Averaging Techniques for A Posteriori Error Control in Finite Element and Boundary Element Analysis

Carsten Carstensen[1] and Dirk Praetorius[2]

[1] Department of Mathematics, Humboldt-Universität zu Berlin, Unter den Linden 6, D-10099 Berlin, Germany
cc@math.hu-berlin.de*
[2] Institute for Analysis and Scientific Computing, Vienna University of Technology, Wiedner Hauptstraße 8-10, A-1040 Vienna, Austria
Dirk.Praetorius@tuwien.ac.at

Summary. Averaging techniques for a posteriori error control are established for differential and integral equations within a unifying setting. The reliability and efficiency of the introduced estimator results from two grids $\mathcal{T}_h$ and $\mathcal{T}_H$ with different polynomial degrees for a smooth exact solution. The proofs are based on first order approximation operators and inverse estimates. For a finer and finer fine mesh $\mathcal{T}_h$, the estimator becomes asymptotically exact. The abstract framework is applicable to a finite element method for the Laplace equation, boundary element methods for Symm's and the hypersingular integral equation or transmission problems.

Key words: averaging, gradient recovery, a posteriori error analysis, finite element method, boundary element method, adaptive algorithm

1 Introduction

The striking simplicity of averaging techniques in a posteriori error control as well as their amazing accuracy in many numerical examples have made them an extremely popular tool in scientific computing over the last decade. Given a discrete stress or flux p_h and a post-processed (smoothened) approximation $\mathcal{A}p_h$, the a posteriori error estimator reads

$$\eta_A := \|p_h - \mathcal{A}p_h\|.$$

There is not even a need for an equation to compute the estimator η_A, and hence averaging techniques are easily employed everywhere. The most promi-

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nent example is occasionally named after Zienkiewicz and Zhu [ZZ], and also called *gradient recovery* but preferably called *averaging technique* in the literature. The most frequently quoted paper is [ZZ] for the $P1$ finite element method for some Laplace equation on some domain ω and some local averaging operator $\mathcal{A}p_h$ on the piecewise constant gradients $p_h = Du_h$ followed by linear interpolation. The estimator $\eta_A = \|p_h - \mathcal{A}p_h\|$ is then computed with respect to the norm $\|\cdot\|$ on $L^2(\Omega)$.
In the work of Zienkiewicz and Zhu [ZZ], there was no rigorous justification to interpret η_A as some computable approximation of the (rigorous) exact error $\|p-p_h\|$ with $p = Du$, but there arose quite some numerical evidence for that. The first mathematical justification of the error estimator η_A as a computable approximation of the (unknown) error $\|p - p_h\|$ involved the concept of superconvergence points. For highly structured meshes and a very smooth exact solution p, the error $\|p - \mathcal{A}p_h\|$ of the post-processed approximation $\mathcal{A}p_h$ may be (much) smaller than the error $\|p - p_h\|$ of the given p_h. Under the assumption that $\|p - \mathcal{A}p_h\|$ is sufficiently small in relative terms, written $\|p - \mathcal{A}p_h\| = \text{h.o.t} =$ higher-order terms, the triangle inequality immediately verifies reliability, i.e.,

$$\|p - p_h\| \le C_{\text{rel}}\, \eta_A + \text{ h.o.t.},$$

and efficiency, i.e.,

$$\eta_A \le C_{\text{eff}}\, \|p - p_h\| + \text{ h.o.t.},$$

of the averaging error estimator η_A (even with $C_{\text{rel}} = C_{\text{eff}} = 1$). However, the required assumptions on the symmetry of the mesh and the smoothness of the solution essentially contradict the use of adaptive grid refinement when p is singular. Moreover, the proper treatment of boundary conditions remains unclear.

The first mathematical verification by Rodriguez on reliability of η_A on unstructured grids has been indicated in the literature [R1, R2, N, BR] but was not mentioned in the (otherwise comprehensive) works [AO, BS, EEHJ, V]. The first author was unaware of Rodriguez's result [R2] when he started to work on the mathematical justification [CV] that ended in the surprizing and new conclusion that, in fact, all averaging techniques are reliable [BC1, BC2, C1, C2, CA, CF1, CF2].

A corresponding technique for the boundary element method was initiated with extraction and recovery techniques in [WSS], [SWe], [SchMW], [SchHW], [SSW] and was proposed thereafter in a small series of works of the two authors [CP1, CP2] and in [FP]. In the latter works, an approximation $\mathcal{A}p_h$ is computed as some best approximation of p_h based on a higher-order spline space on some coarser mesh. For some smooth exact solution, the resulting approximation error is of higher order. The corresponding error estimator is therefore efficient. Reliability follows provided the quotient of the mesh-sizes is sufficiently small. These two arguments, called approximation assumption (AA) and discrete property (DP), allow a unified analysis of reliability and efficiency of η_A.

This paper links the two discretization methods, namely the finite element method and the boundary element method, in that there is one abstract setting provided in which an averaging scheme is seen to be reliable and efficient without any reference to some saturation assumption or superconvergence. The paper is roughly organized in two mayor parts: In Section 2–4, we provide and analyze the analytical setting for our averaging method, while the remaining Sections 5–8 of the paper discuss concrete applications. Namely, in Section 2 we state and prove our abstract main result in Theorem 2.1, which is commented in Section 3. The essential condition for Theorem 2.1 is a discrete property (DP). We stress the difference of (DP) and a saturation assumption and remark on further generalizations of Theorem 2.1. In Section 4, the essential condition is studied in detail and characterized as some strengthened Cauchy inequality of related spaces. Section 5 considers the introduced averaging technique for the finite element method for a model example. Section 6 is an overview of a recent work [CP1] on averaging for Symm's integral equation. In Section 7, we treat the hypersingular integral equation following [CP2, FP]. Finally, the last application of our abstract analysis concerns the boundary integral formulation of a transmission problem in Section 8.

2 Abstract Setting

We consider the abstract framework of the Lax-Milgram lemma with a finite dimensional subspace $\mathcal{S}_h$ of a real Hilbert space $\mathcal{H}$ with corresponding norm $\|\cdot\|_{\mathcal{H}}$. Let $\langle\!\langle \cdot\,,\cdot \rangle\!\rangle$ be an elliptic and bounded (but possibly non-symmetric) bilinear form on $\mathcal{H}$, i.e., there are constants $0 < C_{\rm ell} \le C_{\rm bd}$ such that

$$C_{\rm ell}\|u\|_{\mathcal{H}}^2 \le \langle\!\langle u\,, u \rangle\!\rangle \quad \text{and} \quad \langle\!\langle u\,, v \rangle\!\rangle \le C_{\rm bd}\|u\|_{\mathcal{H}}\|v\|_{\mathcal{H}} \quad \text{for all } u, v \in \mathcal{H}. \tag{1}$$

The (linear) Galerkin projection $\mathbb{G}_h : \mathcal{H} \to \mathcal{S}_h$ is characterized by the Galerkin orthogonality

$$\langle\!\langle v - \mathbb{G}_h v\,, v_h \rangle\!\rangle = 0 \quad \text{for all } v_h \in \mathcal{S}_h \text{ and } v \in \mathcal{H}. \tag{2}$$

An immediate consequence is the quasi-optimal convergence, also known as Céa's lemma:

$$\|v - \mathbb{G}_h v\|_{\mathcal{H}} \le (C_{\rm bd}/C_{\rm ell}) \min_{v_h \in \mathcal{S}_h} \|v - v_h\|_{\mathcal{H}} \quad \text{for all } v \in \mathcal{H}. \tag{3}$$

Given an unknown solution $u \in \mathcal{H}$ for a prescribed right-hand side $f = \langle\!\langle u\,, \cdot \rangle\!\rangle \in \mathcal{H}^*$, the discrete solution $u_h := \mathbb{G}_h u$ is computed. In order to approximate the energy norm of the (unknown) error

$$e := u - u_h, \tag{4}$$

we are given a second finite-dimensional subspace $\mathcal{S}_H$ of $\mathcal{H}$. Then, the a posteriori error estimator for $\|u - u_h\|_{\mathcal{H}}$ reads

$$\eta_M := \min_{v_H \in \mathcal{S}_H} \|u_h - v_H\|_{\mathcal{H}}. \tag{5}$$

The justification below is based on one approximation assumption (AA) and some discrete property (DP) of $\mathcal{S}_h$ and $\mathcal{S}_H$ where, in applications below, $\mathcal{S}_h$ corresponds to a lower polynomial degree ansatz but a finer mesh when compared to $\mathcal{S}_H$, and u is smooth. Moreover, as the triangulation $\mathcal{T}_h$ corresponding to $\mathcal{S}_h$ will be a uniform refinement of the triangulation $\mathcal{T}_H$ corresponding to $\mathcal{S}_H$, we assume that $\mathcal{S}_h$ and $\mathcal{S}_H$ are linked through the mesh-sizes h and H:

$$\delta_{hH} := \min_{v_H \in \mathcal{S}_H} \|u - v_H\|_{\mathcal{H}} / \min_{v_h \in \mathcal{S}_h} \|u - v_h\|_{\mathcal{H}} = o(1), \tag{AA}$$

$$q := \max_{v_H \in \mathcal{S}_H \setminus \{0\}} \min_{v_h \in \mathcal{S}_h} \frac{\|v_H - v_h\|_{\mathcal{H}}}{\|v_H\|_{\mathcal{H}}} < C_{\mathrm{ell}}/C_{\mathrm{bd}}. \tag{DP}$$

Theorem 2.1. *With the notation from* (AA) *and under assumption* (DP) *there holds*

$$\eta_M/(1+\delta_{hH}) \le \|e\|_{\mathcal{H}} \le C_{\mathrm{rel}}(\eta_M + \min_{v_H \in \mathcal{S}_H} \|u - v_H\|_{\mathcal{H}}) \tag{6}$$

with

$$C_{\mathrm{rel}} := C_{\mathrm{bd}}/(C_{\mathrm{ell}} - qC_{\mathrm{bd}}). \tag{7}$$

Proof. The lower estimate (efficiency of η_M) is an immediate consequence of the triangle inequality: For any $v_H \in \mathcal{S}_H$, there holds

$$\eta_M \le \|e\|_{\mathcal{H}} + \|u - v_H\|_{\mathcal{H}}.$$

A passage of v_H to the minimum in (AA) yields

$$\eta_M \le \|e\|_{\mathcal{H}} + \delta_{hH} \min_{v_h \in \mathcal{S}_h} \|u - v_h\|_{\mathcal{H}} \le \|e\|_{\mathcal{H}}(1+\delta_{hH}).$$

This establishes efficiency of η_M. To prove the reliability of η_M, let $e_H \in \mathcal{S}_H$ be the best approximation of e, i.e.

$$\|e - e_H\|_{\mathcal{H}} = \min_{v_H \in \mathcal{S}_H} \|e - v_H\|. \tag{8}$$

By the definition of q in the discrete property (DP), there holds

$$\min_{v_h \in \mathcal{S}_h} \|e_H - v_h\|_{\mathcal{H}} \le q\|e_H\|_{\mathcal{H}}.$$

The Galerkin orthogonality of $\mathbb{G}_h$ and the boundedness of the bilinear form $\langle\!\langle \cdot\,,\cdot \rangle\!\rangle$ followed by the aforementioned estimate lead to

$$\langle\!\langle e\,, e_H\rangle\!\rangle = \min_{v_h\in\mathcal{S}_h}\langle\!\langle e\,, e_H - v_h\rangle\!\rangle \le q\,C_{\mathrm{bd}}\|e\|_{\mathcal{H}}\|e_H\|_{\mathcal{H}}.$$

Combining this with the ellipticity and boundedness of $\langle\!\langle\cdot\,,\cdot\rangle\!\rangle$, we obtain

$$C_{\mathrm{ell}}\|e\|_{\mathcal{H}}^2 \le \langle\!\langle e\,,e\rangle\!\rangle = \langle\!\langle e\,, e-e_H\rangle\!\rangle + \langle\!\langle e\,, e_H\rangle\!\rangle \le C_{\mathrm{bd}}\|e\|_{\mathcal{H}}\big(\|e-e_H\|_{\mathcal{H}} + q\|e_H\|_{\mathcal{H}}\big).$$

Now, the stability estimate $\|e_H\|_{\mathcal{H}} \le \|e\|_{\mathcal{H}}$ proves

$$\|e\|_{\mathcal{H}} \le \frac{C_{\mathrm{ell}}^{-1}C_{\mathrm{bd}}}{1-qC_{\mathrm{ell}}^{-1}C_{\mathrm{bd}}}\,\|e-e_H\|_{\mathcal{H}} = C_{\mathrm{rel}}\min_{v_H\in\mathcal{S}_H}\|e-v_H\|_{\mathcal{H}}.$$

If u_H and u_{hH} denote the best approximations of u resp. u_h in $\mathcal{S}_H$, the special choice of $v_H = u_H - u_{hH}$ and a triangle inequality yield

$$\|e\|_{\mathcal{H}} \le C_{\mathrm{rel}}(\|u-u_H\|_{\mathcal{H}} + \|u_{hH}-u_h\|_{\mathcal{H}}) = C_{\mathrm{rel}}(\min_{v_H\in\mathcal{S}_H}\|u-v_H\|_{\mathcal{H}} + \eta_M).$$

This concludes the proof of the reliability. □

3 Comments

Some remarks are in order before a list of applications enlightens the abstract results of the preceeding chapter.

3.1. Efficiency and Reliability. The discrete property (DP) is *not* necessary for efficiency of η_M. The reliability depends essentially on the discrete property (DP) in that, up to some approximation error

$$\text{h.o.t.} := \min_{v_H\in\mathcal{S}_H}\|u-v_H\|_{\mathcal{H}},$$

there holds reliability in the sense of

$$\|e\|_{\mathcal{H}} \le C_{\mathrm{rel}}(\eta_M + \text{h.o.t.}).$$

However, this is reasonable only if h.o.t. $\sim \delta_{hH}\|e\|_{\mathcal{H}}$ is indeed of higher order. In fact, there holds

$$\|e\|_{\mathcal{H}} \le C_{\mathrm{rel}}(\eta_M + \delta_{hH}\|e\|_{\mathcal{H}}).$$

Then, for $\delta_{hH} < C_{\mathrm{rel}}^{-1}$, there holds

$$\|e\|_{\mathcal{H}} \le C_{\mathrm{rel}}/(1-\delta_{hH}C_{\mathrm{rel}})\,\eta_M.$$

3.2. Constants in the Symmetric Case. In the important case that the bilinear form $\langle\!\langle\cdot\,,\cdot\rangle\!\rangle$ is symmetric, it is a scalar product. The induced norm $|\!|\!|v|\!|\!| := \langle\!\langle v\,,v\rangle\!\rangle^{1/2}$ is an equivalent Hilbert norm on $\mathcal{H}$. Moreover, $\mathbb{G}_h$ is the orthogonal projection onto $\mathcal{S}_h$ with respect to $\langle\!\langle \cdot\,,\cdot \rangle\!\rangle$. Then, (3) holds with $(C_{\rm bd}/C_{\rm ell})^{1/2}$ replacing $C_{\rm bd}/C_{\rm ell}$, and $\mathbb{G}_h$ is characterized by the best approximation property $|\!|\!| v - \mathbb{G}_h v |\!|\!| = \min_{v_h \in \mathcal{S}_h} |\!|\!| v - v_h |\!|\!|$ for all $v \in \mathcal{H}$.

In the symmetric case, one usually states (6) with respect to the energy norm $\|\cdot\|_{\mathcal{H}} = |\!|\!| \cdot |\!|\!|$, i.e. $C_{\rm bd} = 1 = C_{\rm ell}$. Asymptotic exactness of η_M then follows for $q \to 0$ in the sense of $C_{\rm rel} \to 1$. Moreover, the reliability constant $C_{\rm rel} = 1/(1-q)$ from (7) can be improved to $C_{\rm rel} = 1/(1-q^2)^{1/2}$ by the following refined stability estimate: Using the symmetry of orthogonal projections and the same arguments as in the proof of Theorem 2.1, we obtain

$$|\!|\!| e_H |\!|\!|^2 = \langle\!\langle e_H\,, e_H \rangle\!\rangle = \langle\!\langle e\,, e_H \rangle\!\rangle = \min_{v_h \in \mathcal{S}_h} \langle\!\langle e\,, e_H - v_h \rangle\!\rangle \le q |\!|\!| e |\!|\!| \, |\!|\!| e_H |\!|\!|.$$

This implies the refined stability estimate $|\!|\!| e_H |\!|\!| \le q |\!|\!| e |\!|\!|$. Together with the Pythagoras theorem, there holds

$$|\!|\!| e |\!|\!|^2 = |\!|\!| e - e_H |\!|\!|^2 + |\!|\!| e_H |\!|\!|^2 \le |\!|\!| e - e_H |\!|\!|^2 + q^2 |\!|\!| e |\!|\!|^2.$$

This yields $|\!|\!| e |\!|\!| \le |\!|\!| e_H |\!|\!| / (1-q^2)^{1/2}$, and we obtain the reliability of η_M with the improved constant $C_{\rm rel} = 1/(1-q^2)^{1/2}$.

3.3. Remarks on the Saturation Assumption. Assumption (DP) is just a definition of δ_{hH} with the possible interpretation discussed in Section 3.1. A much stronger statement is the *saturation assumption* of the form

$$\delta_{hH} = |\!|\!| u - \mathbb{G}_H u |\!|\!| / |\!|\!| e |\!|\!| \le C_{\rm sat} < 1 \tag{SA}$$

in the symmetric case $\|\cdot\|_{\mathcal{H}} = |\!|\!| \cdot |\!|\!|$ etc. of the preceding subsection. Recall that $\mathbb{G}_H$ denote the Galerkin projection onto $\mathcal{S}_H$. With $u_H := \mathbb{G}_H u$, a triangle inequality for $e = u - u_H + u_H - u_h$ plus (SA) leads to the reliable a posteriori error estimate

$$|\!|\!| e |\!|\!| \le |\!|\!| u_h - u_H |\!|\!| / (1 - C_{\rm sat})$$

for the different hierarchical estimator $|\!|\!| u_h - u_H |\!|\!|$. It has been the starting point of our analysis to avoid a strong assumption on the actual size of δ_{hH} like (SA) because it is hard to check in practise.

3.4. Verification of Assumption (DP). This subsection outlines the arguments sufficient for (DP) in an abstract (and non-local) framework. Examples follow in the remaining applications of this paper. For an appropriate seminorm $|\cdot|$ and the mesh-size parameter $H > 0$ associated with $\mathcal{S}_H$, an inverse estimate is of the form

$$|v_H| \le c_{\rm inv} H^{-\alpha} \|v_H\|_{\mathcal{H}} \quad \text{for all } v_H \in \mathcal{S}_H.$$

The exponent $\alpha > 0$ depends only on the energy (Sobolev) space, e.g., $\mathcal{H} = H^\alpha$ or $\mathcal{H} = H^{-\alpha}$. Moreover, $|\cdot|$ may allow an approximation estimate of the form

$$\min_{v_h \in \mathcal{S}_h} \|v_H - v_h\|_{\mathcal{H}} \le c_{\rm apx} h^\alpha |v_H| \quad \text{for all } v_H \in \mathcal{S}_H.$$

The combination of the two estimates yields

$$q := \max_{v_H \in \mathcal{S}_H \setminus \{0\}} \min_{v_h \in \mathcal{S}_h} \frac{\|v_H - v_h\|_{\mathcal{H}}}{\|v_H\|_{\mathcal{H}}} \le c_{\rm apx} c_{\rm inv} (h/H)^\alpha.$$

Hence, for any mesh-size h sufficiently small relative to H, (DP) follows.

3.5. Other Averaging Techniques. Under assumptions (AA)–(DP), we obtain reliable error estimators η_A whenever we replace the minimum of the best approximation by an *arbitrary* operator $\mathcal{A}_H : \mathcal{H} \to \mathcal{S}_H$,

$$\eta_A := \|u_h - \mathcal{A}_H u_h\|_{\mathcal{H}} \ge \min_{v_H \in \mathcal{S}_H} \|u_h - v_H\|_{\mathcal{H}} =: \eta_M. \tag{9}$$

Thus, *each averaging technique yields a reliable error estimator* [BC1]. Clearly, the efficiency of η_A is some further property of the chosen operator $\mathcal{A}_H$. According to Céa's lemma (3), the Galerkin projection $\mathcal{A}_H = \mathbb{G}_H$ always leads to an efficient and reliable error estimator since

$$(C_{\rm ell}/C_{\rm bd})\, \|v - \mathbb{G}_H v\|_{\mathcal{H}} \le \min_{v_H \in \mathcal{S}_H} \|v - v_H\|_{\mathcal{H}} \le \|v - \mathbb{G}_H v\|_{\mathcal{H}}.$$

3.6. Generalizations. Theorem 2.1 can be generalized in several ways. In the following, we give some simple examples, for which the analysis from Section 2 also works: (i) For the Hilbert space $\mathcal{H}$, there holds $e_H = u_H - u_{hH}$ for the best approximations in the proof of Theorem 2.1. However, the linearity of the best approximation is not needed, and the argument remains valid in the case that $\mathcal{H}$ only is a reflexive Banach space: There still holds the Lax-Milgram lemma, and the best approximation problem (8) still allows for a (in general non-unique) solution e_H. Finally, a triangle inequality proves stability $\|e_H\|_{\mathcal{H}} \le 2\|e\|_{\mathcal{H}}$. We must therefore assume $2qC_{\rm ell}^{-1}C_{\rm bd} < 1$ in (DP) and are led to reliability with $C_{\rm rel} = 2C_{\rm bd}/(C_{\rm ell} - 2qC_{\rm bd})$.
(ii) Theorem 2.1 also holds when we consider weakly non-linear problems. More precisely, let $A : \mathcal{H} \to \mathcal{H}^*$ be a uniformly monotone and Lipschitz continuous operator on the Hilbert space $\mathcal{H}$, i.e. there holds, for all $u, v \in \mathcal{H}$,

$$C_{\rm ell}\|u - v\|_{\mathcal{H}}^2 \le \langle Au - Av\,, u - v\rangle_{\mathcal{H}^* \times \mathcal{H}} \quad \text{and} \quad \|Au - Av\|_{\mathcal{H}^*} \le C_{\rm bd}\|u - v\|_{\mathcal{H}},$$

where $\langle\cdot\,,\cdot\rangle_{\mathcal{H}^* \times \mathcal{H}}$ denote the duality brackets. Also in this context, there holds the Lax-Milgram lemma. The (nonlinear) Galerkin projection $\mathbb{G}_h : \mathcal{H} \to \mathcal{S}_h$ is characterized by the Galerkin orthogonality

$$\langle Av - A(\mathbb{G}_h v)\,, v_h\rangle_{\mathcal{H}^* \times \mathcal{H}} = 0 \quad \text{for all } v_h \in \mathcal{S}_h \text{ and } v \in \mathcal{H}.$$

There still holds Céa's lemma (3), and we prove Theorem 2.1 with the same techniques.
(iii) A generalization of our averaging method in the context of the FEM-BEM coupling and saddle point problems which allow an LBB condition is slightly more involved and shall therefore appear elsewhere [CP3].

4 Characterizations of Discrete Property (DP) in Hilbert Spaces

In this section, let V and W be closed subspaces of the real Hilbert space $\mathcal{H}$ and let $V^\perp$ denote the orthogonal complement of V,

$$V^\perp := \{x \in \mathcal{H} \,:\, \forall v \in V \quad \langle x\,, v\rangle_{\mathcal{H}} = 0\}.$$

The main focus is on the uniform estimate

$$\min_{v\in V} \|v - w\|_{\mathcal{H}} \le c\|w\|_{\mathcal{H}} \quad \text{for all } w \in W. \tag{10}$$

Obviously, there holds $c \le 1$, and we discuss the case of $c < 1$ in the following. This plus the optimal constant is characterized in Theorem 4.1 in terms of

$$\gamma_{V^\perp,W} := \sup_{v^\perp\in V^\perp\setminus\{0\}} \sup_{w\in W\setminus\{0\}} \frac{\langle v^\perp\,, w\rangle_{\mathcal{H}}}{\|v^\perp\|_{\mathcal{H}}\|w\|_{\mathcal{H}}}$$

and

$$q_{V,W} := \sup_{w\in W\setminus\{0\}} \min_{v\in V} \frac{\|v - w\|_{\mathcal{H}}}{\|w\|_{\mathcal{H}}}.$$

Notice that $q_{\mathcal{S}_H,\mathcal{S}_h}$ is called q in the discrete property (DP) of Section 2. The estimate $\gamma_{V^\perp,W} < 1$ is known as *strengthened Cauchy inequality* between $V^\perp$ and W. (In fact $0 \le \cos(\sphericalangle(V^\perp, W)) := \gamma_{V^\perp,W} \le 1$ defines the angle $\sphericalangle(V^\perp, W)$ between the spaces $V^\perp$ and W.)
The following result, which is essentially taken from [B], states that the optimal constant in (10) equals $c = q_{V,W} = \gamma_{V^\perp,W}$ and the estimates (ii)-(iv) are in fact equivalent characterizations of $c < 1$.

Theorem 4.1. *There holds $q_{V,W} = \gamma_{V^\perp,W} \le 1$, and for any constant $c \ge 0$ with $c < 1$ the assertions* (i), (ii), (iii), (iv) *are pairwise equivalent.*
(i) $\gamma_{V^\perp,W} = q_{V,W} \le c$,
(ii) *there holds* $\sqrt{1 - c^2}\, \|v^\perp\|_{\mathcal{H}} \le \min_{w\in W} \|v^\perp - w\|_{\mathcal{H}}$ *for all* $v^\perp \in V^\perp$,
(iii) *there holds* $\sqrt{(1 - c^2)/2}\, (\|v^\perp\|_{\mathcal{H}} + \|w\|_{\mathcal{H}}) \le \|v^\perp + w\|_{\mathcal{H}}$ *for all* $(v^\perp, w) \in V^\perp \times W$,
(iv) *there holds* $\min_{v\in V} \|v - w\|_{\mathcal{H}} \le c\|w\|_{\mathcal{H}}$ *for all* $w \in W$.

Proof. The equivalence of $\gamma_{V^\perp,W} \le c < 1$ with (ii) and (iii), respectively, can be found in [B, Lemma 3.1], where V is substituted by $V^\perp$. The equivalence of $q_{V,W} \le c$ and (iv) is obvious since $q_{V,W}$ is, by definition, the optimal constant in (iv). Thus, it only remains to prove the equality $\gamma_{V^\perp,W} = q_{V,W}$: Given $v^\perp \in V^\perp$, $v \in V$, and $w \in W$, there holds $\langle v^\perp\,, w\rangle_{\mathcal{H}} = \langle v^\perp\,, w - v\rangle_{\mathcal{H}} \le \|v^\perp\|_{\mathcal{H}}\|w - v\|_{\mathcal{H}}$. Since $v \in V$ is arbitrary, we obtain

$$\langle v^\perp\,, w\rangle_{\mathcal{H}} \le \|v^\perp\|_{\mathcal{H}} \min_{v\in V} \|v - w\|_{\mathcal{H}} \le q_{V,W}\|v^\perp\|_{\mathcal{H}}\|w\|_{\mathcal{H}} \quad \text{for all } w \in W,$$

whence $\gamma_{V^\perp,W} \le q_{V,W}$. To prove the converse inequality, we construct sequences $v_j^\perp \in V^\perp\backslash\{0\}$ and $w_j \in W$ such that $\|w_j\|_\mathcal{H} = 1$ and $\lim_{j\to\infty}\langle v_j^\perp, w_j\rangle_\mathcal{H}/\|v_j^\perp\|_\mathcal{H} = q_{V,W}$. Without loss of generality we assume $q_{V,W} \ne 0$ since $q_{V,W} = 0$ implies $V = W$ and thus $\gamma_{V^\perp,W} = 0$ as well. For $q_{V,W} > 0$, let $w_j \in W$ be a sequence with

$$\|w_j\|_\mathcal{H} = 1, \quad \lim_{j\to\infty}\min_{v\in V}\|v - w_j\|_\mathcal{H} = q_{V,W} > 0, \quad \text{and} \quad \min_{v\in V}\|v - w_j\|_\mathcal{H} > 0.$$

Let $\Pi : \mathcal{H} \to V$ denote the orthogonal projection onto V and choose $v_j := \Pi w_j$. Then, there holds

$$\|v_j - w_j\|_\mathcal{H} = \min_{v\in V}\|v - w_j\|_\mathcal{H},$$

and $v_j^\perp := w_j - v_j$ satisfies $v_j^\perp \in V^\perp\backslash\{0\}$ and

$$\langle v_j^\perp, w_j\rangle_\mathcal{H} = \langle v_j^\perp, w_j - v_j\rangle_\mathcal{H} = \|w_j - v_j\|_\mathcal{H}^2 = \|w_j - v_j\|_\mathcal{H}\|v_j^\perp\|_\mathcal{H}.$$

Finally, we obtain

$$\gamma_{V^\perp,W} \ge \lim_{j\to\infty}\frac{\langle v_j^\perp, w_j\rangle_\mathcal{H}}{\|v_j^\perp\|_\mathcal{H}} = \lim_{j\to\infty}\|w_j - v_j\|_\mathcal{H} = q_{V,W}.$$

This concludes the proof. □

5 Finite Element Method for the Laplace Problem

We consider the following model example on a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$,

$$\begin{aligned} -\Delta u &= f && \text{in } \Omega, \\ u &= 0 && \text{on } \Gamma_D \subseteq \partial\Omega, \\ \partial u/\partial\nu &= g && \text{on } \Gamma_N = \partial\Omega\backslash\Gamma_D. \end{aligned} \tag{11}$$

We assume that Γ_D is closed and that the right-hand side f and the given normal flux g allow for a weak solution

$$u \in \mathcal{H} = H_D^1(\Omega) := \{u \in H^1(\Omega) : u|_{\Gamma_D} = 0\}, \tag{12}$$

of (11). Provided Γ_D has positive surface measure, the Friedrichs' inequality shows that

$$\langle\!\langle u, v\rangle\!\rangle = \int_\Omega \nabla u \cdot \nabla v \, dx \tag{13}$$

defines the energy scalar product with equivalent norm $\|\cdot\|_\mathcal{H} := |\!|\!|\cdot|\!|\!| \sim \|\cdot\|_{H^1(\Omega)}$ on $\mathcal{H}$. The weak form of (11) allows for a unique solution $u \in \mathcal{H}$ in the usual sense

$$\langle\!\langle u\,,v\rangle\!\rangle = \int_\Omega f v\,dx + \int_{\Gamma_N} gv\,ds_x \quad \text{for all } v \in \mathcal{H}. \tag{14}$$

The lowest order conforming FE discretization of (14) uses $\mathcal{T}_h$-piecewise affine and globally continuous functions: Let $\mathcal{T}_h$ be a regular triangulation [in the sense of Ciarlet] which consists of triangles, for $d = 2$, and tetrahedra, for $d = 3$, respectively. For $p \in \mathbb{N}$, let $\mathcal{P}^p(\mathcal{T}_h)$ denote the vector space of functions $w_h \in \mathcal{P}^p(\mathcal{T}_h)$ which are polynomials of total degree $\le p$ on each element $T \in \mathcal{T}_h$. Let $h \in L^\infty(\Omega)$ denote the local mesh-size of $\mathcal{T}_h$ defined by $h|_T = \mathrm{diam}(T)$ for $T \in \mathcal{T}_h$.
To apply the averaging technique, let $\mathcal{T}_H$ be a regular triangulation of Ω and let $\mathcal{T}_h$ be obtained from $\ell \in \mathbb{N}$ red-refinements of $\mathcal{T}_H$, i.e., we recursively refine each element $T \in \mathcal{T}_H$ ℓ-times into 4 congruent elements. In particular, $H/h = 2^\ell$. With

$$\mathcal{S}_D^p(\mathcal{T}_h) := \{u_h \in \mathcal{P}^p(\mathcal{T}_h) \cap \mathcal{C}(\Omega) \,:\, u_h|_{\Gamma_D} = 0\} \subset \mathcal{H},$$

set

$$\mathcal{S}_h = \mathcal{S}_D^1(\mathcal{T}_h) \quad \text{and} \quad \mathcal{S}_H = \mathcal{S}_D^2(\mathcal{T}_H). \tag{15}$$

Finally, we denote by $H^s(\mathcal{T})$ the space of all $\mathcal{T}$-piecewise H^s functions for $s \ge 0$.

Theorem 5.1. *Provided $u \in \mathcal{H} \cap H^{2+\varepsilon}(\mathcal{T}_H)$ for some $\varepsilon > 0$ and ℓ large enough, Assumptions* (AA)–(DP) *hold and therefore Theorem 2.1 applies with $\eta_M = |\!|\!| u_h - \mathbb{G}_H u_h |\!|\!|$.*

Proof. Recall the local inverse estimate

$$\|H\, w_H\|_{L^2(\Omega)} \le c_{\mathrm{inv}} \|\nabla w_H\|_{L^2(\Omega)} \quad \text{for all } w_H \in \mathcal{P}^1(\mathcal{T}_H),$$

where $c_{\mathrm{inv}} > 0$ depends only on the shape of the elements in $\mathcal{T}_H$ and the gradient ∇ is evaluated elementwise. In particular, this holds with $w_H = \nabla v_H$ for all $v_H \in \mathcal{P}^2(\mathcal{T}_H)$. Moreover, the Bramble-Hilbert lemma implies

$$\|\nabla v - \nabla(\mathbb{P}_h v)\|_{L^2(\Omega)} \le c_{\mathrm{apx}} \|h\, D^2 v\|_{L^2(\Omega)}$$

for all continuous $v \in H^1(\Omega) \cap H^2(\mathcal{T}_h)$ and $\mathbb{P}_h$ the nodal interpolation operator. Together with $H/h = 2^\ell$, the combination of both estimates proves

$$q := \max_{v_H \in \mathcal{S}_H \setminus \{0\}} \min_{v_h \in \mathcal{S}_h} \frac{|\!|\!| v_H - v_h |\!|\!|}{|\!|\!| v_H |\!|\!|} \le c_{\mathrm{apx}} c_{\mathrm{inv}} / 2^\ell$$

Therefore, (DP) is satisfied for ℓ sufficiently large. Note the best approximation result $|\!|\!| u - \mathbb{G}_h u |\!|\!| = \mathcal{O}(h)$ and $|\!|\!| u - \mathbb{G}_H u |\!|\!| = \mathcal{O}(H^{1+\varepsilon})$. Given a fixed parameter ℓ, (AA) follows. $\square$

Remark 5.1. Since the energy norm is based on the local L^2-norm, we can write η_M as a sum of local contributions

$$\eta_M = \Big(\sum_{T_j \in \mathcal{T}_H} \eta_{M,j}^2 \Big)^{1/2} \quad \text{with} \quad \eta_{M,j} := \|\nabla u_h - \nabla(\mathbb{G}_H u_h)\|_{L^2(T_j)}. \tag{16}$$

The refinement indicators $\eta_{M,j}$ can be used for an adaptive mesh-refining strategy.

Remark 5.2. With $\varPi_H$ the L^2 projection onto $\mathcal{P}^1(\mathcal{T}_H)^d$, we define

$$\mu_\varPi := \min_{q_H \in \mathcal{P}^1(\mathcal{T}_H)^d} \|\nabla u_h - q_H\|_{L^2(\varOmega)} = \|\nabla u_h - \varPi_H(\nabla u_h)\|_{L^2(\varOmega)}. \tag{17}$$

Since $\nabla(\mathbb{G}_H u_h) \in \mathcal{P}^1(\mathcal{T}_H)^d$, there holds $\mu_\varPi \le \eta_M$. Therefore, $\mu_\varPi$ is efficient up to terms of higher order under the assumptions of Theorem 5.1. The mathematical analysis of the reliability of $\mu_\varPi$ — although supported by numerical evidence — remains open.

6 Symm's Integral Equation

In this section, we consider Symm's integral equation

$$Vu = f \quad \text{on } \varGamma \tag{18}$$

with a relatively open subset $\varGamma \subseteq \partial\varOmega$ of the boundary $\partial\varOmega$ of a bounded Lipschitz domain $\varOmega$ in $\mathbb{R}^d$, $d = 2, 3$. The operator V is the single-layer potential

$$Vu(x) = \int_\varGamma \kappa(x,y) u(y)\, ds_y, \tag{19}$$

where ds denotes the integration on the manifold $\varGamma$, and $\kappa(x,y)$ denotes (up to a multiplicative constant) the fundamental solution of the Laplace operator,

$$\kappa(x,y) = \begin{cases} -\dfrac{1}{\pi} \log|x-y| & \text{for } d = 2, \\ +\dfrac{1}{2\pi}\, |x-y|^{-1} & \text{for } d = 3. \end{cases} \tag{20}$$

The variational formulation of (19) needs Sobolev spaces on the boundary. First, the space

$$H^{1/2}(\partial\varOmega) = \{u|_{\partial\varOmega} \,:\, u \in H^1(\mathbb{R}^d)\}$$

of traces of H^1 functions associated with the trace norm

$$\|u\|_{H^{1/2}(\partial\varOmega)} = \inf\{\|\widehat{u}\|_{H^1(\mathbb{R}^d)} \,:\, \widehat{u} \in H^1(\mathbb{R}^d) \text{ with } \widehat{u}|_\varGamma = u\}.$$

Moreover, we consider the subspace

$$H^{1/2}(\Gamma) = \{u|_\Gamma \,:\, u \in H^{1/2}(\partial\Omega)\},$$

where the norm of $u \in H^{1/2}(\Gamma)$ is defined as the minimal norm of any extension, i.e.

$$\|u\|_{H^{1/2}(\Gamma)} = \inf\{\|\widehat{u}\|_{H^{1/2}(\partial\Omega)} \,:\, \widehat{u} \in H^{1/2}(\partial\Omega) \text{ with } \widehat{u}|_\Gamma = u\}.$$

Furthermore, there are Sobolev spaces

$$\widetilde{H}^{1/2}(\Gamma) = \{u \in H^{1/2}(\partial\Omega) \,:\, \mathrm{supp}(u) \subseteq \overline{\Gamma}\}$$

associated with the usual $H^{1/2}(\Gamma)$ norm. Finally, the corresponding spaces of negative order are defined by duality with respect to the extended L^2 scalar product,

$$H^{-1/2}(\Gamma) = \widetilde{H}^{1/2}(\Gamma)^* \quad \text{and} \quad \widetilde{H}^{-1/2}(\Gamma) = H^{1/2}(\Gamma)^*.$$

Remark 6.1. There are other equivalent definitions of the involved Sobolev spaces, e.g., by real oder complex interpolation, a Fourier norm, or Sobolev-Slobodeckij norms [W, McL].

For a particular right-hand side f in (18) and $\Gamma = \partial\Omega$, Symm's integral equation is an equivalent formulation of the Laplace problem (11) with $\Gamma_D = \partial\Omega$, cf. [McL]. For $d = 3$ and provided additionally $\mathrm{diam}(\Omega) < 1$ for $d = 2$, the operator

$$V : \widetilde{H}^{-1/2}(\Gamma) \to H^{1/2}(\Gamma) \tag{21}$$

is an isomorphism between the two Hilbert spaces $\widetilde{H}^{-1/2}(\Gamma)$ and $H^{1/2}(\Gamma)$ which build a dual pairing with respect to the extended L^2 scalar product $\langle\cdot\,,\cdot\rangle$. The energy scalar product

$$\langle\!\langle u\,, v\rangle\!\rangle := \langle Vu\,, v\rangle \quad \text{for } u, v \in \widetilde{H}^{-1/2}(\Gamma) \tag{22}$$

induces an equivalent norm $\|\cdot\|_{\mathcal{H}} := |\!|\!|\cdot|\!|\!|$ on $\mathcal{H} = \widetilde{H}^{-1/2}(\Gamma)$.
Let $\mathcal{T}_h = \{\Gamma_1, \dots, \Gamma_n\}$ be a regular triangulation of Γ with local mesh-size $h \in L^\infty(\Gamma)$, $h|_{\Gamma_j} = \mathrm{diam}(\Gamma_j)$. Each element Γ_j of $\mathcal{T}_h$ is supposed to be a connected (affine) boundary piece for $d = 2$ and a (flat) triangle for $d = 3$, respectively.
For an integer $p \geq 0$, $\mathcal{P}^p(\mathcal{T}_h)$ denotes the space of all piecewise polynomials of degree $\leq p$ [defined on reference elements $\Gamma_{\mathrm{ref}}^{\mathrm{2D}} = [0,1]$ and $\Gamma_{\mathrm{ref},3}^{3D} = \mathrm{conv}\{(0,0),(0,1),(1,0)\}$ and $\Gamma_{\mathrm{ref},4}^{3D} = \mathrm{conv}\{(0,0),(0,1),(1,0),(1,1)\}$ for $d = 2, 3$, respectively].
For the averaging error estimation, we consider again the lowest order case: Let $\mathcal{T}_H$ be a regular triangulation of Γ and obtain $\mathcal{T}_h$ by $\ell \in \mathbb{N}$ red-refinements of $\mathcal{T}_H$. Adopt the aforegoing notations for $\mathcal{T}_H$ and $\mathcal{T}_h$ accordingly and define the discrete spaces

$$\mathcal{S}_h = \mathcal{P}^0(\mathcal{T}_h) \quad \text{and} \quad \mathcal{S}_H = \mathcal{P}^1(\mathcal{T}_H). \tag{23}$$

Theorem 6.1. *Provided* $u \in \mathcal{H} \cap H^{1+\varepsilon}(\mathcal{T}_H)$ *for some* $\varepsilon > 0$ *and* ℓ *large enough, Assumptions* (AA)–(DP) *hold and therefore Theorem 2.1 applies with* $\eta_M = \|\!|u_h - \mathbb{G}_H u_h|\!\|$.

Proof. Local inverse estimates for fractional order Sobolev spaces [DFGHS, GHS] read

$$\|H^{\alpha+\kappa} v_H\|_{L^2(\Gamma)} \le c_{\rm inv}^{H,p} \|H^\kappa v_H\|_{H^{-\alpha}(\Gamma)} \quad \text{for all } v_H \in \mathcal{P}^p(\mathcal{T}_H) \text{ and } \kappa \in \mathbb{R}. \tag{24}$$

The constant $c_{\rm inv}^{H,p} > 0$ depends only on the shape of the elements in $\mathcal{T}_H$, the polynomial degree $p \in \mathbb{N}_0$, and the parameter $\alpha \ge 0$. Since $\widetilde{H}^\alpha(\Gamma)$ is a closed subspace of $H^\alpha(\Gamma)$, the corresponding dual spaces $H^{-\alpha}(\Gamma) = \widetilde{H}^\alpha(\Gamma)^*$ and $\widetilde{H}^{-\alpha}(\Gamma) = H^\alpha(\Gamma)^*$ satisfy $\widetilde{H}^{-\alpha}(\Gamma) \subseteq H^\alpha(\Gamma)^*$ with $\|v\|_{H^{-\alpha}(\Gamma)} \le \|v\|_{\widetilde{H}^{-\alpha}(\Gamma)}$. Therefore, we may apply (24) for the energy norm $\|\!|\cdot|\!\| \sim \|\cdot\|_{\widetilde{H}^\alpha(\Gamma)}$. This leads to

$$\|H^{1/2} v_H\|_{L^2(\Gamma)} \le c_{\rm inv}^{H,p} \|\!|v_H|\!\| \quad \text{for all } v_H \in \mathcal{P}^p(\mathcal{T}_H). \tag{25}$$

[Note that, for a closed boundary $\Gamma = \partial\Omega$, there holds $H^\alpha(\Gamma) = \widetilde{H}^\alpha(\Gamma)$ with equal norms.] Moreover, with the L^2-projection Π_h^p onto $\mathcal{P}^p(\mathcal{T}_h)$, there holds [CP1]

$$\|v - \Pi_h^p v\|_{\widetilde{H}^{-\alpha}(\Gamma)} \le c_{\rm apx}^{h,p} \|h^\alpha v\|_{L^2(\Gamma)} \quad \text{for all } v \in L^2(\Gamma). \tag{26}$$

Here, $c_{\rm apx}^{h,p} > 0$ depends only on the shape of the elements in $\mathcal{T}_h$, the polynomial degree $p \in \mathbb{N}_0$, and $\alpha \ge 0$. Together with $H/h = 2^\ell$, the combination of (25) and (26), for $\alpha = 1/2$ and $\|\!|\cdot|\!\| \sim \|\cdot\|_{\widetilde{H}^{-1/2}(\Gamma)}$, proves

$$q := \max_{v_H \in \mathcal{S}_H \setminus \{0\}} \min_{v_h \in \mathcal{S}_h} \frac{\|\!|v_H - v_h|\!\|}{\|\!|v_H|\!\|} \le c_{\rm apx}^{h,0} c_{\rm inv}^{H,1} / 2^{-\ell/2}.$$

This proves (DP) for ℓ sufficiently large. Assumption (AA) follows from best approximation results $\|\!|u - \mathbb{G}_h u|\!\| = \mathcal{O}(h^{3/2})$, $\|\!|u - \mathbb{G}_H u|\!\| = \mathcal{O}(H^{3/2+\varepsilon})$, cf. [SaS]. $\square$

In contrast to the FE method from the previous section with H^m norms, the energy norm $\|\!|\cdot|\!\| \sim \|\cdot\|_{\widetilde{H}^{-1/2}(\Gamma)}$ is non-local, i.e., it cannot be written as a sum over non-interacting local contributions. The following theorem asserts the equivalence of the energy norm based error estimator η_M and the weighted L^2 norm based error estimator

$$\mu_M := \|H^{1/2}(u_h - \mathbb{G}_H u_h)\|_{L^2(\Gamma)}. \tag{27}$$

This leads to the equivalent error estimators

$$\eta_\Pi := \|\!|u_h - \Pi_H^1 u_h|\!\| \quad \text{and} \quad \mu_\Pi := \|H^{1/2}(u_h - \Pi_H^1 u_h)\|_{L^2(\Gamma)}, \tag{28}$$

where Π_H^1 denotes the L^2 projection onto $\mathcal{P}^1(\mathcal{T}_H)$. Under the assumptions of Theorem 6.1, μ_M, μ_Π, and η_Π are reliable and efficient in the following sense.

Theorem 6.2. *There are constants $C_1, C_2 > 0$ which only depend on the shape of the elements in $\mathcal{T}_H$ and the quotient $H/h = 2^\ell$ such that*

$$\eta_M \le \eta_\Pi \le C_1\, \mu_\Pi \quad \textit{and} \quad \mu_\Pi \le \mu_M \le C_2\, \eta_M. \tag{29}$$

Proof. The estimate $\eta_M \le \eta_\Pi$ follows from the best approximation property of $\mathbb{G}_H$ and was already mentioned in the introduction. Since we consider globally discontinuous polynomials, Π_H^1 is also $\mathcal{T}_H$-elementwise orthogonal. Hence,

$$\|u_h - \Pi_H^1 u_h\|_{L^2(\Gamma_j)} \le \|u_h - \mathbb{G}_H u_h\|_{L^2(\Gamma_j)}.$$

This proves $\mu_\Pi \le \mu_M$. According to the mesh generation of $\mathcal{T}_h$ from $\mathcal{T}_H$, there holds $u_h - \mathbb{G}_H u_h \in \mathcal{P}^1(\mathcal{T}_h)$. An inverse estimate (25) yields $\|h^{1/2}(u_h - \mathbb{G}_H u_h)\|_{L^2(\Gamma)} \le c_{\rm inv}^{h,1} |\!|\!| u_h - \mathbb{G}_H u_h |\!|\!|$ and, therefore, with $H/h = 2^\ell$, that

$$\mu_M = 2^{\ell/2}\, \|h^{1/2}(u_h - \mathbb{G}_H u_h)\|_{L^2(\Gamma)} \le 2^{\ell/2}\, c_{\rm inv}^{h,1} \eta_M.$$

To prove $\eta_\Pi \le c_{\rm apx}^{H,1} \mu_\Pi$, define $v = u_h - \Pi_H^1 u_h \in L^2(\Gamma)$. With $\mathbb{1}$ the identity on $L^2(\Gamma)$, the operator $(\mathbb{1} - \Pi_H^1)$ is a projection, whence $v = (\mathbb{1} - \Pi_H^1)v$. An application of (26) proves

$$\eta_\Pi = |\!|\!| v |\!|\!| = |\!|\!| (\mathbb{1} - \Pi_H^1)v |\!|\!| \le c_{\rm apx}^{H,1} \|H^{1/2} v\|_{L^2(\Gamma)} = \mu_\Pi. \qquad \square$$

Remark 6.2. For an adaptive mesh-refining algorithm, one may localize the error estimators μ_M and μ_Π, respectively, to obtain refinement indicators, e.g.

$$\mu_\Pi = \Big(\sum_{\Gamma_j \in \mathcal{T}_H} \mu_{\Pi,j}^2 \Big)^{1/2} \quad \text{with} \quad \mu_{\Pi,j} = \|H^{1/2}(u_h - \Pi_H^1 u_h)\|_{L^2(\Gamma_j)}. \tag{30}$$

The computation of the error estimators η_M, μ_M, and η_Π needs the computation of dense matrices which stem from the Galerkin projection $\mathbb{G}_H$ [explicitly or implicitly for the computation of the energy norm]. Matrix compression techniques, e.g., hierarchical matrices or panel clustering provide an effective implementation. The error estimator μ_Π avoids the computation of $\mathbb{G}_H$ and can be computed in linear complexity with respect to the number N of elements.

7 Hypersingular Integral Equation

With the notation from Section 6, we consider the hypersingular integral equation

$$Wu = f \quad \text{on } \Gamma \tag{31}$$

and the hypersingular integral operator

$$Wu(x) = -\frac{\partial}{\partial\nu_x}\int_\Gamma \frac{\partial}{\partial\nu_y}\kappa(x,y)u(y)\,ds_y, \tag{32}$$

where ν_x and ν_y denote the outer normal vectors on Γ at x and y, respectively. For particular right-hand sides and $\Gamma = \partial\Omega$, the hypersingular integral equation (31) is equivalent to the Laplace problem (11) with pure Neumann boundary condition $\Gamma_N = \partial\Omega$.
For an open boundary piece $\Gamma \subsetneqq \partial\Omega$, the operator

$$W : \widetilde{H}^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$$

is an isomorphism. For a closed boundary $\Gamma = \partial\Omega$, one has to consider the factor spaces $H_0^\alpha(\Gamma) = H^\alpha/\mathbb{R}(\Gamma) = \{u \in H^\alpha(\Gamma) \,:\, \int_\Gamma u\,ds = 0\}$ to neglect constant functions. Then,

$$W : H_0^{1/2}(\Gamma) \to H_0^{-1/2}(\Gamma)$$

is isomorphic. In both cases, W maps the energy space $\mathcal{H} = \widetilde{H}^{1/2}(\Gamma)$ resp. $\mathcal{H} = H_0^{1/2}(\Gamma)$ onto its dual, and

$$\langle\!\langle u\,, v\rangle\!\rangle := \langle Wu\,, v\rangle \quad \text{for } u, v \in \mathcal{H} \tag{33}$$

defines a scalar product with equivalent norm $\|\cdot\|_{\mathcal{H}} := |\!|\!|\cdot|\!|\!|$ on $\mathcal{H}$. The discretization is based on subspaces of $\mathcal{S}^p(\mathcal{T}_h) := \mathcal{P}^p(\mathcal{T}_h) \cap \mathcal{C}(\Gamma)$ for a regular triangulation $\mathcal{T}_h$ of Γ and

$$S_0^p(\mathcal{T}_h) = \begin{cases} \{v_h \in \mathcal{S}^p(\mathcal{T}_h) \,:\, v_h|_{\partial\Gamma} = 0\} & \text{if } \Gamma \subset \partial\Omega; \\ \{v_h \in \mathcal{S}^p(\mathcal{T}_h) \,:\, \int_\Gamma v_h\,ds = 0\} & \text{if } \Gamma = \partial\Omega. \end{cases}$$

With respect to the abstract setting in Section 2, let $\mathcal{T}_H$ be a shape-regular triangulation of Γ and $\mathcal{T}_h$ obtained from $\mathcal{T}_H$ by $\ell \in \mathbb{N}$ red-refinements and set

$$\mathcal{S}_h = \mathcal{S}_0^1(\mathcal{T}_h) \quad \text{and} \quad \mathcal{S}_H = \mathcal{S}_0^2(\mathcal{T}_H). \tag{34}$$

Theorem 7.1. *Provided $u \in \mathcal{H} \cap H^{2+\varepsilon}(\mathcal{T}_H)$ for some $\varepsilon > 0$ and ℓ large enough, Assumptions* (AA)–(DP) *hold and therefore Theorem 2.1 applies with $\eta_M = |\!|\!|u_h - \mathbb{G}_H u_h|\!|\!|$.*

Proof. Note that there holds the local inverse estimate [CP2]

$$\|H^{1-\alpha}\nabla v_H\|_{L^2(\Gamma)} \le c_{\mathrm{inv}}^{H,p}\|v_H\|_{H^\alpha(\Gamma)} \quad \text{for all } v_H \in \mathcal{S}^p(\mathcal{T}_H), \tag{35}$$

where ∇ denotes the arc-length derivative ∇ for $d = 2$ and the surface gradient for $d = 3$, respectively. The constant $c_{\mathrm{inv}}^{H,p} > 0$ depends only on the shape of the elements in $\mathcal{T}_h$, the polynomial degree $p \in \mathbb{N}$, and the parameter $\alpha \ge 0$.

In [FP] it is proven that the Galerkin projection $\mathbb{G}_h^p$ onto $\mathcal{S}_0^p(\mathcal{T}_h)$ satisfies, for all $v \in \mathcal{H} \cap H^1(\Gamma)$,

$$\|\!|v - \mathbb{G}_h^p v|\!\| \le c_{\rm apx}^{h,p} \min\big\{\|h^{1/2}\nabla v\|_{L^2(\Gamma)}, \|h^{1/2}\nabla(v - \mathbb{G}_h^p v)\|_{L^2(\Gamma)}\big\}. \tag{36}$$

The constant $c_{\rm apx}^{h,p} > 0$ depends only on the shape of the elements in $\mathcal{T}_h$. As before, Assumption (DP) is satisfied, provided ℓ is large enough,

$$q := \max_{v_H \in \mathcal{S}_H \setminus \{0\}} \min_{v_h \in \mathcal{S}_h} \frac{\|\!|v_H - v_h|\!\|}{\|\!|v_H|\!\|} \le c_{\rm apx}^{h,1} c_{\rm inv}^{H,2} / 2^{\ell/2}.$$

Assumption (AA) follows from best approximation results $\|\!|u - \mathbb{G}_h u|\!\| = \mathcal{O}(h^{3/2})$ and $\|\!|u - \mathbb{G}_H u|\!\| = \mathcal{O}(H^{3/2+\varepsilon})$ [SaS]. □

As for Symm's integral equation, the energy norm $\|\!|\cdot|\!\|$ for the hypersingular equation is non-local and has to be localized. This can be done by $H^{1/2}$-weighted H^1-seminorms. The following theorem states the efficiency and reliability of the error estimator

$$\mu_M := \|H^{1/2}\nabla(u_h - \mathbb{G}_H u_h)\|_{L^2(\Gamma)} \tag{37}$$

under the assumptions of Theorem 7.1.

Theorem 7.2. *There are constants $C_3, C_4 > 0$ which only depend on the shape of the elements in $\mathcal{T}_H$ and the quotient $H/h = 2^\ell$ such that*

$$C_3^{-1} \mu_M \le \eta_M \le C_4\, \mu_M. \tag{38}$$

Proof. The follows from an inverse estimate with constant $C_3 = c_{\rm inv}^{h,2}\, \ell^{1/2}$ and the approximation result (36) with $C_4 = c_{\rm apx}^{H,2}$. □

The computation of μ_M involves the dense stiffness matrix corresponding to the Galerkin projection $\mathbb{G}_H$. To avoid this numerical effort, one can consider the estimator

$$\mu_\Pi := \|H^{1/2}(\nabla u_h - \Pi_H^1(\nabla u_h))\|_{L^2(\Gamma)} \tag{39}$$

with the L^2 projection Π_H^1 onto $\mathcal{P}^1(\mathcal{T}_H)$, which is efficient under the assumptions of Theorem 7.1.

Corollary 7.1. *There holds $\mu_\Pi \le \mu_M$.* □

Remark 7.1. The reliability of μ_Π, which is observed numerically [CP2, FP], remains open — as for the finite element method in Section 5.

Another computationally challenging variant might be to consider the H_0^1 projection $\mathbb{P}_H : \mathcal{H} \cap H^1(\Gamma) \to \mathcal{S}_H$, i.e. the gradient L^2 projection defined by

$$\int_\Gamma \nabla(u - \mathbb{P}_H u) \cdot \nabla v_H = 0 \quad \text{for all } v_H \in \mathcal{S}_H. \tag{40}$$

The numerical realization only involves the sparse stiffness matrix from the P^1 finite element method.

$$\eta_{\mathbb{P}} := \|u_h - \mathbb{P}_H u_h\| \quad \text{and} \quad \mu_{\mathbb{P}} := \|H^{1/2}\nabla(u_h - \mathbb{P}_H u_h\|_{L^2(\Gamma)} \tag{41}$$

Clearly, $\eta_M \leq \eta_{\mathbb{P}}$, and therefore $\eta_{\mathbb{P}}$ is reliable under the assumptions of Theorem 7.1. The analysis for fractional order Sobolev spaces $H^\alpha(\Gamma)$ and $\alpha > 0$ is more involved than for $\alpha < 0$, i.e. for Symm's integral equation: For quasi-uniform meshes, there holds $\mu_{\mathbb{P}} \leq C\,\mu_M$ since $\|\nabla(u_h - \mathbb{P}_H u_h)\|_{L^2(\Gamma)} \leq \|\nabla(u_h - \mathbb{G}_H u_h)\|_{L^2(\Gamma)}$. An estimate of the type $\mu_{\mathbb{P}} \leq C\,\mu_M$ remains open for adaptively generated meshes. For $d = 2$, it is proven that $\eta_{\mathbb{P}}$ and $\mu_{\mathbb{P}}$ are equivalent [CP2].

Theorem 7.3. *For $d = 2$, there are constants $C_5, C_6 > 0$ such that*

$$C_5^{-1}\mu_\Pi \leq \eta_{\mathbb{P}} \leq C_6\,\mu_{\mathbb{P}}. \tag{42}$$

Proof. The lower estimate follows as in Theorem 7.2. We recall from [CP2] that the H_0^1 projection $\mathbb{P}_h^p$ onto $\mathcal{S}_0^p(\mathcal{T}_h)$ satisfies, for all $v \in \mathcal{H} \cap H^1(\Gamma)$,

$$\|v - \mathbb{P}_h^p v\| \leq c_{\text{apx}}^{h,p} \min\big\{\|h^{1/2}\nabla v\|_{L^2(\Gamma)}, \|h^{1/2}\nabla(v - \mathbb{P}_h^p v)\|_{L^2(\Gamma)}\big\}. \tag{43}$$

The constant $c_{\text{apx}}^{h,p}$ only depends on p and the local mesh-ratio

$$\varrho(\mathcal{T}_h) := \max\{h_j/h_k \,:\, \Gamma_j, \Gamma_k \in \mathcal{T}_h \text{ s.t. } \Gamma_j \text{ is a neighbour of } \Gamma_k\}. \tag{44}$$

From (43), we obtain the upper estimate with $C_6 = c_{\text{apx}}^{H,2}$. □

Remark 7.2. If $\mathcal{A}_H$ denotes the L^2 projection onto $\mathcal{S}_0^2(\mathcal{T}_H)$, define

$$\eta_A := \|u_h - \mathcal{A}_H u_h\| \quad \text{and} \quad \mu_A := \|H^{1/2}\nabla(u_h - \mathcal{A}_H u_h)\|_{L^2(\Gamma)}.$$

Then, η_A is reliable, and one can prove that η_A and μ_A are equivalent. Unfortunately, the L^2 projection $\mathcal{A}_H$ onto $\mathcal{S}_0^2(\mathcal{T}_H)$ is, in general, not H^1 stable. Thus, one does neither analytically obtain nor numerically observe efficiency of η_A and μ_A, cf. [CP2].

8 Integral Equation for a Transmission Problem

This section is devoted to a transmission problem which involves the integral operators of Section 6 and 7, from where notation is adopted. Given $(f, g) \in H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$ along the boundary $\Gamma = \partial\Omega$ of a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$, the strong form of the transmission problem reads: Find $u^- \in H^1(\Omega)$ and $u^+ \in H^1_{\ell oc}(\Omega)$ with

$$\Delta u^- = 0 \text{ in } \Omega, \quad \Delta u^+ = 0 \text{ in } \mathbb{R}^d \backslash \Omega \tag{45}$$

with some radiation condition on u^+ at infinity and

$$u^- = u^+ + f, \quad \frac{\partial u^-}{\partial \nu} = \frac{\partial u^+}{\partial \nu} + g \quad \text{on } \Gamma. \tag{46}$$

This is equivalently formulated by the boundary integral equation [CoS]

$$A\begin{pmatrix} u \\ \phi \end{pmatrix} = \Big(\frac{1}{2} + A\Big)\begin{pmatrix} f \\ g \end{pmatrix} \quad \text{in } \mathcal{H} \subset H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma) \tag{47}$$

with the Calderón projector (in symbolic form)

$$A = \begin{pmatrix} -K & V \\ W & K' \end{pmatrix}. \tag{48}$$

The operator V is defined in (19), and W is defined in (32) with kernel $\kappa(x,y)$ from (20). Moreover, K denotes the double layer potential operator and K' its adjoint defined by

$$K : H^{1/2}(\Gamma) \to H^{1/2}(\Gamma), \qquad Kv(x) = \int_\Gamma v(y) \frac{\partial}{\partial \nu_y} \kappa(x,y)\, ds_y, \tag{49}$$

$$K' : H^{-1/2}(\Gamma) \to H^{-1/2}(\Gamma), \qquad K'\phi(x) = \int_\Gamma \phi(y) \frac{\partial}{\partial \nu_x} \kappa(x,y)\, ds_y. \tag{50}$$

Duality is understood with respect to the extended L^2 scalar product,

$$\langle \begin{pmatrix} u \\ \phi \end{pmatrix}, \begin{pmatrix} v \\ \psi \end{pmatrix} \rangle_{\mathcal{H}} = \langle u\,, \psi \rangle + \langle v\,, \phi \rangle \quad \text{for } (u,\phi), (v,\psi) \in \mathcal{H} := H_0^{1/2}(\Gamma) \times H_0^{-1/2}(\Gamma). \tag{51}$$

The transmission problem (45)–(46) and the boundary integral formulation (47) are equivalent in the following sense [CoS, CS2]: If $(u^-, u^+) \in H^1(\Omega) \times H^1_{\ell oc}(\mathbb{R}^d \backslash \Omega)$ solves the transmission problem, then $(u,\phi) \in \mathcal{H}$ solves (47), where $u := u^-|_\Gamma - \int_\Gamma u^-\, ds \in H_0^{1/2}(\Gamma)$ and $\phi := \partial u^-/\partial\nu|_\Gamma \in H_0^{-1/2}(\Gamma)$. Conversely, if $(u,\phi) \in \mathcal{H}$ solves (47), then the Cauchy data of u^- are given by $(u^-, \partial u^-/\partial \nu)|_\Gamma = (u + u_0, \phi)$ with

$$u_0 = \frac{\int_\Gamma \big(\frac{1}{2}\,(K-1)f - \frac{1}{2}\,Vg + V\phi - Ku\big)\, ds}{\int_\Gamma 1\, ds} \in \mathbb{R}.$$

The solution (u^-, u^+) is then obtained from the representation formulae in Ω and $\mathbb{R}^d \backslash \Omega$.

The mapping properties of the involved boundary operators [McL] shows that $A : \mathcal{H} \to \mathcal{H}$ is continuous and $\mathcal{H}$-elliptic with respect to the canonical norm $\|(v,\psi)\|^2_{\mathcal{H}} := \|v\|^2_{H^{1/2}(\Gamma)} + \|\psi\|^2_{H^{-1/2}(\Gamma)}$. In fact, elementary calculations show that the (non-symmetric) bilinear form

$$\langle\!\langle (u,\phi)\,,(v,\psi)\rangle\!\rangle = \langle A\binom{u}{\phi}\,,\binom{v}{\psi}\rangle_{\mathcal{H}}, \tag{52}$$

induces an equivalent norm $|\!|\!| \cdot |\!|\!|$ which satisfies

$$|\!|\!|(u,\phi)|\!|\!|^2 = \|\phi\|_V^2 + \|u\|_W^2 \geq C_{\mathrm{ell}}\|(u,\phi)\|_{\mathcal{H}}^2 \quad \text{for all } (u,\phi) \in \mathcal{H} \tag{53}$$

with the energy norms $\|\cdot\|_V$ and $\|\cdot\|_W$ from Section 6 and 7, respectively. Note that $|\!|\!|\cdot|\!|\!|$ is indeed a Hilbert norm, but $\langle\!\langle\cdot\,,\cdot\rangle\!\rangle$ is *not* the corresponding scalar product! Let $\mathcal{T}_H$ be a shape-regular triangulation of Γ and let $\mathcal{T}_h$ be obtained from $\mathcal{T}_H$ by $\ell \in \mathbb{N}$ red-refinements. Set $\mathcal{P}_0^p(\mathcal{T}) := \{v_h \in \mathcal{P}^p(\mathcal{T}) : \int_\Gamma v_h\, ds = 0\}$, set

$$\mathcal{S}_h = \mathcal{S}_0^1(\mathcal{T}_h) \times \mathcal{P}_0^0(\mathcal{T}_h) \quad \text{and} \quad \mathcal{S}_H = \mathcal{S}_0^2(\mathcal{T}_H) \times \mathcal{P}_0^1(\mathcal{T}_H).$$

Theorem 8.1. *Provided* $(u,\phi) \in \mathcal{H}\cap\big(H^{2+\varepsilon}(\mathcal{T}_H)\times H^{1+\varepsilon}(\mathcal{T}_H)\big)$ *for some* $\varepsilon > 0$ *and* ℓ *large enough, Assumptions* (AA) *and* (DP) *hold and therefore Theorem 2.1 applies with* $\eta_M = \min_{(v_H,\psi_H)\in\mathcal{S}_H} |\!|\!|(u_h,\phi_h) - (v_H,\phi_H)|\!|\!|$.

Proof. Assumption (AA) follows from the regularity of (u,ϕ). The inverse estimates (25) and (35) lead to

$$\|H^{1/2}(\nabla v_H,\psi_H)\|_{L^2(\Gamma)} \leq c_{\mathrm{inv}}^{H,2,1}|\!|\!|(v_H,\psi_H)|\!|\!| \quad \text{for all } (v_H,\phi_H) \in \mathcal{S}_H$$

Since the L^2-projection $\Pi_h^0 : L^2(\Gamma) \to \mathcal{P}^0(\mathcal{T}_h)$ preserves the vanishing integral mean (i.e., $\Pi_h^0\psi_H \in \mathcal{P}_0^0(\mathcal{T}_h)$ provided $\int_\Gamma \psi_H\, ds = 0$), (26) and (36) yield

$$|\!|\!|(v_H,\psi_H) - (\mathbb{G}_h^W v_H, \Pi_h^0\psi_H)|\!|\!| \leq c_{\mathrm{apx}}^{h,1,0}\|h^{1/2}(\nabla v_H,\psi_H)\|_{L^2(\Gamma)},$$

where $\mathbb{G}_h^W : H_0^{1/2}(\Gamma) \to \mathcal{S}_0^1(\mathcal{T}_h)$ denotes the Galerkin projection with respect to W from Section 7. The combination of the previous two inequalities results in

$$q := \max_{(v_H,\psi_H)\in\mathcal{S}_H\setminus\{0\}} \min_{(v_h,\psi_h)\in\mathcal{S}_h} \frac{|\!|\!|(v_H,\psi_H) - (v_h,\psi_h)|\!|\!|}{|\!|\!|(v_H,\psi_H)|\!|\!|} \leq c_{\mathrm{apx}}^{h,1,0}c_{\mathrm{inv}}^{H,2,1}/2^{\ell/2},$$

This implies (DP) for sufficiently large ℓ. □

Remark 8.1. For an adaptive mesh-refinement, the non-local energy norm is localized via the localization arguments from the previous sections; further details are straightforward and hence omitted.

9 Numerical Experiments

This section provides some numerical experiments for the proposed error estimation. We only consider the symmetric case, where $\langle\!\langle\cdot\,,\cdot\rangle\!\rangle$ defines a scalar product and give the numerical results with respect to the energy norm, cf. Section 3.1–3.2. Throughout, we compare uniform mesh-refinement with an adaptive mesh-refinement, which is based on the local contributions of our averaging error estimators as refinement indicators.

9.1. Adaptive Mesh-Refinement. The mesh-refinement strategy is formulated in the following adaptive algorithm from [CP1], which is stated for the finite element method from Section 5.

Algorithm 9.1 *Choose a regular initial coarse mesh* $\mathcal{T}_H^{(0)}$, $k=0$, $\ell \in \mathbb{N}$ *and* $0 \le \theta \le 1$.

(i) *Obtain* $\mathcal{T}_h^{(k)} = \{T_1,\dots,T_n\}$ *from* $\mathcal{T}_H^{(k)} = \{\tau_1,\dots,\tau_N\}$ *by* ℓ *uniform refinements.*
(ii) *Compute the approximation* $u_h^{(k)}$ *for the current mesh* $\mathcal{T}_h^{(k)}$.
(iii) *Compute the error estimator* η_M *and the corresponding refinement indicators* $\eta_{M,j}$ *from* (16).
(iv) *Mark element* τ_j *for red-refinement provided the corresponding refinement indicator satisfies* $\eta_{M,j} \ge \theta \max\{\eta_{M,1},\dots,\eta_{M,N}\}$.
(v) *Use a red-green-blue mesh-refinement strategy to obtain a regular coarse mesh* $\mathcal{T}_H^{(k+1)}$, *update* k, *and go to* (i). □

Note that we do the adaptive mesh-refinement on the coarse grid level to obtain a sequence of meshes $\mathcal{T}_H^{(k)}$. Surprisingly, our numerical experiments give empirical evidence that one may choose $\ell = 1$ in Algorithm 9.1. That is, the corresponding fine mesh $\mathcal{T}_h^{(k)}$, on which we compute our discrete solution u_h, is obtained by *one* uniform refinement of $\mathcal{T}_H^{(k)}$. We remark that the choice of $\theta = 0$ leads to uniform mesh-refinement. To obtain an adaptive mesh-refinement, we choose $\theta = 0.5$ in the subsequent experiments.
In the formulation of Algorithm 9.1, we consider the local contributions $\eta_{M,j}$ of η_M as refinement indicators. Alternatively, one may choose the local contributions of the (efficient) error estimator μ_Π from (17),

$$\mu_{\Pi,j} := \min_{q \in \mathcal{P}^1(\tau_j)} \|\nabla u_h - q\|_{L^2(\tau_j)} = \|\nabla u_h - \Pi_H(\nabla u_h)\|_{L^2(\tau_j)}. \tag{54}$$

9.2. Visualization of Numerical Results. In all experiments we plot the Galerkin error $|||u-u_h|||$ and the error estimators η_M and μ_Π against the number $n = \#\mathcal{T}_h$ of fine grid elements for uniform ($\theta = 0$) and adaptive ($\theta = 0.5$) mesh-refinement, respectively. Throughout, we choose the parameter $\ell = 1$ in Algorithm 9.1. The error is computed by use of the Galerkin orthogonality

$$|||u-u_h|||^2 = |||u|||^2 - |||u_h|||^2. \tag{55}$$

The squared energy norm of the discrete solution u_h reads $|||u_h|||^2 = \mathbf{x} \cdot \mathbf{A}\mathbf{x}$ with the stiffness matrix $\mathbf{A}$ and the coefficient vector $\mathbf{x}$ corresponding to u_h.

The norm $\|u\|^2$ can, in principle, be computed exactly. However, we use the value $\|u\|^2$ which is obtained by Aitkin's Δ^2-extrapolation as follows: For a sequence $\mathcal{T}_h^{(k)}$ of uniformly refined meshes, we compute the sequence of energies $E_k := \|u_h^{(k)}\|^2$, where $u_h^{(k)}$ is the discrete solution corresponding to the triangulation $\mathcal{T}_h^{(k)}$. Extrapolation of the sequence E_k then yields a good approximation of $\|u\|^2$.
From our analysis in Section 2 and Section 5, respectively, we know that η_M and μ_Π are efficient, i.e. there holds

$$\mu_\Pi \le \eta_M \le C_{\text{eff}} \|u - u_h\|$$

with efficiency constant $C_{\text{eff}} \le 1+\delta_{hH}$ and the approximation constant $\delta_{hH} = \|u - \mathbb{G}_H u\|/\|u - u_h\|$ from Assumption (AA). Provided δ_{hH} stays bounded, we therefore expect that the curves corresponding to η_M and μ_Π have at least the same slope as the curve corresponding to $\|u - u_h\|$. For smooth u, δ_{hH} tends to zero with h. Therefore, the experimental efficiency constant $C_{\text{eff}} := \eta_M/\|u - u_h\| \le 1 - \delta_{hH}$ is expected to satisfy $C_{\text{eff}} \le 1$ at least for the limit case for a finer and finer mesh-size h. Therefore, the absolute values and hence the curves of the error estimators should be below the curve of the error. Provided η_M is also reliable, i.e. $\|u - u_h\| \le C_{\text{rel}}\eta_M$, the quotient $\|u - u_h\|/\eta_M$ is bounded. In this case, the slopes of the curves corresponding to $\|u - u_h\|$ and η_M are the same, i.e. the curves are parallel.
To study the efficiency and reliability of η_M even in the case that the solution u is non-smooth, we plot the experimental reliability constant $C_{\text{rel}} := \|u - u_h\|/\eta_M$ and the approximation constant δ_{hH} in dependence on the number $n = \#\mathcal{T}_h$ of fine grid elements. The Galerkin error $\|u - \mathbb{G}_H u\|$ for the higher-order method is computed as in (55).

9.3. Finite Element Method with Smooth Solution.
For our first numerical experiment, we adopt the notation from Section 5. We consider the Dirichlet problem (11) on the unit cube $\Omega = [0,1]^2 \subset \mathbb{R}^2$ with $\Gamma_D = \partial\Omega$ and

$$f(x) = (k^2\pi^2/2)\,\sin(x_1 k\pi/2)\,\sin(x_2 k\pi/2).$$

The exact solution is then given by

$$u(x) = \sin(x_1 k\pi/2)\,\sin(x_2 k\pi/2),$$

and therefore u satisfies the smoothness assumptions of Theorem 5.1. According to the Bramble-Hilbert lemma, we expect that uniform mesh-refinement leads to the optimal order of convergence $\mathcal{O}(h)$ for the error $\|u - u_h\|$, which is computed by (55). Aitkin's Δ^2-extrapolation yields $\|u\|^2 = 44.4132$.
In Figure 1 we plot the error $\|u - u_h\|$ as well as the estimators η_M and μ_Π. Note that the optimal order of convergence $\mathcal{O}(h)$ for P^1-elements corresponds

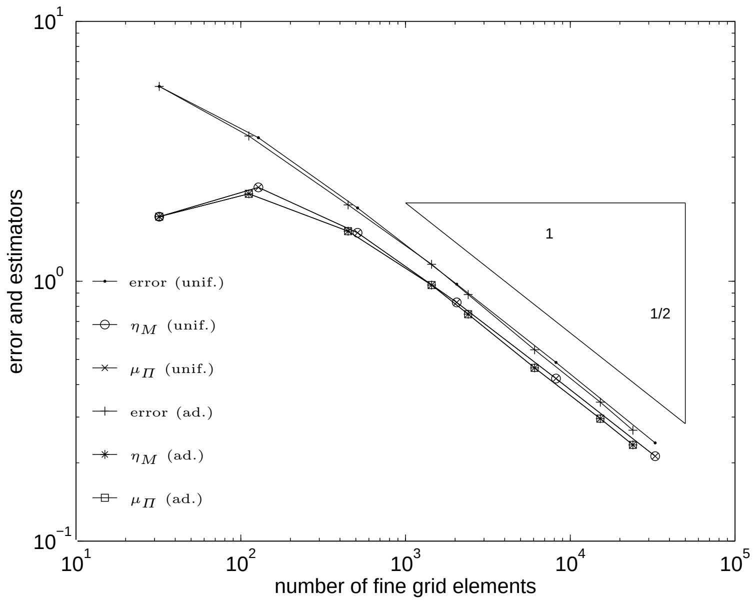

Fig. 1. Error $|||u - u_h|||$ and error estimators η_M and μ_Π in Example 9.3 in dependence on the number of fine grid elements $n = \#\mathcal{T}_h$. We observe optimal order of convergence $\mathcal{O}(n^{-1/2})$ for error and error estimators and independent of uniform [indicated by *unif.*] and adaptive mesh-refinement [indicated by *ad.*]. The values of the error estimators η_M and μ_Π coincide up to rounding errors. The error estimation is reliable and efficient.

to $\mathcal{O}(n^{-1/2})$ in terms of elements $n = \#\mathcal{T}_h$. Both, uniform and adaptive mesh-refinement, lead to the optimal order of convergence for the error. Moreover, we observe that η_M and μ_Π coincide and that both are efficient and reliable. We stress the reliability of η_M which is analytically only predicted for sufficiently large $\ell \in \mathbb{N}$, whereas we use the minimal possible choice $\ell = 1$. Moreover, note that we have only proven $\mu_\Pi \leq \eta_M$. In our experiment, there holds even $\mu_\Pi = \eta_M$ up to rounding errors.
In Figure 2 we plot the approximation quotient δ_{hH}. From standard approximation results and $h \sim H$ for the local mesh-sizes, we know that the nominator converges like $\mathcal{O}(h^2)$, whereas the denominator is $\mathcal{O}(h)$, i.e. we expect $\delta_{hH} = \mathcal{O}(h)$. This is what is observed experimentally in Figure 2. Moreover, we plot the experimental reliability constant $C_{\rm rel} := |||u - u_h|||/\eta_M$. We observe that it is slowly decreasing with absolute values about 1.13 at the end of our computations.

9.4. Finite Element Method with Weakly Singular Solution.
For our second example, we again adopt the notation from Section 5 and consider the Dirichlet problem (11) on the L-shaped domain $\Omega = [-1, 0]^2 \cup$

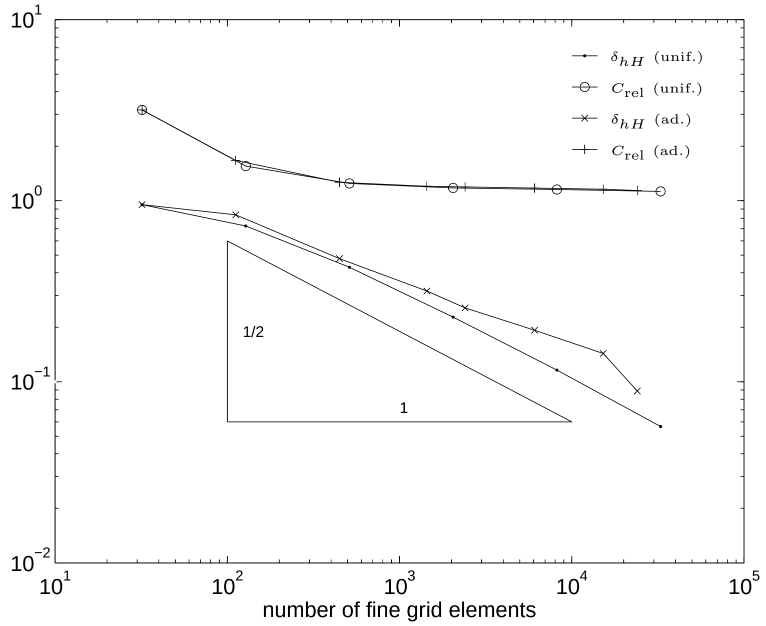

Fig. 2. Quotient $\delta_{hH} = \|u - \mathbb{G}_H u\|/\|u - u_h\|$ in approximation assumption (AA) and experimental reliability constant $C_{\rm rel} := \|u - u_h\|/\eta_M$ for Example 9.3. For both, uniform [indicated by *unif.*] and adaptive mesh-refinement [indicated by *ad.*], δ_{hH} tends to zero with the theoretically expected order $\mathcal{O}(n^{-1/2})$ with $n = \#\mathcal{T}_h$. The experimental reliability constant $C_{\rm rel}$ is slowly decreasing with absolute values ≈ 1.13 at the end of the computations ($n = 32768$ resp. $n = 24016$)

$[-1, 0] \times [0, 1] \cup [0, 1]^2$ with $\Gamma_D = \partial\Omega$, cf. Figure 3 which also shows the initial coarse mesh $\mathcal{T}_H^{(0)}$. The right-hand side is constant $f(x) = 1$. The solution $u(x)$ is known to be a bubble $u \in H^{1+2/3-\varepsilon}(\Omega)$, for all $\varepsilon > 0$, with singularity at the reentrant corner $(0, 0)$. Therefore, uniform mesh-refinement is expected to lead to a suboptimal (experimental) convergence rate for the error $\|u - u_h\| = \mathcal{O}(h^{2/3})$ which can usually be cured by adaptive mesh-refinement.

In Figure 4 we plot the error $\|u - u_h\|$ and the error estimators η_M and μ_Π, where the error is computed by (55) with the extrapolated value $\|u\|^2 = 0.214076$. As in Example 9.3, we observe that for both, uniform and adaptive mesh-refinement, the error estimators η_M and μ_Π coincide up to rounding errors. Independent of the mesh-refining strategy, the error estimators are reliable and efficient. For uniform mesh-refinement, we observe a suboptimal order of convergence $\mathcal{O}(n^{-2/5})$ which corresponds to $\mathcal{O}(h^{4/5})$. This is slightly better than the expected order of $\mathcal{O}(h^{2/3})$. For adaptive mesh-refinement, we retain the optimal order of convergence $\mathcal{O}(n^{-1/2})$ after a preasymptotic

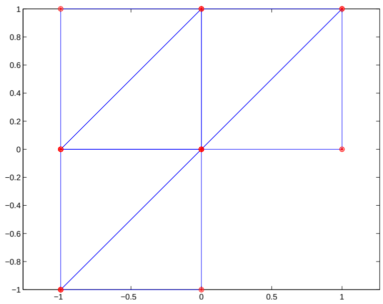

Fig. 3. In Example 9.4, we consider the L-shaped domain $\Omega = [-1,0]^2 \cup [-1,0] \times [0,1] \cup [0,1]^2$. The initial coarse mesh $\mathcal{T}_H^{(0)}$ consists of $N = 6$ rectangular triangles.

phase (up to about $n = 900$ elements), where we observe the same order of convergence as for the uniform refinement.

In Figure 5 we plot the approximation quotient δ_{hH} and the experimental reliability constant $C_{\rm rel} := |||u - u_h|||/\eta_M$. For uniform mesh-refinement, the corner singularity of u dominates the convergence behavior so that we observe $\delta_{hH} = \mathcal{O}(1)$. For adaptive mesh-refinement, however, we obtain the optimal order $\delta_{hH} = \mathcal{O}(n^{-1/2})$. The experimental reliability constant $C_{\rm rel}$ is slowly decreasing in case of adaptive mesh-refinement with absolute value about 1.15 at the end of our computation ($n = 43040$). In contrast, for uniform mesh-refinement, $C_{\rm rel}$ is slowly increasing and is about 1.39 at the end of our computation ($n = 24565$).

9.5. Symm's Integral Equation.

Finally, we consider the integral formulation of the Poisson problem

$$\Delta U = 0 \text{ in } \Omega \quad \text{and} \quad U = g \text{ on } \Gamma = \partial\Omega, \tag{56}$$

which is formulated as Symm's integral equation [McL]

$$Vu = (K+1)g, \tag{57}$$

where V is the single-layer and K is the double-layer potential from (19) and (49), respectively. Then, the exact solution of (57) is just the normal derivative $u = \partial U/\partial n$ of the solution U from (56) on the boundary Γ.

We adopt the notation from Section 6. The presented numerical results are taken from [CP1]: We consider a rotated L-shaped domain shown in Figure 6. The Dirichlet data are chosen such that the exact solution $U \in H^1(\Omega)$ of (56) reads

$$U(x) = r^{2/3}\cos(2\varphi/3) \quad \text{in polar coordinates} \quad x = r\,(\cos\varphi, \sin\varphi).$$

Then, the exact solution $u \in H^{-1/2}(\Gamma)$ of Symm's integral equation (57) is given by

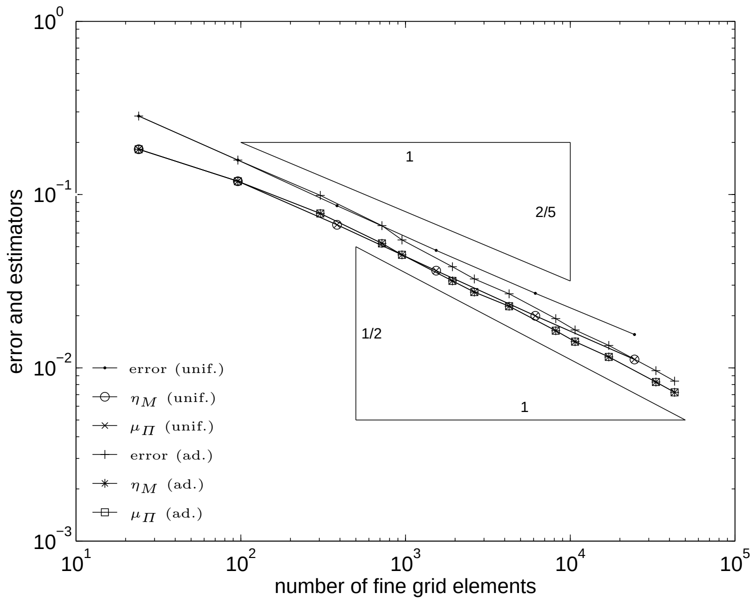

Fig. 4. Error $\|\!|u-u_h\|\!|$ and error estimators η_M and μ_Π in Example 9.4 in dependence on the number of fine grid elements $n = \#\mathcal{T}_h$. For uniform mesh-refinement [indicated by *unif.*], we observe a suboptimal order of convergence $\mathcal{O}(n^{-2/5})$ for error and error estimators. This is cured by our adaptive mesh-refining strategy [indicated by *ad.*], which leads to optimal order of convergence $\mathcal{O}(n^{-1/2})$. The values of the error estimators η_M and μ_Π coincide up to rounding errors. Independent of the mesh-refinement, the error estimation is reliable and efficient.

$$u(x) = \frac{2}{3}\left(w(\varphi)\cdot n(x)\right) r^{-1/3} \tag{58}$$

with

$$w(\varphi) := \begin{pmatrix} \cos(\varphi)\cos(2\varphi/3) + \sin(\varphi)\sin(2\varphi/3) \\ \sin(\varphi)\cos(2\varphi/3) - \cos(\varphi)\sin(2\varphi/3) \end{pmatrix}. \tag{59}$$

Figure 6 shows the initial coarse mesh $\mathcal{T}_H^{(0)}$ as well as the exact solution u from (58) plotted against the arclength of Γ. The singularity of u at $(0,0)$ is visible at arc-length parameter $s = 0$ and $s = 2$ by periodicity. Aitkin's Δ^2-method gives $\|\!|u\|\!|^2 = 0.404116$.

We consider uniform ($\theta = 0$) and adaptive mesh-refinement ($\theta = 1/2$), where we use the local contributions of the error estimator μ_Π from (30) as refinement indicators in Algorithm 9.1. Again, we restrict to the minimal choice $\ell = 1$ to obtain $\mathcal{T}_h$ from $\mathcal{T}_H$.

Figure 7 shows the numerical results on the convergence of the error $\|\!|u-u_h\|\!|$ and of the error estimators $\eta_M = \|\!|u_h - \mathbb{G}_H u_h\|\!|$ and μ_M, η_Π and μ_Π from (27)–(28), respectively. We plot the error and the error estimators in dependence

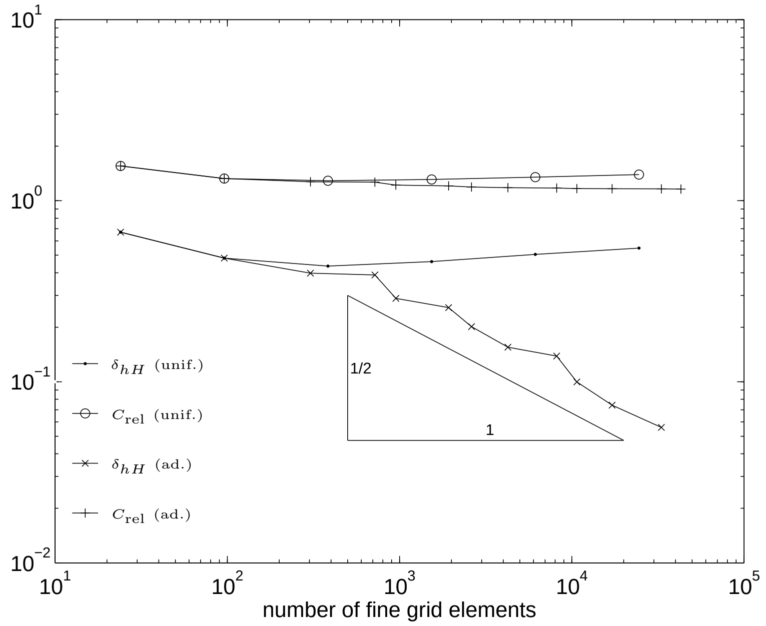

Fig. 5. Quotient $\delta_{hH} = \|u - \mathbb{G}_H u\|/\|u - u_h\|$ in approximation assumption (AA) and experimental reliability constant $C_{\rm rel} := \|u - u_h\|/\eta_M$ for Example 9.4. For uniform mesh-refinement [indicated by *unif.*], the corner singularity of u dominates the convergence behavior so that we observe $\delta_{hH} = \mathcal{O}(1)$. For adaptive mesh-refinement [indicated by *ad.*], we observe optimal convergence of $\delta_{hH} = \mathcal{O}(n^{-1/2})$. The experimental reliability constant $C_{\rm rel}$ is slowly decreasing in case of adaptive mesh-refinement with absolute value ≈ 1.15 at the end of the computation ($n = 43040$). However, for uniform mesh-refinement, $C_{\rm rel}$ is slowly increasing with absolute value ≈ 1.39 at the end of the computation ($n = 24576$).

on the number of fine grid elements $n = \#\mathcal{T}_h$. Note that an experimental convergence rate $\mathcal{O}(h^\kappa)$ now corresponds to $\mathcal{O}(n^{-\kappa})$ in terms of fine grid elements, since we are dealing with a 1D discretization.

Uniform mesh-refinement leads to a suboptimal order of convergence $\mathcal{O}(h^{2/3})$ which is due to the singularity of the exact solution at the reentrant corner and which can be predicted theoretically. The fact that the slope of the corresponding error estimators even is $2/3$ gives empirical evidence that the estimators are reliable and efficient although the solution lacks the regularity assumed in Section 6. The proposed adaptive algorithm cures that shortcoming in the sense that it leads to the optimal order of convergence $\mathcal{O}(n^{-3/2})$ for the error, where we used the local contributions of μ_Π as refinement indicators. Due to numerical instabilities in the computation of the matrices corresponding to $\mathbb{G}_H$, we can only present the results for μ_M, η_M and η_Π up to about $n = 300$ elements in the case of adaptive mesh-refinement. This cor-

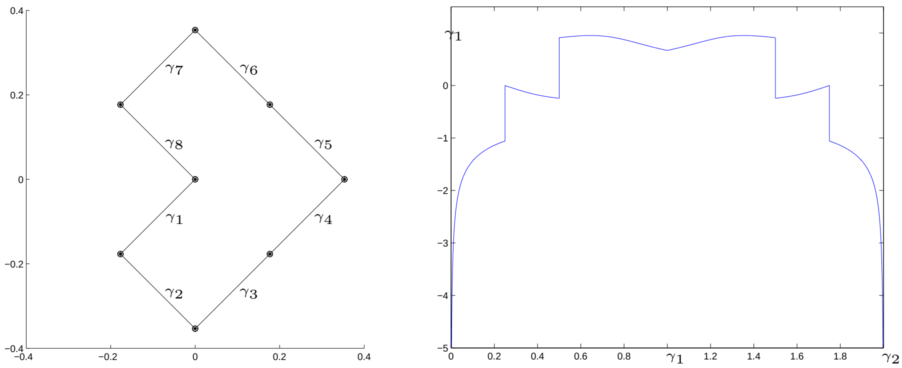

Fig. 6. In Example 9.5, we consider a rotated L-shaped domain Ω (left). Furthermore, the plot shows the initial coarse mesh $\mathcal{T}_H^{(0)}$ with $N = 8$ elements and uniform mesh size $H = 1/4$. The exact solution u from (58) is plotted over the arc-length $s = 0, \ldots, 2$ (right), where $s = 0$ and $s = 2$ correspond to the reentrant corner $(0, 0)$, where u is singular.

responds to an error about $10^{-7/2}$ for the higher order method. The explicit values of η_M and η_Π as well as the explicit values of μ_M and μ_Π coincide up to 2% so that there is no difference visible in the corresponding curves. Moreover, all four estimators show numerical evidence for efficiency and reliability. The computation of μ_Π is stable as it only involves the computation of some L^2-mass matrices, and the condition numbers of which are $\mathcal{O}(1)$ under some mild restrictions on the triangulation. The μ_Π steered mesh-refinement retains the optimal order of convergence $\mathcal{O}(n^{-3/2})$.

10 Conclusions

In this paper we provided an abstract analytical setting for the study of the reliability and efficiency of a posteriori averaging error estimators. The abstract setting applies to the Galerkin method for both, differential and integral equations under weak assumptions on the finite elements or boundary elements gave the analytical fundament that these error estimators are reliable and efficient estimators for the (unknown) error $|||u - u_h|||$. The strongest assumption is a (piecewise) high regularity of the exact solution u. We recalled an adaptive algorithm from [CP1] which steers the mesh-refinement with respect to some localized error estimators. In the numerical experiments we considered examples with different regularity. In our experiments and in the experiments of [CP1, CP2, FP] the adaptive strategy retains the optimal order of convergence and is therefore superior to uniform mesh-refinement.
However, there are still some gaps in the analysis: First, the introduced error estimators are only proven to be reliable if the parameter $\ell \in \mathbb{N}$ in Algo-

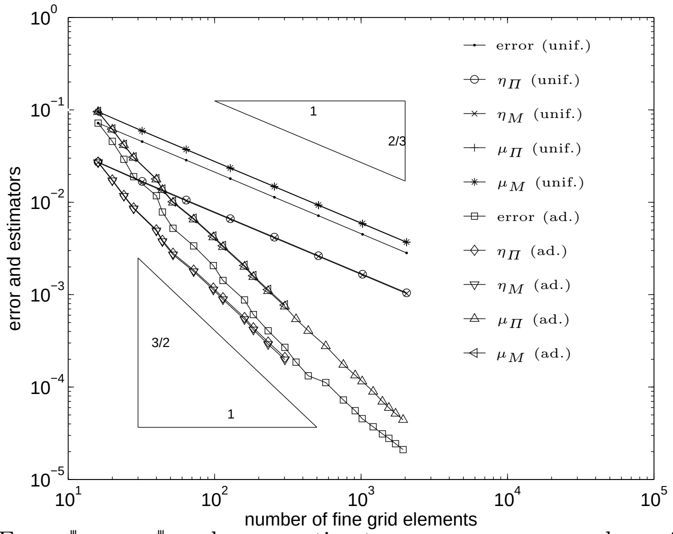

Fig. 7. Error $\|u - u_h\|$ and error estimators η_M, η_Π, μ_M, and μ_Π for uniform [indicated by *unif.*] and μ_Π-adaptive [indicated by *ad.*] mesh-refinement in Example 9.5. Uniform mesh-refinement leads to a suboptimal order of convergence. This is improved by the proposed adaptive strategy, which retains the optimal order of convergence. In both cases, the error estimation is reliable and efficient. The error estimators η_M and η_Π as well as μ_M and μ_Π coincide up to 2%.

rithm 9.1 is large enough. In the experiments we used the minimal choice $\ell = 1$ throughout. Nevertheless, we always observed the reliability. Second, the analytical verification of the introduced error estimators needs a high regularity assumption on u. However, this regularity assumption might be nonsatisfied in practice. Since our numerical experiments indicate that this assumption can be weakened, it would be desirable to have a refined analysis that covers these cases as well, i.e. which either avoids a regularity assumption on u or explains the good performance of the indicator-based adaptive strategy analytically.

References

[AO] Ainsworth, M. and Oden, J.T. (2000). *A posteriori error estimation in finite element analysis*, Wiley-Interscience [John Wiley & Sons], New York. xx+240.

[BS] Babuška, I. and Strouboulis, T. (2001). *The finite element method and its reliability*, The Clarendon Press Oxford University Press, New York, xii+802.

[B] Braess, D. (1998). *Enhanced assumed strain elements and locking in membrane problems.* Comput. Methods Appl. Mech. Engrg., **165**, 1-4, 155–174.

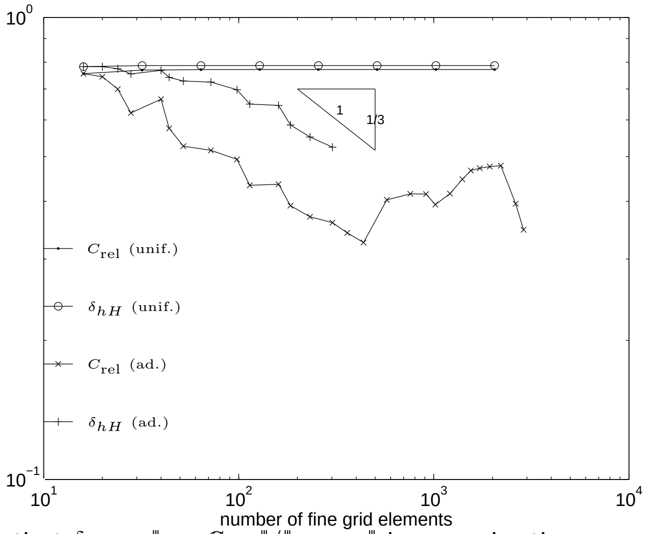

Fig. 8. Quotient $\delta_{hH} = \|u - \mathbb{G}_H u\|/\|u - u_h\|$ in approximation assumption (AA) and experimental reliability constant $C_{\rm rel} := \|u - u_h\|/\eta_M$ for uniform [indicated by *unif.*] and μ_Π-adaptive [indicated by *ad.*] mesh-refinement in Example 9.5. Note that according to the scaling of the y-axis, $C_{\rm rel}$ is almost constant.

[BC1] Bartels, S. and Carstensen, C. (2002). *Each averaging technique yields reliable a posteriori error control in FEM on unstructured grids. I. Low order conforming, nonconforming, and mixed FEM.* Math. Comp., **71**, 239, 945–969.
II. Higher order FEM. Math. Comp., **71**, 239, 971–994.

[BC2] Bartels, S. and Carstensen, C. (2004). *Averaging techniques yield reliable a posteriori finite element error control for obstacle problems.*, Numer. Math., **99**, 2, 225–249.

[BR] Becker, R. and Rannacher, R. (1996). *A feed-back approach to error control in finite element methods: basic analysis and examples*, East-West J. Numer. Math, **4**, 4, 237–264.

[C1] Carstensen, C. (2004). *Some remarks on the history and future of averaging techniques in a posteriori finite element error analysis.* ZAMM Z. Angew. Math. Mech., **84**, 1, 3–21.

[C2] Carstensen, C. (2006). *Reliable and efficient averaging techniques as universal tool for a posteriori finite element error control on unstructured grids.* International Journal of Numerical Analysis and Modeling, **3**, No. 3, pp. 333–347.

[CA] Carstensen, C. and Alberty, J. (2003). *Averaging techniques for reliable a posteriori FE-error control in elastoplasticity with hardening.* Comput. Methods Appl. Mech. Engrg., **192**, 11-12, 1435–1450.

[CF1] Carstensen, C. and Funken, S.A. (2001). *Averaging technique for FE a posteriori error control in elasticity. I. Conforming FEM.* Comput. Methods Appl. Mech. Engrg., **190**, 18-19, 2483–2498.

II. λ-independend estimates. Comput. Methods Appl. Mech. Engrg., **190**, 35–36, 4663–4675.
III. Locking-free nonconforming FEM. Comput. Methods Appl. Mech. Engrg., **191**, 8–10, 861–877.

[CF2] Carstensen, C. and Funken (2001), S. A. *A posteriori error control in low-order finite element discretisations of incompressible stationary flow problems*, Math. Comp., **70**, 236, 1353–1381.

[CP1] Carstensen, C. and Praetorius, D. (2006). *Averaging Techniques for the Effective Numerical Solution of Symm's Integral Equation of the First Kind*, SIAM J.Sci.Comp., **27**, 4, 1226–1260.

[CP2] Carstensen, C. and Praetorius, D. (2006). *Averaging Techniques for the A Posteriori BEM Error Control for a Hypersingular Integral Equation in 2D*, SIAM J.Sci.Comp., accepted for publication.

[CP3] Carstensen, C. and Praetorius, D. *A Unified Theory on Averaging Techniques for the Effective Numerical Solution of Differential and Integral Equations*, work in progress (2006).

[CS1] Carstensen, C. and Stephan, E.P. (1995). *Adaptive coupling of boundary elements and finite elements.* RAIRO Modél. Math. Anal. Numér., **29**, 7, 779–817.

[CS2] Carstensen, C. and Stephan, E.P. (1997). *Adaptive boundary element methods for transmission problems.* J. Austr. Math. Soc. Ser. B, **38**, 3, 336–367.

[CV] Carstensen, C. and Verfürth, R. (1999). *Edge residuals dominate a posteriori error estimates for low order finite element methods*, SIAM J. Numer. Anal., **36**, 5, 1571–1587.

[CoS] Costabel, M. and Stephan, E. (1985). *A direct boundary integral equation method for transmission problems.* J. Math. Anal. Appl., **106**, 2, 367–413.

[DFGHS] Dahmen, W., Faermann, B., Graham, I. G., Hackbusch, W. and Sauter, S. A. (2004). *Inverse inequalities on non-quasi-uniform meshes and application to the mortar element method.* Math. Comp., **73**, 247, 1107–1138.

[EEHJ] Eriksson, K., Estep, D., Hansbo, P. and Johnson, C. (1996) *Computational differential equations.* Cambridge University Press, 1996.

[FP] Funken, S.A. and Praetorius, D. *Averaging on Large Patches for First Kind Integral Equations in 3D.* Work in progress (2006).

[GHS] Graham, I. G., Hackbusch, W. and Sauter, S. A. (2005). *Finite elements on degenerate meshes: inverse-type inequalities and applications.* IMA J. Numer. Anal., **25**, 2, 379–407.

[HSWW] Hoffmann, W. and Schatz, A. H. and Wahlbin, L. B. and Wittum, G. (2001). *Asymptotically exact a posteriori estimators for the pointwise gradient error on each element in irregular meshes. I. A smooth problem and globally quasi-uniform meshes* , Math. Comp., **70**, 235, 897–909.

[McL] McLean, W. (2000). *Strongly elliptic systems and boundary integral equations.* Cambridge University Press, Cambridge, xiv+357.

[N] Nochetto, R.H. (1994). *Removing the saturation assumption in a posteriori error analysis*, Istit. Lombardo Accad. Sci. Lett. Rend. A, **127**, 1, 67–82.

[R1] Rodriguez, R. (1994). *Some remarks on Zienkiewicz-Zhu estimator*, Numer. Methods Partial Differential Equations, **10**, 5, 625–635.

[R2] Rodríguez, R. (1994). *A posteriori error analysis in the finite element method*, (Finite element methods (Jyväskylä, 1993). Lecture Notes in Pure and Appl. Math., Dekker, New York, **164**, 389–397.

[SaS] Sauter, S.A. and Schwab, C. (2004). *Randelemente, Analyse und Implementierung schneller Algorithmen.* Teubner Verlag, Stuttgart.

[SchMW] Schulz, M. and Wendland, W.L. (1998). *A general approach to a posteriori error estimates for strictly monotone and Lipschitz continuous nonlinear operators illustrated in elasto-plasticity*, ENUMATH **97**, World Sci. Publishing, 572–579.

[SchHW] Schulz, H. and Wendland, W.L. (1998). *Local a posteriori error estimates for boundary element methods*, ENUMATH **97**, World Sci. Publishing, 564–571.

[SSW] Schulz, H., Schwab, C., and Wendland, W.L. (1996). *An extraction technique for boundary element methods*, Boundary elements: implementation and analysis of advanced algorithms, Notes Numer. Fluid Mech. **54**, Vieweg, 219–231.

[SWe] Schwab, C. and Wendland, W.L. (1999). *On the extraction technique in boundary integral equations*, Math. Comp., **68**, No. 225, 91–122.

[V] Verfürth, R. (1996). *A review of a posteriori error estimation and adaptive mesh-refinement techniques*, Wiley-Teubner.

[WSS] Wendland, W.L., Schulz, H. and Schwab, C. (1997). *On the computation of derivatives up to the boundary and recovery techniques in BEM*, IUTAM Symposium on Discretization Methods in Structural Mechanics (Vienna, 1997), Solid Mech. Appl., **68**, Kluwer, 155–164.

[W] Wendland, W.L. (1979). *Elliptic systems in the plane*, Monographs and Studies in Mathematics, 3, Boston / London.

[ZZ] Zienkiewicz, O.C. and Zhu, J.Z. (1987). *A simple error estimator and adaptive procedure for practical engineering analysis.* Internat. J. Numer. Methods Engrg., **24**, 2, 337–357.7.