SIAM J. NUMER. ANAL.
Vol. 51, No. 5, pp. 2935–2955

DISCRETE RELIABILITY FOR CROUZEIX–RAVIART FEMs*

CARSTEN CARSTENSEN†, DIETMAR GALLISTL‡, AND MIRA SCHEDENSACK‡

Dedicated to Professor Piotr Matus on the occasion of his 60th birthday

Abstract. The proof of optimal convergence rates of adaptive finite element methods relies on Stevenson's concept of discrete reliability. This paper proves the general discrete reliability for the nonconforming Crouzeix–Raviart finite element method on multiply connected domains in *any* space dimension. A novel discrete quasi-interpolation operator of first-order approximation involves an intermediate triangulation and acts as the identity on unrefined simplices, to circumvent any Helmholtz decomposition. Besides the generalization of the known application to any dimension and multiply connected domains, this paper outlines the optimality proof for uniformly convex minimization problems. This discrete reliability implies reliability for the explicit residual-based a posteriori error estimator in *any* space dimension and for multiply connected domains.

Key words. nonconforming finite element, discrete reliability, adaptive FEM

AMS subject classifications. 65K10, 65M12, 65M60

DOI. 10.1137/130915856

1. Introduction.

The key ingredient in the proof of optimal convergence rates of adaptive finite element methods (AFEMs) based on a loop with the steps SOLVE, ESTIMATE, MARK, REFINE is the concept of discrete reliability, which is the seminal contribution of Stevenson [27] for conforming FEM. The discrete reliability states that the difference of the discrete solutions on two arbitrary levels u_ℓ and $u_{\ell+m}$ with respect to triangulations $\mathcal{T}_\ell$ and $\mathcal{T}_{\ell+m}$ is bounded by the contributions of the residual-based error estimator on the refined simplices $\mathcal{T}_\ell\setminus\mathcal{T}_{\ell+m}$ only. After some natural split of the error, the additional difficulty for the nonconforming FEMs is the proof of an estimate of the distance in the form

$$(1.1)\qquad \min_{v_{\ell+m}\in \mathrm{CR}_0^1(\mathcal{T}_{\ell+m})} \|\mathrm{D}_{\mathrm{NC}}(u_\ell - v_{\ell+m})\|^2_{L^2(\Omega)} \le C_{\mathrm{ddc}} \sum_{F\in\mathcal{F}(\mathcal{T}_\ell\setminus\mathcal{T}_{\ell+m})} h_F^{-1}\,\|[u_\ell]_F\|^2_{L^2(F)}\,.$$

Here and throughout this paper, $[\cdot]_F$ denotes the jump across a hyper-surface $F\in\mathcal{F}(T)$ of the simplex T with diameter h_F (more details on the notation of triangulations follow in section 2) and the sum runs over the set $\mathcal{F}(\mathcal{T}_\ell\setminus\mathcal{T}_{\ell+m})$ of hyper-surfaces of simplices in $\mathcal{T}_\ell\setminus\mathcal{T}_{\ell+m}$.

The proofs of (1.1) in the literature [4, 12, 15, 17, 26] utilize the discrete Helmholtz decomposition [1] and so focus on simply connected domains in dimension $n = 2$. The remaining contributions leave doubts: [3] obtains a constant $C_{\mathrm{ddc}}(m)$ in (1.1) which may depend on the number m of refinement steps as pointed out in [15, p. 292], while the authors of this paper seriously question lines 15–16 of [24, p. 140].

This paper provides a rigorous proof of the discrete distance control (1.1) for multiply connected domains $\Omega\subseteq\mathbb{R}^n$ in any space dimension $n\ge 2$. The main tool is the

*Received by the editors April 5, 2013; accepted for publication (in revised form) September 13, 2013; published electronically October 30, 2013. This work was supported by the DFG Research Center Matheon and the Berlin Mathematical School.

http://www.siam.org/journals/sinum/51-5/91585.html

†Institut für Mathematik, Humboldt-Universität zu Berlin, D-10099 Berlin, Germany, and Department of CSE, Yonsei University Seoul, Korea (cc@math.hu-berlin.de).

‡Institut für Mathematik, Humboldt-Universität zu Berlin, D-10099 Berlin, Germany (gallistl@math.hu-berlin.de, schedens@math.hu-berlin.de).

definition of a transfer operator which is based on an intermediate triangulation. This enables the discrete reliability for a couple of model problems like the Poisson problem, eigenvalue problems, Stokes equations, and linear elasticity and thereby shows optimal convergence of the AFEM for those problems in the general case. With (1.1) the analysis of the aforementioned papers [3, 4, 12, 15, 17, 24, 26] shows convergence of the respective AFEMs also for multiply connected domains in two, three, or even higher dimensions. For $m \to \infty$ the result (1.1) of this paper immediately leads to the reliability in the sense that

$$\min_{v \in H_0^1(\Omega)} \|\mathrm{D}_{\mathrm{NC}}(u_\ell - v)\|_{L^2(\Omega)}^2 \leq C_{\mathrm{ddc}} \sum_{F \in \mathcal{F}_\ell} h_F^{-1} \|[u_\ell]_F\|_{L^2(F)}^2 . \tag{1.2}$$

This generalizes [22] in two dimensions and [2] in three dimensions to multiply connected domains in any space dimension. The efficiency is the converse of (1.1) (resp., (1.2)) and is rather immediate [12, 31] via Verfürth's discrete test function technique [31].

As a novel application of the discrete reliability, this paper outlines the optimality proof for nonconforming FEM for uniformly convex minimization problems. The proof of the contraction property relies on the observation that the error of the FEM is equivalent to the difference of the exact and discrete energies up to some computable data term. For nonconforming FEMs, this technique seems to be a new argument.

The remaining parts of this paper are organized as follows. Section 2 provides the necessary preliminaries on regular triangulations into simplices and their refinement in any space dimension from [28]. The main result is stated in section 3 and proved in section 4 by means of a carefully designed transfer operator which is a discrete quasi interpolation for nonconforming finite element functions. Section 5 discusses applications to various model problems like linear problems, eigenvalue problems, the Stokes equations, and the Navier–Lamé equations of linear elasticity in the generalization of [10, 14]. Section 6 concludes the paper with a sketch of the proof of the optimality of a convex minimization problem. This illustrates how the discrete reliability (1.1) enters the analysis and also provides a novel application of nonconforming FEMs for a class of nonlinear problems.

Throughout this paper, standard notation on Lebesgue and Sobolev spaces and their norms is employed and $P_k(\omega)$ denotes the space of polynomials of degree $\leq k$. The piecewise action of the differential operators D and div is denoted by D_{NC} and $\mathrm{div}_{\mathrm{NC}}$. The formula $A \lesssim B$ represents an inequality $A \leq CB$ for some mesh-size independent, positive generic constant C; $A \approx B$ abbreviates $A \lesssim B \lesssim A$. By convention, all generic constants $C \approx 1$ depend neither on the mesh-size nor on the level of a triangulation but may depend on the fixed coarse triangulation $\mathcal{T}_0$ and its interior angles as well as on the space dimension n.

2. Triangulations and refinements.

This section recalls the concepts of triangulations and some suitable refinement strategies from [28] (which trace back to [25, 30]) and proves some properties of the refinement strategies for self-contained convenient reading.

2.1. Triangulations.

This section recalls the concepts of local mesh-refinements from [28] as a natural generalization of the newest-vertex-bisection in $\mathbb{R}^n$.

A tagged simplex $(z_0, \ldots, z_n; \gamma)$ is an $(n+2)$-tuple with vertices $z_0, \ldots, z_n \in \mathbb{R}^n$, which do not lie on an $(n-1)$-dimensional hyperplane, and a type $\gamma \in \{0, \ldots, n-1\}$. The mapping $\mathrm{dom} : \mathbb{R}^n \times \cdots \times \mathbb{R}^n \times \{0, \ldots, n-1\} \to 2^{\mathbb{R}^n}$ extracts the corresponding (closed) simplex $\mathrm{dom}(z_0, \ldots, z_n; \gamma) := \mathrm{conv}\{z_0, \ldots, z_n\}$ from a tagged simplex

$(z_0,\dots,z_n;\gamma)$. Given tagged simplices T,T', define for abbreviation $\partial T := \partial\,\mathrm{dom}(T)$, $T\cap T' := \mathrm{dom}(T)\cap\mathrm{dom}(T')$, $T\cup T' := \mathrm{dom}(T)\cup\mathrm{dom}(T')$, $v|_T := v|_{\mathrm{dom}(T)}$, and $\mathrm{int}(T) := \mathrm{int}(\mathrm{dom}(T))$.

Let $\mathcal{T}$ be a regular triangulation of the polyhedral bounded Lipschitz domain $\Omega\subseteq\mathbb{R}^n$ into simplices in the sense of Ciarlet. This means that the corresponding simplices $\mathrm{dom}(\mathcal{T}) := \{\mathrm{dom}(T) \mid T\in\mathcal{T}\}$ cover the domain $\overline{\Omega}$ and two distinct simplices $\mathrm{dom}(T) = \mathrm{conv}\{y_0,\dots,y_n\}$ and $\mathrm{dom}(T') := \mathrm{conv}\{z_0,\dots,z_n\}$ for $T,T'\in\mathcal{T}$ are either disjoint or share exactly one surface (e.g., an edge or a side) in the sense that there exist $0\le j_1<\cdots<j_N\le n$ and $0\le k_1<\cdots<k_N\le n$ for some $N\in\{1,\dots,n\}$ such that

$$T\cap T' = \mathrm{conv}\{y_{j_1},\dots,y_{j_N}\} = \mathrm{conv}\{z_{k_1},\dots,z_{k_N}\}.$$

The set of hyper-surfaces of a tagged simplex $T=(z_0,\dots,z_n;\gamma)\in\mathcal{T}$ with vertices $\mathcal{N}(T) := \{z_0,\dots,z_n\}$ is

$$\mathcal{F}(T) := \big\{\mathrm{conv}\{z_0,\dots,z_{k-1},z_{k+1},\dots,z_n\}\subseteq\mathbb{R}^n \;\big|\; k=0,\dots,n\big\}.$$

Let $\mathcal{F}(\mathcal{T})$ denote the set of all hyper-surfaces $\mathcal{F}(\mathcal{T}) := \bigcup_{T\in\mathcal{T}}\mathcal{F}(T)$ (e.g., the set of edges for $n=2$ and the set of faces for $n=3$) and let $\mathcal{N}(\mathcal{T}) := \bigcup_{T\in\mathcal{T}}\mathcal{N}(T)$ denote the set of all vertices. The set of simplices that share a vertex $z\in\mathcal{N}(\mathcal{T})$ reads

$$\mathcal{T}(z) := \{T\in\mathcal{T} \;\big|\; z\in\mathcal{N}(T)\}.$$

Any $F\in\mathcal{F}(\mathcal{T})$ is associated to a fixed orientation of the unit normal ν_F on F; on the boundary, ν_F is the outer unit normal of Ω. For an interior hyper-surface $F\not\subseteq\partial\Omega$ the orientation is fixed through the choice of the simplices $T_+\in\mathcal{T}$ and $T_-\in\mathcal{T}$ with $F=T_+\cap T_-$ and $\nu_F=\nu_{T_+}|_F$ (i.e., ν_F points outward of T_+). In this situation, $[v]_F := v|_{T_+} - v|_{T_-}$ denotes the jump across F. For a hyper-surface $F\subseteq\partial\Omega$ on the boundary, the jump across this hyper-surface F is $[v]_F := v$ (in the case of homogeneous Dirichlet data on $\partial\Omega$ at hand).

2.2. Bisection. The bisection of a tagged simplex $(z_0,\dots,z_n;\gamma)$ generates the two tagged simplices

$$\left(z_0,\frac{z_0+z_n}{2},z_1,\dots,z_\gamma,z_{\gamma+1},\dots,z_{n-1};(\gamma+1)\bmod n\right)\quad\text{and}$$
$$\left(z_n,\frac{z_0+z_n}{2},z_1,\dots,z_\gamma,z_{n-1},\dots,z_{\gamma+1};(\gamma+1)\bmod n\right). \tag{2.1}$$

(By convention, the finite sequence $(z_{\gamma+1},\dots,z_{n-1})$ and $(z_1,\dots,z_\gamma)$ is void for $\gamma=n-1$ and $\gamma=0$, respectively.) The two new tagged simplices are called the children of the tagged simplex $(z_0,\dots,z_n;\gamma)$ and any child of some child of a tagged simplex is called a grandchild; conversely, in this situation, $(z_0,\dots,z_n;\gamma)$ is called a parent (resp., grandparent) of each of its two children (resp., four grandchildren).

The following proposition ensures that grandchildren do not share hyper-surfaces with their grandparents.

PROPOSITION 2.1. *Any grandchild T of a tagged simplex K satisfies $\mathcal{F}(T)\cap\mathcal{F}(K)=\emptyset$.*

Proof. Let the tagged simplex $K = (z_0, \dots, z_n; \gamma)$ be the grandparent of T; that is, T is a child of some $\widetilde{K}$ and $\widetilde{K}$ is a child of K. The bisection rule (2.1) implies that the child $\widetilde{K}$ of K contains the new vertex $(z_0+z_n)/2$. Moreover, the child T of $\widetilde{K}$ contains the vertex $(z_0+z_n)/2$ and the new vertex $(z_0+z_{n-1})/2$ or $(z_n+z_{\gamma+1})/2$ (depending on whether $\widetilde{K}$ is the first or the second tagged simplex in (2.1)). Consequently, the tagged simplex T contains two vertices outside of $\mathcal{N}(K)$. Each hyper-surface $F \in \mathcal{F}(T)$ is the convex combination of n vertices from the $n+1$ vertices from the simplex T, and therefore F contains at least one new vertex. This proves $F \notin \mathcal{F}(K)$. ∎

2.3. Initial conditions. The initial condition (C) below from [28, p. 232] guarantees that successive refinements of a regular triangulation $\mathcal{T}$ lead to regular triangulations. The notion of a reflected neighbor is required for the statement of (C). Note that given a tagged simplex $T = (z_0, \dots, z_n; \gamma)$, the simplex

$$T_R := (z_n, z_1, \dots, z_\gamma, z_{n-1}, z_{n-2}, \dots, z_{\gamma+1}, z_0; \gamma)$$

with $\mathrm{dom}(T_R) = \mathrm{dom}(T)$ has the same children as T. Two tagged simplices T, K are called neighbors if they share a common $(n-1)$-dimensional hyper-surface. Two neighboring tagged simplices T and K are called *reflected neighbors* [28] if the ordered sequence of vertices of either T or T_R coincides with that of K on all but one position; for graphical illustrations see [28].

The following initial condition from [28] is crucial for the regularity of refinements.

Condition (C). All simplices in $\mathcal{T}$ are of the same type γ. Any two neighboring tagged simplices $T = (y_0, \dots, y_n; \gamma)$ and $K = (z_0, \dots, z_n; \gamma)$ satisfy the following:

(a) If $\mathrm{conv}\{y_0, y_n\} \subseteq T \cap K$ or $\mathrm{conv}\{z_0, z_n\} \subseteq T \cap K$, then T and K are reflected neighbors.

(b) If $\mathrm{conv}\{y_0, y_n\} \not\subseteq T \cap K \neq \emptyset$ and $\mathrm{conv}\{z_0, z_n\} \not\subseteq T \cap K$, then any two neighboring children of T and K are reflected neighbors.

Condition (C) guarantees that uniform refinements of a triangulation $\mathcal{T}$ are regular [28, Theorem 4.3], which transfers to the refinement routine of the following subsection.

2.4. Admissible triangulations. Throughout the paper, the initial regular triangulation $\mathcal{T}_0$ of Ω is assumed to satisfy Condition (C). A regular triangulation $\mathcal{T}$ is called an admissible triangulation of $\mathcal{T}_0$ if it is a regular triangulation and it was created by refining $\mathcal{T}_0$ with a successive application of the bisection rule (2.1).

The set of all admissible triangulations is denoted by $\mathbb{T}$. This set is known to be uniformly shape regular [28], i.e., the ratio of the diameter and the radius of the largest inscribed ball is uniformly bounded only dependent on $\mathcal{T}_0$. For any $\mathcal{T} \in \mathbb{T}$,

$$\mathbb{T}(\mathcal{T}) := \{\mathcal{T}' \in \mathbb{T} \mid \mathcal{T}' \text{ is an admissible refinement of } \mathcal{T}\}.$$

Notice that $\mathcal{T}_1 \in \mathbb{T}(\mathcal{T}_2)$ and $\mathcal{T}_2 \in \mathbb{T}(\mathcal{T}_1)$ imply $\mathcal{T}_1 = \mathcal{T}_2$. For any $T \in \mathcal{T}$, the routine $\texttt{refine}(\mathcal{T}, T)$ from [28, p. 235] computes a refinement $\widehat{\mathcal{T}} \in \mathbb{T}(\mathcal{T})$ such that $T \in \mathcal{T} \setminus \widehat{\mathcal{T}}$. The following proposition ensures the minimality of this routine. In the case that $T \notin \mathcal{T}$ set $\texttt{refine}(\mathcal{T}, T) := \mathcal{T}$.

PROPOSITION 2.2. *The output* $\widehat{\mathcal{T}} := \texttt{refine}(\mathcal{T}, T)$ *is minimal in the sense that any other refinement* $\widetilde{\mathcal{T}} \in \mathbb{T}(\mathcal{T})$ *with* $T \in \mathcal{T} \setminus \widetilde{\mathcal{T}}$ *is a refinement* $\widetilde{\mathcal{T}} \in \mathbb{T}(\widehat{\mathcal{T}})$ *of* $\widehat{\mathcal{T}}$.

Proof. The minimality of $\texttt{refine}(\mathcal{T}, T)$ with respect to the cardinality is stated in [28, Theorem 5.1] and the proposition follows from the arguments of that paper. The concept of binary trees [5] behind the notion of admissible refinements clarifies that the minimality with respect to the number of new elements is indeed equivalent to the minimality in the sense of the proposition. ◻

For a set of simplices $\mathcal{M} \subseteq \mathcal{T}$, the routine $\texttt{refine}(\mathcal{T}, \mathcal{M})$ runs the following loop.

ALGORITHM 2.3 ($\texttt{refine}(\mathcal{T}, \mathcal{M})$).

Input: $\mathcal{M}$ and $\widetilde{\mathcal{T}} := \mathcal{T}$.
while $\mathcal{M} \cap \widetilde{\mathcal{T}} \neq \emptyset$ do
  choose $T \in \mathcal{M} \cap \widetilde{\mathcal{T}}$,
  compute $\widetilde{\mathcal{T}} := \texttt{refine}(\widetilde{\mathcal{T}}, T)$ od
Output: $\texttt{refine}(\mathcal{T}, \mathcal{M}) := \widetilde{\mathcal{T}}$.

This loop computes a refinement $\widehat{\mathcal{T}} \in \mathbb{T}(\mathcal{T})$ of $\mathcal{T}$ by applying $\texttt{refine}(\widetilde{\mathcal{T}}, T)$ for simplices in $\mathcal{M}$ and results in a triangulation in which all simplices of $\mathcal{M} \subseteq \mathcal{T} \backslash \widehat{\mathcal{T}}$ are refined. The following proposition guarantees that the result is independent of the order of $T \in \mathcal{M} \cap \widetilde{\mathcal{T}}$ in the loop of refine.

PROPOSITION 2.4. *The output* $\widehat{\mathcal{T}} := \texttt{refine}(\mathcal{T}, \mathcal{M})$ *does not depend on the selection of* $T \in \mathcal{M} \cap \widetilde{\mathcal{T}}$ *in Algorithm* 2.3.

Proof. Let $T_a, T_b \in \mathcal{T}$ be tagged simplices and set $\mathcal{T}_a := \texttt{refine}(\mathcal{T}, T_a)$, $\mathcal{T}_b := \texttt{refine}(\mathcal{T}, T_b)$. The overlay $\mathcal{T}_a \otimes \mathcal{T}_b$ is defined as the smallest common refinement of $\mathcal{T}_a$ and $\mathcal{T}_b$ in the sense that any triangulation $\widetilde{\mathcal{T}} \in \mathbb{T}(\mathcal{T}_a) \cap \mathbb{T}(\mathcal{T}_b)$ satisfies $\widetilde{\mathcal{T}} \in \mathbb{T}(\mathcal{T}_a \otimes \mathcal{T}_b)$. Since $\texttt{refine}(\mathcal{T}_a, T_b) \in \mathbb{T}(\mathcal{T})$ and $T_b \in \mathcal{T} \backslash \texttt{refine}(\mathcal{T}_a, T_b)$, the minimality of Proposition 2.2 leads to $\texttt{refine}(\mathcal{T}_a, T_b) \in \mathbb{T}(\mathcal{T}_b)$. Since $\texttt{refine}(\mathcal{T}_a, T_b) \in \mathbb{T}(\mathcal{T}_a)$, the minimality of the overlay implies $\texttt{refine}(\mathcal{T}_a, T_b) \in \mathbb{T}(\mathcal{T}_a \otimes \mathcal{T}_b)$.

On the other hand, if $T_b \notin \mathcal{T}_a$, then $\texttt{refine}(\mathcal{T}_a, T_b) = \mathcal{T}_a$ and so $\mathcal{T}_a \otimes \mathcal{T}_b \in \mathbb{T}(\texttt{refine}(\mathcal{T}_a, T_b))$. If $T_b \in \mathcal{T}_a$, then $\mathcal{T}_a \otimes \mathcal{T}_b$ is a refinement of $\mathcal{T}_a$ with $T_b \notin \mathcal{T}_a \otimes \mathcal{T}_b$. Proposition 2.2 guarantees $\mathcal{T}_a \otimes \mathcal{T}_b \in \mathbb{T}(\texttt{refine}(\mathcal{T}_a, T_b))$.

Altogether, $\texttt{refine}(\mathcal{T}_a, T_b) \in \mathbb{T}(\mathcal{T}_a \otimes \mathcal{T}_b)$ and $\mathcal{T}_a \otimes \mathcal{T}_b \in \mathbb{T}(\texttt{refine}(\mathcal{T}_a, T_b))$ imply

$$\texttt{refine}(\mathcal{T}_a, T_b) = \mathcal{T}_a \otimes \mathcal{T}_b.$$

The symmetry of a and b also proves $\mathrm{refine}(\mathcal{T}_b, T_a) = \mathcal{T}_b \otimes \mathcal{T}_a$ and so $\mathcal{T}_a \otimes \mathcal{T}_b = \mathcal{T}_b \otimes \mathcal{T}_a$ implies

$$\texttt{refine}(\mathcal{T}_a, T_b) = \texttt{refine}(\mathcal{T}_b, T_a).$$

It follows that the order of two consecutive selections in Algorithm 2.3 does not change the output. This concludes the proof. ◻

The following proposition states that the minimality of $\texttt{refine}$ for one simplex implies the minimality of $\texttt{refine}$ for any input set $\mathcal{M} \subseteq \mathcal{T}$.

PROPOSITION 2.5. *The output* $\widehat{\mathcal{T}} := \texttt{refine}(\mathcal{T}, \mathcal{M})$ *is minimal in the sense that any other refinement* $\mathcal{T}' \in \mathbb{T}(\mathcal{T})$ *with* $\mathcal{M} \subseteq \mathcal{T} \backslash \mathcal{T}'$ *is a refinement* $\mathcal{T}' \in \mathbb{T}(\widehat{\mathcal{T}})$.

Proof. The proof of Proposition 2.4 shows for $\mathcal{M} = \{T_1, \ldots, T_{\mathrm{card}(\mathcal{M})}\}$

$$\texttt{refine}(\mathcal{T}, \mathcal{M}) = \texttt{refine}(\mathcal{T}, T_1) \otimes \cdots \otimes \texttt{refine}(\mathcal{T}, T_{\mathrm{card}(\mathcal{M})}).$$

The minimality of $\texttt{refine}$ for one simplex and the minimality of the overlay prove the assertion. ◻

3. Main result. This section defines the Crouzeix–Raviart FEM space and piecewise H^1 spaces, and the main result of the paper is stated in subsection 3.2. In the subsequent chapters $\mathcal{T}_\ell \in \mathbb{T}$ is an admissible refinement from $\mathcal{T}_0$ with hyper-surfaces $\mathcal{F}_\ell := \mathcal{F}(\mathcal{T}_\ell)$. In the following three chapters the piecewise constant mesh-size function h_ℓ reads $h_\ell|_T = \mathrm{diam}(T)$ for all $T \in \mathcal{T}_\ell$.

3.1. Crouzeix–Raviart finite element space. For $k \geq 0$ the space of the piecewise polynomial functions of degree $\leq k$ reads

$$P_k(\mathcal{T}_\ell) := \{v_\ell \in L^2(\Omega) \mid v_\ell|_T \in P_k(T) \text{ for all } T \in \mathcal{T}_\ell\}.$$

The nonconforming finite element space after Crouzeix and Raviart [19, 21] with respect to $\mathcal{T}_\ell \in \mathbb{T}$ is defined as

$$\mathrm{CR}_0^1(\mathcal{T}_\ell) := \{v_\ell \in P_1(\mathcal{T}_\ell) \mid \forall F \in \mathcal{F}_\ell,\ \ [v_\ell]_F(\mathrm{mid}(F)) = 0\}$$

for the barycenter $\mathrm{mid}(F) := n^{-1} \sum_{j=1}^n y_j$ of a hyper-surface F with vertices $y_1, \dots, y_n$.

For piecewise H^1 functions (with respect to $\mathcal{T}_\ell$) the piecewise differential operators D_{NC} and $\mathrm{div}_{\mathrm{NC}}$ exist and act as $(\mathrm{D}_{\mathrm{NC}} v_{\mathrm{NC}})|_T = \mathrm{D}(v_{\mathrm{NC}}|_T)$ and $(\mathrm{div}_{\mathrm{NC}}\, v_{\mathrm{NC}})|_T = \mathrm{div}(v_{\mathrm{NC}}|_T)$ for all $T \in \mathcal{T}_\ell$. Define the spaces $P_1(\mathcal{T}_\ell; \mathbb{R}^k) := [P_1(\mathcal{T}_\ell)]^k$, $\mathrm{CR}_0^1(\mathcal{T}_\ell; \mathbb{R}^k) := [\mathrm{CR}_0^1(\mathcal{T}_\ell)]^k$.

3.2. Discrete distance control. The following main result states the discrete distance control (1.1) for the Crouzeix–Raviart FEM. The point is that $C_{\mathrm{ddc}} \approx 1$ depends only on the initial triangulation $\mathcal{T}_0$ but not on either $\ell \in \mathbb{N}_0$ or on $m \in \mathbb{N}$. The proof follows in section 4.

THEOREM 3.1 (discrete distance control). *Let $\mathcal{T}_{\ell+m} \in \mathbb{T}(\mathcal{T}_\ell)$ be a refinement of $\mathcal{T}_\ell$ created by the refinement rules of section* 2 *and recall*

$$\mathcal{F}(\mathcal{T}_\ell \backslash \mathcal{T}_{\ell+m}) = \{F \in \mathcal{F}_\ell \;\big|\; \exists T \in \mathcal{T}_\ell \backslash \mathcal{T}_{\ell+m},\ \ F \in \mathcal{F}(T)\}.$$

Any function $u_\ell \in \mathrm{CR}_0^1(\mathcal{T}_\ell)$ satisfies

$$\min_{v_{\ell+m} \in \mathrm{CR}_0^1(\mathcal{T}_{\ell+m})} \|\mathrm{D}_{\mathrm{NC}}(u_\ell - v_{\ell+m})\|^2_{L^2(\Omega)} \leq C_{\mathrm{ddc}} \sum_{F \in \mathcal{F}(\mathcal{T}_\ell \backslash \mathcal{T}_{\ell+m})} h_F^{-1} \left\| [u_\ell]_F \right\|^2_{L^2(F)}. \tag{3.1}$$

Figure 3.1 illustrates possible triangulations $\mathcal{T}_\ell \in \mathbb{T}$ and $\mathcal{T}_{\ell+m} \in \mathbb{T}(\mathcal{T}_\ell)$ and emphasizes the hyper-surfaces which appear in the sum in the right-hand side in (3.1). The point is that hyper-surfaces $F \in \mathcal{F}_\ell$ for which all adjacent simplices $T \in \mathcal{T}_\ell$ with $F \in \mathcal{F}(T)$ are not refined can be neglected.

3.3. Main tool. The methodology behind the discrete distance control as the main result of this paper is the design of a discrete quasi interpolation.

THEOREM 3.2 (discrete quasi interpolation). *Given $\mathcal{T}_\ell \in \mathbb{T}$ and some refinement $\mathcal{T}_{\ell+m} \in \mathbb{T}(\mathcal{T}_\ell)$, there exists an operator $\mathcal{J} : \mathrm{CR}_0^1(\mathcal{T}_\ell) \to \mathrm{CR}_0^1(\mathcal{T}_{\ell+m})$ such that for any $u_\ell \in \mathrm{CR}_0^1(\mathcal{T}_\ell)$, $\mathcal{J} u_\ell|_T = u_\ell|_T$ for all $T \in \mathcal{T}_\ell \cap \mathcal{T}_{\ell+m}$ and*

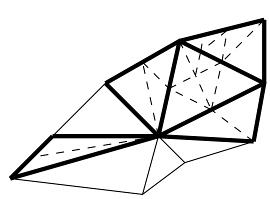

FIG. 3.1. *Illustration of a triangulation $\mathcal{T}_\ell$ (solid) with its refinement $\mathcal{T}_{\ell+m}$ (dashed) and thick edges which appear in the sum in* (3.1).

$$\|\mathrm{D}_{\mathrm{NC}}(u_\ell - \mathcal{J}u_\ell)\|^2_{L^2(\Omega)} \lesssim \sum_{F\in\mathcal{F}(\mathcal{T}_\ell\setminus\mathcal{T}_{\ell+m})} h_F^{-1}\,\|[u_\ell]_F\|^2_{L^2(F)}\,.$$

4. Proofs. This section is devoted to the proof of Theorems 3.1 and 3.2 based on an intermediate triangulation $\widehat{\mathcal{T}}_\ell$ with $\mathcal{T}_\ell \cap \mathcal{T}_{\ell+m} = \mathcal{T}_\ell \cap \widehat{\mathcal{T}}_\ell$.

4.1. Intermediate triangulation. Given $\mathcal{T}_\ell$ and $\mathcal{T}_{\ell+m}$ of Theorem 3.1, the following algorithm computes some intermediate triangulation $\widehat{\mathcal{T}}_\ell \in \mathbb{T}(\mathcal{T}_\ell)$.

ALGORITHM 4.1.
Input: $\mathcal{S} := \mathcal{F}_\ell\setminus\mathcal{F}_{\ell+m}$ and $\widetilde{\mathcal{T}} := \mathcal{T}_\ell$.
while $\mathcal{S} \neq \emptyset$ do
$\quad\widetilde{\mathcal{T}}(\mathcal{S}) := \{T \in \widetilde{\mathcal{T}} \mid \exists F \in \mathcal{S} \text{ with } F \in \mathcal{F}(T)\}$,
$\quad\widetilde{\mathcal{T}} := \texttt{refine}(\widetilde{\mathcal{T}}, \widetilde{\mathcal{T}}(\mathcal{S}))$,
$\quad\mathcal{S} := \mathcal{S} \cap \mathcal{F}(\widetilde{\mathcal{T}})$ od
Output: $\widehat{\mathcal{T}}_\ell := \widetilde{\mathcal{T}}$.

Algorithm 4.1 is the natural generalization of the refinement of Algorithm 2.3 to the case of marked hyper-surfaces. For any marked hyper-surface $F \in \mathcal{F}_\ell$, Algorithm 2.3 is applied to the adjacent simplices $T \in \mathcal{T}_\ell$ with $F \in \mathcal{F}(T)$ until the hyper-surface F is refined and, hence, excluded from the current set $\mathcal{F}(\widetilde{\mathcal{T}})$.

LEMMA 4.2. *Algorithm* 4.1 *terminates after at most two runs of the while loop. Furthermore, any two simplices $K \in \mathcal{T}_\ell$ and $T \in \widehat{\mathcal{T}}_\ell$ with $T \subseteq K$ have comparable sizes $h_K \approx h_T$, $|K| \approx |T|$, etc.*

Proof. The termination after two loops follows from Proposition 2.1. The comparability of the mesh-sizes follows from the fact that each simplex $T \in \widetilde{\mathcal{T}}(\mathcal{S})$ is split into at least two and at most $C_{\mathrm{desc}} \geq 2$ descendants. The proof of $C_{\mathrm{desc}} \lesssim 1$ is trivial for $n = 2$ and nontrivial for $n \geq 3$. The latter follows indeed with the arguments from Corollary 4.6 and Theorems 5.1 and 5.2 of [28] as pointed out by Stevenson [29]. ∎

Figure 4.1 illustrates the definition of the intermediate triangulation $\mathcal{T}_\ell$ with $\mathbb{T}(\mathcal{T}_{\ell+m}) \subsetneqq \mathbb{T}(\widehat{\mathcal{T}}_\ell) \subsetneqq \mathbb{T}(\mathcal{T}_\ell)$.

4.2. Properties of $\widehat{\mathcal{T}}_\ell$. This subsection provides three lemmas on the intermediate triangulation $\widehat{\mathcal{T}}_\ell$ computed by Algorithm 4.1 with vertices $\widehat{\mathcal{N}}_\ell := \mathcal{N}(\widehat{\mathcal{T}}_\ell)$ and hyper-surfaces $\widehat{\mathcal{F}}_\ell := \mathcal{F}(\widehat{\mathcal{T}}_\ell)$. Recall from Lemma 4.2 that in Algorithm 4.1 the number of bisections for one simplex is bounded independently of the possibly large number $m \in \mathbb{N}$.

LEMMA 4.3. *Algorithm* 4.1 *is minimal in the sense that any $\mathcal{T}' \in \mathbb{T}(\mathcal{T}_\ell)$ with hyper-surfaces $\mathcal{F}'$ and $(\mathcal{F}_\ell\setminus\mathcal{F}_{\ell+m}) \cap \mathcal{F}' = \emptyset$ satisfies $\mathcal{T}' \in \mathbb{T}(\widehat{\mathcal{T}}_\ell)$. In other words $\widehat{\mathcal{T}}_\ell$ is the unique smallest admissible refinement of $\mathcal{T}_\ell$ where at least the faces $\mathcal{F}_\ell\setminus\mathcal{F}_{\ell+m}$ are refined.*

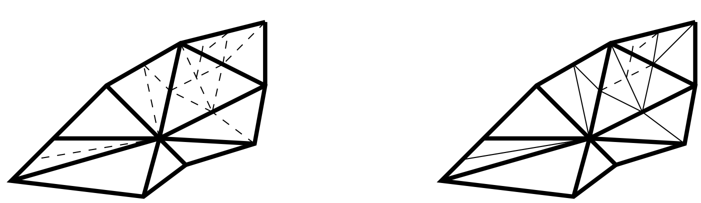

FIG. 4.1. *A triangulation $\mathcal{T}_\ell$ (thick) with refinement $\mathcal{T}_{\ell+m}$ (dashed) and the intermediate triangulation $\widehat{\mathcal{T}}_\ell$ (solid, right).*

Proof. Let $\mathcal{T}' \in \mathbb{T}(\mathcal{T}_\ell)$ be any refinement of $\mathcal{T}_\ell$ with hyper-surfaces $\mathcal{F}'$ such that $(\mathcal{F}_\ell \backslash \mathcal{F}_{\ell+m}) \cap \mathcal{F}' = \emptyset$. The first loop of Algorithm 4.1 computes the set

$$\widetilde{\mathcal{T}}_1 := \mathtt{refine}(\mathcal{T}_\ell, \{T \in \mathcal{T}_\ell \mid \exists F \in \mathcal{F}_\ell \backslash \mathcal{F}_{\ell+m} \text{ with } F \in \mathcal{F}(T)\})$$

with a set of hyper-surfaces $\widetilde{\mathcal{F}}_1$. Since $(\mathcal{F}_\ell \backslash \mathcal{F}_{\ell+m}) \cap \mathcal{F}' = \emptyset$, any $T \in \mathcal{T}_\ell$ with some hyper-surface $F \in \mathcal{F}(T) \cap (\mathcal{F}_\ell \backslash \mathcal{F}_{\ell+m})$ satisfies $T \notin \mathcal{T}'$. Proposition 2.5 therefore shows $\mathcal{T}' \in \mathbb{T}(\widetilde{\mathcal{T}}_1)$. This establishes the lemma in case that Algorithm 4.1 terminates after one loop with $\widehat{\mathcal{T}}_\ell = \widetilde{\mathcal{T}}_1$.

Otherwise, the second loop computes $\mathcal{M}_2 := \{T \in \widetilde{\mathcal{T}}_1 \mid \exists F \in (\mathcal{F}_\ell \backslash \mathcal{F}_{\ell+m}) \cap \widetilde{\mathcal{F}}_1 \text{ with } F \in \mathcal{F}(T)\} \neq \emptyset$ and terminates with $\widehat{\mathcal{T}}_\ell := \mathtt{refine}(\widetilde{\mathcal{T}}_1, \mathcal{M}_2)$. Since $\mathcal{T}' \in \mathbb{T}(\widetilde{\mathcal{T}}_1)$ and any $T \in \mathcal{T}'$ satisfies $\mathcal{F}(T) \cap (\mathcal{F}_\ell \backslash \mathcal{F}_{\ell+m}) = \emptyset$, Proposition 2.5 shows $\mathcal{T}' \in \mathbb{T}(\widehat{\mathcal{T}}_\ell)$. This and Lemma 4.2 conclude the proof. ∎

LEMMA 4.4. *It holds* $\widehat{\mathcal{F}}_\ell \cap \mathcal{F}_\ell = \mathcal{F}_{\ell+m} \cap \mathcal{F}_\ell$.

Proof. The minimality of $\widehat{\mathcal{T}}_\ell$ in Lemma 4.3 shows that $\mathcal{T}_{\ell+m}$ is an admissible refinement of $\widehat{\mathcal{T}}_\ell$. It follows $\mathcal{F}_{\ell+m} \cap \mathcal{F}_\ell \subseteq \widehat{\mathcal{F}}_\ell \cap \mathcal{F}_\ell$. Conversely, given any $F \in \mathcal{F}_\ell \cap \widehat{\mathcal{F}}_\ell$, F cannot belong to the input set $\mathcal{S} = \mathcal{F}_\ell \backslash \mathcal{F}_{\ell+m}$ of Algorithm 4.1. Therefore, $F \in \mathcal{F}_{\ell+m}$. Since F is arbitrary, this proves $\widehat{\mathcal{F}}_\ell \cap \mathcal{F}_\ell \subseteq \mathcal{F}_{\ell+m} \cap \mathcal{F}_\ell$. ∎

LEMMA 4.5. *It holds that* $\widehat{\mathcal{T}}_\ell \cap \mathcal{T}_\ell = \mathcal{T}_\ell \cap \mathcal{T}_{\ell+m}$.

Proof. The minimality of $\widehat{\mathcal{T}}_\ell$ in Lemma 4.3 shows that $\mathcal{T}_{\ell+m}$ is an admissible refinement of $\widehat{\mathcal{T}}_\ell$. Hence, $\mathcal{T}_{\ell+m} \cap \mathcal{T}_\ell \subseteq \mathcal{T}_\ell \cap \widehat{\mathcal{T}}_\ell$. Conversely, given any $T \in \mathcal{T}_\ell \cap \widehat{\mathcal{T}}_\ell$, all hyper-surfaces of T belong to $\mathcal{F}_\ell \cap \widehat{\mathcal{F}}_\ell$ and, by Lemma 4.4, to $\mathcal{F}_\ell \cap \mathcal{F}_{\ell+m}$. Therefore $T \in \mathcal{T}_\ell \cap \mathcal{T}_{\ell+m}$. ∎

4.3. Transfer operator. Consider the vertex $z \in \mathcal{N}(T)$ of a tagged simplex $T \in \widehat{\mathcal{T}}_\ell$ in the intermediate triangulation $\widehat{\mathcal{T}}_\ell$ and define the set of the hyper-surface-connected refined simplices at z by $\mathcal{Z}(z;T) := \{T\}$ for $T \in \widehat{\mathcal{T}}_\ell \cap \mathcal{T}_\ell$ and otherwise (i.e., for $T \in \widehat{\mathcal{T}}_\ell \backslash \mathcal{T}_\ell$) set

$$\begin{aligned}\mathcal{Z}(z;T) := \big\{K \in \widehat{\mathcal{T}}_\ell \backslash \mathcal{T}_\ell \mid \exists J \in \mathbb{N}\, \exists T_1, \dots, T_J \in \widehat{\mathcal{T}}_\ell(z) \backslash \mathcal{T}_\ell \text{ with } T = T_1 \text{ and } K = T_J \\ \text{such that } T_j \cap T_{j+1} \in \widehat{\mathcal{F}}_\ell \text{ for } j = 1, \dots, J-1\big\}.\end{aligned}$$

If $T \in \widehat{\mathcal{T}}_\ell \cap \mathcal{T}_\ell$ is unrefined, $\mathcal{Z}(z;T)$ consists of T only. Any refined $T \in \widehat{\mathcal{T}}_\ell(z) \backslash \mathcal{T}_\ell$ belongs to $\mathcal{Z}(z;T)$ as well as possibly some other neighboring $K \in \widehat{\mathcal{T}}_\ell(z) \backslash \mathcal{T}_\ell$, plus the chain $T_1, \dots, T_J$ which connects T and K and which consists of hyper-surface-connected neighbors of this type. Figure 4.2 illustrates this definition of $\mathcal{Z}(z;T)$ and its dependence on $T \in \widehat{\mathcal{T}}_\ell$.

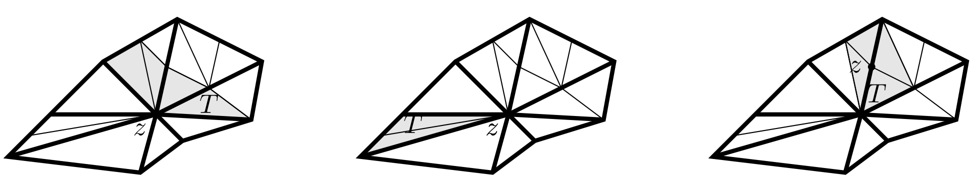

FIG. 4.2. *A triangulation $\mathcal{T}_\ell$ (thick) and the refinement $\widehat{\mathcal{T}}_\ell$ (solid) and $\mathcal{Z}(z,T)$ (gray) for three different z and T.*

Recall $1 \le \operatorname{card}(\mathcal{Z}(z;T)) \le \operatorname{card}(\widehat{\mathcal{T}}_\ell(z)) \lesssim 1$ and define the averaging operator $J^* : \mathrm{CR}^1_0(\mathcal{T}_\ell) \to P_1(\widehat{\mathcal{T}}_\ell)$ for $z \in \widehat{\mathcal{N}}_\ell \cap \Omega$ and $T \in \widehat{\mathcal{T}}_\ell(z)$ by

$$J^* u_\ell|_T(z) := \sum_{K \in \mathcal{Z}(z;T)} u_\ell|_K(z) / \operatorname{card}(\mathcal{Z}(z;T)),$$

while $J^* u_\ell(z) := 0$ for $z \in \widehat{\mathcal{N}}_\ell \cap \partial\Omega$.

Given $u_\ell \in \mathrm{CR}^1_0(\mathcal{T}_\ell)$, define $\mathcal{J} u_\ell \in P_1(\widehat{\mathcal{T}}_\ell)$ as a combination of the averaging operator J^* and the identity for simplices $T \in \mathcal{T}_\ell \cap \widehat{\mathcal{T}}_\ell$, i.e., for $T \in \widehat{\mathcal{T}}_\ell$ and $F \in \mathcal{F}(T)$, set

$$\mathcal{J} u_\ell|_T(\operatorname{mid}(F)) := \begin{cases} u_\ell(\operatorname{mid}(F)) & \text{if } F \in \mathcal{F}_\ell \cap \widehat{\mathcal{F}}_\ell, \\ J^* u_\ell|_T(\operatorname{mid}(F)) & \text{if } F \in \widehat{\mathcal{F}}_\ell \backslash \mathcal{F}_\ell. \end{cases}$$

The first observation is that $\mathcal{J} u_\ell$ is well defined as a function in $\mathrm{CR}^1_0(\widehat{\mathcal{T}}_\ell)$ and (surprisingly at first glance) in $\mathrm{CR}^1_0(\mathcal{T}_{\ell+m})$ as well.

Theorem 4.6. *It holds that* $\mathcal{J} u_\ell \in \mathrm{CR}^1_0(\mathcal{T}_{\ell+m}) \cap \mathrm{CR}^1_0(\widehat{\mathcal{T}}_\ell)$ *and* $\mathcal{J} u_\ell|_T = u_\ell|_T$ *for all* $T \in \mathcal{T}_\ell \cap \mathcal{T}_{\ell+m}$.

The remaining parts of this subsection are devoted to the proof of Theorem 4.6. Figures 4.2(a) and 4.2(b) illustrate that $J^* u_\ell$ is possibly not continuous on $\operatorname{dom}(\widehat{\mathcal{T}}_\ell \backslash \mathcal{T}_\ell)$, where $\mathcal{Z}(z;T)$ is different for different T.

Lemma 4.7. *The function* $J^* u_\ell$ *is continuous on* $\operatorname{int}(\cup(\widehat{\mathcal{T}}_\ell \backslash \mathcal{T}_\ell))$ *and vanishes on* $\cup(\widehat{\mathcal{T}}_\ell \backslash \mathcal{T}_\ell) \cap \partial\Omega$.

Proof. Consider an interior hyper-surface $F = \operatorname{conv}\{y_1, \dots, y_n\} \in \widehat{\mathcal{F}}_\ell$, $F \not\subseteq \partial\Omega$, shared by two simplices T_+ and T_- of $\widehat{\mathcal{T}}_\ell$. If $T_+ \in \widehat{\mathcal{T}}_\ell \cap \mathcal{T}_\ell$ or $T_- \in \widehat{\mathcal{T}}_\ell \cap \mathcal{T}_\ell$, then continuity is not asserted. Hence, suppose $T_+, T_- \in \widehat{\mathcal{T}}_\ell \backslash \mathcal{T}_\ell$ and so the vertices of F satisfy $\mathcal{Z}(y_j;T_+) = \mathcal{Z}(y_j;T_-)$ for all $j = 1, \dots, n$. The definition of J^* defines $(J^* u_\ell)|_{T_+}(y_j) = (J^* u_\ell)|_{T_-}(y_j)$ uniquely. Since $J^* u_\ell$ is affine on T_+ and T_-, $(J^* u_\ell)|_{T_+}$ and $(J^* u_\ell)|_{T_-}$ coincide on $F = T_+ \cap T_-$. In the case that $F \subseteq \partial\Omega$ is a boundary hyper-surface, the definition of J^* implies $J^* u_\ell|_F \equiv 0$. ∎

Proof of $\mathcal{J} u_\ell \in \mathrm{CR}^1_0(\widehat{\mathcal{T}}_\ell)$. Lemma 4.7 guarantees that $J^* u_\ell$ is continuous along any interior hyper-surface $F \in \widehat{\mathcal{F}}_\ell \backslash \mathcal{F}_\ell$, $F \not\subseteq \partial\Omega$, and equals zero along any boundary hyper-surface $F \in \widehat{\mathcal{F}}_\ell \backslash \mathcal{F}_\ell$, $F \subseteq \partial\Omega$. This means that $\mathcal{J} u_\ell$ is continuous at $\operatorname{mid}(F)$ (resp., zero if $F \subseteq \partial\Omega$). The point is that for all other $F \in \widehat{\mathcal{F}}_\ell \cap \mathcal{F}_\ell$, $(\mathcal{J} u_\ell)(\operatorname{mid}(F)) = u_\ell(\operatorname{mid}(F))$ is continuous at $\operatorname{mid}(F)$ (resp., zero if $F \subseteq \partial\Omega$). ∎

Lemma 4.8. *If* $S \in \mathcal{T}_{\ell+m} \backslash \widehat{\mathcal{T}}_\ell$ *and* $\operatorname{dom}(S) \subsetneqq \operatorname{dom}(T) \subseteq \operatorname{dom}(K)$ *for simplices* $T \in \widehat{\mathcal{T}}_\ell$ *and* $K \in \mathcal{T}_\ell$, *then* $|T| \le |K|/4$ *(i.e., T is at least a grandchild of K).*

Proof. The simplex $S \in \mathcal{T}_{\ell+m}$ is generated by a series of bisections from the simplex $K \in \mathcal{T}_\ell$. This means that there exist a sequence of simplices $K_0, \dots, K_J$ with $K = K_0$ and $S = K_J$ and

$$\operatorname{dom}(K_J) \subsetneqq \operatorname{dom}(K_{J-1}) \subsetneqq \cdots \subsetneqq \operatorname{dom}(K_0)$$

such that the simplex K_j is a child of K_{j-1} for $j = 1, \dots, J$. If $J = 0$, then $S \in \mathcal{T}_\ell$ and the condition $S \in \mathcal{T}_{\ell+m}$ leads to $S \in \widehat{\mathcal{T}}_\ell$, which is a contradiction to $S \in \mathcal{T}_{\ell+m} \backslash \widehat{\mathcal{T}}_\ell$. If $J = 1$, then S is a child of K and $\mathcal{F}(K) \backslash \mathcal{F}_{\ell+m} \neq \emptyset$. This implies $S \in \widehat{\mathcal{T}}_\ell$, which contradicts $S \in \mathcal{T}_{\ell+m} \backslash \widehat{\mathcal{T}}_\ell$. It follows that $J \ge 2$. ∎

Proof of $\mathcal{J}u_\ell \in \mathrm{CR}^1_0(\mathcal{T}_{\ell+m})$. The proof verifies the continuity of $\mathcal{J}u_\ell$ at the midpoints $\mathrm{mid}(\mathcal{F}_{\ell+m})$ of the hypersurfaces $\mathcal{F}_{\ell+m}$ of $\mathcal{T}_{\ell+m}$ (and the stated boundary conditions) and distinguishes four cases.

Case 1. Let $F \in \mathcal{F}_{\ell+m} \cap \widehat{\mathcal{F}}_\ell$. Then the function $\mathcal{J}u_\ell \in \mathrm{CR}^1_0(\widehat{\mathcal{T}}_\ell)$ is continuous in $\mathrm{mid}(F)$ (and vanishes in $\mathrm{mid}(F)$ in case of $F \subseteq \partial\Omega$).

Case 2. Let $F \in \mathcal{F}_{\ell+m} \backslash \widehat{\mathcal{F}}_\ell$ and let $\mathrm{mid}(F) \in \mathrm{int}(\mathrm{dom}(T))$ belong to the interior of some simplex $T \in \widehat{\mathcal{T}}_\ell$. Since $\mathcal{J}u_\ell$ is affine on T, $\mathcal{J}u_\ell$ is continuous in $\mathrm{mid}(F)$.

Case 3. Let $F \in \mathcal{F}_{\ell+m} \backslash \widehat{\mathcal{F}}_\ell$, $F \not\subseteq \partial\Omega$ be an interior hyper-surface and let there exist an interior hyper-surface $\widehat{F} \in \widehat{\mathcal{F}}_\ell$ shared by two simplices $T_+, T_- \in \widehat{\mathcal{T}}_\ell$ with $F \subseteq \widehat{F} = \partial T_+ \cap \partial T_-$. Any simplex $S_\pm \in \mathcal{T}_{\ell+m}$ with $F \in \mathcal{F}(S_\pm)$ does not belong to $\widehat{\mathcal{T}}_\ell$. Lemma 4.8 therefore implies that T_+ and T_- are grandchildren or refinements of grandchildren of simplices in $\mathcal{T}_\ell$. Hence, Proposition 2.1 guarantees $\mathcal{F}(T_\pm) \cap \mathcal{F}_\ell = \emptyset$. This and the definition of $\mathcal{J}$ imply $\mathcal{J}u_\ell|_{T_\pm}(\mathrm{mid}(F')) = J^*u_\ell|_{T_\pm}(\mathrm{mid}(F'))$ for all $F' \in \mathcal{F}(T_\pm)$. Since $\mathcal{J}u_\ell$ and J^*u_ℓ are affine on $\mathrm{dom}(T_\pm)$, this implies $\mathcal{J}u_\ell|_{T_\pm} \equiv J^*u_\ell|_{T_\pm}$ on $\mathrm{dom}(T_\pm)$. Lemma 4.7 and $T_\pm \in \widehat{\mathcal{T}}_\ell \backslash \mathcal{T}_\ell$ show that J^*u_ℓ is continuous along $\mathrm{int}(\widehat{F}) = \mathrm{int}(\partial T_+ \cap \partial T_-)$ for the relative interior $\mathrm{int}(\widehat{F})$ of $\widehat{F}$. Hence, $\mathcal{J}u_\ell$ equals J^*u_ℓ on $T_\pm$ and is continuous along $\mathrm{int}(\widehat{F})$ as well. In particular, $\mathcal{J}u_\ell$ is continuous at $\mathrm{mid}(F)$.

Case 4. Let $F \in \mathcal{F}_{\ell+m} \backslash \widehat{\mathcal{F}}_\ell$ belong to the boundary, $F \subseteq \partial\Omega$, and let there exist $\widehat{F} \in \widehat{\mathcal{F}}_\ell$ with $F \subsetneqq \widehat{F}$. For $T_+ \in \widehat{\mathcal{T}}_\ell$ with $\widehat{F} \in \mathcal{F}(T_+)$ the arguments of Case 3 lead to $\mathcal{F}(T_+) \cap \mathcal{F}_\ell = \emptyset$ and furthermore to $\mathcal{J}u_\ell|_{T_+} = J^*u_\ell|_{T_+}$. Since $T_+ \in \widehat{\mathcal{T}}_\ell \backslash \mathcal{T}_\ell$, $J^*u_\ell = 0$ along $\widehat{F}$ and so $(\mathcal{J}u_\ell)(\mathrm{mid}(F)) = 0$. $\square$

Proof of $\mathcal{J}u_\ell|_T = u_\ell|_T$ *for all* $T \in \mathcal{T}_\ell \cap \mathcal{T}_{\ell+m}$. This follows from the definition of $\mathcal{J}u_\ell$ and Lemma 4.4. $\square$

4.4. Error estimates for the transfer operator. The following theorem estimates the distance between u_ℓ and the quasi interpolant $\mathcal{J}u_\ell$. This theorem generalizes [11, Theorem 5.1] to a local estimate and to space dimensions $n \geq 2$.

For any $T \in \widehat{\mathcal{T}}_\ell$ and $z \in \mathcal{N}(T)$, the set of hyper-surfaces of $\widehat{\mathcal{F}}_\ell$ that contain z and belong to $\mathcal{Z}(z;T)$ is defined as

$$\widehat{\mathcal{F}}_\ell(z,T) := \{F \in \widehat{\mathcal{F}}_\ell \;\big|\; z \in F \text{ and } \exists K \in \mathcal{Z}(z;T) \text{ with } F \in \mathcal{F}(K)\}.$$

THEOREM 4.9 (error estimate for J^* and $\mathcal{J}$). *Any* $T \in \widehat{\mathcal{T}}_\ell \backslash \mathcal{T}_\ell$ *satisfies*

$$\|\mathrm{D}_{\mathrm{NC}}(u_\ell - J^*u_\ell)\|^2_{L^2(T)} + \|\mathrm{D}_{\mathrm{NC}}(u_\ell - \mathcal{J}u_\ell)\|^2_{L^2(T)} \lesssim \sum_{z \in \mathcal{N}(T)} \sum_{F \in \widehat{\mathcal{F}}_\ell(z,T)} h_F^{-1} \|[u_\ell]_F\|^2_{L^2(F)}.$$

Proof. Given $F \in \widehat{\mathcal{F}}_\ell$, let $\psi_F \in \mathrm{CR}^1_0(\widehat{\mathcal{T}}_\ell)$ denote the Crouzeix–Raviart basis function defined by $\psi_F(\mathrm{mid}(F)) = 1$ and $\psi_F(\mathrm{mid}(E)) = 0$ for $E \in \widehat{\mathcal{F}}_\ell \backslash \{F\}$. Given $T \in \widehat{\mathcal{T}}_\ell \backslash \mathcal{T}_\ell$, the affine function $u_\ell - J^*u_\ell$ reads

$$(u_\ell - J^*u_\ell)|_T = \sum_{F \in \mathcal{F}(T)} \big(u_\ell|_T(\mathrm{mid}(F)) - J^*u_\ell|_T(\mathrm{mid}(F))\big)\psi_F.$$

The triangle inequality proves

$$\|\mathrm{D}_{\mathrm{NC}}(u_\ell - J^*u_\ell)\|_{L^2(T)} \leq \sum_{F \in \mathcal{F}(T)} \big|u_\ell|_T(\mathrm{mid}(F)) - J^*u_\ell|_T(\mathrm{mid}(F))\big| \;\|\mathrm{D}\psi_F\|_{L^2(T)}.$$

Analogous arguments prove

$$\|\mathrm{D}_{\mathrm{NC}}(u_\ell - \mathcal{J}u_\ell)\|_{L^2(T)} \le \sum_{F\in\mathcal{F}(T)} \left| u_\ell|_T(\mathrm{mid}(F)) - J^* u_\ell|_T(\mathrm{mid}(F))\right| \, \|\mathrm{D}\psi_F\|_{L^2(T)} .$$

The shape regularity leads to the scaling $\|\mathrm{D}\psi_F\|_{L^2(T)} \approx h_F^{(n-2)/2}$ of the Crouzeix–Raviart basis functions. Since $\sum_{K\in\mathcal{Z}(y_j;T)} 1 = \mathrm{card}(\mathcal{Z}(y_j;T))$, the definition of J^* leads for $(u_\ell - J^*u_\ell)|_T \in P_1(F)$ on $F = \mathrm{conv}\{y_1,\dots,y_n\} \in \mathcal{F}(T)$ to

$$\left| u_\ell|_T(\mathrm{mid}(F)) - J^* u_\ell|_T(\mathrm{mid}(F))\right| \le \frac{\sum_{j=1}^n \left| \sum_{K\in\mathcal{Z}(y_j,T)} (u_\ell|_T(y_j) - u_\ell|_K(y_j))\right|}{n\ \mathrm{card}(\mathcal{Z}(y_j,T))}.$$

For a fixed $K \in \mathcal{Z}(y_j,T)$ let $N\in\mathbb{N}$ and $T_1,\dots,T_N \in \mathcal{Z}(y_j,T)$ with $T = T_1$, $K = T_N$, and $T_k \cap T_{k+1} \in \widehat{\mathcal{F}}_\ell$ for $k = 1,\dots,N-1$. This shows

$$u_\ell|_T(y_j) - u_\ell|_K(y_j) = \sum_{k=1}^{N-1} (u_\ell|_{T_k}(y_j) - u_\ell|_{T_{k+1}}(y_j)). \tag{4.1}$$

Consider $F = T_j \cap T_{j+1} \in \widehat{\mathcal{F}}_\ell$. Let $\varphi_j \in P_1(F)$ denote the barycentric coordinates on F with $\varphi_j(y_k) = \delta_{jk}$ for $j,k = 1,\dots,n$. Any $v \in P_1(F)$ with coefficient vector $x = (v(y_1),\dots,v(y_n))$ satisfies

$$\|v\|_{L^2(F)}^2 = \sum_{j,k=1}^n v(y_j)v(y_k)\int_F \varphi_j\varphi_k\, ds = x\cdot Mx$$

for the mass matrix $M \in \mathbb{R}^{n\times n}$. Elementary calculations reveal

$$M_{jk} = (1+\delta_{jk})|F|(n-1)!/(n+1)! \quad \text{for } j,k = 1,2,\dots,n.$$

Since the lowest eigenvalue of the symmetric matrix

$$1_{n\times n} + (1,\dots,1)\otimes(1,\dots,1) = (1+\delta_{jk})_{j,k=1,\dots,n}$$

is one, it follows that

$$v(y_j)^2 \le x\cdot x \le |F|^{-1}\, n(n+1)\|v\|_{L^2(F)}^2.$$

With $v := [u_\ell]_F \in P_1(F)$ for $F = \partial T_k \cap \partial T_{k+1}$ and $k = 1,\dots,N-1$, this proves

$$|u_\ell|_{T_k}(y_j) - u_\ell|_{T_{k+1}}(y_j)|^2 \le |F|^{-1}\, n(n+1)\|[u_\ell]_F\|_{L^2(F)}^2.$$

This reveals in (4.1) that

$$|u_\ell|_T(y_j) - u_\ell|_K(y_j)|^2 \lesssim \sum_{F\in\widehat{\mathcal{F}}_\ell(y_j,T)} h_F^{1-n}\, \|[u_\ell]_F\|_{L^2(F)}^2 .$$

The shape regularity implies $\mathrm{card}(\widehat{\mathcal{T}}_\ell(z)) \lesssim 1$. The combination of the aforementioned estimates leads to

$$\|\mathrm{D}_{\mathrm{NC}}(u_\ell - J^*u_\ell)\|_{L^2(T)}^2 + \|\mathrm{D}_{\mathrm{NC}}(u_\ell - \mathcal{J}u_\ell)\|_{L^2(T)}^2 \lesssim \sum_{z\in\mathcal{N}(T)} \sum_{F\in\widehat{\mathcal{F}}_\ell(z,T)} h_F^{-1}\, \|[u_\ell]_F\|_{L^2(F)}^2 .$$

This concludes the proof of Theorem 4.9. ∎

4.5. Proof of Theorem 3.2. Theorem 4.6 implies $\mathcal{J}u_\ell \in \mathrm{CR}^1_0(\mathcal{T}_{\ell+m})$ and so

$$\min_{v_{\ell+m}\in \mathrm{CR}^1_0(\mathcal{T}_{\ell+m})} \|\mathrm{D}_{\mathrm{NC}}(u_\ell - v_{\ell+m})\|_{L^2(\Omega)} \leq \|\mathrm{D}_{\mathrm{NC}}(u_\ell - \mathcal{J}u_\ell)\|_{L^2(\Omega)}\,.$$

Since $u_\ell = \mathcal{J}u_\ell$ on $\mathcal{T}_\ell \cap \widehat{\mathcal{T}}_\ell$, it follows that

$$\|\mathrm{D}_{\mathrm{NC}}(u_\ell - \mathcal{J}u_\ell)\|_{L^2(\Omega)} = \|\mathrm{D}_{\mathrm{NC}}(u_\ell - \mathcal{J}u_\ell)\|_{L^2(\cup(\widehat{\mathcal{T}}_\ell\setminus\mathcal{T}_\ell))}\,.$$

Lemma 4.2 implies $h_F \approx h_G$ for $G \in \mathcal{F}_\ell$, $F \in \widehat{\mathcal{F}}_\ell$ with $F \subseteq G$. Therefore, the finite overlap of the nodal patches in $\widehat{\mathcal{T}}_\ell$ and Theorem 4.9 imply

$$\|\mathrm{D}_{\mathrm{NC}}(u_\ell - \mathcal{J}u_\ell)\|^2_{L^2(\Omega)} \lesssim \sum_{F\in\mathcal{F}(\mathcal{T}_\ell\setminus\mathcal{T}_{\ell+m})} h_F^{-1}\,\|[u_\ell]_F\|^2_{L^2(F)}\,. \qquad \square$$

5. Applications. This section deduces the discrete reliability from the discrete distance control. This is done in an abstract framework in subsection 5.1, while subsections 5.2–5.5 discuss immediate applications of the abstract result to various model problems.

5.1. Abstract residual-based error control. Let $N \in \{1, n\}$ and $\mathcal{L}_\ell := P_0(\mathcal{T}_\ell;\mathbb{R}^n)$ if $N = 1$ and $\mathcal{L}_\ell := \{\tau_\ell \in P_0(\mathcal{T}_\ell;\mathbb{R}^{n\times n}) \mid \int_\Omega \mathrm{tr}(\tau_\ell)\,dx = 0\}$ if $N = n$. Let $\mathcal{H}_\ell := \mathrm{CR}^1_0(\mathcal{T}_\ell;\mathbb{R}^N)$ and $\mathcal{X}_\ell := \mathcal{L}_\ell \times \mathcal{H}_\ell$. Let $A \in P_0(\mathcal{T}_0;\mathbb{R}^{(n\times N)\times(n\times N)})$ with $\mathrm{tr}(A\tau_\ell) = \alpha\,\mathrm{tr}(\tau_\ell)$ for some $\alpha \in \mathbb{R}$ and all $\tau_\ell \in \mathcal{L}_\ell$ if $N = n$. Define the linear operator $\mathcal{A}_\ell : \mathcal{X}_\ell \to \mathcal{X}^*_\ell$ through

$$\big(\mathcal{A}_\ell(\tau_\ell, v_\ell)\big)(\xi_\ell, w_\ell) := (A\tau_\ell, \xi_\ell)_{L^2(\Omega)} - (\tau_\ell, \mathrm{D}_{\mathrm{NC}} w_\ell)_{L^2(\Omega)} - (\xi_\ell, \mathrm{D}_{\mathrm{NC}} v_\ell)_{L^2(\Omega)}.$$

Given $f_\ell \in L^2(\Omega;\mathbb{R}^N)$ and some approximation $(\widetilde{\sigma}_\ell, \widetilde{u}_\ell) \in \mathcal{L}_\ell \times \mathcal{H}_\ell$ to the solution (σ_ℓ, u_ℓ) of the equation

$$\big(\mathcal{A}_\ell(\sigma_\ell, u_\ell)\big)(\tau_\ell, v_\ell) = -(f_\ell, v_\ell)_{L^2(\Omega)} \quad \text{for all } (\tau_\ell, v_\ell) \in \mathcal{L}_\ell \times \mathcal{H}_\ell \tag{5.1}$$

the residuals read

$$\begin{aligned} \mathrm{Res}_{\mathcal{L}_\ell}(\widetilde{\sigma}_\ell, \widetilde{u}_\ell; \tau_\ell) &:= (A\widetilde{\sigma}_\ell, \tau_\ell)_{L^2(\Omega)} - (\tau_\ell, \mathrm{D}_{\mathrm{NC}}\widetilde{u}_\ell)_{L^2(\Omega)} && \text{for all } \tau_\ell \in \mathcal{L}_\ell,\\ \mathrm{Res}_{\mathcal{H}_\ell}(\widetilde{\sigma}_\ell; v_\ell) &:= (f_\ell, v_\ell)_{L^2(\Omega)} - (\widetilde{\sigma}_\ell, \mathrm{D}_{\mathrm{NC}} v_\ell)_{L^2(\Omega)} && \text{for all } v_\ell \in \mathcal{H}_\ell. \end{aligned}$$

The operator norms of the residuals read

$$\begin{aligned} \|\mathrm{Res}_{\mathcal{L}_\ell}(\widetilde{\sigma}_\ell, \widetilde{u}_\ell; \bullet)\|_{\mathcal{L}^*_\ell} &:= \sup_{\tau_\ell\in\mathcal{L}_\ell\setminus\{0\}} \frac{\mathrm{Res}_{\mathcal{L}_\ell}(\widetilde{\sigma}_\ell, \widetilde{u}_\ell; \tau_\ell)}{\|\tau_\ell\|_{L^2(\Omega)}},\\ \|\mathrm{Res}_{\mathcal{H}_\ell}(\widetilde{\sigma}_\ell; \bullet)\|_{\mathcal{H}^*_\ell} &:= \sup_{v_\ell\in\mathcal{H}_\ell\setminus\{0\}} \frac{\mathrm{Res}_{\mathcal{H}_\ell}(\widetilde{\sigma}_\ell; v_\ell)}{\|\mathrm{D}_{\mathrm{NC}} v_\ell\|_{L^2(\Omega)}}\,. \end{aligned}$$

Suppose that the discrete problem is well-posed in that $\mathcal{A}_\ell$ is bijective and bounded with bounded inverse. As in the abstract theory of [10], this implies the following equivalence:

$$\|\sigma_\ell - \widetilde{\sigma}_\ell\|_{L^2(\Omega)} + \|\mathrm{D}_{\mathrm{NC}}(u_\ell - \widetilde{u}_\ell)\|_{L^2(\Omega)} \approx \|\mathrm{Res}_{\mathcal{L}_\ell}(\widetilde{\sigma}_\ell, \widetilde{u}_\ell; \bullet)\|_{\mathcal{L}^*_\ell} + \|\mathrm{Res}_{\mathcal{H}_\ell}(\widetilde{\sigma}_\ell; \bullet)\|_{\mathcal{H}^*_\ell}\,. \tag{5.2}$$

Define the error estimator

$$\mu_\ell(f_\ell, u_\ell, \mathcal{T}_\ell\setminus\mathcal{T}_{\ell+m})^2 := \sum_{T\in\mathcal{T}_\ell\setminus\mathcal{T}_{\ell+m}} \|h_T f_\ell\|^2_{L^2(T)} + \sum_{F\in\mathcal{F}(\mathcal{T}_\ell\setminus\mathcal{T}_{\ell+m})} h_F^{-1}\|\,[u_\ell]_F\|^2_{L^2(F)}.$$

The following discrete reliability combines the discrete distance control with a control of $\|\mathrm{Res}_{\mathcal{H}_\ell}(\widetilde{\sigma}_\ell;\bullet)\|_{\mathcal{H}_\ell^*}$.

THEOREM 5.1 (discrete reliability). *The discrete solutions* $(\sigma_\ell, u_\ell) \in \mathcal{L}_\ell \times \mathcal{H}_\ell$ *and* $(\sigma_{\ell+m}, u_{\ell+m}) \in \mathcal{L}_{\ell+m} \times \mathcal{H}_{\ell+m}$ *of* (5.1) *on the levels* ℓ *and* $\ell+m$ *for the right-hand-sides* f_ℓ *and* $f_{\ell+m}$ *satisfy* $A\sigma_\ell = \mathrm{D}_{\mathrm{NC}} u_\ell$ *and* $A\sigma_{\ell+m} = \mathrm{D}_{\mathrm{NC}} u_{\ell+m}$ *and the following discrete reliability holds:*

$$\|\sigma_{\ell+m} - \sigma_\ell\|_{L^2(\Omega)} \lesssim \mu_\ell(f_\ell, u_\ell, \mathcal{T}_\ell \backslash \mathcal{T}_{\ell+m}) + \|f_{\ell+m} - f_\ell\|_{L^2(\Omega)}.$$

Proof. The definition of A implies $A\tau_\ell \in \mathcal{L}_\ell$ for all $\tau_\ell \in \mathcal{L}_\ell$. For $N = n$ a piecewise integration by parts reveals for $v_\ell \in \mathcal{H}_\ell$

$$\int_\Omega \mathrm{tr}(\mathrm{D}_{\mathrm{NC}} v_\ell)\, dx = \int_\Omega \mathrm{div}_{\mathrm{NC}}\, v_\ell\, dx = \sum_{F \in \mathcal{F}_\ell} \int_F [v_\ell]_F \cdot \nu_F\, ds = 0.$$

Hence, for $N = n$ and (obviously) for $N = 1$, it holds that $\mathrm{D}_{\mathrm{NC}} v_\ell \in \mathcal{L}_\ell$ for all $v_\ell \in \mathcal{H}_\ell$. This implies $A\sigma_\ell = \mathrm{D}_{\mathrm{NC}} u_\ell$ (and analogously $A\sigma_{\ell+m} = \mathrm{D}_{\mathrm{NC}} u_{\ell+m}$).

Set $\widetilde{\sigma}_{\ell+m} = \sigma_\ell$ and $\widetilde{u}_{\ell+m} := \mathrm{argmin}_{v_{\ell+m} \in \mathcal{L}_{\ell+m}} \|\mathrm{D}_{\mathrm{NC}}(u_\ell - v_{\ell+m})\|_{L^2(\Omega)}$. The equivalence (5.2) shows that it suffices to bound the residuals

$$\left\|\mathrm{Res}_{\mathcal{L}_{\ell+m}}(\sigma_\ell, \widetilde{u}_{\ell+m};\bullet)\right\|_{\mathcal{L}_{\ell+m}^*} \quad \text{and} \quad \left\|\mathrm{Res}_{\mathcal{H}_{\ell+m}}(\sigma_\ell;\bullet)\right\|_{\mathcal{H}_{\ell+m}^*}.$$

The nonconforming interpolation operator $I_\ell : \mathrm{CR}_0^1(\mathcal{T}_{\ell+m};\mathbb{R}^N) \to \mathrm{CR}_0^1(\mathcal{T}_\ell;\mathbb{R}^N)$ is defined for $v_{\ell+m} \in \mathrm{CR}_0^1(\mathcal{T}_{\ell+m};\mathbb{R}^N)$ on each midpoint of an interior hyper-surface by $I_\ell(\mathrm{mid}(F)) := \fint_F v_{\ell+m}\, ds$ for all $F \in \mathcal{F}(\mathcal{T}_\ell)$. It satisfies the well-known projection property

$$\mathrm{D}_{\mathrm{NC}}(I_\ell v_{\ell+m})|_T = \fint_T \mathrm{D}_{\mathrm{NC}} v_{\ell+m}\, dx \qquad \text{for all } T \in \mathcal{T}_\ell. \tag{5.3}$$

This and the discrete Friedrichs inequality (a direct generalization of [8] and [9, Theorem 10.6.12] to higher dimensions) for the function $v_{\ell+m} - I_\ell v_{\ell+m}$ prove, for any simplex $T \in \mathcal{T}_\ell$ and $v_{\ell+m} \in \mathrm{CR}_0^1(\mathcal{T}_{\ell+m};\mathbb{R}^N)$, the approximation and stability properties

$$\|h_T^{-1}(v_{\ell+m} - I_\ell v_{\ell+m})\|_{L^2(T)} \lesssim \|\mathrm{D}_{\mathrm{NC}}(v_{\ell+m} - I_\ell v_{\ell+m})\|_{L^2(T)} \le \|\mathrm{D}_{\mathrm{NC}} v_{\ell+m}\|_{L^2(T)}. \tag{5.4}$$

The integral mean property (5.3) and the discrete problem (5.1) prove

$$(\sigma_\ell, \mathrm{D}_{\mathrm{NC}} v_{\ell+m})_{L^2(\Omega)} = (\sigma_\ell, \mathrm{D}_{\mathrm{NC}} I_\ell v_{\ell+m})_{L^2(\Omega)} = (f_\ell, I_\ell v_{\ell+m})_{L^2(\Omega)}.$$

Since $I_\ell v_{\ell+m} = v_{\ell+m}$ on $\mathcal{T}_{\ell+m} \cap \mathcal{T}_\ell$, the approximation property (5.4) and the discrete Friedrichs inequality show

$$\begin{aligned}
&\mathrm{Res}_{\mathcal{H}_{\ell+m}}(\sigma_\ell; v_{\ell+m}) \\
&\quad = (f_{\ell+m}, v_{\ell+m})_{L^2(\Omega)} - (f_\ell, I_\ell v_{\ell+m})_{L^2(\Omega)} \\
&\quad = (f_{\ell+m} - f_\ell, v_{\ell+m})_{L^2(\Omega)} - (f_\ell, I_\ell v_{\ell+m} - v_{\ell+m})_{L^2(\Omega)} \\
&\quad \lesssim \left(\sqrt{\sum_{T \in \mathcal{T}_\ell \backslash \mathcal{T}_{\ell+m}} \|h_T f\|_{L^2(T)}^2} + \|f_{\ell+m} - f_\ell\|_{L^2(\Omega)} \right) \|\mathrm{D}_{\mathrm{NC}} v_{\ell+m}\|_{L^2(\Omega)}.
\end{aligned}$$

The residual $\operatorname{Res}_{\mathcal{L}_{\ell+m}}(\sigma_\ell, \widetilde{u}_{\ell+m}; \bullet)$ satisfies

$$\begin{aligned}\operatorname{Res}_{\mathcal{L}_{\ell+m}}(\sigma_\ell, \widetilde{u}_{\ell+m}; \tau_{\ell+m}) &= (A\sigma_\ell - \mathrm{D}_{\mathrm{NC}}\widetilde{u}_{\ell+m}, \tau_{\ell+m})_{L^2(\Omega)}\\ &= (\mathrm{D}_{\mathrm{NC}}(u_\ell - \widetilde{u}_{\ell+m}), \tau_{\ell+m})_{L^2(\Omega)}.\end{aligned}$$

Therefore, the definition of $\widetilde{u}_{\ell+m}$ and (5.2) imply

$$\|\sigma_{\ell+m} - \sigma_\ell\|_{L^2(\Omega)} \lesssim \min_{v_{\ell+m}\in \mathrm{CR}^1_0(\mathcal{T}_{\ell+m})} \|\mathrm{D}_{\mathrm{NC}}(v_{\ell+m} - u_\ell)\|_{L^2(\Omega)} + \|\operatorname{Res}_{\mathcal{H}_{\ell+m}}(\sigma_\ell; \bullet)\|_{\mathcal{H}^*}.$$

The combination of the previous estimates with Theorem 3.1 concludes the proof. $\square$

5.2. Linear model problem. Given $f \in L^2(\Omega)$, the Crouzeix–Raviart finite element discretization of the problem $\operatorname{div} L\mathrm{D}u + f = 0$ for a symmetric positive definite tensor field $L \in P_0(\mathcal{T}_0; \mathbb{R}^{n\times n})$ and homogeneous Dirichlet boundary conditions seeks $u_\ell \in \mathrm{CR}^1_0(\mathcal{T}_\ell)$ with

$$(\mathrm{D}_{\mathrm{NC}}v_\ell, L\mathrm{D}_{\mathrm{NC}}u_\ell)_{L^2(\Omega)} = (f, v_\ell)_{L^2(\Omega)} \qquad \text{for all } v_\ell \in \mathrm{CR}^1_0(\mathcal{T}_\ell).$$

For $N = 1$, $A := L^{-1}$ and $f_\ell := f$, this problem is equivalent to (5.1). Theorem 5.1 implies $\|L\mathrm{D}_{\mathrm{NC}}(u_{\ell+m} - u_\ell)\|_{L^2(\Omega)} \lesssim \mu_\ell(f_\ell, u_\ell, \mathcal{T}_\ell\backslash\mathcal{T}_{\ell+m})$ and so generalizes [4, 26] to multiply connected $\Omega \subseteq \mathbb{R}^n$ for $n \geq 2$.

5.3. Eigenvalue problems. The discretization of the eigenvalue problem corresponding to the linear problem of subsection 5.2 seeks the first eigenpair $(\lambda_\ell, u_\ell) \in \mathbb{R} \times \mathrm{CR}^1_0(\mathcal{T}_\ell)$ with

$$(\mathrm{D}_{\mathrm{NC}}v_\ell, L\mathrm{D}_{\mathrm{NC}}u_\ell)_{L^2(\Omega)} = (\lambda_\ell u_\ell, v_\ell)_{L^2(\Omega)} \qquad \text{for all } v_\ell \in \mathrm{CR}^1_0(\mathcal{T}_\ell)$$

with L as above. With $N = 1$, $A = L^{-1}$, and $f_\ell = \lambda_\ell u_\ell$, Theorem 5.1 leads to

$$\|L\mathrm{D}_{\mathrm{NC}}(u_\ell - u_{\ell+m})\|_{L^2(\Omega)} \lesssim \mu_\ell(\lambda_\ell u_\ell, u_\ell, \mathcal{T}_\ell\backslash\mathcal{T}_{\ell+m}) + \|\lambda_\ell u_\ell - \lambda_{\ell+m}u_{\ell+m}\|_{L^2(\Omega)}.$$

The term $\|\lambda_\ell u_\ell - \lambda_{\ell+m}u_{\ell+m}\|_{L^2(\Omega)}$ is of higher order [6, 12] and can be absorbed in the proof of optimality. This generalizes the discrete reliability of [12] to multiply connected $\Omega \subseteq \mathbb{R}^n$ for $n \geq 2$.

5.4. Stokes equations. For $n = 2, 3$ the nonconforming FEM for the Stokes equations $-\Delta u + \mathrm{D}p = f$ with homogeneous Dirichlet boundary conditions and $f \in L^2(\Omega; \mathbb{R}^n)$ seeks $u_\ell \in \mathrm{CR}^1_0(\mathcal{T}_\ell; \mathbb{R}^n)$ and $p_\ell \in P_0(\mathcal{T}_\ell) \cap L^2_0(\Omega)$ (for $L^2_0(\Omega) := \{q \in L^2(\Omega) \mid \int_\Omega q\,dx = 0\}$) such that

$$\begin{aligned}(\mathrm{D}_{\mathrm{NC}}u_\ell, \mathrm{D}_{\mathrm{NC}}v_\ell)_{L^2(\Omega)} - (p_\ell, \operatorname{div}_{\mathrm{NC}} v_\ell)_{L^2(\Omega)} &= (f, v_\ell)_{L^2(\Omega)} &&\text{for all } v_\ell \in \mathrm{CR}^1_0(\mathcal{T}_\ell; \mathbb{R}^n),\\ (q_\ell, \operatorname{div}_{\mathrm{NC}} u_\ell)_{L^2(\Omega)} &= 0 &&\text{for all } q_\ell \in P_0(\mathcal{T}_\ell) \cap L^2_0(\Omega).\end{aligned}$$

The substitution $\sigma_\ell := \mathrm{D}_{\mathrm{NC}}u_\ell - p_\ell 1_{n\times n} \in \mathcal{L}_\ell$ leads to an equivalent formulation with $N = n$, $A := \operatorname{dev}$ (defined by $\operatorname{dev} M := M - (\operatorname{tr}(M)/n)1_{n\times n}$ for $M \in \mathbb{R}^{n\times n}$) and $f_\ell := f$. Since $\mathrm{D}_{\mathrm{NC}}(u_{\ell+m} - u_\ell) = \operatorname{dev}(\sigma_{\ell+m} - \sigma_\ell)$, Theorem 5.1 implies

$$\|\mathrm{D}_{\mathrm{NC}}(u_{\ell+m} - u_\ell)\|_{L^2(\Omega)} \leq \|\sigma_{\ell+m} - \sigma_\ell\|_{L^2(\Omega)} \lesssim \mu_\ell(f_\ell, u_\ell, \mathcal{T}_\ell\backslash\mathcal{T}_{\ell+m})$$

and so generalizes [15] to multiply connected $\Omega \subseteq \mathbb{R}^n$ for $n \geq 2$.

5.5. Linear elasticity. For $\Omega \subseteq \mathbb{R}^n$ $(n = 2, 3)$ and $f \in L^2(\Omega; \mathbb{R}^n)$ the nonconforming discretization of the Navier–Lamé equations for linear elasticity (with full gradient) seeks $u_\ell \in \mathrm{CR}_0^1(\mathcal{T}_\ell; \mathbb{R}^n)$ with

$$(\mathrm{D}_{\mathrm{NC}} v_\ell, \widetilde{\mathbb{C}} \mathrm{D}_{\mathrm{NC}} u_\ell)_{L^2(\Omega)} = (f, v_\ell)_{L^2(\Omega)} \qquad \text{for all } v_\ell \in \mathrm{CR}_0^1(\mathcal{T}_\ell; \mathbb{R}^n).$$

The fourth-order elasticity tensor $\widetilde{\mathbb{C}}$ acts as $\widetilde{\mathbb{C}}A := \mu A + (\mu + \lambda)\operatorname{tr}(A) 1_{n\times n}$ for Lamé parameters $\mu, \lambda > 0$. This problem is equivalent to (5.1) for $N = n$, $A := \widetilde{\mathbb{C}}^{-1}$ and $f_\ell := f$.

The arguments of [10, Lemma 4.1] and the projection property (5.3) easily prove that the operator $\mathcal{A}_\ell : \mathcal{X}_\ell \to \mathcal{X}_\ell^\star$ is linear, bounded, and bijective and the operator norms of $\mathcal{A}_\ell$ and $\mathcal{A}_\ell^{-1}$ are λ-independent. Hence, Theorem 5.1 implies $\|\sigma_{\ell+m} - \sigma_\ell\|_{L^2(\Omega)} \lesssim \mu_\ell(f_\ell, u_\ell, \mathcal{T}_\ell \backslash \mathcal{T}_{\ell+m})$ and so generalizes [17] to multiply connected $\Omega \subseteq \mathbb{R}^n$ for $n \geq 2$.

6. Example for optimal convergence of AFEM. As an application of the discrete reliability, this section discusses the proof of optimal convergence rates of an AFEM for uniformly convex minimization problems. This section utilizes a modified definition $h_\ell|_T := h_T := |T|^{1/n}$ for a simplex $T \in \mathcal{T}_\ell$. The shape regularity implies $|T|^{1/n} \approx \operatorname{diam}(T)$ and therefore the results of sections 1–5 remain valid with this definition.

6.1. AFEM for uniformly convex minimization. Let $W \in C^1(\mathbb{R}^n)$ be a uniformly convex energy density with Lipschitz continuous derivative, i.e., there exist positive constants α, $L > 0$ such that

$$\alpha|\sigma - \tau|^2 \leq W(\sigma) - W(\tau) - \mathrm{D}W(\tau)\cdot(\sigma - \tau) \qquad \text{and} \tag{6.1a}$$

$$|\mathrm{D}W(\sigma) - \mathrm{D}W(\tau)| \leq L|\sigma - \tau| \qquad \text{for all } \sigma, \tau \in \mathbb{R}^n. \tag{6.1b}$$

Explicit applications and precise examples can be found in the literature [32, 33]. Given $f \in L^2(\Omega)$, the minimizer $u \in V := H_0^1(\Omega)$ of the energy functional

$$E(v) := \int_\Omega W(\mathrm{D}v)\, dx - \int_\Omega f v\, dx \quad \text{for all } v \in V$$

satisfies [32, 33] the Euler–Lagrange equation $f + \operatorname{div} \mathrm{D}W(\mathrm{D}u) = 0$ in $H^{-1}(\Omega)$. For a regular triangulation $\mathcal{T}_\ell$, the discrete problem seeks the minimizer $u_\ell \in \mathrm{CR}_0^1(\mathcal{T}_\ell)$ of the discrete energy

$$E_{\mathrm{NC}}(v_\ell) := \int_\Omega W(\mathrm{D}_{\mathrm{NC}} v_\ell)\, dx - \int_\Omega f v_\ell\, dx.$$

Given any triangulation $\mathcal{T}_\ell \in \mathbb{T}$, the adaptive algorithm (AFEM) makes use of the error estimator $\eta_\ell^2 := \eta_\ell^2(u_\ell, \mathcal{T}_\ell)$ defined by

$$\eta_\ell^2(u_\ell, T) := \|h_T f\|_{L^2(T)}^2 + \sum_{F\in\mathcal{F}(T)} h_F^{-1} \|[u_\ell]_F\|_{L^2(F)}^2 \quad \text{for } T \in \mathcal{T}_\ell$$

$$\text{and} \quad \eta_\ell^2(u_\ell, \mathcal{K}) := \sum_{T\in\mathcal{K}} \eta_\ell^2(u_\ell, T) \quad \text{for any subset } \mathcal{K} \subseteq \mathcal{T}_\ell.$$

Algorithm 6.1 (AFEM).
Input: $\mathcal{T}_0$, bulk parameter $0 < \theta < \theta_0 \le 1$.
Loop: For $\ell = 0, 1, 2, \dots$
Solve Compute discrete solution u_ℓ with respect to $\mathcal{T}_\ell$.
Estimate Compute $\eta_\ell^2 = \eta_\ell^2(u_\ell, \mathcal{T}_\ell)$.
Mark a minimal subset $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell$ with $\theta\eta_\ell^2 \le \eta_\ell^2(u_\ell, \mathcal{M}_\ell)$.
Refine Compute $\mathcal{T}_{\ell+1} := \mathtt{refine}(\mathcal{T}_\ell, \mathcal{M}_\ell)$.
Output: Sequence of triangulations $(\mathcal{T}_\ell)_\ell$ and discrete solutions $(u_\ell)_\ell$.

The concept of optimality relies on the nonlinear approximation class $\mathbb{A}_s$ which involves the data resolution $\operatorname{osc}^2(f, \mathcal{T}) := \|h_\mathcal{T}(1 - \Pi_\mathcal{T})f\|_{L^2(\Omega)}$ and the best-approximation error $\|(1 - \Pi_\mathcal{T})\mathrm{D}u\|^2_{L^2(\Omega)}$ for $\Pi_\mathcal{T}$ the L^2 projection onto piecewise constant functions. For any subset $\mathcal{K} \subseteq \mathcal{T}$, the oscillations of f read

$$\operatorname{osc}^2(f, \mathcal{K}) := \|h_\mathcal{T}(f - \Pi_\mathcal{T} f)\|^2_{L^2(\cup\mathcal{K})}.$$

Define the seminorm

$$|(u,f)|_{\mathbb{A}_s} := \sup_{N\in\mathbb{N}} N^s \inf_{\substack{\mathcal{T}\in\mathbb{T}\\ \operatorname{card}(\mathcal{T})-\operatorname{card}(\mathcal{T}_0)\le N}} \left(\|(1-\Pi_\mathcal{T})\mathrm{D}u)\|^2_{L^2(\Omega)} + \operatorname{osc}^2(f,\mathcal{T})\right)^{1/2}$$

and the approximation class

$$\mathbb{A}_s := \left\{(u,f) \in V \times L^2(\Omega) \;\middle|\; u \text{ minimizes } E \text{ with respect to } f \text{ and } |(u,f)|_{\mathbb{A}_s} < \infty\right\}.$$

Theorem 6.2 (optimal convergence rates). *For sufficiently small $0 < \theta \le \theta_0$, and any $s > 0$ with $|(u,f)|_{\mathbb{A}_s} < \infty$, AFEM computes sequences of triangulations $(\mathcal{T}_\ell)_\ell$ and discrete solutions $(u_\ell)_\ell$ of optimal rate of convergence in the sense that for some C_{opt} (which depends on θ, s and $\mathcal{T}_0$) and all $\ell \in \mathbb{N}_0$ it holds that*

$$(\operatorname{card}(\mathcal{T}_\ell) - \operatorname{card}(\mathcal{T}_0))^s \left(\|\mathrm{D}_{\mathrm{NC}}(u-u_\ell)\|^2_{L^2(\Omega)} + \operatorname{osc}^2(f,\mathcal{T}_\ell)\right)^{1/2} \le C_{\mathrm{opt}}|(u,f)|_{\mathbb{A}_s}.$$

The following best-approximation result is an immediate consequence of the results of [16, 23] and implies convergence for a sequence of uniform refinements.

Lemma 6.3 (best-approximation up to oscillations). *For any $\mathcal{T}_\ell \in \mathbb{T}$ the discrete solution $u_\ell \in \mathrm{CR}^1_0(\mathcal{T}_\ell)$ satisfies*

$$\|\mathrm{D}_{\mathrm{NC}}(u-u_\ell)\|^2_{L^2(\Omega)} \lesssim \|(1-\Pi_\ell)\mathrm{D}u\|^2_{L^2(\Omega)} + \operatorname{osc}^2(f,\mathcal{T}_\ell). \qquad \square$$

The main tool in the proof of Theorem 6.2 is the discrete reliability.

Theorem 6.4 (discrete reliability, reliability, and efficiency). *For any $\mathcal{T}_\ell \in \mathrm{CR}^1_0(\mathcal{T}_\ell)$ and any refinement $\mathcal{T}_{\ell+m} \in \mathbb{T}(\mathcal{T}_\ell)$ the discrete solutions $u_\ell \in \mathrm{CR}^1_0(\mathcal{T}_\ell)$ and $u_{\ell+m} \in \mathrm{CR}^1_0(\mathcal{T}_{\ell+m})$ satisfy for constants $C_{\mathrm{drel}} \approx C_{\mathrm{rel}} \approx C_{\mathrm{eff}} \approx 1$ that*

$$\begin{aligned}
&\|\mathrm{D}_{\mathrm{NC}}(u_{\ell+m}-u_\ell)\|^2_{L^2(\Omega)} \le C_{\mathrm{drel}}\eta_\ell^2(u_\ell, \mathcal{T}_\ell\backslash\mathcal{T}_{\ell+m}) \quad \textit{and}\\
&C_{\mathrm{rel}}^{-1}\|\mathrm{D}_{\mathrm{NC}}(u-u_\ell)\|^2_{L^2(\Omega)} \le \eta_\ell^2(u_\ell,\mathcal{T}_\ell) \le C_{\mathrm{eff}}\left(\|\mathrm{D}_{\mathrm{NC}}(u-u_\ell)\|^2_{L^2(\Omega)} + \operatorname{osc}^2(f,\mathcal{T}_\ell)\right).
\end{aligned}$$

Proof. Let $v_{\ell+m} := \operatorname{argmin}_{w_{\ell+m}\in\mathrm{CR}^1_0(\mathcal{T}_{\ell+m})} \|\mathrm{D}_{\mathrm{NC}}(u_\ell - w_{\ell+m})\|_{L^2(\Omega)}$. The discrete Euler–Lagrange equation reads

$$(\mathrm{D}W(\mathrm{D}_{\mathrm{NC}}u_\ell), \mathrm{D}_{\mathrm{NC}}v_\ell)_{L^2(\Omega)} = (f, v_\ell)_{L^2(\Omega)} \quad \text{for all } v_\ell \in \mathrm{CR}^1_0(\mathcal{T}_\ell).$$

The monotonicity

$$\|\mathrm{D}_{\mathrm{NC}}(u_{\ell+m}-u_\ell)\|^2_{L^2(\Omega)} \lesssim (\mathrm{D}W(\mathrm{D}_{\mathrm{NC}}u_{\ell+m}) - \mathrm{D}W(\mathrm{D}_{\mathrm{NC}}u_\ell), \mathrm{D}_{\mathrm{NC}}(u_{\ell+m}-u_\ell))_{L^2(\Omega)}$$

is a direct consequence of the uniform convexity (6.1a). This and the discrete problem lead for $\sigma_{\ell+m} := \mathrm{D}W(\mathrm{D}_{\mathrm{NC}} u_{\ell+m})$ and $\sigma_\ell := \mathrm{D}W(\mathrm{D}_{\mathrm{NC}} u_\ell)$ to

$$\begin{aligned}&\|\mathrm{D}_{\mathrm{NC}}(u_{\ell+m} - u_\ell)\|^2_{L^2(\Omega)} \lesssim \big(\sigma_{\ell+m} - \sigma_\ell, \mathrm{D}_{\mathrm{NC}}(u_{\ell+m} - v_{\ell+m} + v_{\ell+m} - u_\ell)\big)_{L^2(\Omega)} \\ &\quad = \big(f, u_{\ell+m} - v_{\ell+m} - I_\ell(u_{\ell+m} - v_{\ell+m}\big)_{L^2(\Omega)} + \big(\sigma_{\ell+m} - \sigma_\ell, \mathrm{D}_{\mathrm{NC}}(v_{\ell+m} - u_\ell)\big)_{L^2(\Omega)}.\end{aligned}$$

The Cauchy inequality, the projection property (5.3), the approximation property (5.4), the Lipschitz continuity of $\mathrm{D}W$, and the Pythagoras theorem show that this can be bounded from above by

$$\big(\|h_\ell f\|_{L^2(\cup(\mathcal{T}_\ell \setminus \mathcal{T}_{\ell+m}))} + \|\mathrm{D}_{\mathrm{NC}}(v_{\ell+m} - u_\ell)\|_{L^2(\Omega)}\big)\|\mathrm{D}_{\mathrm{NC}}(u_{\ell+m} - u_\ell)\|_{L^2(\Omega)}.$$

Theorem 3.1 proves the discrete reliability. Lemma 6.3 implies convergence on a sequence of uniformly refined triangulations. Hence, reliability follows from the discrete reliability. The proof of efficiency follows the standard arguments of [31] and hence is omitted. ∎

6.2. Contraction property. The contraction property considers an appropriate linear combination of the energy difference and an error estimator term. Let $\kappa \approx 1$ denote the constant (only dependent on the shape regularity; cf. (5.4) and [13])) that satisfies for all $T \in \mathcal{T}_\ell$ and all $v_{\mathrm{NC}} \in H^1_0(\Omega) \cup \mathrm{CR}^1_0(\mathcal{T}_{\ell+m})$ that

$$\|h_T^{-1}(v_{\mathrm{NC}} - I_\ell v_{\mathrm{NC}})\|_{L^2(T)} \le \kappa \|\mathrm{D}_{\mathrm{NC}}(v_{\mathrm{NC}} - I_\ell v_{\mathrm{NC}})\|_{L^2(T)}. \tag{6.2}$$

Choose $\gamma > \kappa^2 2^{2/n}/(4\alpha(2^{2/n} - 1))$ and define

$$\delta_\ell^2 := \delta(\mathcal{T}_\ell)^2 := E(u) - E_{\mathrm{NC}}(u_\ell) + \gamma\|h_\ell f\|^2_{L^2(\Omega)}.$$

THEOREM 6.5 (contraction property). *There exist* $0 < \beta < \infty$ *and* $0 < \rho_1 < 1$ *which only depend on* $\mathcal{T}_0$, γ, *and* θ_0 *such that for any* $\ell \in \mathbb{N}_0$ *for the refinement* $\mathcal{T}_{\ell+1}$ *of* $\mathcal{T}_\ell$ *generated by AFEM on two consecutive levels* ℓ *and* $\ell + 1$, *the term*

$$\xi_\ell^2 := \eta_\ell^2 + \beta\delta_\ell^2 \quad \textit{satisfies} \quad \xi_{\ell+1} \le \rho_1 \xi_\ell.$$

The following lemma proves (together with Theorem 6.4) the equivalence of δ_ℓ^2, $\|\mathrm{D}_{\mathrm{NC}}(u - u_\ell)\|^2_{L^2(\Omega)}$, and η_ℓ^2 up to oscillations.

LEMMA 6.6. *There exist constants* $C_1 \approx 1 \approx C_2$ *such that any* $\mathcal{T}_\ell \in \mathbb{T}$ *satisfies*

$$C_1^{-1}\|\mathrm{D}_{\mathrm{NC}}(u - u_\ell)\|^2_{L^2(\Omega)} \le \delta_\ell^2 \le C_2 \eta_\ell^2(u_\ell, \mathcal{T}_\ell). \tag{6.3}$$

Proof. The uniform convexity, the projection property (5.3), and the discrete Euler–Lagrange equations imply

$$\begin{aligned}&\alpha\|\mathrm{D}_{\mathrm{NC}}(u - u_\ell)\|^2_{L^2(\Omega)} \\ &\quad \le \int_\Omega W(\mathrm{D}u)\,dx - \int_\Omega W(\mathrm{D}_{\mathrm{NC}} u_\ell)\,dx - \big(\mathrm{D}W(\mathrm{D}_{\mathrm{NC}} u_\ell), \mathrm{D}_{\mathrm{NC}}(u - u_\ell)\big)_{L^2(\Omega)} \\ &\quad = E(u) - E_{\mathrm{NC}}(u_\ell) + (f, u - I_\ell u)_{L^2(\Omega)}.\end{aligned}$$

The approximation property (6.2) and the Young inequality prove

$$(f, u - I_\ell u)_{L^2(\Omega)} \le \gamma\|h_\ell f\|^2_{L^2(\Omega)} + \kappa^2/(4\gamma)\|\mathrm{D}_{\mathrm{NC}}(u - I_\ell u)\|^2_{L^2(\Omega)}.$$

This implies the first inequality of (6.3) with $C_1 := (\alpha - \kappa^2/(4\gamma))^{-1} > 0$.

The uniform convexity and $(DW(Du), \cdot)_{L^2(\Omega)} = (f, \cdot)_{L^2(\Omega)}$ in $H^{-1}(\Omega)$ yield

$$\begin{aligned} &E(u) - E_{\mathrm{NC}}(u_\ell) + \alpha \|\mathrm{D}_{\mathrm{NC}}(u - u_\ell)\|^2_{L^2(\Omega)} \\ &\quad \le (DW(\mathrm{D}u), \mathrm{D}_{\mathrm{NC}}(u - u_\ell))_{L^2(\Omega)} - (f, u - u_\ell)_{L^2(\Omega)} \\ &\quad \le \|(DW(\mathrm{D}u), \mathrm{D}_{\mathrm{NC}}\cdot)_{L^2(\Omega)} - (f, \cdot)_{L^2(\Omega)}\|^2_{\mathrm{CR}^1_0(\mathcal{T}_\ell)^\star}/2 + \|\mathrm{D}_{\mathrm{NC}}(u - u_\ell)\|^2_{L^2(\Omega)}/2. \end{aligned}$$

For any $v_\ell \in \mathrm{CR}^1_0(\mathcal{T}_\ell)$ there exists [7, 20] some conforming quasi interpolation $v_{C,\ell} \in P_1(\mathcal{T}_\ell) \cap V$ such that

$$\|h_T^{-1}(v_\ell - v_{C,\ell})\|_{L^2(\Omega)} + \|\mathrm{D}_{\mathrm{NC}}(v_\ell - v_{C,\ell})\|_{L^2(\Omega)} \lesssim \min_{v \in H^1_0(\Omega)} \|\mathrm{D}_{\mathrm{NC}}(v_\ell - v)\|_{L^2(\Omega)}.$$

Hence, for any $v_\ell \in \mathrm{CR}^1_0(\mathcal{T}_\ell)$ with $\|\mathrm{D}_{\mathrm{NC}} v_\ell\|_{L^2(\Omega)} = 1$, (6.1b) proves

$$\begin{aligned} &(DW(\mathrm{D}u), \mathrm{D}_{\mathrm{NC}} v_\ell)_{L^2(\Omega)} - (f, v_\ell)_{L^2(\Omega)} \\ &\quad = (DW(\mathrm{D}u), \mathrm{D}_{\mathrm{NC}}(v_\ell - v_{C,\ell}))_{L^2(\Omega)} - (f, v_\ell - v_{C,\ell})_{L^2(\Omega)} \lesssim \|\mathrm{D}_{\mathrm{NC}}(u - u_\ell)\|_{L^2(\Omega)}. \end{aligned}$$

The reliablilty from Theorem 6.4 concludes the proof. ∎

Proof of Theorem 6.5. The error estimator reduction property [15, 26] leads to constants $0 < \rho_0 < 1$ and $0 < \Lambda < \infty$ (which only depend on $\mathcal{T}_0$ and θ_0) such that

$$\eta^2_{\ell+1} \le \rho_0 \eta^2_\ell + \Lambda \|\mathrm{D}_{\mathrm{NC}}(u_{\ell+1} - u_\ell)\|^2_{L^2(\Omega)}.$$

The arguments from the proof of Lemma 6.6 and the observation that $I_\ell u_{\ell+1} = u_{\ell+1}$ on $\mathcal{T}_\ell \cap \mathcal{T}_{\ell+1}$ prove that $C_3 := (\alpha - \kappa^2 2^{2/n}/(4\gamma(2^{2/n} - 1)))^{-1} > 0$ satisfies

$$\begin{aligned} \text{(6.4)} \qquad &\|\mathrm{D}_{\mathrm{NC}}(u_{\ell+1} - u_\ell)\|^2_{L^2(\Omega)} \\ &\quad \le C_3(E_{\mathrm{NC}}(u_{\ell+1}) - E_{\mathrm{NC}}(u_\ell) + (1 - 2^{-2/n})\gamma \, \|h_\ell f\|^2_{L^2(\cup(\mathcal{T}_\ell \setminus \mathcal{T}_{\ell+1}))}). \end{aligned}$$

The relation $h^n_{\ell+1} \le h^n_\ell/2$ on $\mathcal{T}_\ell \setminus \mathcal{T}_{\ell+1}$ proves

$$\|h_\ell f\|^2_{L^2(\cup(\mathcal{T}_\ell \setminus \mathcal{T}_{\ell+1}))} \le (1 - 2^{-2/n})^{-1}(\|h_\ell f\|^2_{L^2(\Omega)} - \|h_{\ell+1} f\|^2_{L^2(\Omega)}).$$

Hence,

$$\begin{aligned} &E_{\mathrm{NC}}(u_{\ell+1}) - E_{\mathrm{NC}}(u_\ell) + (1 - 2^{-2/n})\gamma \|h_\ell f\|_{L^2(\cup(\mathcal{T}_\ell \setminus \mathcal{T}_{\ell+1}))} \\ &\quad = \delta^2_\ell - \delta^2_{\ell+1} + \gamma\big(- \|h_\ell f\|^2_{L^2(\Omega)} + \|h_{\ell+1} f\|^2_{L^2(\Omega)} + (1 - 2^{-2/n}) \, \|h_\ell f\|^2_{L^2(\cup(\mathcal{T}_\ell \setminus \mathcal{T}_{\ell+1}))}\big) \\ &\quad \le \delta^2_\ell - \delta^2_{\ell+1}. \end{aligned}$$

The combination of the preceding estimates yields

$$\eta^2_{\ell+1} \le \rho_0 \eta^2_\ell + C_3 \Lambda(\delta_\ell - \delta_{\ell+1}).$$

Lemma 6.6 proves for any $\lambda > 0$, $\rho_1 := \max\{\rho_0 + \Lambda\lambda C_3 C_2, 1 - \lambda\}$, and $\beta := \Lambda C_3$ that

$$\eta^2_{\ell+1} + \beta\delta^2_{\ell+1} \le \rho_1(\eta^2_\ell + \beta\delta^2_\ell).$$

The choice of a sufficiently small λ leads to $\rho_1 < 1$. ∎

6.3. Proof of optimality. The results of the foregoing subsections allow us to adapt the strategy from [18, 27] to the present situation and to prove Theorem 6.2.

For a triangulation $\mathcal{T} \in \mathbb{T}$ with mesh-size $h_\mathcal{T} \in P_0(\mathcal{T})$ let $u_\mathcal{T} \in \mathrm{CR}_0^1(\mathcal{T})$ denote the minimizer of E_{NC} in $\mathrm{CR}_0^1(\mathcal{T})$ with respect to f and define

$$\delta(\mathcal{T}, u, f) := \sqrt{E(u) - E_{\mathrm{NC}}(u_\mathcal{T}) + \gamma \|h_\mathcal{T} f\|^2_{L^2(\Omega)}}\,.$$

The proof of Theorem 6.2 introduces the modified approximation class

$$\mathbb{A}'_s := \big\{(u, f) \in V \times L^2(\Omega) \;\big|\; u \text{ minimises } E \text{ with respect to } f \text{ and } |(u, f)|_{\mathbb{A}'_s} < \infty\big\}$$
$$\text{with} \qquad |(u, f)|_{\mathbb{A}'_s} := \sup_{N\in\mathbb{N}} N^s \inf_{\substack{\mathcal{T}\in\mathbb{T}\\ \mathrm{card}(\mathcal{T})-\mathrm{card}(\mathcal{T}_0)\le N}} \delta(\mathcal{T}, u, f).$$

Lemma 6.3, Theorem 6.4, and Lemma 6.6 show that $\mathbb{A}_s = \mathbb{A}'_s$ with equivalent seminorms.

The proof of Theorem 6.2 excludes the pathological case $\xi_0 = 0$ for ξ_ℓ from Theorem 6.5. Choose $0 < \tau \le |(u, f)|^2_{\mathbb{A}'_s}/\xi_0^2$, and set $\varepsilon(\ell)^2 := \tau\xi_\ell^2$. Let $N(\ell) \in \mathbb{N}$ be minimal with the property

$$|(u, f)|_{\mathbb{A}'_s} \le \varepsilon(\ell)\, N(\ell)^s.$$

The definition of $|(u, f)|_{\mathbb{A}'_s}$ as a supremum over N shows for $N = N(\ell)$ that there exists some optimal triangulation $\widetilde{\mathcal{T}}$ (which is possibly not related to $\mathcal{T}_\ell$) of cardinality $\mathrm{card}(\widetilde{\mathcal{T}}_\ell) \le \mathrm{card}(\mathcal{T}_0) + N(\ell)$ with discrete solution $\widetilde{u}_\ell \in \mathrm{CR}_0^1(\widetilde{\mathcal{T}}_\ell)$ and

$$\delta(\widetilde{\mathcal{T}}_\ell, \widetilde{u}_\ell, f)^2 \le N(\ell)^{-2s} |(u, f)|^2_{\mathbb{A}_s} \le \varepsilon(\ell)^2.$$

The overlay $\widehat{\mathcal{T}}_\ell := \mathcal{T}_\ell \otimes \widetilde{\mathcal{T}}_\ell$ is known [18, 28] as the smallest common refinement of $\mathcal{T}_\ell$ and $\widetilde{\mathcal{T}}_\ell$. Let $\widehat{u}_\ell \in \mathrm{CR}_0^1(\widehat{\mathcal{T}}_\ell)$ denote the discrete solution with respect to $\widehat{\mathcal{T}}_\ell$.

Key argument. There exists $C_4 \approx 1$ with $\eta_\ell^2 \le C_4\, \eta_\ell^2(u_\ell, \mathcal{T}_\ell \backslash \widehat{\mathcal{T}}_\ell)$.

Proof. The efficiency reads

$$C_{\mathrm{eff}}^{-1} \eta_\ell^2 \le \|\mathrm{D}_{\mathrm{NC}}(u - u_\ell)\|^2_{L^2(\Omega)} + \mathrm{osc}^2(f, \mathcal{T}_\ell).$$

Young's inequality, Lemma 6.6, the definition of $\varepsilon(\ell)$, and the discrete reliability imply

$$\begin{aligned}
&\|\mathrm{D}_{\mathrm{NC}}(u - u_\ell)\|^2_{L^2(\Omega)}\\
&\quad \le 3(\|\mathrm{D}_{\mathrm{NC}}(u - \widetilde{u}_\ell)\|^2_{L^2(\Omega)} + \|\mathrm{D}_{\mathrm{NC}}(\widetilde{u}_\ell - \widehat{u}_\ell)\|^2_{L^2(\Omega)} + \|\mathrm{D}_{\mathrm{NC}}(\widehat{u}_\ell - u_\ell)\|^2_{L^2(\Omega)})\\
&\quad \le 3(C_1\varepsilon(\ell)^2 + C_{\mathrm{drel}}\eta_\ell^2(\widetilde{u}_\ell, \widetilde{\mathcal{T}}_\ell \backslash \widehat{\mathcal{T}}_\ell) + C_{\mathrm{drel}}\eta_\ell^2(u_\ell, \mathcal{T}_\ell \backslash \widehat{\mathcal{T}}_\ell)).
\end{aligned}$$

The efficiency proves

$$\eta_\ell^2(\widetilde{u}_\ell, \widetilde{\mathcal{T}}_\ell \backslash \widehat{\mathcal{T}}_\ell) \le C_{\mathrm{eff}}(\|\mathrm{D}_{\mathrm{NC}}(u - \widetilde{u}_\ell)\|^2_{L^2(\Omega)} + \mathrm{osc}^2(f, \widetilde{\mathcal{T}}_\ell)) \le C_{\mathrm{eff}}(C_1 + 1)\varepsilon(\ell)^2.$$

The oscillations are controlled through

$$\mathrm{osc}^2(f, \mathcal{T}_\ell) = \mathrm{osc}^2(f, \mathcal{T}_\ell \backslash \widehat{\mathcal{T}}_\ell) + \mathrm{osc}^2(f, \mathcal{T}_\ell \cap \widetilde{\mathcal{T}}_\ell) \le \eta_\ell^2(u_\ell, \mathcal{T}_\ell \backslash \widehat{\mathcal{T}}_\ell) + \varepsilon(\ell)^2.$$

Hence, the combination of the preceding formulas reveals

$$C_{\mathrm{eff}}^{-1}\eta_\ell^2 \le (1 + 3C_1 + 3C_{\mathrm{drel}}C_{\mathrm{eff}}(C_1 + 1))\varepsilon(\ell)^2 + (1 + 3C_{\mathrm{drel}})\eta_\ell^2(u_\ell, \mathcal{T}_\ell \backslash \widehat{\mathcal{T}}_\ell).$$

The choice $\tau < (1 + 3C_1 + 3C_{\mathrm{drel}}C_{\mathrm{eff}}(C_1 + 1))^{-1}(1 + \beta C_2)^{-1}C_{\mathrm{eff}}^{-1}$ proves the assertion. ∎

Finish of the proof. The choice $0 < \theta \leq 1/C_4$, the bounded overhead [5, 28], and a geometric series argument [27] eventually prove

$$\big(\mathrm{card}(\mathcal{T}_\ell) - \mathrm{card}(\mathcal{T}_0)\big)^s \delta_\ell \leq C|(u, f)|_{\mathbb{A}'_s}.$$

Further details can be found in [4, 26]. ∎

Acknowledgment. The authors are thankful to Rob Stevenson for the clarification that one-level refinements produce a finite number of descendants in any space dimension.

REFERENCES

[1] D. N. Arnold and R. S. Falk, *A uniformly accurate finite element method for the Reissner-Mindlin plate*, SIAM J. Numer. Anal., 26 (1989), pp. 1276–1290.

[2] S. Bartels, C. Carstensen, and G. Dolzmann, *Inhomogeneous Dirichlet conditions in a priori and a posteriori finite element error analysis*, Numer. Math., 99 (2004), pp. 1–24.

[3] R. Becker and S. Mao, *Quasi-optimality of adaptive nonconforming finite element methods for the Stokes equations*, SIAM J. Numer. Anal., 49 (2011), pp. 970–991.

[4] R. Becker, S. Mao, and Z.-C. Shi, *A convergent nonconforming adaptive finite element method with quasi-optimal complexity*, SIAM J. Numer. Anal., 47 (2010), pp. 4639–4659.

[5] P. Binev, W. Dahmen, and R. DeVore, *Adaptive finite element methods with convergence rates*, Numer. Math., 97 (2004), pp. 219–268.

[6] D. Boffi, *Finite element approximation of eigenvalue problems*, Acta Numer., 19 (2010), pp. 1–120.

[7] S. C. Brenner, *Two-level additive Schwarz preconditioners for nonconforming finite element methods*, Math. Comp., 65 (1996), pp. 897–921.

[8] S. C. Brenner, *Poincaré-Friedrichs inequalities for piecewise H^1 functions*, SIAM J. Numer. Anal., 41 (2003), pp. 306–324.

[9] S. C. Brenner and L. R. Scott, *The Mathematical Theory of Finite Element Methods*, 3rd ed., Springer-Verlag, New York, 2008.

[10] C. Carstensen, *A unifying theory of a posteriori finite element error control*, Numer. Math., 100 (2005), pp. 617–637.

[11] C. Carstensen, M. Eigel, R. H. W. Hoppe, and C. Löbhard, *A review of unified a posteriori finite element error control*, Numer. Math. Theory Methods Appl., 5 (2012), pp. 509–558.

[12] C. Carstensen, D. Gallistl, and M. Schedensack, *Adaptive nonconforming Crouzeix-Raviart FEM for eigenvalue problems*, Math. Comp. (2013), in press.

[13] C. Carstensen, J. Gedicke, and D. Rim, *Explicit error estimates for Courant, Crouzeix-Raviart and Raviart-Thomas finite element methods*, J. Comput. Math., 30 (2012), pp. 337–353.

[14] C. Carstensen and J. Hu, *A unifying theory of a posteriori error control for nonconforming finite element methods*, Numer. Math., 107 (2007), pp. 473–502.

[15] C. Carstensen, D. Peterseim, and H. Rabus, *Optimal adaptive nonconforming FEM for the Stokes problem*, Numer. Math., 123 (2013), pp. 291–308.

[16] C. Carstensen, D. Peterseim, and M. Schedensack, *Comparison results of finite element methods for the Poisson model problem*, SIAM J. Numer. Anal., 50 (2012).

[17] C. Carstensen and H. Rabus, *The adaptive nonconforming FEM for the pure displacement problem in linear elasticity is optimal and robust*, SIAM J. Numer. Anal., 50 (2012), pp. 1264–1283.

[18] J. M. Cascon, Ch. Kreuzer, R. H. Nochetto, and K. G. Siebert, *Quasi-optimal convergence rate for an adaptive finite element method*, SIAM J. Numer. Anal., 46 (2008), pp. 2524–2550.

[19] Ph. G. Ciarlet, *The Finite Element Method for Elliptic Problems*, North-Holland, Amsterdam, 1978.

[20] Ph. Clément, *Approximation by finite element functions using local regularization*, Rev. Française Automat. Informat. Recherche Operationnelle, 9 (1975), pp. 77–84.

[21] M. Crouzeix and P.-A. Raviart, *Conforming and nonconforming finite element methods for solving the stationary Stokes equations.* I, Rev. Française Automat. Informat. Recherche Opérationnelle Sér. Rouge, 7 (1973), pp. 33–75.

[22] E. Dari, R. Duran, C. Padra, and V. Vampa, *A posteriori error estimators for nonconforming finite element methods*, RAIRO Modél. Math. Anal. Numér., 30 (1996), pp. 385–400.
[23] T. Gudi, *A new error analysis for discontinuous finite element methods for linear elliptic problems*, Math. Comp., 79 (2010), pp. 2169–2189.
[24] J. Hu and J. Xu, *Convergence and optimality of the adaptive nonconforming linear element method for the Stokes problem*, J. Sci. Comput., 55 (2013), pp. 125–148.
[25] J. M. Maubach, *Local bisection refinement for n-simplicial grids generated by reflection*, SIAM J. Sci. Comput., 16 (1995), pp. 210–227.
[26] H. Rabus, *A natural adaptive nonconforming FEM of quasi-optimal complexity*, Comput. Methods Appl. Math., 10 (2010), pp. 315–325.
[27] R. Stevenson, *Optimality of a standard adaptive finite element method*, Found. Comput. Math., 7 (2007), pp. 245–269.
[28] R. Stevenson, *The completion of locally refined simplicial partitions created by bisection*, Math. Comp., 77 (2008), pp. 227–241.
[29] R. Stevenson, *Private communication*, Korteweg-de Vries (KdV) Institute for Mathematics, University of Amsterdam, 2013.
[30] C. T. Traxler, *An algorithm for adaptive mesh refinement in n dimensions*, Computing, 59 (1997), pp. 115–137.
[31] R. Verfürth, *A Review of a Posteriori Error Estimation and Adaptive Mesh-Refinement Techniques*, Adv. Numer. Math., Wiley, New York, 1996.
[32] E. Zeidler, *Nonlinear Functional Analysis and Its Applications.* III, Springer-Verlag, New York, 1985.
[33] E. Zeidler, *Nonlinear Functional Analysis and Its Applications.* IV, Springer-Verlag, New York, 1988.