^{p-2} |u_j|^2 dx \leq \|\gamma + |u_j|\|_{L^p(\Omega)}^p \leq (1 + \gamma|\Omega|^{1/p})^p.$$

Hence, for all  $1 < p < \infty$ ,  $\int_{\Omega} G(a_j/\delta_j, |u_j|) dx$  is bounded. A scaling of (4.1) then shows

$$(4.5) \quad \lim_{j \rightarrow \infty} \int_{\Omega} G(a_j/\delta_j, |\nabla u_j|) dx = \lim_{j \rightarrow \infty} G(a_j/\delta_j, |f(u_j)|) = 0.$$

If  $1 < p \leq 2$ , a Hölder inequality with exponents  $2/p$  and  $2/(2-p)$  leads to

$$(4.6) \quad \begin{aligned} \|\nabla u_j\|_{L^p(\Omega)}^p &= \int_{\Omega} |\nabla u_j|^p (a_j/\delta_j + |\nabla u_j|)^{p(p-2)/2} (a_j/\delta_j + |\nabla u_j|)^{p(2-p)/2} dx \\ &\leq \left( \int_{\Omega} G(a_j/\delta_j, |\nabla u_j|) dx \right)^{p/2} \left( \int_{\Omega} (a_j/\delta_j + |\nabla u_j|)^p dx \right)^{1-p/2}. \end{aligned}$$

The last factor is bounded as  $j \rightarrow \infty$  by (4.2)–(4.3) and the second to last factor tends to zero by (4.5). Again, for  $1 < p \leq 2$  (when  $G(\cdot, |f(u_j)|)$  is monotone decreasing), (4.5) shows that  $G(\gamma, |f(u_j)|)$  tends to zero and, hence, so does  $|f(u_j)|$ . Consequently,

$$(4.7) \quad \lim_{j \rightarrow \infty} \|\nabla u_j\|_{L^p(\Omega)} = \lim_{j \rightarrow \infty} |f(u_j)| = 0.$$

So far we established (4.7) for  $1 < p \leq 2$ . For  $2 < p < \infty$ ,  $|\nabla u_j|^p \leq G(a_j/\delta_j, |\nabla u_j|)$  and  $|f(u_j)|^p \leq G(a_j/\delta_j, |f(u_j)|)$  and so (4.5) implies (4.7) directly. From (4.7) we deduce a contradiction to (4.2): Since  $W^{1,q}(\Omega)$  is compactly embedded in  $L^q(\Omega)$  we have  $u_j \rightarrow u$  in  $L^q(\Omega)$ . With (4.7),  $u_j \rightarrow u$  in  $W^{1,q}(\Omega)$  and so  $\|u\|_{W^{1,q}(\Omega)} = 1$ . Conversely,  $u$  is constant (as  $\nabla u_j \rightarrow 0$  in  $L^q(\Omega)$ ). Since  $f$  is a bounded linear form,  $f(u_j) \rightarrow f(u)$  and  $f(u) = 0$ . Since  $u \in P_0(\Omega) \cap \text{Ker} f$ , we have  $u = 0$ . This contradiction with  $\|u\|_{W^{1,q}(\Omega)} = 1$  concludes the proof in case (4.3).

In the remaining second case we suppose that  $a_j/\delta_j$  is not bounded (even not for a subsequence). Hence,  $\lim_{j \rightarrow \infty} a_j/\delta_j = +\infty$ . One can assume that

$$(4.8) \quad \delta_j \leq \gamma a_j \text{ for } q = \min\{2, p\} \text{ and for } j = 1, 2, 3, \dots$$

for a constant  $\gamma$  (and at least for sufficiently large  $j$  which we have not relabelled). If  $1 < p \leq 2$ , we use  $(1 + \delta_j/a_j |u_j|)^{p-2} \leq 1$ . If  $2 \leq p < \infty$ , we use  $\delta_j/a_j \leq \gamma$ . This leads to

$$(4.9) \quad \begin{aligned} 1/j \int_{\Omega} (1 + \delta_j/a_j |u_j|)^{p-2} |u_j|^2 dx &\leq \begin{cases} \|u_j\|_{L^2(\Omega)}^2/j & \text{if } 1 < p \leq 2, \\ (\|u_j\|_{L^p(\Omega)} + \gamma|\Omega|^{1/p})^p/j & \text{if } 2 \leq p < \infty. \end{cases} \end{aligned}$$

Since  $q = \min\{2, p\}$  and  $\|u_j\|_{W^{1,q}(\Omega)} = 1$ , we conclude that (4.9) tends to zero as  $j \rightarrow \infty$  from embedding. Therefore, a scaling of (4.1) yields

$$(4.10) \quad \begin{aligned} \lim_{j \rightarrow \infty} \int_{\Omega} (1 + \delta_j/a_j |\nabla u_j|)^{p-2} |\nabla u_j|^2 dx &= \lim_{j \rightarrow \infty} (1 + \delta_j/a_j |f(u_j)|)^{p-2} |f(u_j)|^2 = 0. \end{aligned}$$

If  $2 \leq p < \infty$ , we directly deduce (4.7) for  $q = 2$  and finish the proof as in the first case since  $\|u_j\|_{H^1(\Omega)} = 1$ . If  $1 < p \leq 2$ , we argue with a Hölder inequality analogy to (4.6) and infer

$$\|\nabla u_j\|_{L^p(\Omega)}^2 \leq \int_{\Omega} (1 + \delta_j/a_j |\nabla u_j|)^{p-2} |\nabla u_j|^2 dx \left( \int_{\Omega} (1 + \delta_j/a_j |\nabla u_j|)^p dx \right)^{2-p}.$$

The last factor is bounded according to (4.8) and  $\|u_j\|_{W^{1,p}(\Omega)} = 1$ . This and (4.10) show (4.7) with  $p = q \leq 2$ . The proof is then finished as in the first case.  $\square$

*Remark 4.2.* Lemma 4.1 is employed in connection with a scaling argument. If we scale the domain  $\Omega$  from a reference size 1 to a patch-size  $h$ , the first term obtains the factor  $h^2$  from a change of variables while the last term in values  $ah |\nabla v|$  instead of  $|\nabla v|$ . With a different  $a$ , this yields

$$\int_{\Omega} G(a, v) dx \lesssim \int_{\Omega} G(a, h|\nabla v|) dx$$

for all  $v \in W^{1,p}(\Omega) \cap \text{Ker}(f)$  and  $h = \text{diam}(\Omega)$ ; the generic constant depends on the shape of  $\Omega$  but is  $h$ -independent.

Recal a weighted Clement-type interpolation on the finite element space  $V_0^h$ .

**Definition 4.2** ([Ca]). Let  $D$  be the set of nodes,

$$\Lambda = \{z \in D : z \cap \partial\Omega = \emptyset\}.$$

Given the nodal basis function  $\varphi_z$  of  $z$  in  $V^h$ , set  $\omega_z = \{x \subset \Omega : \varphi_z(x) > 0\}$ ,

$$\psi_z = \varphi_z / \psi \quad \text{and} \quad \psi = \sum_{z \in \Lambda} \varphi_z.$$

Then, for all  $v \in W_0^{1,p}(\Omega)$ , define the interpolation of  $v$  by

$$\pi v = \sum_{z \in \Lambda} v_z \varphi_z \in V_0^h, \quad v_z = \int_{\Omega} \psi_z v dx / \int_{\Omega} \varphi_z dx.$$

It is essential for later analysis to establish approximation error estimates in the quasi-norm for the operator  $\pi$ .

**Lemma 4.2.** For any  $1 < p < \infty$  and positive integers  $d$  and  $n$  there exists a constant  $c_2 = c(p, d, n)$  such that, for all  $a_1, a_2, \dots, a_n \in \mathbb{R}^d$ , it holds that

$$\sum_{j=1}^n \sum_{k=1}^{j-1} G(|a_j|, |a_j - a_k|) \leq c_2 \sum_{\ell=1}^{n-1} \min_{m=1, \dots, n} G(|a_m|, |a_{\ell+1} - a_{\ell}|).$$

*Proof.* Let  $\alpha := (a_1 + \dots + a_n)/n \in \mathbb{R}^d$  and  $b_j := a_j - \alpha \in \mathbb{R}^d$  so that  $b_1 + \dots + b_n = 0$ . Define

$$f(\alpha; b_1, \dots, b_n) := \sum_{j=1}^n \sum_{k=1}^{j-1} G(|\alpha + b_j|, |b_j - b_k|),$$

$$g(\alpha; b_1, \dots, b_n) := \sum_{\ell=1}^{n-1} \min_{m=1, \dots, n} G(|\alpha + b_m|, |b_{\ell+1} - b_{\ell}|).$$

Observe that  $g(\alpha, \cdot)$  is positive for nonzero arguments on

$$X := \{(b_1, \dots, b_n) \in \mathbb{R}^{d \times n} : b_1 + \dots + b_n = 0\},$$

since  $g(\alpha; b_1, \dots, b_n) = 0$  implies  $b_1 = b_2 = \dots = b_n$ . Let  $B := \{(b_1, \dots, b_n) \in X : |b_1|^2 + \dots + |b_n|^2 = 1\}$  denote the unit ball surface in  $X$ . Then, for any  $\beta \in \mathbb{R}^d$ ,

$$c(\beta) := \max_{(b_1, \dots, b_n) \in B} f(\beta; b_1, \dots, b_n) / g(\beta; b_1, \dots, b_n) < \infty,$$

since the denominator is positive and  $f(\alpha; \cdot), g(\alpha; \cdot)$  are continuous on the compact set  $B$ . The same argument shows

$$c_\infty := \max_{(b_1, \dots, b_n) \in X \setminus \{0\}} \sum_{j=1}^n \sum_{k=1}^{j-1} |b_j - b_k|^2 / \sum_{\ell=1}^{n-1} |b_{\ell+1} - b_\ell|^2 < \infty.$$

Note that  $\limsup_{|\beta| \rightarrow \infty} c(\beta) \leq c_\infty < \infty$  and so  $c(\beta)$  is a bounded continuous function in  $\beta \in \mathbb{R}^d$ . For all  $a_1, \dots, a_n \in \mathbb{R}^d$ , we have  $\alpha \in \mathbb{R}^d$ , and  $(b_1, \dots, b_n) \in X$  as above. Since  $f$  and  $g$  are positively homogeneous functions we have, for  $\lambda := (|b_1|^2 + \dots + |b_n|^2)^{1/2} > 0$ ,

$$\begin{aligned} f(\alpha; b_1, \dots, b_n) &= \lambda f(\alpha/\lambda; b_1/\lambda, \dots, b_n/\lambda) \\ &\lesssim \lambda g(\alpha/\lambda, b_1/\lambda, \dots, b_n/\lambda) \\ &\lesssim g(\alpha; b_1, \dots, b_n). \end{aligned} \quad \square$$

Let  $[w]_l := w|_{K_l^1} - w|_{K_l^2}$  along  $l = \bar{K}_l^1 \cap \bar{K}_l^2$ . Then, we have the following interpolation error estimates for the operator  $\pi$  in the quasi-norm.

**Lemma 4.3.** *Let  $\pi$  be the operator of Definition 4.2,  $1 < p < \infty$ ,  $u_h \in V_h$ ,  $v \in W_0^{1,p}(\Omega)$ , and  $K \in T^h$ . Set  $\mathcal{T}_z := \{K \in T^h : K \subseteq \bar{\omega}_z\}$ ,  $\cup \mathcal{E}_z := \cup \{\partial K : K \in \mathcal{T}_z\}$ , and let  $[\partial u_h / \partial n_\varepsilon]$  denote the jump of the discrete normal fluxes across inner element boundaries. Then it holds that*

$$\begin{aligned} (4.11) \quad &\int_K G(|\nabla u_h|, |v - \pi v|/h_K) dx + \int_K G(|\nabla u_h|, |\nabla(v - \pi v)|) dx \\ &\lesssim \sum_{z \in \Lambda \cap K} \left( \int_{\omega_z} G(|\nabla u_h|, |\nabla v|) dx + \min_{T \in \mathcal{T}_z} \int_{\cup \mathcal{E}_z} G(|\nabla u_h|_T, |[\partial u_h / \partial n_\varepsilon]|) ds \right) \end{aligned}$$

and (where  $\sum_{l \cap \partial \Omega = \emptyset}$  denotes a sum over all interior edges  $l$ )

$$\begin{aligned} (4.12) \quad &\sum_{K \in T^h} \int_K G(|\nabla u_h|, |v - \pi v|/h_K) dx + \int_\Omega G(|\nabla u_h|, |\nabla(v - \pi v)|) dx \\ &\lesssim \int_\Omega G(|\nabla u_h|, |\nabla v|) dx \\ &\quad + \sum_{l \cap \partial \Omega = \emptyset} \min_{\bar{K} \cap l \neq \emptyset} \int_K (|\nabla u_h| + |[\frac{\partial u_h}{\partial n}]_l|)^{p-2} |[\frac{\partial u_h}{\partial n}]_l|^2 dx. \end{aligned}$$

*Proof.* In the first step, we show that the first term of the left-hand side of (4.11) is bounded by the right-hand side. Fix  $a := \nabla u_h|_K$  and set  $v_z := (\pi v)(z)$  for all  $z \in D$ . Since  $(\psi_z)_{z \in \Lambda \cap K}$  is a partition of unity on  $K$ , and  $G$  satisfies a triangle inequality in the sense of Lemma 2.2(C),

$$\begin{aligned} (4.13) \quad &\int_K G(|\nabla u_h|, |v - \pi v|/h_K) dx = \int_K G(|a|, |\sum_{z \in \Lambda \cap K} (v\psi_z - v_z\varphi_z)|/h_K) dx \\ &\lesssim \sum_{z \in \Lambda \cap K} \int_K G(|a|, |v\psi_z - v_z\varphi_z|/h_K) dx. \end{aligned}$$

Since  $K \subseteq \bar{\omega}_z$ , we have, for any fixed  $z \in \Lambda \cap K$ ,

$$(4.14) \quad \int_K G(|a|, |v\psi_z - v_z\varphi_z|/h_K) dx \leq \int_{\omega_z} G(|a|, |v\psi_z - v_z\varphi_z|/h_K) dx.$$

In the first case, we suppose  $\psi_z = \varphi_z$ , i.e., all nodes in  $\bar{\omega}_z$  are free nodes. A scaled version of Lemma 4.1 with  $w := (v - v_z)/h_K$ ,  $f(w) = \int_{\omega_z} \varphi_z w dx$ , and  $\psi_z = \varphi_z$  and  $|\varphi_z| \leq 1$  yield

$$(4.15) \quad \begin{aligned} \int_{\omega_z} G(|a|, |v\psi_z - v_z\varphi_z|/h_K) dx &\leq \int_{\omega_z} G(|a|, |v - v_z|/h_K) dx \\ &\lesssim \int_{\omega_z} G(|a|, |\nabla v|) dx. \end{aligned}$$

In the remaining second case,  $\psi \neq 1$  on  $\omega_z$  and so,  $\partial\omega_z \cap \partial\Omega$  includes at least one outer edge  $E$ . With  $f(w) = \int_E w dx$ , we deduce from Lemma 4.1 that

$$\int_{\omega_z} G(|a|, |v\psi_z|/h_K) dx \leq \int_{\omega_z} G(|a|, |v|/h_K) dx \lesssim \int_{\omega_z} G(|a|, |\nabla v|) dx.$$

Since  $|v_z| \lesssim \int_{\omega_z} \psi_z v dx$  and  $G(|a|, \cdot/h_K)$  is convex, Jensen's inequality shows that

$$(4.16) \quad \begin{aligned} \int_{\omega_z} G(|a|, |v_z|/h_K) dx &\lesssim \int_{\omega_z} G(|a|, \int_{\omega_z} \psi_z v dx / h_K) dy \\ &\leq \int_{\omega_z} \int_{\omega_z} G(|a|, |\psi_z v|/h_K) dx dy = \int_{\omega_z} G(|a|, |\psi_z v|/h_K) dx. \end{aligned}$$

Here and throughout, the sign  $\int_{\omega_z}$  represents the integral average over  $\omega_z$ . The triangle inequality of Lemma 2.2(C), (4.15)–(4.16), the monotonicity in the second argument of  $G$ , and the inequalities  $0 \leq \varphi_z \leq 1, 0 \leq \psi \leq 1$  prove that

$$(4.17) \quad \begin{aligned} \int_{\omega_z} G(|a|, |v\psi_z - v_z\varphi_z|/h_K) dx &\lesssim \int_{\omega_z} G(|a|, |v\psi_z|/h_K) dx \\ &\quad + \int_{\omega_z} G(|a|, |v_z\varphi_z|/h_K) dx \lesssim \int_{\omega_z} G(|a|, |\nabla v|) dx. \end{aligned}$$

Notice that  $a + |\nabla u_h - a| \approx a + |\nabla u_h| \approx |\nabla u_h| + |a - \nabla u_h|$  and so

$$(4.18) \quad \begin{aligned} G(|a|, |\nabla v|) &\leq G(|a|, |\nabla v| + |a - \nabla u_h|) \\ &= (|a| + |a - \nabla u_h| + |\nabla v|)^{p-2} (|\nabla v| + |a - \nabla u_h|)^2 \\ &\lesssim (|\nabla u_h| + |\nabla v| + |a - \nabla u_h|)^{p-2} (|\nabla v| + |a - \nabla u_h|)^2 \\ &= G(|a|, |\nabla v| + |a - \nabla u_h|). \end{aligned}$$

The triangle inequality of Lemma 2.2(C) shows that

$$(4.19) \quad G(|a|, |\nabla v|) \lesssim G(|\nabla u_h|, |\nabla v|) + G(|\nabla u_h|, |a - \nabla u_h|) dx.$$

This, (4.15), and (4.17) result in

$$(4.20) \quad \begin{aligned} \int_{\omega_z} G(|a|, |v\psi_z - v_z\varphi_z|/h_K) dx \\ \lesssim \int_{\omega_z} G(|\nabla u_h|, |\nabla v|) dx + \int_{\omega_z} G(|\nabla u_h|, |a - \nabla u_h|) dx. \end{aligned}$$

This and Lemma 4.2 with  $a_j = \nabla u_h|_{K_j}$  for  $K_1, \dots, K_n \in T^h$  with  $K_1 \cup \dots \cup K_n = \bar{\omega}_z$  and  $\{K_1 \cap K_2, \dots, K_{n-1} \cap K_n\} = \{E \in \mathcal{E} : E \subseteq \bar{\omega}_z \setminus \partial\Omega\}$  proves that (4.13) is

bounded by the right-hand side of (4.11). The second step is to show that the second term on the left-hand side of (4.11) is bounded in this way as well. To this end, we let  $c$  be the integral mean of  $v$  on  $K$ . The triangle inequality of Lemma 2.2(C) shows that

$$(4.21) \quad \int_K G(|\nabla u_h|, |\nabla(v - \pi v)|) dx \\ \lesssim \int_K G(|\nabla u_h|, |\nabla v|) dx + \int_K G(|\nabla u_h|, |\nabla(\pi v - c)|) dx.$$

The first term on the right-hand side of (4.21) is already bounded as asserted. To estimate the second term, note that  $\pi v - c$  is an affine function on  $K$ . Then an inverse estimate shows that

$$(4.22) \quad |\nabla(\pi v - c)| \lesssim \int_K |\pi v - c| dx / h_K$$

( $\int$  represents the integral average). Jensen's inequality leads to

$$\int_K G(|\nabla u_h|, |\nabla(\pi v - c)|) dx \lesssim \int_K \int_K G(|\nabla u_h|, |\pi v - c|/h_K) dx dy \\ = \int_K G(|\nabla u_h|, |\pi v - c|/h_K) dx.$$

The triangle inequality of Lemma 2.2(C) yields

$$(4.23) \quad \int_K G(|\nabla u_h|, |\pi v - c|/h_K) dx \\ \lesssim \int_K G(|\nabla u_h|, |v - \pi v|/h_K) dx + \int_K G(|\nabla u_h|, |v - c|/h_K) dx.$$

The first term on the right-hand side of (4.23) is already shown to be bounded by the right-hand side of (4.11). The same conclusion for the second term follows from Lemma 4.1 with  $f(w) = \int_K w dx$  and  $\Omega = K$  as in (4.15):

$$(4.24) \quad \int_K G(|\nabla u_h|, |v - c|/h_K) dx \lesssim \int_K G(|\nabla u_h|, |\nabla v|) dx.$$

Then (4.21)–(4.24) imply that

$$(4.25) \quad \int_K G(|\nabla u_h|, |\nabla(v - \pi v)|) dx \\ \lesssim \int_K G(|\nabla u_h|, |\nabla v|) dx + \int_K G(|\nabla u_h|, |v - \pi v|/h_K) dx.$$

Hence the desired estimate of the second term on the left-hand side of (4.11) follows from (4.25) and the first step of the proof.  $\square$

The next lemma establishes a quasi-norm estimate for the inner product of a function and an interpolation error. Let  $h_z = \max_{K \subset \omega_z} h_K$  denote the maximal mesh-size in the patch  $\omega_z$ .

**Lemma 4.4.** For any  $\delta > 0, 1 < p, p' < \infty$  with  $1/p + 1/p' = 1$ ,  $u_h \in V_h$ ,  $v \in W_0^{1,p}(\Omega)$ , and  $f \in W^{1,p'}(\Omega)$  it holds that

$$\begin{aligned}
 \int_{\Omega} f(v - \pi v) dx &\leq C\delta \int_{\Omega} G(|\nabla u_h|, |\nabla v|) dx \\
 (4.26) \quad &+ C(\delta) \sum_{z \in \Lambda} \int_{\omega_z} (|\nabla u_h|^{p-1} + h_z^2 |\nabla f|)^{p'-2} h_z^4 |\nabla f|^2 dx \\
 &+ C\delta \sum_{z \in \Lambda} \min_{K \in \mathcal{T}_z} \int_{\cup \mathcal{E}_z} G(|\nabla u_h|_K, [\partial u_h / \partial n_{\varepsilon}]) ds.
 \end{aligned}$$

With  $\tilde{\eta}$  of Lemma 4.3, it holds that

$$\begin{aligned}
 \int_{\Omega} f(v - \pi v) dx &\leq C(\delta) \sum_{K \in \mathcal{T}^h} \int_K (|\nabla u_h|^{p-1} + h_K^2 |\nabla f|)^{p'-2} h_K^4 |\nabla f|^2 dx \\
 (4.27) \quad &+ C\delta \sum_{K \in \mathcal{T}^h} \int_K (|\nabla u_h| + |\nabla v|)^{p-2} |\nabla v|^2 dx + C\delta \tilde{\eta}^2.
 \end{aligned}$$

*Proof.* First note that  $\int (v\psi_z - v_z\varphi_z) dx = 0$ . Thus, with the integral mean  $f_z := \int_{\omega_z} f(x) dx$ ,

$$\begin{aligned}
 \int_{\Omega} f(v - \pi v) dx &= \sum_{z \in \Lambda} \int_{\Omega} f(v\psi_z - v_z\varphi_z) dx \\
 (4.28) \quad &= \sum_{z \in \Lambda} \int_{\omega_z} (f - f_z)h_z (v\psi_z - v_z\varphi_z)/h_z dx.
 \end{aligned}$$

Lemma 2.2(B) allows an estimate of the product and so

$$\begin{aligned}
 &\int_{\omega_z} (f - f_z)h_z (v\psi_z - v_z\varphi_z)/h_z dx \\
 (4.29) \quad &\leq \delta^{-\beta} \int_{\omega_z} (|a|^{p-1} + |f - f_z|h_z)^{p'-2} h_z^2 |f - f_z|^2 dx \\
 &+ \delta \int_{\omega_z} G(|a|, |v\psi_z - v_z\varphi_z|/h_z) dx.
 \end{aligned}$$

Here  $a$  is one of the discrete gradients  $|\nabla u_h|$  on  $\omega_z$ . Lemma 4.1 will be employed for  $f - f_z$  and the functional  $g(w) = \int_{\omega_z} w$  (so it vanishes for  $w := f - f_z$ ). Notice that  $a$  is replaced by  $|a|^{p-1}$  and  $p$  is replaced by  $p'$ . Then we obtain

$$\begin{aligned}
 (4.30) \quad \int_{\omega_z} (|a|^{p-1} + |f - f_z|h_z)^{p'-2} h_z^2 |f - f_z|^2 dx \\
 \lesssim \int_{\omega_z} (|a|^{p-1} + h_z^2 |\nabla f|)^{p'-2} h_z^4 |\nabla f|^2 dx.
 \end{aligned}$$

Arguing as in proving (4.14)–(4.17), we deduce from (4.29)–(4.30) that

$$\begin{aligned}
 (4.31) \quad \int_{\omega_z} (f - f_z)h_z (v\psi_z - v_z\varphi_z)/h_z dx \\
 \lesssim \delta \int_{\omega_z} G(|a|, |\nabla v|) dx + \delta^{-\beta} \int_{\omega_z} (|a|^{p-1} + h_z^2 |\nabla f|)^{p'-2} h_z^4 |\nabla f|^2 dx.
 \end{aligned}$$

So far,  $a$  is a constant vector on  $\omega_z$ . Depending on  $p'$ , we choose  $a$  so that  $|a|$  is minimal or maximal amongst  $(|\nabla u_h|_K : K \in \mathcal{T}_z)$ , and thus

$$\begin{aligned} \int_{\omega_z} (|a|^{p-1} + h_z^2 |\nabla f|)^{p'-2} h_z^4 |\nabla f|^2 dx \\ \leq \int_{\omega_z} (|\nabla u_h|^{p-1} + h_z^2 |\nabla f|)^{p'-2} h_z^4 |\nabla f|^2 dx. \end{aligned}$$

Arguing as in the first step of the proof of Lemma 4.3, we have

$$\begin{aligned} \int_{\omega_z} G(|a|, |\nabla v|) dx &\lesssim \int_{\omega_z} G(|\nabla u_h|, |\nabla v|) dx \\ &+ \min_{K \in \mathcal{T}_z} \int_{\cup \mathcal{E}_z} G(|\nabla u_h|_K, [\partial u_h / \partial n_\varepsilon]) ds. \end{aligned}$$

Thus the desired estimate follows from (4.31) and the above two inequalities.  $\square$

*Remark 4.3.* It follows from the above proofs that Lemmas 4.1–4.4 hold for any continuous function  $G(\cdot, \cdot)$  such that it is increasing (decreasing) as  $p \geq 2$  ( $p \leq 2$ ) in the first argument, and is convex and increasing in the second argument.

### 5. QUASI-NORM A POSTERIORI ERROR ESTIMATORS

In this section we construct a class of a posteriori error estimators based on gradient recovery and derive a posteriori error estimates.

**Definition 5.1.** For any  $v_h \in V^h$ , let its gradient recovery  $G_h v_h$  be defined by

$$G_h v_h = \sum_{z \in D} G_h v_h(z) \varphi_z \quad \text{with} \quad G_h v_h(z) = \sum_{j=1}^{J_z} \alpha_z^j (\nabla v_h)_{K_z^j}.$$

Here  $\cup_{j=1}^{J_z} \bar{K}_z^j = \bar{\omega}_z$ ,  $\varphi_z$ ,  $\omega_z$ ,  $D$  are defined in Definition 4.2, and  $0 \leq \alpha_z^j \leq 1$ ,  $j = 1, \dots, J_z$ , are such that  $\sum_{j=1}^{J_z} \alpha_z^j = 1$ .

*Remark 5.1.* There are several possible selections on  $\alpha_z^j$ . For example,

- (1)  $\alpha_z^j = \text{meas}(K_z^j) / \text{meas}(\omega_z)$ . That is,  $G_h u_h(z) = \int_{\omega_z} \nabla u_h / \text{meas}(\omega_z)$ .
- (2)  $\alpha_z^j = 1 / J_z$ . That is,  $G_h u_h(z) = (\sum_{j=1}^{J_z} (\nabla u_h)_{K_z^j}) / J_z$ .

Using the above gradient recovery, we can construct some a posteriori error estimators on any regular meshes, with upper and lower error bounds.

**Theorem 5.1.** *Let  $u$  and  $u_h$  be the solutions of (3.1) and (3.2), respectively. Let  $1 < p, p' < \infty$  with  $1/p + 1/p' = 1$  and  $f \in W^{1,p'}(\Omega)$ . Then,*

$$|u - u_h|_{(p)}^2 \lesssim \eta^2 + \epsilon^2,$$

where

$$\begin{aligned} (5.1) \quad \eta^2 &= \sum_{K \in \mathcal{T}^h} \int_K (|\nabla u_h| + |\nabla u_h - G_h u_h|)^{p-2} |\nabla u_h - G_h u_h|^2 dx, \\ \epsilon^2 &= \sum_{K \in \mathcal{T}^h} \int_K (|\nabla u_h|^{p-1} + h_K^2 |\nabla f|)^{p'-2} h_K^4 |\nabla f|^2 dx. \end{aligned}$$

*Proof.* Let  $e = u - u_h$  and let  $\pi e \in V_0^h$  be defined in Definition 4.2. It follows from (3.1), (3.2), and (3.6) that

$$\begin{aligned}
 c|u - u_h|_{(p)}^2 &\leq \int_{\Omega} (|\nabla u|^{p-2} \nabla u - |\nabla u_h|^{p-2} \nabla u_h) \nabla e \, dx \\
 (5.2) \qquad &= \int_{\Omega} (|\nabla u|^{p-2} \nabla u - |\nabla u_h|^{p-2} \nabla u_h) \nabla (e - \pi e) \, dx \\
 &= \int_{\Omega} f(e - \pi e) - \sum_{K \in T^h} \int_{\partial K} |\nabla u_h|^{p-2} \frac{\partial u_h}{\partial n} (e - \pi e) \, dx \\
 &= I_1 + I_2.
 \end{aligned}$$

Lemma 4.4 leads to

$$\begin{aligned}
 I_1 &\lesssim \sum_{K \in T^h} \int_K (|\nabla u_h|^{p-1} + h_K^2 |\nabla f|)^{p'-2} h_K^4 |\nabla f|^2 \, dx \\
 (5.3) \qquad &+ \delta_1 \sum_{K \in T^h} \int_K (|\nabla u_h| + |\nabla e|)^{p-2} |\nabla e|^2 \, dx \\
 &+ \delta_1 \sum_{l \cap \partial \Omega = \emptyset} \min_{\bar{K} \cap l \neq \emptyset} \int_K (|\nabla u_h| + |[\frac{\partial u_h}{\partial n}]_l|)^{p-2} |[\frac{\partial u_h}{\partial n}]_l|^2 \, dx \\
 &\lesssim \epsilon^2 + \delta_1 (|u - u_h|_{(p)}^2 \\
 &+ \sum_{l \cap \partial \Omega = \emptyset} \min_{\bar{K} \cap l \neq \emptyset} \int_K (|\nabla u_h| + |[\frac{\partial u_h}{\partial n}]_l|)^{p-2} |[\frac{\partial u_h}{\partial n}]_l|^2 \, dx).
 \end{aligned}$$

Define  $A_l$  as in (3.9). With Lemmas 2.1, 2.2, and 4.3, we infer that

$$\begin{aligned}
 I_2 &= - \sum_{K \in T^h} \int_{\partial K} |\nabla u_h|^{p-2} \frac{\partial u_h}{\partial n} (e - \pi e) \, dx \\
 &= - \sum_{l \cap \partial \Omega = \emptyset} \int_l A_l (e - \pi e) \, ds \\
 &\lesssim \sum_{l \cap \partial \Omega = \emptyset} \int_{K_l^1 \cup K_l^2} |A_l| (h_{K_l}^{-1} |e - \pi e| + |\nabla(e - \pi e)|) \, dx \\
 &\lesssim \delta_2^{-\beta} \sum_{l \cap \partial \Omega = \emptyset} \int_{K_l^1 \cup K_l^2} (|\nabla u_h|^{p-1} + |A_l|)^{p'-2} A_l^2 \, dx \\
 &+ \delta_2 \sum_{K \in T^h} \int_K (|\nabla u_h| + h_K^{-1} |e - \pi e|)^{p-2} h_K^{-2} |e - \pi e|^2 \, dx \\
 &+ \delta_2 \sum_{K \in T^h} \int_K (|\nabla u_h| + |\nabla(e - \pi e)|)^{p-2} |\nabla(e - \pi e)|^2 \, dx \\
 &\lesssim \delta_2^{-\beta} \sum_{l \cap \partial \Omega = \emptyset} \int_{K_l^1 \cup K_l^2} (|\nabla u_h|^{p-1} + |A_l|)^{p'-2} A_l^2 + \delta_2 |u - u_h|_{(p)}^2 \, dx \\
 &+ \delta_2 \sum_{l \cap \partial \Omega = \emptyset} \min_{\bar{K} \cap l \neq \emptyset} \int_K (|\nabla u_h| + |[\frac{\partial u_h}{\partial n}]_l|)^{p-2} |[\frac{\partial u_h}{\partial n}]_l|^2 \, dx.
 \end{aligned}$$

Noting that for all  $K$  with  $\bar{K} \cap l \neq \emptyset$ , it holds that

$$\begin{aligned}
 & \int_K (|\nabla u_h|^{p-1} + |A_l|)^{p'-2} A_l^2 dx \\
 &= \int_K (|\nabla u_h|^{p-1} + |[\nabla u_h]_l|)^{p'-2} [|\nabla u_h|^{p-2} \frac{\partial u_h}{\partial n}]_l^2 dx \\
 &\leq \int_K (|\nabla u_h|^{p-1} + |[\nabla u_h]_l|)^{p'-2} |[\nabla u_h]_l|^2 dx \\
 &\lesssim \int_K (|\nabla u_h|^{p-1} + (|\nabla u_h| + |[\nabla u_h]_l|)^{p-2} |[\nabla u_h]_l|)^{p'-2} \\
 &\quad \times (|\nabla u_h| + |[\nabla u_h]_l|)^{2(p-2)} |[\nabla u_h]_l|^2 dx \\
 &= \int_K Q (|\nabla u_h| + |[\nabla u_h]_l|)^{p-2} |[\nabla u_h]_l|^2 dx,
 \end{aligned}$$

where

$$\begin{aligned}
 Q &:= (a^{p-1} + (a+b)^{p-2}b)^{p'-2} (a+b)^{p-2} \\
 &= ((1+b/a)^{1-p} + 1/(1+a/b))^{p'-2}
 \end{aligned}$$

and

$$a := |\nabla u_h|_l, \quad b := |[\nabla u_h]_l|.$$

It is directly verified that  $Q \leq 2^{p'-2}$  for  $1 < p \leq 2 \leq p' < \infty$ . When  $1 < p' \leq 2 \leq p < \infty$  and  $a \leq b$ , it holds that

$$Q \leq (1+a/b)^{2-p'} \leq 2^{2-p'}.$$

Otherwise, when  $1 < p' \leq 2 \leq p < \infty$  and  $a > b$ ,

$$Q \leq (1+a/b)^{(1-p)(p'-2)} = (1+b/a)^{p-2} \leq 2^{p-2}.$$

Hence,  $Q \lesssim 1$ . Utilizing  $Q \lesssim 1$  above, we infer that

$$\begin{aligned}
 & \int_{K_l^1 \cup K_l^2} (|\nabla u_h|^{p-1} + |A_l|)^{p'-2} A_l^2 dx \\
 &\lesssim \int_{K_l^1 \cup K_l^2} (|\nabla u_h| + |[\nabla u_h]_l|)^{p-2} |[\nabla u_h]_l|^2 dx.
 \end{aligned}$$

It follows from Lemma 4.2 that

$$\begin{aligned}
 & \int_{K_l^1 \cup K_l^2} (|\nabla u_h| + |[\nabla u_h]_l|)^{p-2} |[\nabla u_h]_l|^2 dx \\
 &\lesssim \min_{\bar{K} \cap l \neq \emptyset} \int_K (|\nabla u_h| + |[\nabla u_h]_l|)^{p-2} |[\nabla u_h]_l|^2 dx.
 \end{aligned}$$

Moreover, note that  $G_h u_h$  is continuous on  $\Omega$ . It follows from Lemma 2.1 that for any edge  $l$  with  $l \cap \partial\Omega = \emptyset$ , we have

$$\begin{aligned}
 (5.4) \quad & \min_{\bar{K} \cap l \neq \emptyset} \int_K (|\nabla u_h| + |[\nabla u_h]_l|)^{p-2} |[\nabla u_h]_l|^2 dx \\
 & = \min_{\bar{K} \cap l \neq \emptyset} \int_K (|\nabla u_h| + |[\nabla u_h - G_h u_h]_l|)^{p-2} |[\nabla u_h - G_h u_h]_l|^2 dx \\
 & \lesssim \int_{K_l^1 \cup K_l^2} (|\nabla u_h| + |\nabla u_h - G_h u_h|)^{p-2} |\nabla u_h - G_h u_h|^2 dx.
 \end{aligned}$$

Then, it follows from (5.3)–(5.4) that

$$(5.5) \quad I_1 \lesssim \epsilon^2 + \delta_1 (|u - u_h|_{(p)}^2 + \eta^2),$$

$$(5.6) \quad I_2 \lesssim \eta^2 + \delta_2 (|u - u_h|_{(p)}^2 + \eta^2).$$

Then (5.1) follows from (5.2), (5.5)–(5.6), and  $\delta_1 = \delta_2 = c/(4C)$ . □

*Remark 5.2.* The above estimates require  $f \in W^{1,p'}(\Omega)$ , which may not be always true in an application. It follows from the proofs of Lemma 4.4 and Theorem 5.1 that the estimates still hold for the general case  $f \in L^{p'}(\Omega)$  as long as one replaces the term  $\epsilon^2$  in Theorem 5.1 by

$$\epsilon_1^2 = \sum_{K \in T^h} \int_K (|\nabla u|^{p-1} + |f - \bar{f}| h_K)^{p'-2} h_K^2 |f - \bar{f}|^2 dx$$

with  $\bar{f} = \int_K f dx / \text{meas}(K)$ .

**Theorem 5.2.** *Let  $u$  and  $u_h$  be the solutions of (3.1) and (3.2), respectively. Then we have the lower bound*

$$(5.7) \quad \eta \lesssim |u - u_h|_{(p)} + \epsilon^* \quad \text{with } \epsilon^* = \inf_{v_h^k \in V_h^k} |u - v_h^k|_{(p)},$$

$\eta$  defined in Theorem 5.1, the space  $P_k$  of polynomials of the  $k$ -degree, and

$$V_h^k = \{v_h^k \in C^1(\Omega) : \forall K \in T^h, v_h^k|_K \in P_k\}.$$

*Proof.* Set  $S_K = \bigcup_{\bar{K}' \cap \bar{K} \neq \emptyset} K'$  for any  $K \in T^h$ . Note that for each  $z \in D$ ,

$$G_h u_h(z) = \sum_{j=1}^{J_z} \alpha_z^j (\nabla u_h)_{K_z^j},$$

where  $\sum_{j=1}^{J_z} \alpha_z^j = 1$ . Then, on the element  $K$ ,

$$\begin{aligned}
 |\nabla u_h - G_h u_h| & = |(\nabla u_h)_K - \sum_{z \cap \bar{K} \neq \emptyset} \varphi_z (\sum_{j=1}^{J_z} \alpha_z^j (\nabla u_h)_{K_z^j})| \\
 & = | \sum_{z \cap \bar{K} \neq \emptyset} \varphi_z (\sum_{j=1}^{J_z} \alpha_z^j ((\nabla u_h)_K - (\nabla u_h)_{K_z^j})) | \\
 & \lesssim \sum_{K' \subset S_K} |(\nabla u_h)_K - (\nabla u_h)_{K'}|.
 \end{aligned}$$

Moreover,  $|\nabla u_h - \sigma_h \nabla u_h|$  is less than or equal to a convex combination of  $|(\nabla u_h)_K - (\nabla u_h)_K|$ . Since  $\sigma(x, y)$  is monotone increasing and convex in  $y$ , we have

$$\begin{aligned} \eta^2 &= \int_{\Omega} (|\nabla u_h| + |\nabla u_h - G_h u_h|)^{p-2} |\nabla u_h - G_h u_h|^2 dx \\ &\lesssim \sum_{K \in T^h} \sum_{K' \subset S_K} \int_K (|\nabla u_h| + |\nabla u_h - (\nabla u_h)_{K'}|)^{p-2} |\nabla u_h - (\nabla u_h)_{K'}|^2 dx. \end{aligned}$$

It follows from Lemma 4.2 that

$$\begin{aligned} \eta^2 &\lesssim \sum_{K \in T^h} \sum_{l \subset S_K} \min_{\bar{K} \cap l \neq \emptyset} \int_K (|\nabla u_h| + |[\nabla u_h]_l|)^{p-2} |[\nabla u_h]_l|^2 dx \\ &\lesssim \sum_{l \cap \partial\Omega = \emptyset} \min_{\bar{K} \cap l \neq \emptyset} \int_K (|\nabla u_h| + |[\nabla u_h]_l|)^{p-2} |[\nabla u_h]_l|^2 dx. \end{aligned}$$

It follows from Lemma 2.1 that, for any  $v_h^k \in V_h^k$ ,

$$\begin{aligned} \eta^2 &\lesssim \sum_{l \cap \partial\Omega = \emptyset} \min_{\bar{K} \cap l \neq \emptyset} \int_K (|\nabla u_h| + |[\nabla u_h]_l|)^{p-2} |[\nabla u_h]_l|^2 dx \\ &= \sum_{l \cap \partial\Omega = \emptyset} \min_{\bar{K} \cap l \neq \emptyset} \int_K (|\nabla u_h| + |[\nabla(u_h - v_h^k)]_l|)^{p-2} |[\nabla(u_h - v_h^k)]_l|^2 dx \\ &\lesssim \sum_{l \cap \partial\Omega = \emptyset} \int_{K_l^1 \cup K_l^2} (|\nabla u_h| + |\nabla(u_h - v_h^k)|)^{p-2} |\nabla(u_h - v_h^k)|^2 dx \\ &\lesssim \sum_{K \in T^h} \int_K (|\nabla u_h| + |\nabla(u_h - v_h^k)|)^{p-2} |\nabla(u_h - v_h^k)|^2 dx. \end{aligned}$$

Hence

$$\begin{aligned} \eta^2 &\lesssim \sum_{K \in T^h} \int_K (|\nabla u| + |\nabla(u_h - u)|)^{p-2} |\nabla(u_h - u)|^2 dx \\ &\quad + \sum_{K \in T^h} \int_K (|\nabla u| + |\nabla(v_h^k - u)|)^{p-2} |\nabla(v_h^k - u)|^2 dx \\ &\lesssim |u - u_h|_{(p)}^2 + (\epsilon^*)^2. \quad \square \end{aligned}$$

*Remark 5.3.* By combining Theorems 5.1 and 5.2, we have that  $|u - u_h|_{(p)} - \epsilon \lesssim \eta \leq |u - u_h|_{(p)} + \epsilon^*$ , where  $\eta$  is the a posteriori error estimator defined in Theorem 5.1,

$$\epsilon^2 = \sum_{K \in T^h} \int_K (|\nabla u_h|^{p-1} + h_K^2 |\nabla f|)^{p'-2} h_K^4 |\nabla f|^2 dx,$$

and, with  $V_h^k$  from Theorem 5.2,  $\epsilon^* = \inf_{v_h^k \in V_h^k} |u - v_h^k|_{(p)}$ ,  $f \in W^{1,p'}(\Omega)$ . Then,

$$\epsilon^2 \lesssim \begin{cases} (|u_h|_{W^{1,p}(\Omega)}^{2-p} + h^{p'-2} |f|_{1,p',\Omega}^{p'-2}) h^4 |f|_{W^{1,p'}(\Omega)}^2 \lesssim h^4 & \text{for } 1 < p < 2, \\ h^{2p'} |f|_{W^{1,p'}(\Omega)}^{p'} \lesssim h^{2p'} & \text{for } 2 \leq p. \end{cases}$$

Similarly, if  $u \in W^{3,p}(\Omega)$  and  $v_h^k$  is the piecewise quadratic interpolation of  $u$ ,

$$(\epsilon^*)^2 \lesssim \begin{cases} \|\nabla(u - v_h^k)\|_{L^p(\Omega)}^p \lesssim h^{2p} |u|_{W^{3,p}(\Omega)}^p & \text{for } 1 < p < 2, \\ \|\nabla(u - v_h^k)\|_{L^p(\Omega)}^2 \lesssim h^4 |u|_{W^{3,p}(\Omega)}^2 & \text{for } 2 \leq p. \end{cases}$$

The conclusions above can be proved under weaker conditions. For example, it can be proved that (see [LY2])

$$\begin{aligned}\epsilon^* &= o(h^2) \quad \text{if } u \in W^{1+2/p,p}(\Omega), 1 < p < 2 \text{ or } u \in W^{2,p}(\Omega), p \geq 2, \\ \epsilon &= o(h^2) \quad \text{if } f \in L^{p'}(\Omega), 1 < p < 2 \text{ or } f \in W^{2/p'-1,p'}(\Omega), p \geq 2.\end{aligned}$$

Furthermore, using the results in [EL], it can be shown that  $\epsilon = o(h^2)$  and  $\epsilon^* = o(h^2)$  if

$$\int_{\Omega} |\nabla u|^{p-2} |D^2 u|^2 dx < \infty,$$

and this condition is indeed achievable; see [EL] for details.

*Remark 5.4.* The idea used in constructing  $\eta$  can be generalized to obtain new a posteriori error estimators. For example, one could define the a posteriori error estimator

$$(\eta^*)^2 = \sum_{K \in T^h} \int_K (|\nabla u_h| + |\nabla(u_h - u_h^*)|)^{p-2} |\nabla(u_h - u_h^*)|^2 dx,$$

where  $u_h^*$  is the solution of a local approximation problem of (WP) as defined in the linear case [V2].

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