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ON THE STRONG CONVERGENCE OF GRADIENTS IN STABILIZED DEGENERATE CONVEX MINIMIZATION PROBLEMS*

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Abstract. Infimizing sequences in nonconvex variational problems typically exhibit enforced finer and finer oscillations called microstructures such that the infimal energy is *not* attained. Although those oscillations are physically meaningful, finite element approximations experience difficulty in their reconstruction. The relaxation of the nonconvex minimization problem by (semi) convexification leads to a macroscopic model for the effective energy. The resulting discrete macroscopic problem is degenerate in the sense that it is convex but not strictly convex. This paper studies a modified discretization by adding a stabilization term to the discrete energy. It will be proven that for a wide class of problems, this stabilization technique leads to strong H^1 convergence of the macroscopic variables even on unstructured triangulations. In contrast to the work [C. Carstensen, P. Plecháĉ, S. Bartels, and A. Prohl, *Interfaces Free Bound.*, 6 (2004), pp. 253–269] on quasi-uniform triangulations, this paper allows for general unstructured shape-regular triangulations and so enables the use of adaptive algorithms for the stabilized formulations.

Key words. adaptive finite element methods, calculus of variations, convexification, degenerate convex problems, energy reduction, nonconvex minimization, partial differential equation, relaxation, stabilization, strong convergence, variational problem

AMS subject classifications. 35J70, 65K10, 65N12, 65N30

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1. Introduction.

It is well known that variational problems with non (quasi-) convex energy density have, in general, no classical solution [8], [5], [2], [6], [9]. Minimizing sequences develop enforced finer and finer oscillations with a Young-measure valued limit. Moreover, those oscillations make it hard for numerical methods to solve releated discrete problems.

Relaxation methods provide macroscopic models, which replace the nonconvex energy density by a (semi) convex envelope. A range of examples and applications includes the nonlinear Laplacian, some optimal design problems, several convexified model problems in computational microstructure like the three-well problem, and elastoplasticity, discussed in [5] and [2]. The lack of strict convexity yields situations where the Hessian matrix is not definite, and so minimization algorithms encounter severe difficulties—e.g., the (unstabilized) three-well problem does not allow a Newton solve.

A straightforward remedy is a stabilization technique, where the problem is regularized by adding a positive semidefinite stabilization term to the discrete energy. Such methods (and especially different choices for the stabilization) have been proposed in [3], where it is proven that proper stabilization can even yield strong H^1 convergence when working with quasi-uniform triangulations.

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This paper suggests a modified stabilization of [3, equation (6.1)] which yields H^1 convergence even on possibly nonquasi-uniform triangulations and hence enables adaptive mesh-refinement.

Strong convergence is a promising property, for it enables the computation of quantities related to the solution. For example, volume fractions of Young-measure valued solutions (of the original nonconvex problem) are nonlinearly dependent of the solution (of the convexified problem) and thus cannot be derived from weakly converging approximations.

Based on this result, the next challenge is finding reliable and efficient error estimators in order to control the adaptive grid refinement.

The model problem analyzed is defined as follows: Given a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$ ($n = 2, 3$) with polygonal boundary, a fixed $p \geq 2$, and $m \in \mathbb{N}$, let $u_D \in W^{1,p}(\Omega; \mathbb{R}^m) \cap C(\overline{\Omega}; \mathbb{R}^m)$ be piecewise $W^{2,p}$ in Ω and piecewise H^2 on the boundary $\partial\Omega$ (see below or section 2.1 for details). The set of *admissible functions* reads $\mathcal{A} := V + u_D$ for $V = W_0^{1,p}(\Omega; \mathbb{R}^m)$.

With a continuous convex energy density $W^{**} : \mathbb{R}^{m\times n} \to \mathbb{R}$ and some lower-order term $\mathcal{A} \to \mathbb{R}, v \mapsto \int_\Omega L(x, v(x))\,\mathrm{d}x$ with a continuous convex $L : \Omega \times \mathbb{R}^m \to \mathbb{R}$, the (convex) minimization problem reads

$$\text{(1.1)} \quad \text{minimize} \quad E(v) := \int_\Omega W^{**}(\nabla v(x))\,\mathrm{d}x + \int_\Omega L(x, v(x))\,\mathrm{d}x \quad \text{amongst } v \in \mathcal{A}.$$

A finite element approximation of (1.1) is associated with a family of regular triangulations $(\mathcal{T}_\ell)_{\ell\in\mathbb{N}_0}$ of Ω in the sense of Ciarlet [7] (e.g., for $n = 2$, the intersection of two triangles is either a common node or a common edge or empty), $\mathcal{F}_\ell^\Omega$ the set of inner edges or faces of $\mathcal{T}_\ell$, and $H_\ell := \max_{T\in\mathcal{T}_\ell} \operatorname{diam}(T)$. Let $u_{D,\ell} \in \mathcal{S}^1(\mathcal{T}_\ell; \mathbb{R}^m)$ be the nodal interpolation of u_D, where $\mathcal{S}^1(\mathcal{T}_\ell; \mathbb{R}^m)$ contains the lowest-order conforming $\mathbb{R}^m$-valued finite element functions on $\mathcal{T}_\ell$, and

$$\mathcal{A}_\ell := V_\ell + u_{D,\ell} \quad \text{for } V_\ell = V \cap P_1(\mathcal{T}_\ell, \mathbb{R}^m).$$

For later error estimates, we assume u_D to be $\mathcal{T}_0$-piecewise $W^{2,p}$ and $\mathcal{F}_0^{\partial\Omega}$-piecewise H^2.

For functions on Ω which are discontinuous on the edges or faces of the triangulations, the *jump* of such a function v on a face or edge $F \in \mathcal{F}_\ell^\Omega$ shared by two triangles or tetrahedra T_+ and T_- reads

$$[v](x) = [v]_F(x) = \lim_{T_+\ni y\to x} v(y) - \lim_{T_-\ni y\to x} v(y) \quad \text{for } x \in F.$$

The stabilization reads

$$a_\ell(v, w) := \sum_{F\in\mathcal{F}_\ell^\Omega} \frac{H_\ell^2}{\operatorname{diam}(F)} \int_F [\nabla v] : [\nabla w]\,\mathrm{d}s,$$

where ":" denotes the scalar product in $\mathbb{R}^{m\times n}$. The discrete problem reads

$$\text{(1.2)} \qquad \text{minimize } E_\ell(v) := E(v) + \frac{1}{2} a_\ell(v, v) \quad \text{amongst } v \in \mathcal{A}_\ell.$$

We denote $S(X) := DW^{**}(X)$ for $X \in \mathbb{R}^{m\times n}$,

$$J(v; w) = \int_\Omega DL(x, v(x); w(x))\,\mathrm{d}x \quad \text{for } v, w \in W^{1,p}(\Omega; \mathbb{R}^m)$$

(where DL is the derivative of L with respect to the second argument), and

$$J_\ell(v; w) := J(v; w) + a_\ell(v, w).$$

The Euler–Lagrange equations of (1.1)–(1.2) consist in finding $u \in \mathcal{A}$ and $u_\ell \in \mathcal{A}_\ell$ with

$$\int_\Omega S(\nabla u) : \nabla v \,\mathrm{d}x + J(u; v) = 0 \quad \text{for all } v \in V, \tag{1.3}$$

$$\int_\Omega S(\nabla u_\ell) : \nabla v_\ell \,\mathrm{d}x + J_\ell(u_\ell; v_\ell) = 0 \quad \text{for all } v_\ell \in V_\ell. \tag{1.4}$$

For the convexity conditions of section 2, the main result of this paper is Theorem 4.4, which states that a unique continuous solution $u \in \mathcal{A} \cap W^{2,p}(\mathcal{T}_0) \cap H^{3/2+\varepsilon}(\Omega)$ (with $\varepsilon > 0$) satisfies

$$\|u - u_\ell\|^2_{H^1(\Omega)} \lesssim H_\ell. \tag{1.5}$$

The remainder of this paper is devoted to the proof of (1.5) and is organized as follows: Section 2 presents the general framework and lists all the regularity that is demanded of the continuous problem. Section 3 discusses the general error estimate of [3, Theorem 2.1]. Finally, section 4 presents the proof of the main result.

Here and throughout this paper, we employ standard notation on Lebesgue and Sobolev spaces, and we will often abbreviate spaces of vector- or matrix-valued functions, whenever the dimensionality is clear, e.g.,

$$|v|_{W^{1,p}(\Omega)} = \|\nabla v\|_{L^p(\Omega)} = \|\nabla v\|_{L^p(\Omega;\mathbb{R}^{m\times n})} \quad \text{for } v \in W^{1,p}(\Omega) = W^{1,p}(\Omega; \mathbb{R}^m).$$

Furthermore,

$$W^{2,p}(\mathcal{T}_\ell; \mathbb{R}^m) := \left\{ v : \Omega \to \mathbb{R} : v|_T \in W^{2,p}(T; \mathbb{R}^m) \quad \text{for all } T \in \mathcal{T}_\ell \right\}$$

and $H^2(\mathcal{T}_\ell; \mathbb{R}^m) = W^{2,2}(\mathcal{T}_\ell; \mathbb{R}^m)$.

In the following, "$a \lesssim b$" abbreviates $a \le Cb$ with a constant $C > 0$ independent of ℓ, and "$a \approx b$" abbreviates $a \lesssim b \lesssim a$.

2. Prerequisites and assumptions.

In this section we will state some assumptions on the given triangulations and functions which will be needed in the following proofs.

2.1. Triangulations.

Let $(\mathcal{T}_\ell)_{\ell\in\mathbb{N}_0}$ be a family of *regular* triangulations of Ω in the sense of Ciarlet [7]. For each triangulation $\mathcal{T}_\ell$, we denote the set of its edges or faces with $\mathcal{F}_\ell$, as well as its subsets $\mathcal{F}_\ell^{\partial\Omega} = \bigcup\{F \in \mathcal{F}_\ell : F \subset \partial\Omega\}$ and $\mathcal{F}_\ell^{\Omega} = \mathcal{F}_\ell \setminus \mathcal{F}_\ell^{\partial\Omega}$. For a given $F \in \mathcal{F}_\ell^{\Omega}$, we denote the patch of F with $\omega_F = \bigcup_{\substack{T\in\mathcal{T}_\ell \\ F\subset T}} T$. We further define $h_T := \operatorname{diam}(T)$ and $h_F := \operatorname{diam}(F)$ for $T \in \mathcal{T}_\ell$ and $F \in \mathcal{F}_\ell$, so $H_\ell = \max_{T\in\mathcal{T}_\ell}(h_T)$. We assume the family $(\mathcal{T}_\ell)_{\ell\in\mathbb{N}_0}$ to be *shape-regular* in the sense that $h_T \approx h_F$ for all $T \in \mathcal{T}_\ell$, $F \in \mathcal{F}_\ell$ with $F \subset T$.

The initial triangulation $\mathcal{T}_0$ is assumed to be fine enough, and it is also assumed that $u_D \in W^{2,p}(\mathcal{T}_0)$ satisfies $u_D|_F \in H^2(F)$ for each $F \in \mathcal{F}_0^{\partial\Omega}$.

The nodal interpolation operator $I_\ell : C(\overline{\Omega}) \to \mathcal{S}^1(\mathcal{T}_\ell)$ on $\mathcal{S}^1(\mathcal{T}_\ell)$ is defined by $I_\ell v(z) = v(z)$ for all $v \in C(\overline{\Omega})$ and $z \in \mathcal{N}_\ell$ (e.g., $u_{D,\ell} = I_\ell u_D$).

The *trace inequality* (a consequence of [1, Theorem 1.6.6]) reads as follows: For any $v \in H^1(\Omega)$ and $T \in \mathcal{T}_\ell$ with $F \subset T$,

$$\|v\|^2_{L^2(F)} \lesssim \|v\|_{L^2(T)} \left(\|\nabla v\|_{L^2(T)} + h_F^{-1} \|v\|_{L^2(T)}\right)$$
$$\lesssim h_F \|\nabla v\|^2_{L^2(T)} + h_F^{-1} \|v\|^2_{L^2(T)}.$$

The following estimates can be found in [1, Theorem 4.4.4]. Let T be a triangle or tetrahedron with diameter $h_T = \operatorname{diam}(T)$, and for $t > 1$, let $v \in W^{2,t}(T)$ with $v = 0$ on each node of T; then

$$\|v\|_{L^t(T)} + h_T \|\nabla v\|_{L^t(T)} \lesssim h_T^2 \left\|\nabla^2 v\right\|_{L^t(T)}. \tag{2.1}$$

2.2. Assumptions on the energy density. This subsection presents the assumptions on the energy density W^{**}.

The following assumption is similar to [3, (H1)] with the Frobenius matrix norm $|\cdot|$ induced by the scalar product ":" in $\mathbb{R}^{m\times n}$.

Assumption 1 (convexity control). There are $\alpha, r, s > 0$ with $1 < r \le 2$ and $s + r + p \le rp$ such that, for all $X, Y \in \mathbb{R}^{m\times n}$, it holds that

$$|S(X) - S(Y)|^r \le \alpha \left(1 + |X|^s + |Y|^s\right) \left(S(X) - S(Y)\right) : (X - Y). \tag{2.2}$$

Since $S = DW^{**}$ is the derivative of the energy density function $W^{**} : \mathbb{R}^{m\times n} \to \mathbb{R}$, condition (2.2) follows from

$$|S(X) - S(Y)|^r \le \alpha \left(1 + |X|^s + |Y|^s\right) \left(W^{**}(X) - W^{**}(Y) - S(Y) : (X - Y)\right).$$

Assumption 2 (growth conditions).

$$|X|^p - 1 \lesssim W^{**}(X) \lesssim 1 + |X|^p \quad \text{for all } X \in \mathbb{R}^{m\times n}.$$

There is a fixed q with $0 < q < p$ such that the lower-order term satisfies

$$\left|\int_\Omega L(x, v(x))\,\mathrm{d}x\right| \lesssim 1 + |v|^q_{W^{1,p}(\Omega)} \quad \text{for all } v \in \mathcal{A}.$$

The partial derivatives $\partial_{\xi_j} L(x, \xi)$ of the integrand are *Carathéodory functions* in the sense of [8, Definition 3.5] (i.e., $\partial_{\xi_j} L(x, \cdot)$ is continuous for almost every $x \in \Omega$, and $\partial_{\xi_j} L(\cdot, \xi)$ is measurable for every $\xi \in \mathbb{R}^m$), the Jacobian $DL(x, \xi; \cdot) \in \mathbb{R}^m$ with respect to the second argument of L satisfies for every $\xi \in \mathbb{R}^m$, and almost every $x \in \Omega$

$$|DL(x, \xi; \cdot)| \lesssim \zeta(x) + |\xi|^{p-1},$$

with an $\zeta \in L^{p'}(\Omega)$, where p' is the conjugate exponent of p, $1/p + 1/p' = 1$.

COROLLARY 2.1. *The set of minimizers of the continuous and discrete problems* (1.1) *and* (1.2) *are the set of solutions of the corresponding Euler–Lagrange equations* (1.3) *and* (1.4).

Proof. This is a consequence of the last paragraph in Assumption 2 and [8, Theorem 3.37]. ∎

2.3. Stabilization via jumps of gradients. As described in the introduction, we implement a *stabilization function* $a_\ell : H^2(\mathcal{T}_\ell;\mathbb{R}^m) \times H^2(\mathcal{T}_\ell;\mathbb{R}^m) \to \mathbb{R}$. In this paper, we will study a class of stabilization functions which penalize jumps of gradients and which are defined by

$$a_\ell(v,w) := \sum_{F\in\mathcal{F}_\ell^\Omega} \rho_F \int_F [\nabla v] : [\nabla w]\,\mathrm{d}s, \tag{2.3}$$

where $\rho_F := H_\ell^{1+\gamma}/h_F$ for $F \in \mathcal{F}_\ell$, and $-1 < \gamma < 3$. These stabilization functions are modifications of [3, equation (6.1)].

With this, we define $J_\ell := J + a_\ell$ on $H^2(\mathcal{T}_\ell;\mathbb{R}^m) \times H^2(\mathcal{T}_\ell;\mathbb{R}^m)$. Furthermore, we define a seminorm by $|\cdot|_\ell^2 = a_\ell(\cdot,\cdot)$.

Here and throughout the paper we assume there exist $u \in \mathcal{A} \cap W^{2,p}(\mathcal{T}_0;\mathbb{R}^m) \cap H^{3/2+\varepsilon}(\Omega;\mathbb{R}^m)$ (with $\varepsilon > 0$) and $u_\ell \in \mathcal{A}_\ell$ such that $\sigma := S(\nabla u)$ and $\sigma_\ell := S(\nabla u_\ell)$ satisfy (1.3)–(1.4), which reads

$$\begin{aligned} \int_\Omega \sigma : \nabla v\,\mathrm{d}x + J(u;v) = 0 \quad &\text{for all } v \in V,\\ \int_\Omega \sigma_\ell : \nabla v_\ell\,\mathrm{d}x + J_\ell(u_\ell;v_\ell) = 0 \quad &\text{for all } v_\ell \in V_\ell. \end{aligned} \tag{2.4}$$

We denote $e_\ell := u - u_\ell$ and $\delta_\ell := \sigma - \sigma_\ell$.

2.4. Assumptions on low-order terms. The following assumptions on the derivative J of the low-order term are similar to [3, (H4)–(H5)], where u, σ and u_ℓ, σ_ℓ are defined as in the preceding subsection. Only one of these alternative assumptions needs to be satisfied for the main theorem to be applicable.

Assumption 3. There exist $0 < m \le M < \infty$ such that

$$\begin{aligned} \beta_\ell := m\,\|e_\ell\|_{L^2(\Omega)}^2 &\le J(u;e_\ell) - J(u_\ell;e_\ell),\\ J(u;v) - J(u_\ell;v) \;&\le\; M\,\|e_\ell\|_{L^2(\Omega)}\,\|v\|_{L^2(\Omega)} \quad \text{for all } v \in W^{1,p}(\Omega;\mathbb{R}^m). \end{aligned}$$

Furthermore, we define $\zeta := r/(r-1)$.

Assumption 4. There exist $M > 0$, a constant vector $z \in \mathbb{R}^2$, $|z| = 1$, and a constant $C_z > 0$ such that

$$\begin{aligned} \beta_\ell := 0 &\le J(u;e_\ell) - J(u_\ell;e_\ell),\\ J(u;v) - J(u_\ell;v) \;&\le\; M\,\|e_\ell\|_{L^2(\Omega)}\,\|v\|_{L^2(\Omega)} \quad \text{for all } v \in W^{1,p}(\Omega;\mathbb{R}^m), \end{aligned}$$

$$\|z\cdot\nabla e_\ell\|_{L^2(\Omega)}^2 \;\le\; C_z \int_\Omega \delta_\ell : \nabla e_\ell\,\mathrm{d}x. \tag{2.5}$$

Furthermore, we define $\zeta := 2$.

2.5. Example: Scalar two-well problem. Let $X_1, X_2 \in \mathbb{R}^2$ with $X_1 \neq X_2$ and define the *energy density*

$$W : \mathbb{R}^2 \to \mathbb{R},\ X \mapsto |X - X_1|^2\,|X - X_2|^2.$$

The space of admissible functions is here given by $\mathcal{A} := W_0^{1,4}(\Omega) + u_D$ with a fixed function $u_D \in W^{1,4}(\Omega)$ with $u_D|_{\partial\Omega} \in H^2(\partial\Omega)$. Let W^{**} be the convex hull of W; then the (convexified) energy function $E : \mathcal{A} \to \mathbb{R}$ is

$$E(v) = \int_\Omega W^{**}(\nabla v)\,\mathrm{d}x + \lambda\,\|f - v\|_{L^2(\Omega)}^2 - \int_\Omega g v\,\mathrm{d}x,$$

where $f, g \in L^2(\Omega)$ are fixed functions and $\lambda \geq 0$ is some fixed constant. Here the derivative J of the lower-order term reads as follows: $J(v;w) = \int_\Omega (2\lambda(v-f)-g)\, w \,\mathrm{d}x$.

The minimization of E in $\mathcal{A}$ can be discretized according to (1.2), with the stabilization function (2.3) for $\gamma = 1$; that is

$$a_\ell(v,w) := \sum_{F\in\mathcal{F}_\ell^\Omega} \frac{H_\ell^2}{h_F} \int_F [\nabla v] : [\nabla w] \,\mathrm{d}s.$$

According to [4, Corollary 1], for any $X, Y \in \mathbb{R}^2$ and $Y_1 = (X_2 - X_1)/2$ and $Y_2 = (X_2 + X_1)/2$,

$$\begin{aligned} |S(Y) - S(X)|^2 \leq 16 \max\left\{1, |Y_1|^2 + 2\,|Y_2|^2\right\} \\ \times \left(1 + |X|^2 + |Y|^2\right) (S(Y) - S(X)) \cdot (Y - X). \end{aligned}$$

The result [4, Corollary 1] also shows

$$|X|^4 - 1 \lesssim W(X) \lesssim 1 + |X|^4 \quad \text{for all } X \in \mathbb{R}^2.$$

This is inherited by W^{**}.

Young's and Friedrichs' inequalities yield

$$\begin{aligned} \left|\lambda \left\|f - v\right\|_{L^2(\Omega)}^2 - \int_\Omega gv \,\mathrm{d}x\right| \lesssim 1 + \left\|v\right\|_{L^2(\Omega)}^2 \\ \lesssim 1 + \left\|v\right\|_{L^4(\Omega)}^2 \lesssim 1 + |v|_{W^{1,4}(\Omega)}^2 \quad \text{for all } v \in \mathcal{A}, \end{aligned}$$

where $\|\cdot\|_{L^2(\Omega)} \lesssim \|\cdot\|_{L^4(\Omega)}$ is a consequence of Hölder's inequality.

With $\zeta(x) = |2\lambda f(x) + g(x)| + \sup_{\xi>0}\left(2\lambda\xi - |\xi|^3\right)$ we also have

$$|DL(x,\xi;\cdot)| = |2\lambda(\xi - f(x)) - g(x)| \leq \zeta(x) + |\xi|^3.$$

Hence the problem at hand fulfills Assumptions 1 and 2.

By applying a Cauchy–Schwarz inequality to the definition of J, it follows that the continuous solution u of (1.3) and the discrete solutions u_ℓ of (1.4) satisfy, for all $v \in W^{1,4}(\Omega)$,

$$\begin{aligned} \lambda \left\|u - u_\ell\right\|_{L^2(\Omega)}^2 &\leq J(u; u - u_\ell) - J(u_\ell; u - u_\ell), \\ J(u;v) - J(u_\ell;v) &\leq \lambda \left\|u - u_\ell\right\|_{L^2(\Omega)} \left\|v\right\|_{L^2(\Omega)}. \end{aligned}$$

This proves Assumptions 3 (for $\lambda > 0$) or 4 (for $\lambda = 0$), the latter up to condition (2.5), which is shown in the proof of [3, Lemma 9.1] for the problem at hand. Together with the preceding results, this proves all the assumptions necessary in order to apply Theorem 4.4.

THEOREM 2.2. *There exists at most one minimum* $u \in \mathcal{A} \cap W^{2,p}(\mathcal{T}_0) \cap H^{3/2+\varepsilon}(\Omega)$ *(with* $\varepsilon > 0$*). Such a solution* u *satisfies* $\|u - u_\ell\|_{H^1(\Omega)}^2 = \mathcal{O}(H_\ell)$.

3. A general error estimate. This section generalizes [3] from quasi-uniform triangulations to general shape-regular triangulations with proper weights.

LEMMA 3.1. *Let* $v \in W^{2,t}(\mathcal{T}_0)$ *for some* $1 < t < \infty$. *Then,* $w_\ell = v - I_\ell v$ *satisfies*

$$\|w_\ell\|_{L^t(\Omega)} + H_\ell \left\|\nabla w_\ell\right\|_{L^t(\Omega)} \lesssim H_\ell^2 \left\|\nabla_\ell^2 v\right\|_{L^t(\Omega)}.$$

Proof. The estimates in (2.1) with $\nabla_\ell^2 w_\ell = \nabla_\ell^2 v$ lead to

$$\|w_\ell\|_{L^t(\Omega)}^t = \sum_{T\in\mathcal{T}_\ell} \|w_\ell\|_{L^t(T)}^t \lesssim \sum_{T\in\mathcal{T}_\ell} h_T^{2t} \left\|\nabla^2 v\right\|_{L^t(T)}^t \lesssim H_\ell^{2t} \left\|\nabla_\ell^2 v\right\|_{L^t(\Omega)}^t,$$
$$\|\nabla w_\ell\|_{L^t(\Omega)}^t = \sum_{T\in\mathcal{T}_\ell} \|\nabla w_\ell\|_{L^t(T)}^t \lesssim \sum_{T\in\mathcal{T}_\ell} h_T^{t} \left\|\nabla^2 v\right\|_{L^t(T)}^t \lesssim H_\ell^{t} \left\|\nabla_\ell^2 v\right\|_{L^t(\Omega)}^t. \qquad \square$$

LEMMA 3.2. *Let $v \in H^2(\mathcal{T}_0)$. Then, $w_\ell = v - I_\ell v$ satisfies*

$$|w_\ell|_\ell^2 \lesssim H_\ell^{1+\gamma} \left\|\nabla_\ell^2 v\right\|_{L^2(\Omega)}^2.$$

Proof. The trace inequality, (2.1), and $h_T \approx h_F$ yield

$$\begin{aligned}\|[\nabla w_\ell]\|_{L^2(F)}^2 &\lesssim \left\|\nabla w_\ell|_{T_+}\right\|_{L^2(F)}^2 + \left\|\nabla w_\ell|_{T_-}\right\|_{L^2(F)}^2 \\ &\lesssim h_F \left\|\nabla_\ell^2 w_\ell\right\|_{L^2(\omega_F)}^2 + h_F^{-1} \|\nabla w_\ell\|_{L^2(\omega_F)}^2 \\ &\lesssim h_F \left\|\nabla_\ell^2 w_\ell\right\|_{L^2(\omega_F)}^2 + h_F^{-1} h_T^2 \left\|\nabla_\ell^2 w_\ell\right\|_{L^2(\omega_F)}^2 \\ &\approx h_F \left\|\nabla_\ell^2 v\right\|_{L^2(\omega_F)}^2.\end{aligned}$$

Recall that ω_F denotes the patch of $F \in \mathcal{F}_\ell^\Omega$, i.e., the union set of the two triangles (or tetrahedra) having the common edge (or face) F. The definition of $|\cdot|_\ell$ then gives

$$|w_\ell|_\ell^2 = \sum_{F\in\mathcal{F}_\ell^\Omega} \rho_F \|[\nabla w_\ell]\|_{L^2(F)}^2 \lesssim \sum_{F\in\mathcal{F}_\ell^\Omega} \rho_F h_F \left\|\nabla_\ell^2 v\right\|_{L^2(\omega_F)}^2 \lesssim H_\ell^{1+\gamma} \left\|\nabla_\ell^2 v\right\|_{L^2(\Omega)}^2. \qquad \square$$

LEMMA 3.3. *The discrete solutions are bounded in $W^{1,p}$, $\|u_\ell\|_{W^{1,p}(\Omega)} \lesssim 1$.*

Proof (see [5, Lemma 4.1]). With Assumption 2, the triangle inequality, and Lemmas 3.1 and 3.2, we have for $w_\ell = u_D - u_{D,\ell}$,

$$\begin{aligned}E(u_\ell) &\le E_\ell(u_\ell) \le E_\ell(u_{D,\ell}) \\ &= \int_\Omega W^{**}(\nabla u_{D,\ell}(x))\,\mathrm{d}x + \int_\Omega L(x, u_{D,\ell}(x))\,\mathrm{d}x + \frac{1}{2}\left|u_{D,\ell}\right|_\ell^2 \\ &\lesssim 1 + |u_{D,\ell}|_{W^{1,p}(\Omega)}^p + |u_{D,\ell}|_{W^{1,p}(\Omega)}^q + |u_{D,\ell}|_\ell^2 \\ &\lesssim 1 + |u_D|_{W^{1,p}(\Omega)}^p + |w_\ell|_{W^{1,p}(\Omega)}^p + |u_D|_{W^{1,p}(\Omega)}^q + |w_\ell|_{W^{1,p}(\Omega)}^q + |w_\ell|_\ell^2 \\ &\lesssim 1 + H_\ell^p + H_\ell^q + H_\ell^{1+\gamma} \lesssim 1.\end{aligned}$$

On the other hand, Assumption 2 also yields

$$\begin{aligned}E(u_\ell) &= \int_\Omega W^{**}(\nabla u_\ell(x))\,\mathrm{d}x + \int_\Omega L(x, u_\ell(x))\,\mathrm{d}x \\ &\gtrsim |u_\ell|_{W^{1,p}(\Omega)}^p - |u_\ell|_{W^{1,p}(\Omega)}^q - 1.\end{aligned}$$

Hence there exists an ℓ-independent constant $C > 0$ such that

$$|u_\ell|_{W^{1,p}(\Omega)}^p - |u_\ell|_{W^{1,p}(\Omega)}^q - 1 \le C.$$

Since $p > q$, $|u_\ell|_{W^{1,p}(\Omega)}$ is bounded by the positive root of $x^p - x^q - 1 - C = 0$.

The boundedness of $|u_\ell|_{W^{1,p}(\Omega)}$ and Friedrichs' inequality also lead to the boundedness of $\|u_\ell\|_{L^p(\Omega)}$. ∎

Remark 3.4. Similar arguments can be used to prove the boundedness of any exact solution u, i.e., $\sup_{u \text{ solves (1.1)}} \|u\|_{W^{1,p}(\Omega)} < \infty$. However, since we fix some solution u throughout the arguments, the analysis focuses on the behavior of constants as $\ell \longrightarrow \infty$.

The following result is essentially [3, Lemma 4.1] with the same outline of the proof but somehow different constants.

LEMMA 3.5. *Assumptions* 1 *and* 2 *yield*

$$\|\delta_\ell\|^r_{L^{p'}(\Omega)} \le c_1 \int_\Omega \delta_\ell : \nabla e_\ell \,\mathrm{d}x,$$

where p' is the conjugate exponent of p, $1/p + 1/p' = 1$, and c_1 is given by

$$c_1 = 3^{\frac{p-s}{p}} \alpha |\Omega|^{\frac{r}{p'} - \frac{s}{p} - 1} \left(|\Omega| + |u|^p_{W^{1,p}(\Omega)} + |u_\ell|^p_{W^{1,p}(\Omega)} \right)^{\frac{s}{p}},$$

which is bounded with respect to ℓ.

Proof. The proof follows that of [4, Theorem 2]. Define $t := 1 + s/p$. Applying the tth root to (2.2) shows

$$|\delta_\ell|^{r/t} \le \alpha^{1/t} \left(1 + |\nabla u|^s + |\nabla u_\ell|^s\right)^{1/t} \left(\delta_\ell : \nabla e_\ell\right)^{1/t}.$$

Integrating and applying Hölder's inequality with exponents t' and t (with $1/t + 1/t' = 1$) results in

$$\begin{aligned} \|\delta_\ell\|^{r/t}_{L^{r/t}(\Omega)} &\le \alpha^{1/t} \int_\Omega \left(1 + |\nabla u|^s + |\nabla u_\ell|^s\right)^{1/t} \left(\delta_\ell : \nabla e_\ell\right)^{1/t} \mathrm{d}x \\ &\le \alpha^{1/t} \left(\int_\Omega \left(1 + |\nabla u|^s + |\nabla u_\ell|^s\right)^{t'/t} \mathrm{d}x \right)^{1/t'} \left(\int_\Omega \delta_\ell : \nabla e_\ell \,\mathrm{d}x \right)^{1/t}. \end{aligned}$$

Here $r/t > 1$ is enforced by Assumption 1. With $tp = st'$ and $\|\cdot\|_{L^{p'}(\Omega)} \le |\Omega|^{\frac{1}{p'} - \frac{t}{r}} \|\cdot\|_{L^{r/t}(\Omega)}$ (a consequence of Hölder's inequality), this yields

$$\|\delta_\ell\|^r_{L^{p'}(\Omega)} \le |\Omega|^{\frac{r}{p'} - t} \alpha \left(\int_\Omega \left(1 + |\nabla u|^s + |\nabla u_\ell|^s\right)^{p/s} \mathrm{d}x \right)^{s/p} \int_\Omega \delta_\ell : \nabla e_\ell \,\mathrm{d}x.$$

Assumption 1 implies $p > s$. The $L^{p/s}$-norm on the right-hand side is estimated by applying another Hölder's inequality with exponents $p/(p-s)$ and p/s in $\mathbb{R}^3$, i.e.,

$$\begin{aligned} 1 + |\nabla u|^s + |\nabla u_\ell|^s &= \langle [1,1,1] \,,\, [1, |\nabla u|^s, |\nabla u_\ell|^s] \rangle_{\ell^2} \\ &\le 3^{\frac{p-s}{p}} \left(1 + |\nabla u|^p + |\nabla u_\ell|^p\right)^{\frac{s}{p}}. \end{aligned}$$

Therefore

$$\int_\Omega \left(1 + |\nabla u|^s + |\nabla u_\ell|^s\right)^{\frac{p}{s}} \mathrm{d}x \le 3^{\frac{p-s}{s}} \left(|\Omega| + |u|^p_{W^{1,p}(\Omega)} + |u_\ell|^p_{W^{1,p}(\Omega)} \right).$$

Combining the last inequalities yields the claimed inequality. The constant c_1 is bounded due to Lemma 3.3. ∎

LEMMA 3.6 (see [3, Lemma 4.3]). *Let $r' > 0$ be such that $1/r + 1/r' = 1$; then Assumptions* 1 *and* 2 *yield with* $c_2 = \left(2^{r'} c_1^{r'-1}\right)/r'$ *for every* $v_\ell \in V_\ell$,

$$2 \int_\Omega \delta_\ell : \nabla(e_\ell - v_\ell)\,\mathrm{d}x \le r^{-1} \int_\Omega \delta_\ell : \nabla e_\ell \,\mathrm{d}x + c_2 \left| e_\ell - v_\ell \right|_{W^{1,p}(\Omega)}^{r'}.$$

The preceding lemma is proven in [3, Lemma 4.3].

LEMMA 3.7 (see [3, Lemma 4.2]). *Assumption* 3 *or* 4 *implies for every* $v_\ell \in V_\ell$,

$$\int_\Omega \delta_\ell : \nabla e_\ell \,\mathrm{d}x + \frac{1}{2} \left| e_\ell \right|_\ell^2 + \beta_\ell \\ \le \int_\Omega \delta_\ell : \nabla(e_\ell - v_\ell)\,\mathrm{d}x + M \left\| e_\ell \right\|_{L^2(\Omega)} \left\| e_\ell - v_\ell \right\|_{L^2(\Omega)} + \frac{1}{2} \left| e_\ell - v_\ell \right|_\ell^2.$$

The proof of the preceding lemma is analogous to the proof of [3, Lemma 4.2].

THEOREM 3.8 (see [3, Theorem 2.1]). *With the conjugate exponent* p' *of* p, *Assumptions* 1 *and* 2 *together with Assumption* 3 *or* 4 *imply for every* $v_\ell \in V_\ell$,

$$(1 - r^{-1}) \int_\Omega \delta_\ell : \nabla e_\ell \,\mathrm{d}x + c_1^{-1} \left\| \delta_\ell \right\|_{L^{p'}(\Omega)}^r + \left| e_\ell \right|_\ell^2 + 2\beta_\ell \\ \le c_2 \left| e_\ell - v_\ell \right|_{W^{1,p}(\Omega)}^{r/(r-1)} + 2M \left\| e_\ell \right\|_{L^2(\Omega)} \left\| e_\ell - v_\ell \right\|_{L^2(\Omega)} + \left| e_\ell - v_\ell \right|_\ell^2.$$

All terms on the left-hand side are nonnegative. The constants $c_1, c_2 > 0$ *(as given in Lemmas* 3.5 *and* 3.6*) are bounded with respect to* ℓ.

Proof. This follows from the preceding lemmas. ∎

4. Convergence results. In the following we assume that Assumptions 1 and 2 hold. Further, we assume Assumption 3 or 4 to be true and that β_ℓ and ζ are defined accordingly. Furthermore, we will denote $v_\ell = I_\ell e_\ell$ and $w_\ell = e_\ell - v_\ell$.

Based on the two cases for β_ℓ and ζ, our next step is proving the L^2-convergence of u_ℓ.

LEMMA 4.1 (L^2-convergence). *Assumption* 3 *implies*

$$\left\| e_\ell \right\|_{L^2(\Omega)}^2 + \left| e_\ell \right|_\ell^2 \lesssim H_\ell^\xi \quad \text{for } \xi := \min\left\{ 1 + \gamma \ , \ \frac{r}{r-1} \right\}.$$

Proof. Theorem 3.8 and Young's inequality lead to

$$2m \left\| e_\ell \right\|_{L^2(\Omega)}^2 + \left| e_\ell \right|_\ell^2 \le c_2 \left| w_\ell \right|_{W^{1,p}(\Omega)}^{r/(r-1)} + m \left\| e_\ell \right\|_{L^2(\Omega)}^2 + \frac{M^2}{m} \left\| w_\ell \right\|_{L^2(\Omega)}^2 + \left| w_\ell \right|_\ell^2.$$

We can apply the estimates provided by Lemmas 3.1 and 3.2 (with $v = e_\ell$) after subtracting $m \left\| e_\ell \right\|_{L^2(\Omega)}^2$, which concludes the proof. ∎

LEMMA 4.2 (L^2-convergence). *Assumption* 4 *implies*

$$\left\| e_\ell \right\|_{L^2(\Omega)}^2 + \left| e_\ell \right|_\ell^2 \lesssim H_\ell^\xi \quad \text{for } \xi := \min\left\{ 1 + \gamma \ , \ 2 \right\}.$$

Proof. As $v_\ell = 0$ on the boundary $\partial\Omega$, a one-dimensional Friedrichs inequality shows $\left\| v_\ell \right\|_{L^2(\Omega)} \lesssim \left\| z \cdot \nabla v_\ell \right\|_{L^2(\Omega)}$, with z given in Assumption 4. This assumption then yields

$$\begin{aligned}\|e_\ell\|^2_{L^2(\Omega)} &\lesssim \|w_\ell\|^2_{L^2(\Omega)} + \|v_\ell\|^2_{L^2(\Omega)}\\ &\lesssim \|w_\ell\|^2_{L^2(\Omega)} + \|z\cdot\nabla v_\ell\|^2_{L^2(\Omega)}\\ &\lesssim \|w_\ell\|^2_{L^2(\Omega)} + \|z\cdot\nabla e_\ell\|^2_{L^2(\Omega)} + \|z\cdot\nabla w_\ell\|^2_{L^2(\Omega)}\\ &\lesssim \|w_\ell\|^2_{L^2(\Omega)} + \|\nabla w_\ell\|^2_{L^2(\Omega)} + \int_\Omega \delta_\ell : \nabla e_\ell.\end{aligned}$$

We estimate the last term with Theorem 3.8, which gives us

$$\|e_\ell\|^2_{L^2(\Omega)} \lesssim \|w_\ell\|^2_{L^2(\Omega)} + \|\nabla w_\ell\|^2_{L^2(\Omega)} + |w_\ell|^{r/(r-1)}_{W^{1,p}(\Omega)} + \|e_\ell\|_{L^2(\Omega)}\,\|w_\ell\|_{L^2(\Omega)} + |w_\ell|^2_\ell\,.$$

We absorb $\|e_\ell\|_{L^2(\Omega)}$ with the left-hand side, which leaves another $\|w_\ell\|^2_{L^2(\Omega)}$. All remaining terms can be estimated with Lemmas 3.1 and 3.2 (with $v = e_\ell$). Observing $r/(r-1) \geq 2$ proves the estimate on $\|e_\ell\|_{L^2(\Omega)}$. Using this and the above estimates in Theorem 3.8 again, together with Young's inequality, we obtain

$$|e_\ell|^2_\ell \lesssim |w_\ell|^{r/(r-1)}_{W^{1,p}(\Omega)} + \|e_\ell\|^2_{L^2(\Omega)} + \|w_\ell\|^2_{L^2(\Omega)} + |w_\ell|^2_\ell\,,$$

which leads to the claimed estimate on $|e_\ell|_\ell$. ◻

COROLLARY 4.3 (uniqueness of the continuous solution). *The solution u given by* (2.4) *is unique.*

Proof. With Lemma 4.1 or 4.2, respectively, the discrete solutions u_ℓ L^2 converge to the continuous solution u for $H_\ell \longrightarrow 0$. This holds for all continuous solutions u, but the limit is unique. ◻

The following main result implies the H^1 error estimate $\|u - u_\ell\|_{H^1(\Omega)} = \mathcal{O}(H_\ell^{1/2})$ for $\gamma = 1$.

THEOREM 4.4 (H^1-convergence). *Either Assumption* 3 *or Assumption* 4 *implies*

$$\|\nabla e_\ell\|_{L^2(\Omega)} \lesssim H_\ell^{\xi/2} \quad \text{for } \xi = \min\left\{\frac{1+\gamma}{2}, \zeta - \frac{1+\gamma}{2}\right\}.$$

Proof. An integration by parts yields

$$\begin{aligned}\|\nabla e_\ell\|^2_{L^2(\Omega)} &= \sum_{T\in\mathcal{T}_\ell}\int_T \nabla e_\ell : \nabla e_\ell \,\mathrm{d}x\\ &= \sum_{T\in\mathcal{T}_\ell}\left(\int_{\partial T}(\nabla e_\ell\cdot\nu)e_\ell\,\mathrm{d}s - \int_T e_\ell\,\Delta e_\ell\,\mathrm{d}x\right)\\ &= \int_{\partial\Omega}(\nabla e_\ell\cdot\nu)e_\ell\,\mathrm{d}s - \int_\Omega e_\ell\,\Delta_\ell e_\ell\,\mathrm{d}x\\ &\quad + \sum_{F\in\mathcal{F}_\ell^\Omega}\int_F([\nabla e_\ell]\cdot\nu)e_\ell\,\mathrm{d}s\\ &=: A + B + C.\end{aligned}$$

Estimate on A. Since $e_\ell = u_D - I_\ell u_D$ on the boundary $\partial\Omega$, we may apply (2.1) to each edge or face $F \in \mathcal{F}_\ell^{\partial\Omega}$, which leads to

$$\|e_\ell\|_{L^2(F)} \lesssim h_F^2\left\|\nabla^2 e_\ell|_F\right\|_{L^2(F)} = h_F^2\left\|\nabla^2 u_D|_F\right\|_{L^2(F)}.$$

Here $\nabla^2 u_D|_F$ denotes the $(n-1)$-dimensional Hessian of u_D on the face or edge F. We approximate the normal derivative $\nabla e_\ell \cdot \nu$ with the strong trace inequality. With $F \subset T$, this yields

$$\|\nabla e_\ell \cdot \nu\|^2_{L^2(F)} \leq \|\nabla e_\ell\|^2_{L^2(F)} \lesssim \|\nabla e_\ell\|_{L^2(T)} \left(\|\nabla e_\ell\|_{L^2(T)} + h_T^{-1} \left\|\nabla^2 u\right\|_{L^2(T)} \right).$$

Summing up all $F \in \mathcal{F}_\ell^{\partial\Omega}$ and applying several Cauchy inequalities proves

$$\begin{aligned}
|A| &\lesssim \sum_{F \in \mathcal{F}_\ell^{\partial\Omega}} \|e_\ell\|_{L^2(F)} \|\nabla e_\ell \cdot \nu\|_{L^2(F)} \\
&\lesssim \sum_{\substack{F \in \mathcal{F}_\ell^{\partial\Omega} \\ F \subset T \in \mathcal{T}_\ell}} h_F^2 \left\|\nabla^2 u_D|_F\right\|_{L^2(F)} \|\nabla e_\ell\|^{1/2}_{L^2(T)} \left(\|\nabla e_\ell\|^{1/2}_{L^2(T)} + h_T^{-1/2} \left\|\nabla^2 u\right\|^{1/2}_{L^2(T)} \right) \\
&\lesssim H_\ell^2 \left\|\nabla_\ell^2 u_D|_{\partial\Omega}\right\|_{L^2(\partial\Omega)} \|\nabla e_\ell\|_{L^2(\Omega)} \\
&\quad + H_\ell^{3/2} \left\|\nabla_\ell^2 u_D|_{\partial\Omega}\right\|_{L^2(\partial\Omega)} \|\nabla e_\ell\|^{1/2}_{L^2(\Omega)} \left\|\nabla^2 u\right\|^{1/2}_{L^2(\Omega)}.
\end{aligned}$$

Estimate on B. Since $u_\ell \in \mathcal{S}^1(\mathcal{T}_\ell)$, it holds that $\nabla_\ell^2 e_\ell = \nabla_\ell^2 u$. Therefore

$$|B| = \left| \int_\Omega e_\ell \, \Delta_\ell e_\ell \, \mathrm{d}x \right| = \left| \int_\Omega e_\ell \, \Delta_\ell u \, \mathrm{d}x \right| \lesssim \|e_\ell\|_{L^2(\Omega)}.$$

Estimate on C. With ρ_F and a Cauchy inequality,

$$\begin{aligned}
|C| &= \left| \sum_{F \in \mathcal{F}_\ell^\Omega} \int_F \left(\rho_F^{1/2} ([\nabla e_\ell] \cdot \nu) \right) \left(\rho_F^{-1/2} e_\ell \right) \mathrm{d}s \right| \\
&\leq \sum_{F \in \mathcal{F}_\ell^\Omega} \left\| \rho_F^{1/2} [\nabla e_\ell] \cdot \nu \right\|_{L^2(F)} \left\| \rho_F^{-1/2} e_\ell \right\|_{L^2(F)} \\
&\leq \sqrt{\sum_{F \in \mathcal{F}_\ell^\Omega} \rho_F \, \|[\nabla e_\ell] \cdot \nu\|^2_{L^2(F)}} \sqrt{\sum_{F \in \mathcal{F}_\ell^\Omega} \rho_F^{-1} \|e_\ell\|^2_{L^2(F)}} \\
&= |e_\ell|_\ell \sqrt{\sum_{F \in \mathcal{F}_\ell^\Omega} \rho_F^{-1} \|e_\ell\|^2_{L^2(F)}}.
\end{aligned}$$

The trace inequality yields

$$\begin{aligned}
\sum_{F \in \mathcal{F}_\ell^\Omega} \rho_F^{-1} \|e_\ell\|^2_{L^2(F)} &\lesssim \sum_{F \in \mathcal{F}_\ell^\Omega} \rho_F^{-1} \left(h_F \|\nabla e_\ell\|^2_{L^2(\omega_F)} + h_F^{-1} \|e_\ell\|^2_{L^2(\omega_F)} \right) \\
&\leq \max_{F \in \mathcal{F}_\ell^\Omega} \left(\rho_F^{-1} h_F \right) \sum_{F \in \mathcal{F}_\ell^\Omega} \|\nabla e_\ell\|^2_{L^2(\omega_F)} \\
&\quad + \max_{F \in \mathcal{F}_\ell^\Omega} \left(\rho_F^{-1} h_F^{-1} \right) \sum_{F \in \mathcal{F}_\ell^\Omega} \|e_\ell\|^2_{L^2(\omega_F)} \\
&\lesssim H_\ell^{1-\gamma} \|\nabla e_\ell\|^2_{L^2(\Omega)} + H_\ell^{-1-\gamma} \|e_\ell\|^2_{L^2(\Omega)}.
\end{aligned}$$

By combining the preceding estimates and absorbing $\|\nabla e_\ell\|_{L^2(\Omega)}$ and $\|\nabla e_\ell\|^{1/2}_{L^2(\Omega)}$, we obtain

$$\|\nabla e_\ell\|^2_{L^2(\Omega)} \lesssim H_\ell^4 + H_\ell^2 + \|e_\ell\|_{L^2(\Omega)} + H_\ell^{1-\gamma} |e_\ell|^2_\ell + H_\ell^{-\frac{1+\gamma}{2}} \|e_\ell\|_{L^2(\Omega)} |e_\ell|_\ell.$$

Assumption 3 and Lemma 4.1 imply $\|\nabla e_\ell\|_{L^2(\Omega)} \lesssim H_\ell^{\xi/2}$ with

$$\xi = \min\left\{ 4 \ , \ 2 \ , \ \frac{r}{2(r-1)} \ , \ \frac{1+\gamma}{2} \ , \ \frac{r}{r-1} - \frac{1+\gamma}{2} \ , \ \frac{r}{r-1} + 1 - \gamma \ \right\}.$$

Since $-\frac{1+\gamma}{2} \le 1-\gamma$ and $\frac{\gamma+1}{2} \le 2$ as well as $\frac{r}{2(r-1)} = \frac{1}{2}((\frac{r}{r-1} - \frac{1+\gamma}{2}) + (\frac{1+\gamma}{2}))$, this representation of ξ leads to the theorem.

Assumption 4 implies that the term $\frac{r}{r-1}$ is replaced with 2, due to Lemma 4.2, which concludes the proof. ∎

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CORRECTION TO "ON THE STRONG CONVERGENCE OF GRADIENTS IN STABILIZED DEGENERATE CONVEX MINIMIZATION PROBLEMS"

Because of a production error, section 4 in "On the Strong Convergence of Gradients in Stabilized Degenerate Convex Minimization Problems" by Wolfgang Boiger and Carsten Carstensen is incorrect. The correct version is as follows.

4. Convergence results. In the following we assume that Assumptions 1 and 2 hold. Further, we assume Assumption 3 or 4 to be true and that β_ℓ and ζ are defined accordingly. Furthermore, we will denote $v_\ell = I_\ell e_\ell$ and $w_\ell = e_\ell - v_\ell$.

Based on the two cases for β_ℓ and ζ, our next step is proving the L^2-convergence of u_ℓ.

LEMMA 4.1 (L^2-convergence). *Assumption* 3 *implies*

$$\|e_\ell\|^2_{L^2(\Omega)} + |e_\ell|^2_\ell \lesssim H_\ell^\xi \quad \textit{for } \xi := \min\left\{1+\gamma\ ,\ \frac{r}{r-1}\right\}.$$

Proof. Theorem 3.8 and Young's inequality lead to

$$2m\,\|e_\ell\|^2_{L^2(\Omega)} + |e_\ell|^2_\ell \le c_2\,|w_\ell|^{r/(r-1)}_{W^{1,p}(\Omega)} + m\,\|e_\ell\|^2_{L^2(\Omega)} + \frac{M^2}{m}\,\|w_\ell\|^2_{L^2(\Omega)} + |w_\ell|^2_\ell\,.$$

We can apply the estimates provided by Lemmas 3.1 and 3.2 (with $v = e_\ell$) after subtracting $m\,\|e_\ell\|^2_{L^2(\Omega)}$, which concludes the proof. ∎

LEMMA 4.2 (L^2-convergence). *Assumption* 4 *implies*

$$\|e_\ell\|^2_{L^2(\Omega)} + |e_\ell|^2_\ell \lesssim H_\ell^\xi \quad \textit{for } \xi := \min\{1+\gamma\ ,\ 2\}.$$

Proof. As $v_\ell = 0$ on the boundary $\partial\Omega$, a one-dimensional Friedrichs inequality shows $\|v_\ell\|_{L^2(\Omega)} \lesssim \|z\cdot\nabla v_\ell\|_{L^2(\Omega)}$, with z given in Assumption 4. This assumption then yields

$$\begin{aligned}\|e_\ell\|^2_{L^2(\Omega)} &\lesssim \|w_\ell\|^2_{L^2(\Omega)} + \|v_\ell\|^2_{L^2(\Omega)}\\ &\lesssim \|w_\ell\|^2_{L^2(\Omega)} + \|z\cdot\nabla v_\ell\|^2_{L^2(\Omega)}\\ &\lesssim \|w_\ell\|^2_{L^2(\Omega)} + \|z\cdot\nabla e_\ell\|^2_{L^2(\Omega)} + \|z\cdot\nabla w_\ell\|^2_{L^2(\Omega)}\\ &\lesssim \|w_\ell\|^2_{L^2(\Omega)} + \|\nabla w_\ell\|^2_{L^2(\Omega)} + \int_\Omega \delta_\ell : \nabla e_\ell.\end{aligned}$$

We estimate the last term with Theorem 3.8, which gives us

$$\|e_\ell\|^2_{L^2(\Omega)} \lesssim \|w_\ell\|^2_{L^2(\Omega)} + \|\nabla w_\ell\|^2_{L^2(\Omega)} + |w_\ell|^{r/(r-1)}_{W^{1,p}(\Omega)} + \|e_\ell\|_{L^2(\Omega)}\,\|w_\ell\|_{L^2(\Omega)} + |w_\ell|^2_\ell\,.$$

We absorb $\|e_\ell\|_{L^2(\Omega)}$ with the left-hand side, which leaves another $\|w_\ell\|^2_{L^2(\Omega)}$. All remaining terms can be estimated with Lemmas 3.1 and 3.2 (with $v = e_\ell$). Observing $r/(r-1) \ge 2$ proves the estimate on $\|e_\ell\|_{L^2(\Omega)}$. Using this and the above estimates in Theorem 3.8 again, together with Young's inequality, we obtain

$$|e_\ell|^2_\ell \lesssim |w_\ell|^{r/(r-1)}_{W^{1,p}(\Omega)} + \|e_\ell\|^2_{L^2(\Omega)} + \|w_\ell\|^2_{L^2(\Omega)} + |w_\ell|^2_\ell\,,$$

which leads to the claimed estimate on $|e_\ell|_\ell$. ◻

COROLLARY 4.3 (uniqueness of the continuous solution). *The solution u given by (2.4) is unique.*

Proof. With Lemma 4.1 or 4.2, respectively, the discrete solutions u_ℓ L^2 converge to the continuous solution u for $H_\ell \longrightarrow 0$. This holds for all continuous solutions u, but the limit is unique. ◻

The following main result implies the H^1 error estimate $\|u - u_\ell\|_{H^1(\Omega)} = \mathcal{O}(H_\ell^{1/2})$ for $\gamma = 1$.

THEOREM 4.4 (H^1-convergence). *With ξ from Lemma 4.1 (respectively, Lemma 4.2), Assumption 3 or Assumption 4 implies*

$$\|\nabla e_\ell\|_{L^2(\Omega)} \lesssim H_\ell^\eta \quad \text{for } \eta = \min\left\{\frac{\xi}{4}, \frac{\xi}{2} - \frac{1+\gamma}{4}\right\}.$$

Proof. An integration by parts yields

$$\begin{aligned}
\|\nabla e_\ell\|^2_{L^2(\Omega)} &= \sum_{T\in\mathcal{T}_\ell} \int_T \nabla e_\ell : \nabla e_\ell \,\mathrm{d}x \\
&= \sum_{T\in\mathcal{T}_\ell} \left(\int_{\partial T} (\nabla e_\ell \cdot \nu) e_\ell \,\mathrm{d}s - \int_T e_\ell \,\Delta e_\ell \,\mathrm{d}x \right) \\
&= \int_{\partial\Omega} (\nabla e_\ell \cdot \nu) e_\ell \,\mathrm{d}s - \int_\Omega e_\ell \,\Delta_\ell e_\ell \,\mathrm{d}x \\
&\quad + \sum_{F\in\mathcal{F}_\ell^\Omega} \int_F ([\nabla e_\ell] \cdot \nu) e_\ell \,\mathrm{d}s \\
&=: \ A + B + C.
\end{aligned}$$

Estimate of A. Since $e_\ell = u_D - I_\ell u_D$ on the boundary $\partial\Omega$, we may apply (2.1) to each edge or face $F \in \mathcal{F}_\ell^{\partial\Omega}$, which leads to

$$\|e_\ell\|_{L^2(F)} \lesssim h_F^2 \left\|\nabla^2 e_\ell|_F\right\|_{L^2(F)} = h_F^2 \left\|\nabla^2 u_D|_F\right\|_{L^2(F)}.$$

Here $\nabla^2 u_D|_F$ denotes the $(n-1)$-dimensional Hessian of u_D on the face or edge F. We approximate the normal derivative $\nabla e_\ell \cdot \nu$ with the strong trace inequality. With $F \subset T$, this yields With $F \subset T$, this yields

$$\|\nabla e_\ell \cdot \nu\|^2_{L^2(F)} \le \|\nabla e_\ell\|^2_{L^2(F)} \lesssim \|\nabla e_\ell\|_{L^2(T)} \left(\left\|\nabla^2 u\right\|_{L^2(T)} + h_T^{-1} \|\nabla e_\ell\|_{L^2(T)} \right).$$

The sum over all $F \in \mathcal{F}_\ell^{\partial\Omega}$ and several Cauchy inequalities prove

$$\begin{aligned}
|A| &\lesssim \sum_{F\in\mathcal{F}_\ell^{\partial\Omega}} \|e_\ell\|_{L^2(F)} \|\nabla e_\ell \cdot \nu\|_{L^2(F)} \\
&\lesssim \sum_{\substack{F\in\mathcal{F}_\ell^{\partial\Omega} \\ F\subset T\in\mathcal{T}_\ell}} h_F^2 \left\|\nabla^2 u_D|_F\right\|_{L^2(F)} \|\nabla e_\ell\|^{1/2}_{L^2(T)} \left(\left\|\nabla^2 u\right\|^{1/2}_{L^2(T)} + h_T^{-1/2} \|\nabla e_\ell\|^{1/2}_{L^2(T)} \right) \\
&\lesssim H_\ell^{3/2} \left\|\nabla_\ell^2 u_D|_{\partial\Omega}\right\|_{L^2(\partial\Omega)} \|\nabla e_\ell\|_{L^2(\Omega)} \\
&\quad + H_\ell^2 \left\|\nabla_\ell^2 u_D|_{\partial\Omega}\right\|_{L^2(\partial\Omega)} \|\nabla e_\ell\|^{1/2}_{L^2(\Omega)} \left\|\nabla^2 u\right\|^{1/2}_{L^2(\Omega)}.
\end{aligned}$$

Estimate of B. Since $u_\ell \in \mathcal{S}^1(\mathcal{T}_\ell)$, it holds $\nabla_\ell^2 e_\ell = \nabla_\ell^2 u$. Therefore

$$|B| = \left| \int_\Omega e_\ell \, \Delta_\ell e_\ell \,\mathrm{d}x \right| = \left| \int_\Omega e_\ell \, \Delta_\ell u \,\mathrm{d}x \right| \lesssim \|e_\ell\|_{L^2(\Omega)} .$$

Estimate of C. With ρ_F and a Cauchy inequality,

$$\begin{aligned}
|C| &= \left| \sum_{F\in\mathcal{F}_\ell^\Omega} \int_F \left(\rho_F^{1/2}([\nabla e_\ell]\cdot\nu)\right)\left(\rho_F^{-1/2} e_\ell\right) \,\mathrm{d}s \right| \\
&\le \sum_{F\in\mathcal{F}_\ell^\Omega} \left\|\rho_F^{1/2}[\nabla e_\ell]\cdot\nu\right\|_{L^2(F)} \left\|\rho_F^{-1/2} e_\ell\right\|_{L^2(F)} \\
&\le \sqrt{\sum_{F\in\mathcal{F}_\ell^\Omega} \rho_F \left\|[\nabla e_\ell]\cdot\nu\right\|_{L^2(F)}^2} \sqrt{\sum_{F\in\mathcal{F}_\ell^\Omega} \rho_F^{-1} \|e_\ell\|_{L^2(F)}^2} \\
&= |e_\ell|_\ell \sqrt{\sum_{F\in\mathcal{F}_\ell^\Omega} \rho_F^{-1} \|e_\ell\|_{L^2(F)}^2}.
\end{aligned}$$

The trace inequality yields

$$\begin{aligned}
\sum_{F\in\mathcal{F}_\ell^\Omega} \rho_F^{-1} \|e_\ell\|_{L^2(F)}^2 &\lesssim \sum_{F\in\mathcal{F}_\ell^\Omega} \rho_F^{-1} \left(h_F \|\nabla e_\ell\|_{L^2(\omega_F)}^2 + h_F^{-1} \|e_\ell\|_{L^2(\omega_F)}^2 \right) \\
&\le \max_{F\in\mathcal{F}_\ell^\Omega} \left(\rho_F^{-1} h_F\right) \sum_{F\in\mathcal{F}_\ell^\Omega} \|\nabla e_\ell\|_{L^2(\omega_F)}^2 \\
&\quad + \max_{F\in\mathcal{F}_\ell^\Omega} \left(\rho_F^{-1} h_F^{-1}\right) \sum_{F\in\mathcal{F}_\ell^\Omega} \|e_\ell\|_{L^2(\omega_F)}^2 \\
&\lesssim H_\ell^{1-\gamma} \|\nabla e_\ell\|_{L^2(\Omega)}^2 + H_\ell^{-1-\gamma} \|e_\ell\|_{L^2(\Omega)}^2 .
\end{aligned}$$

By combining the preceding estimates and absorbing $\|\nabla e_\ell\|_{L^2(\Omega)}$ and $\|\nabla e_\ell\|_{L^2(\Omega)}^{1/2}$, we obtain

$$\|\nabla e_\ell\|_{L^2(\Omega)}^2 \lesssim H_\ell^4 + H_\ell^2 + \|e_\ell\|_{L^2(\Omega)} + H_\ell^{1-\gamma} |e_\ell|_\ell^2 + H_\ell^{-\frac{1+\gamma}{2}} \|e_\ell\|_{L^2(\Omega)} |e_\ell|_\ell .$$

Lemma 4.1 or 4.2 (with a different ξ) implies $\|\nabla e_\ell\|_{L^2(\Omega)} \lesssim H_\ell^\eta$ with

$$\eta = \min\left\{ \frac{3}{2}, \frac{4}{3}, \frac{\xi}{4}, \frac{\xi}{2} + \frac{1-\gamma}{2}, \frac{\xi}{2} - \frac{1+\gamma}{4} \right\}.$$

Since $\frac{\xi}{4} \le \frac{1+\gamma}{4} \le \frac{4}{3} \le \frac{3}{2}$ and $-\frac{1+\gamma}{4} < \frac{1-\gamma}{2}$ this representation of η leads to the theorem. $\square$