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REFINED FULLY EXPLICIT A POSTERIORI RESIDUAL-BASED ERROR CONTROL*

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Abstract. The explicit residual-based a posteriori error estimator for elliptic partial differential equations is reliable up to the multiplication of some generic constant which needs to be involved for full error control. The present mathematical literature takes this constant from the stability and approximation properties of Clément-type quasi-interpolation operators and so results in an overestimation of the error which is bigger than for implicit and more expensive a posterori error estimators. This paper propagates a paradigm shift to start with an equilibration error estimator technique followed by its efficiency analysis. The outcome is a refined residual-based a posteriori error estimate with explicit constants which leads to slightly sharper error control than the work of Veeser and Verfürth in 2009. A first application to guaranteed explicit error estimation for two-dimensional nonconforming and a generalization to higher-order finite element methods conclude the paper.

Key words. finite element method, adaptive finite element method, a posteriori error estimation, reliability

AMS subject classifications. 65N15, 65N30

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1. Introduction.

Explicit a posteriori error estimators [BS94, Ver96, Bra07] are an important and sufficient tool for adaptive finite element methods in the numerical treatment of elliptic second-order partial differential equations, since they allow cheap calculation of refinement indicators and of some quantity η that is equivalent to the energy error $|||e|||$ up to reliability and efficiency constants C_{rel} and C_{eff}. That is,

$$C_{\mathrm{eff}}\eta \leqslant |||e||| := \left\|\nabla e\right\|_{L^p(\Omega)} \leqslant C_{\mathrm{rel}}\eta.$$

The proof for the reliability of the explicit residual-based a posteriori error estimator η usually involves Clement-type quasi-interpolation operators [Clé75, SZ90, CF00] and is nowadays explained even in textbooks [Bra07] on the finite element method.

The finite element method discretizes some domain $\Omega \subset \mathbb{R}^n$ $(n = 2, 3)$ with closed Dirichlet boundary Γ_D and Neumann boundary $\Gamma_N := \partial\Omega\backslash\Gamma_D$ by some regular triangulation $\mathcal{T}$ and computes some discrete flux $\sigma_h \in P_0(\mathcal{T};\mathbb{R}^n)$ for given data $f \in L^{p'}(\Omega)$ and $g \in L^{p'}(\Gamma_N)$. The central object of estimation is then some residual [Car05, VV09, CEHL12] of the form

$$\mathrm{Res}(v) = \int_\Omega f v dx + \int_{\Gamma_N} g v ds - \int_\Omega \sigma_h \cdot \nabla v dx \qquad \forall\ v \in V := W_0^{1,p}(\Omega)$$

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with $1 < p < \infty$ and its conjugate p' such that $1/p + 1/p' = 1$. The dual norm $|||\mathrm{Res}|||_\star := \sup_{v \in V\setminus\{0\}} \mathrm{Res}(v)/|||v|||$ is equivalent to the energy error $|||e|||$ for well-posed problems.

The new approach in this paper utilizes equilibrated fluxes that allow a split in the form

$$|||\mathrm{Res}|||_\star \leqslant |||f + \operatorname{div} q|||_\star + |||\gamma_N^\star(g - q\cdot\nu)|||_\star + |||\operatorname{div}(q - \sigma_h)|||_\star \tag{1.1}$$

for any $q \in H^{p'}(\operatorname{div},\Omega) := \{q \in L^{p'}(\Omega;\mathbb{R}^n) \mid \operatorname{div} q \in L^{p'}(\Omega)\}$ with $q\cdot\nu \in L^{p'}(\Gamma_N)$ to permit

$$|||\gamma_N^\star(g - q\cdot\nu)|||_\star := \sup_{v\in V\setminus\{0\}} \int_{\Gamma_N} (g - q\cdot\nu) v ds / |||v|||.$$

The three terms on the right-hand side of (1.1) no longer enjoy any Galerkin orthogonality in the sense that they vanish for nodal basis functions as test functions. Instead the additional fluxes allow for an immediate application of Poincaré and Friedrichs inequalities to derive an a posteriori error estimate in one strike. This circumvents the usage of the approximation and stability estimates for some quasi interpolant with multiplicative constants. The proposed paradigm shift is the estimate in one direct approach without explicit quasi interpolation.

The special design of some equilibrated Raviart–Thomas element $q \in RT_0(\mathcal{T}^\star)$ and some approximations $f^\star \in P_0(\mathcal{T}^\star)$ of f and $g^\star \in P_0(\mathcal{E}^\star(\Gamma_N))$ of g with $\operatorname{div} q + f^\star = 0$ in Ω and $q\cdot\nu = g^\star$ along Γ_N on the dual mesh $\mathcal{T}^\star$ allows the control of the first two terms of (1.1) by

$$\begin{aligned}
|||f + \operatorname{div} q|||_\star &\leqslant C_p(\mathcal{T}) \left\| h_\mathcal{T}(f - f^\star)\right\|_{L^{p'}(\Omega)} \lesssim \operatorname{osc}(f,\mathcal{T}) := \left\|h_\mathcal{T}(f - f_\mathcal{T})\right\|_{L^{p'}(\Omega)},\\
|||\gamma_N^\star(g - q\cdot\nu)|||_\star &\leqslant C_p(\mathcal{E}(\Gamma_N)) \left\| h_\mathcal{T}^{1/p'}(g - g^\star)\right\|_{L^{p'}(\Omega)} \lesssim \operatorname{osc}(f,\mathcal{E})\\
&:= \left\| h_\mathcal{T}^{1/p'}(g - g_\mathcal{E})\right\|_{L^{p'}(\Omega)}.
\end{aligned}$$

Here, $f_\mathcal{T} \in P_0(\mathcal{T})$ denotes the piecewise integral mean of f, $g_\mathcal{E} \in P_0(\mathcal{E}(\Gamma_N))$ denotes the piecewise integral mean on the Neumann sides $\mathcal{E}(\Gamma_N)$, and $h_\mathcal{T} \in P_0(\mathcal{T})$ is the local element diameter. Our main result, Theorem 3.1 below, controls the last term of (1.1) by the novel explicit residual-based a posteriori error estimator

$$|||\operatorname{div}(q - \sigma_h)|||_\star \leqslant \eta := \left(\sum_{z\in\mathcal{N}} \left(c_1(z)\eta(z) + c_2(z)\eta(\mathcal{E}(z))\right)^{p'}\right)^{1/p'}$$

with $\eta(z) := \operatorname{diam}(\omega_z^\star)\left\|\varphi_z^{1/p'}(f - f_{\omega_z})\right\|_{L^{p'}(\omega_z)}$ and

$$\eta(\mathcal{E}(z)) := \left(\sum_{E\in\mathcal{E}(z)} |E|^{1/(n-1)} \left\|\varphi_z^{1/p'}[\sigma_h\cdot\nu_E]_E\right\|_{L^{p'}(E)}^{p'}\right)^{1/p'}.$$

Here, f_{ω_z} is the integral mean of f over the node patch ω_z if z is a free node, and $f_{\omega_z} = 0$ for nodes on the Dirichlet boundary. Hence, for constant f the quantity $\eta(z)$ vanishes for free nodes $z \in \mathcal{M}$.

This makes this paper the first attempt to quantify the reliability constant in the refined explicit residual-based a posteriori error estimator [CV99] which involves patch-oriented data oscillations rather than the (possibly much larger) volume contribution of the form mesh-size times source function. This opens the door to explicit bounds for reliability constants for averaging schemes [CB02] as well, which are extremely popular in the applied computational sciences. Moreover, this is a two-fold improvement compared to the result of Veeser and Verfürth in [VV09] and gives rise to further developments. First, the error estimator no longer contains the volume contribution mesh-size times the source term and solely utilizes data oscillation terms. Hence for a large global constant f, the novel upper bounds might be far superior. Second, the multiplicative constants are significantly improved as illustrated in the subsequent benchmark example. The upper bound of [VV09] reads

$$|||\mathrm{Res}|||_\star \leqslant \left(\sum_{z\in\mathcal{N}} \left(c_p(\omega_z)\mu(z) + c_p(\sigma_z)\mu(\mathcal{E}(z))\right)^{p'}\right)^{1/p'}$$

with $\mu(z) := \mathrm{diam}(\omega_z)\big\|\varphi_z^{1/p'} f\big\|_{L^{p'}(\omega_z)}$ and $h_E^\perp := \fint_{\omega_E}\varphi_z dx/\fint_E \varphi_z dx$ in

$$\mu(\mathcal{E}(z)) := \mathrm{diam}(\omega_z)\left(\sum_{E\in\mathcal{E}(z)} (h_E^\perp)^{1-p'}\big\|\varphi_z^{1/p'}[\sigma_h\cdot\nu]_E\big\|_{L^{p'}(E)}^{p'}\right)^{1/p'}.$$

As a small example, uniform cross refinement of the L-shaped domain $\Omega = (-1,1)^2\backslash((0,1)\times(-1,0))$ with right-isosceles triangles of area $|T| = h^2/2$ and a diagonal parallel to the main diagonal results in five different patches depicted in Figure 3 in section 5. Calculations for $n = p = 2$ and $\Gamma_D = \partial\Omega$ in section 6 below allow the simplified guaranteed upper bound

$$|||\mathrm{Res}|||_\star \leqslant 0.31831\big\|h_\mathcal{T}(f - f^\star)\big\|_{L^2(\Omega)} + 0.6366\left(\sum_{z\in\mathcal{N}}\eta(z)^2\right)^{1/2} + 2.3952\left(\sum_{E\in\mathcal{E}} |E|\,\big\|[\sigma_h\cdot\nu_E]_E\big\|_{L^2(E)}^2\right)^{1/2}.$$

The interchange of notation on the local geometry leads to a possibly unfair comparison. However, the results of [VV09] plus some straightforward computation yield the aformentioned estimate with some volume term with $f^* \equiv 0$ and the constant in front of this standard edge-jump term is 17.3721 (or even 38.0007 if patch (E) from Figure 3 is included) instead of 2.3952.

The remaining parts of the paper are organized as follows. Section 2 explains the design of the equilibrated fluxes used for the derivation of the novel explicit residual-based error estimator in section 3. Section 4 proves the main result, Theorem 3.1 from section 3. Section 5 shows that our main result implies the explicit bounds from [VV09] with even better constants for some benchmark example. Section 6 compares three guaranteed upper error bounds for some Poisson model problem. Section 7 applies the results to the nonconforming finite element method. Section 8 concludes the paper with an extension to higher-order finite element approximations.

2. Modified equilibrated fluxes. This section explains the design of the equilibrated fluxes via some Raviart–Thomas function $q \in RT_0(\mathcal{T}^\star)$ on the dual mesh $\mathcal{T}^\star$. Consider a regular triangulation $\mathcal{T}$ of $\Omega \subseteq \mathbb{R}^n$ $(n = 2,3)$ into n-dimensional simplices with sides $\mathcal{E}$, nodes $\mathcal{N}$, and free nodes $\mathcal{M}$. For $n = 2$ dimensions, $\mathcal{T}$ con-

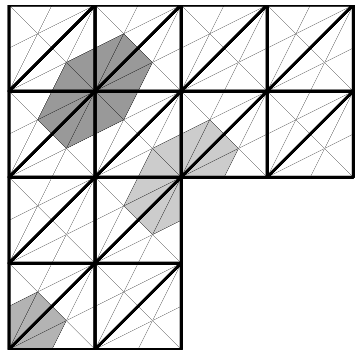

FIG. 1. *Triangulation $\mathcal{T}$ (thick lines) and its dual $\mathcal{T}^\star$ (thin lines) plus node patches $\omega_z^\star$ (light gray) for $n = 2$.*

sists of triangles and $\mathcal{E}$ consists of edges, while in $n = 3$ dimensions, $\mathcal{T}$ consists of tetrahedra and $\mathcal{E}$ consists of faces. The boundary consists of Dirichlet sides $\mathcal{E}(\Gamma_D)$ and Neumann sides $\mathcal{E}(\Gamma_N)$ such that $\bigcup_{E\in\mathcal{E}(\Gamma_D)} E = \Gamma_D$. The set $\mathcal{N}(T)$ contains the $n+1$ vertices of the simplex $T \in \mathcal{T}$, while $\mathcal{E}(T)$ contains the $n+1$ sides along its boundary ∂T and $\mathcal{N}(E)$ consists of the n vertices of the side $E \in \mathcal{E}$. All elements that share the same vertex $z \in \mathcal{N}$ form the set $\mathcal{T}(z) := \{T \in \mathcal{T} \mid z \in \mathcal{N}(T)\}$. Similarly, $\mathcal{T}(E) := \{T \in \mathcal{T} \mid E \in \mathcal{E}(T)\}$ contains all elements that share the same side $E \in \mathcal{E}$ and $\mathcal{E}(z) := \{E \in \mathcal{E} \mid z \in \mathcal{N}(E)\}$ contains all sides that have z in common. Furthermore, there are the neighborhoods $\omega_z := \operatorname{int}(\bigcup \mathcal{T}(z))$ of $z \in \mathcal{N}$ and $\omega_E := \operatorname{int}(\bigcup \mathcal{T}(E))$ of $E \in \mathcal{E}$. Finally, let $P_k(T)$ denote the polynomials of degree $\leqslant k$ on $T \in \mathcal{T}$ and

$$P_k(\mathcal{T}) := \{v_h \in L^2(\Omega) \mid \forall T \in \mathcal{T},\ v_h|_T \in P_k(T)\}.$$

The lowest-order space of Raviart–Thomas functions on $\mathcal{T}$ reads

$$RT_0(\mathcal{T}) := \big\{q \in H(\operatorname{div},\Omega) \mid \forall T \in \mathcal{T}\ \exists\, a \in P_0(T;\mathbb{R}^{n+1})\ \forall x \in T,\ q(x) = a_{n+1}x + (a_1,\dots,a_n)\big\}.$$

The dual mesh $\mathcal{T}^\star$ is well established in the finite volume methodology and connects each center $\operatorname{mid}(T)$ of an element $T \in \mathcal{T}$ with the side midpoints $\operatorname{mid}(\mathcal{E}(T))$ (and the edge midpoints for $n = 3$) and nodes $\mathcal{N}(T)$ and so divides each element $T \in \mathcal{T}$ into $(n+1)!$ subelements of volume $|T|/(n+1)!$ and every side $E \in \mathcal{E}$ into $n!$ many subsides of volume $|E|/n!$. Figure 1 displays some triangulation $\mathcal{T}$ and its dual mesh $\mathcal{T}^\star$ for the L-shaped domain, while Figure 2 illustrates the partition $\mathcal{T}^\star(T) := \{T^\star \in \mathcal{T}^\star \mid T \subseteq T\}$ on some triangle $T \in \mathcal{T}$ for $n = 2$.

Consider some node $z \in \mathcal{N}(\mathcal{T})$ and its nodal basis function $\varphi_z^\star$ with the fine patch $\omega_z^\star := \{\varphi_z^\star > 0\}$ of the dual mesh $\mathcal{T}^\star$ and its neighboring elements $\mathcal{T}^\star(z) := \{T^\star \in \mathcal{T}^\star \mid z \in \mathcal{N}^\star(T^\star)\}$. Since $\sigma_h \in P_0(\mathcal{T};\mathbb{R}^n)$ is continuous along $\partial\omega_z^\star \cap T$ for any $T \in \mathcal{T}$, the condition $q \cdot \nu = \sigma_h \cdot \nu \in P_0(\mathcal{E}^\star(\partial\omega_z^\star))$ is well-defined on the exterior boundary sides $\mathcal{E}^\star(\partial\omega_z^\star)$ of $\omega_z^\star$.

The suggested design employs an interpolation $f^\star \in P_0(\mathcal{T}^\star)$ of $f \in L^{p'}(\Omega)$ defined by

$$f^\star|_{T^\star} := (n+1)\int_T f\varphi_z dx/|T| \quad \text{for the } n! \text{ many } T^\star \in \mathcal{T}^\star$$

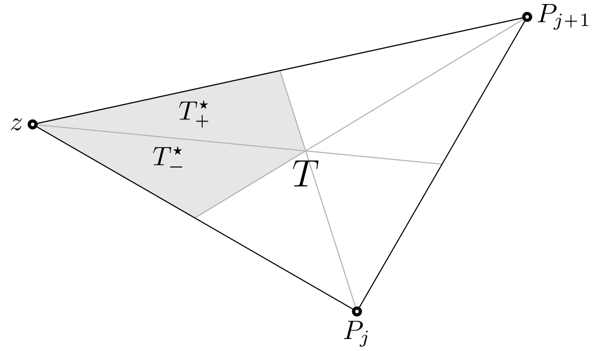

FIG. 2. *The two fine triangles $T_\pm^\star \in \mathcal{T}^\star$ inside $T \in \mathcal{T}$ with $\mathcal{N}(T) \cap \mathcal{N}^\star(T^\star) = \{z\}$ for $n = 2$ dimensions.*

with $\mathcal{N}^\star(T^\star) \cap \mathcal{N}(T) = \{z\}$. (The two triangles $T_\pm^\star$ in case $n = 2$ are depicted in Figure 2.) Similarly, in the case of Neumann data $g \in L^{p'}(\Gamma_N)$, define $g^\star \in P_0(\mathcal{E}^\star(\Gamma_N))$ by

$$g^\star|_{E^\star} := n \int_E g\varphi_z dx/|E| \quad \text{for the } (n-1)! \text{ many } E^\star \in \mathcal{E}^\star$$

with $\mathcal{N}^\star(E^\star) \cap \mathcal{N}(E) = \{z\}$. This suffices to define the set

$$Q(\mathcal{T}^\star(z)) := \big\{\tau_h \in RT_0(\mathcal{T}^\star(z)) \,\big|\, \operatorname{div} \tau_h + f^\star = 0 \text{ in } \omega_z^\star \;\&\; \tau_h \cdot \nu = \sigma_h \cdot \nu \text{ along } \partial\omega_z^\star \backslash \partial\Omega \;\&\; \tau_h \cdot \nu = g^\star \text{ along } \partial\omega_z^\star \cap \Gamma_N\big\}.$$

Remark 2.1. By design, any q with $q|_{\omega_z^\star} \in Q(\mathcal{T}^\star(z))$ is in $RT(\mathcal{T}^\star) \subset H(\operatorname{div}, \Omega)$. The patchwise minimization computes

$$\sigma_h^\star|_{\omega_z^\star} = \operatorname{argmin}_{\tau \in Q(\mathcal{T}^\star(z))} \|\tau - \sigma_h\|_{L^{p'}(\omega_z^\star)}. \tag{2.1}$$

This leads to some equilibration error estimator similar to the Luce–Wohlmuth error estimator [LW04] and very close to the recent suggestions of [Voh11]. Our design mainly differs in the choice of the divergence to allow for a very easy estimation of $|||f + \operatorname{div} q|||_\star = |||f - f^\star|||_\star$.

THEOREM 2.2. *If $Res(\varphi_z) = 0$ for all $z \in \mathcal{M}$, the affine space $Q(\mathcal{T}^\star(z))$ is nonempty. Furthermore, the constants*

$$C_p(\mathcal{T}) := \max_{T \in \mathcal{T}} \sup_{v \in W^{1,p}(T) \backslash P_0(T)} \|v - v_T\|_{L^{p'}(T)} / \|h_\mathcal{T} \nabla v\|_{L^{p'}(T)} \quad \textit{where } v_T := \fint_T v dx,$$

$$C_p(\mathcal{E}(\Gamma_N)) := \max_{E \in \mathcal{E}(\Gamma_N)} \left(\frac{|E| \operatorname{diam}(\omega_E)}{|\omega_E|} \left(C_p(\mathcal{T}(E))^p + C_p(\mathcal{T}(E))^{p-1} p/n \right) \right)^{1/p},$$

$$C_n := \left(1 + (n+1)^{(2+p')/p'} \left(n!\Gamma(p)/\Gamma(p+n)\right)^{1/p}\right)$$

satisfy

$$|||f - f^\star|||_\star \leqslant C_p(\mathcal{T}) \|h_\mathcal{T}(f - f^\star)\|_{L^{p'}(\Omega)} \leqslant C_p(\mathcal{T}) C_n \operatorname{osc}(f, \mathcal{T}),$$

$$|||\gamma_N(g - g^\star)|||_\star \leqslant C_p(\mathcal{E}(\Gamma_N)) \|h_\mathcal{T}^{1/p'}(g - g^\star)\|_{L^{p'}(\Gamma_N)} \leqslant C_p(\mathcal{E}(\Gamma_N)) C_{n-1} \operatorname{osc}(g, \mathcal{E}(\Gamma_N)).$$

Remark 2.3. The first constant $C_p(\mathcal{T})$ is the Poincaré constant on convex triangles or tetrahedra and known results in [AD04] show $C_1(\mathcal{T}) \leqslant 1/2$ and $C_\infty(\mathcal{T}) \leqslant 1$. Operator interpolation arguments as in [BL76, BS94] then allow the bound $C_p(\mathcal{T}) \leqslant (1/2)^{1/p} \leqslant 1$ for all $p \in [1,\infty]$. For $p = 2 = n$, [LS10] recently showed the refined result $C_2(\mathcal{T}) \leqslant 1/j_{1,1}$ with the first positive root $j_{1,1}$ of the Bessel function J_1.

Remark 2.4. For $n = 2$ there is an elementwise design similar to [LW04, BS08]. For interior nodes there is one additional degree of freedom, because $\operatorname{Curl}\varphi_z^\star \in RT_0(\mathcal{T}^\star(z))$ is divergence-free.

Proof. Existence of q. The search for $q|_{\omega_z^\star} \in Q(\mathcal{T}^\star(z))$ for a free node $z \in \mathcal{M}$ describes some Neumann problem and requires the equilibration condition of the constraints, namely,

$$\int_{\partial\omega_z^\star\cap\Gamma_N} g^\star ds + \int_{\partial\omega_z^\star\setminus\Gamma_N} \sigma_h\cdot\nu ds = -\int_{\omega_z} f\varphi_z dx = -\int_{\omega_z^\star} f^\star dx. \tag{2.2}$$

This property is well-known and proved, e.g., in [Voh11, Lemma 3.8]. An integration by parts, $\operatorname{div}\sigma_h = 0$ on every $T \in \mathcal{T}^\star$, and $\int_E \varphi_z ds = |E|/n$, yield for the piecewise constant σ_h that

$$\begin{aligned}
&\int_{\partial\omega_z^\star\cap\Gamma_N} g^\star ds + \int_{\partial\omega_z^\star\setminus\Gamma_N} \sigma_h\cdot\nu ds\\
&\quad= \int_{\partial\omega_z^\star\cap\Gamma_N} g^\star - \sigma_h\cdot\nu ds + \sum_{T\in\mathcal{T}^\star(z)}\int_{\partial T}\sigma_h\cdot\nu ds - \sum_{E\in\mathcal{E}^\star(z)\setminus(\mathcal{E}^\star(\Gamma_N))}\int_E[\sigma_h\cdot\nu_E]_E ds\\
&\quad= \int_{\partial\omega_z\cap\Gamma_N}\varphi_z(g-\sigma_h\cdot\nu)ds - \sum_{E\in\mathcal{E}(z)\setminus(\mathcal{E}(\Gamma_N))}\int_E\varphi_z[\sigma_h\cdot\nu_E]_E ds\\
&\quad= \int_{\Gamma_N} g\varphi_z ds - \int_\Omega \sigma_h\cdot\nabla\varphi_z dx.
\end{aligned}$$

Hence, (2.2) is equivalent to $\operatorname{Res}(\varphi_z) = 0$. For Dirichlet nodes $z \in \mathcal{N}(\Gamma_D)$, $\omega_z^\star$ has at least one unconstrained Dirichlet side and the equilibration condition (2.2) is not necessary.

Proof of upper bound for $|||f-f^\star|||_\star$. With $|T\cap\omega_z^\star| = |T|/(n+1)$ and the partition of unity property, it holds that

$$\int_T f^\star dx = \sum_{z\in\mathcal{N}(T)}\int_{T\cap\omega_z^\star} f^\star dx = \sum_{z\in\mathcal{N}(T)}\frac{(n+1)\,|T\cap\omega_z^\star|}{|T|}\int_T f\varphi_z dx = \int_T f dx.$$

This orthogonality and elementwise Hölder and Poincaré inequalities with Poincaré constant $C_p(\mathcal{T})$ result in

$$\int_\Omega (f-f^\star)v dx = \int_\Omega (f-f^\star)(v-v_\mathcal{T})dx \leqslant C_p(\mathcal{T})\left\|h_\mathcal{T}(f-f^\star)\right\|_{L^{p'}(\Omega)}|||v|||.$$

A triangle inequality yields

$$\left\|f-f^\star\right\|_{L^{p'}(T)} \leqslant \left\|f-f_T\right\|_{L^{p'}(T)} + \left(\sum_{z\in\mathcal{N}(T)}\left\|f_T-f^\star\right\|_{L^{p'}(T\cap\omega_z^\star)}^{p'}\right)^{1/p'}.$$

Elementary calculations with $\|\varphi_z\|^p_{L^p(T)} = n!\,|T|\,\Gamma(p)/\Gamma(p+n)$ and $|T\cap\omega_z^\star| = |T|/(n+1)$ show

$$\begin{aligned}\|f_T - f^\star\|^{p'}_{L^{p'}(T\cap\omega_z^\star)} &= (n+1)^{p'}\left(\int_T (f-f_T)\varphi_z dx\right)^{p'} |T\cap\omega_z^\star|/|T|^{p'}\\ &= (n+1)^{1+p'}\|(f-f_T)\varphi_z\|^{p'}_{L^1(T)}\,|T|^{1-p'}\\ &\leqslant (n+1)^{1+p'}\|\varphi_z\|^{p'}_{L^p(T)}\|f-f_T\|^{p'}_{L^{p'}(T)}\,|T|^{-p'/p}\\ &= (n+1)^{1+p'}(n!\Gamma(p)/\Gamma(p+n))^{p'/p}\|f-f_T\|^{p'}_{L^{p'}(T)}.\end{aligned}$$

The summation over all $n+1$ many nodes $z\in\mathcal{N}(T)$ concludes the proof

$$\sum_{z\in\mathcal{N}(T)}\|f_T - f^\star\|^{p'}_{L^{p'}(T\cap\omega_z^\star)} \leqslant (n+1)^{2+p'}(n!\Gamma(p)/\Gamma(p+n))^{p'/p}\,\|f-f_T\|^{p'}_{L^{p'}(T)}.$$

Proof of $|||g-g^\star|||_\star \leqslant C_p(\mathcal{E}(\Gamma_N))\|h_{\mathcal{T}}^{1/p'}(g-g^\star)\|_{L^{p'}(\Gamma_N)}$. Consider some Neumann boundary $E\in\mathcal{E}(\Gamma_N)$ and some test function $v\in V$. Since $\int_E(g-g^\star)ds = 0$, one can subtract an arbitrary constant $v_E\in P_0(E)$ such that

$$\int_E (g-g^\star)v \leqslant \|g-g^\star\|_{L^{p'}(E)}\|v-v_E\|_{L^p(E)}.$$

The trace identity on the element $\omega_E = T = \operatorname{conv}\{E,p\}$ (the proof is a simple integration by parts plus elementary geometric calculations) shows

$$\begin{aligned}\|v-v_E\|^p_{L^p(E)} &= \frac{|E|}{|\omega_E|}\int_{\omega_E}(v-v_E)^p dx + \frac{|E|}{n\,|\omega_E|}\int_{\omega_E}(x-P)\cdot\nabla((v-v_E)^p)dx\\ &\leqslant \frac{|E|}{|\omega_E|}\|v-v_E\|_{L^p(\omega_E)} + \frac{|E|}{n\,|\omega_E|}\operatorname{diam}(\omega_E)\|\nabla((v-v_E)^p)\|_{L^1(\omega_E)}.\end{aligned}$$

The chain rule and the Poincaré inequality $\|v - v_E\|_{L^p(\omega_E)} \leqslant C_p(\mathcal{T}(E))\operatorname{diam}(\omega_E)\|\nabla v\|_{L^p(\omega_E)}$ show

$$\begin{aligned}\|\nabla((v-v_E)^p)\|_{L^1(\omega_E)} &= \|p(v-v_E)^{p-1}\nabla v\|_{L^1(\omega_E)}\\ &\leqslant p\|(v-v_E)^{p/p'}\|_{L^{p'}(\omega_E)}\|\nabla v\|_{L^p(\omega_E)}\\ &\leqslant pC_p(\mathcal{T}(E))^{p/p'}\operatorname{diam}(\omega_E)^{p/p'}\|\nabla v\|^p_{L^p(\omega_E)}.\end{aligned}$$

A sum over all $E\in\mathcal{E}(\Gamma_N)$ and a Cauchy inequality conclude the proof. The proof of the remaining inequality $\|h_{\mathcal{T}}^{1/p'}(g-g^\star)\|_{L^{p'}(\Gamma_N)} \leqslant C_{n-1}\operatorname{osc}(g,\mathcal{E}(\Gamma_N))$ runs as the one above for $f^\star$, just with one dimension less. ∎

3. Explicit residual-based a posteriori error estimator. This section establishes the reliability of the explicit residual-based error estimator for the Poisson model problem with explicit reliablity constants. Since the seminal work [Ver96], it is well-known that the upper bound is efficient. Moreover, the estimates from Theorems 2.2 and 3.1 lead to a sharp control of the residual which improves other results in the literature [CF00, VV09] without the use of any quasi-interpolation operator.

The new proposed guaranteed upper bound has the form

$$\begin{aligned}(3.1)\qquad \|\!|\mathrm{Res}|\!\|_\star \leqslant C_p(\mathcal{T})\big\|h_\mathcal{T}(f-f^\star)\big\|_{L^{p'}(\Omega)} + C_p(\mathcal{E}(\Gamma_N))\big\|h_\mathcal{T}^{1/p'}(g-g^\star)\big\|_{L^{p'}(\Gamma_N)}\\ + \left(\sum_{z\in\mathcal{N}}\big(c_1(z)\eta(z)+c_2(z)\eta(\mathcal{E}(z))\big)^{p'}\right)^{1/p'}\end{aligned}$$

with constants $C_p(\mathcal{T})$, $C_p(\mathcal{E}(\Gamma_N))$ from Theorem 2.2, $c_1(z)$, $c_2(z)$ from Theorem 3.1, and

$$\eta(z) := \mathrm{diam}(\omega_z^\star)\big\|\varphi_z^{1/p'}(f-f_{\omega_z})\big\|_{L^{p'}(\omega_z)} \quad\text{and}$$

$$\eta(\mathcal{E}(z))^{p'} := \sum_{E\in\mathcal{E}(z)} |E|^{1/(n-1)}\left\|\varphi_z^{1/p'}[\sigma_h\cdot\nu_E]_E\right\|_{L^{p'}(E)}^{p'}.$$

The jump $[\sigma_h\cdot\nu_E]_E$ of σ_h over some side $E\in\mathcal{E}$ is defined by

$$[\sigma_h\cdot\nu_E]_E := \begin{cases}(\sigma_h|_{T_-}-\sigma_h|_{T_+})\cdot\nu_{T_-} & \text{for } E=\partial T_-\cap\partial T_+\in\mathcal{E}(\Omega),\\ \sigma_h\cdot\nu - g & \text{for } E\in\mathcal{E}(\Gamma_N),\\ \sigma_h\cdot\nu-g^\star & \text{for } E\in\mathcal{E}^\star(\Gamma_N),\\ 0 & \text{for } E\in\mathcal{E}(\Gamma_D).\end{cases}$$

The Poincaré–Friedrichs constants play a dominant role in the subsequent theorem,

$$\begin{aligned}&C_{\mathrm{PF}}(p,\omega_z^\star)\\ &:= \begin{cases}\sup\left\{\|f\|_{L^p(\omega_z^\star)} : f\in V \ \&\ \|\nabla f\|_{L^p(\omega_z^\star)}=1\right\} & \text{for } z\in\mathcal{N}(\Gamma_D),\\ \sup\left\{\big\|f-⨍_{\omega_z^\star} f\,dx\big\|_{L^p(\omega_z^\star)} : f\in V\ \&\ \|\nabla f\|_{L^p(\omega_z^\star)}=1\right\} & \text{for } z\in\mathcal{M}:=\mathcal{N}\setminus\mathcal{N}(\Gamma_D).\end{cases}\end{aligned}$$

Throughout this section, the piecewise constant function $\mathrm{mid}(\mathcal{T})\in P_0(\mathcal{T};\mathbb{R}^n)$ reads $\mathrm{mid}(\mathcal{T})|_T := \mathrm{mid}(T)$ for $T\in\mathcal{T}$ and

$$f_{\omega_z} := \int_{\omega_z} f\,dx/|\omega_z|\,dx \text{ for } z\in\mathcal{N}(\Gamma_D) \quad\text{and}\quad f_{\omega_z}:=0 \text{ for } z\in\mathcal{M}.$$

Section 2 introduces the design of $\sigma_h^\star\in RT_0(\mathcal{T}^\star)$ with $\sigma_h^\star|_{\omega_z^\star}\in Q(\mathcal{T}^\star(z))$ for all $z\in\mathcal{N}$ and all those allow a quantified estimate through the main result of this paper.

THEOREM 3.1. *Any $\sigma_h^\star\in RT_0(\mathcal{T}^\star)$ with $\sigma_h^\star|_{\omega_z^\star}\in Q(\mathcal{T}^\star(z))$ for all $z\in\mathcal{N}$ satisfies*

$$\|\!|\,\mathrm{div}(\sigma_h^\star-\sigma_h)|\!\|_\star^{p'} \leqslant \sum_{z\in\mathcal{N}}\big(c_1(z)\eta(z)+c_2(z)\eta(\mathcal{E}(z))\big)^{p'}.$$

With $\mathcal{F}(z) := \{F\in\mathcal{E}^\star(z)\,|\,F\subseteq\bigcup(\mathcal{E}(z)\setminus\mathcal{E}(\Gamma_D))\}$, the constants are $c_1(z)\leqslant C_{\mathrm{PF}}(p,\omega_z^\star)/\mathrm{diam}(\omega_z^\star)\lesssim 1$ and

$$\begin{aligned}c_2(z) \leqslant \frac{n^{1/p'}2^{1/p}}{(n!)^{n/((n-1)p')}}\max_{F\in\mathcal{F}(z)}|F|^{1-\frac{n}{p'(n-1)}}\\ \times\left(C_{\mathrm{PF}}(p,\omega_z^\star)^{p'}|\omega_F^\star|^{1-p'} + \frac{\|\bullet-\mathrm{mid}(\mathcal{T})\|_{L^{p'}(\omega_F^\star)}^{p'}}{n^{p'}|\omega_F^\star|^{p'}}\right)^{1/p'} \lesssim 1.\end{aligned}$$

4. Proof of Theorem 3.1.

Proof. Given any $v \in V$, set the integral mean $v_{\omega_z^\star} = \int_{\omega_z^\star} v dx / |\omega_z^\star|$ of v over $\omega_z^\star$ in case of a free node $z \in \mathcal{M}$ and $v_{\omega_z^\star} := 0$ in case of a Dirichlet boundary node $z \in \mathcal{N}(\Gamma_D)$. Since $(\sigma_h^\star - \sigma_h) \cdot \nu = 0$ along $\partial\omega_z^\star \backslash \partial\Omega$, an integration by parts shows

$$\begin{aligned}\int_{\omega_z^\star} (\sigma_h^\star - \sigma_h) \cdot \nabla v dx &= \int_{\omega_z^\star} (\sigma_h^\star - \sigma_h) \cdot \nabla (v - v_{\omega_z^\star}) dx \\ &= -\int_{\omega_z^\star} (v - v_{\omega_z^\star}) \operatorname{div} \sigma_h^\star dx - \sum_{F \in \mathcal{E}^\star(z)} \int_F (v - v_{\omega_z^\star}) [\sigma_h \cdot \nu_F]_F ds \\ &\quad + \int_{\partial\omega_z^\star \cap \Gamma_D} (v - v_{\omega_z^\star})(\sigma_h^\star - \sigma_h) \cdot \nu ds =: \mathbb{I} + \mathbb{II} + \mathbb{III}.\end{aligned}$$

Since $v = v_{\omega_z^\star} = 0$ along $\partial\omega_z^\star \cap \partial\Gamma_D$, the third expression $\mathbb{III} = 0$ vanishes. Since $\int_{\omega_z^\star} (v - v_{\omega_z^\star}) dx = 0$ in case of $z \in \mathcal{N}(\Omega)$, one can add f_{ω_z} in the first expression $\mathbb{I}$. Then, the Poincaré or Friedrichs inequality yields

(4.1)

$$\mathbb{I} := -\int_{\omega_z^\star} (v - v_{\omega_z^\star})(\operatorname{div} \sigma_h^\star + f_{\omega_z}) dx \leqslant C_{\mathrm{PF}}(p, \omega_z^\star) \left\| \operatorname{div} \sigma_h^\star + f_{\omega_z} \right\|_{L^{p'}(\omega_z^\star)} \left\| \nabla v \right\|_{L^p(\omega_z^\star)}.$$

For each (of the $n!$ many) $T^\star \in \mathcal{T}^\star(z)$ with $\mathcal{N}(T) \cap \mathcal{N}^\star(T^\star) = \{z\}$ (depicted in Figure 2 for $n = 2$), $\operatorname{div} \sigma_h^\star + f^\star = 0$ leads to

$$\begin{aligned}\left\| \operatorname{div} \sigma_h^\star + f_{\omega_z} \right\|_{L^{p'}(T^\star)}^{p'} &= \left\| f^\star - f_{\omega_z} \right\|_{L^{p'}(T^\star)}^{p'} = (n+1)^{p'} |T^\star| / |T|^{p'} \left| \int_T (f - f_{\omega_z}) \varphi_z dx \right|^{p'} \\ &\leqslant \frac{(n+1)^{p'}}{(n+1)!} |T|^{1-p'} \left\| \varphi_z^{1/p} \right\|_{L^p(T)}^{p'} \left\| \varphi_z^{1/p'} (f - f_{\omega_z}) \right\|_{L^{p'}(T)}^{p'}.\end{aligned}$$

Since $\left\| \varphi_z^{1/p} \right\|_{L^p(T)}^p = |T| / (n+1)$, this proves

$$\left\| f^\star - f_{\omega_z} \right\|_{L^{p'}(T^\star)}^{p'} \leqslant \frac{1}{n!} \left\| \varphi_z^{1/p'} (f - f_{\omega_z}) \right\|_{L^{p'}(T)}^{p'}. \tag{4.2}$$

Since each $T \in \mathcal{T}(z)$ contains $n!$ many $T^\star \in \mathcal{T}^\star(z)$,

$$\left\| f^\star - f_{\omega_z} \right\|_{L^{p'}(\omega_z^\star)}^{p'} \leqslant \sum_{T \in \mathcal{T}(z)} \left\| \varphi_z^{1/p'} (f - f_{\omega_z}) \right\|_{L^{p'}(T)}^{p'} = \int_{\omega_z} \varphi_z \left| f - f_{\omega_z} \right|^{p'} dx.$$

The combination of the previous four estimates leads to

$$\mathbb{I} \leqslant C_{\mathrm{PF}}(p, \omega_z^\star) \left\| \varphi_z^{1/p'} (f - f_{\omega_z}) \right\|_{L^{p'}(\omega_z)} \left\| \nabla v \right\|_{L^p(\omega_z^\star)}.$$

Similar to (4.2), for every $F \in \mathcal{E}^\star(\Gamma_N) \cap \mathcal{E}^\star(z)$ and $E \in \mathcal{E}(\Gamma_N)$ with $F \subseteq E$, it holds that

$$\left\| g^\star - \sigma_h \cdot \nu_F \right\|_{L^{p'}(F)}^{p'} \leqslant \frac{1}{(n-1)!} \left\| \varphi_z^{1/p'} (g - \sigma_h \cdot \nu_E) \right\|_{L^{p'}(E)}^{p'}.$$

For any non-Neumann side $F \in \mathcal{E}^\star(z)$, elementary calculations show

$$|F| \left| [\sigma_h \cdot \nu_F]_F \right|^{p'} = \frac{1}{(n-1)!} \left\| \varphi_z^{1/p'} [\sigma_h \cdot \nu_E]_E \right\|_{L^{p'}(E)}^{p'}.$$

Since $[\sigma_h \cdot \nu_F]_F = 0$ on every side $F \in \mathcal{E}^\star(z)\backslash\mathcal{F}(z)$, the second term reduces to

$$\begin{aligned}\mathbb{III} &:= -\sum_{F\in\mathcal{E}^\star(z)} \int_F (v - v_{\omega_z^\star})[\sigma_h\cdot\nu_F]_F ds = -\sum_{F\in\mathcal{E}^\star(z)} \int_{F\cap\bigcup\mathcal{E}(z)} [\sigma_h\cdot\nu_F]_F (v - v_{\omega_z^\star}) ds\\
&\leqslant \sum_{F\in\mathcal{F}(z)} \left(|F|^{n/((n-1)p')}\,|[\sigma_h\cdot\nu_F]_F|\right)\left(|F|^{1-n/((n-1)p')}\left|\fint_F (v - v_{\omega_z^\star})ds\right|\right)\\
&\leqslant \left(\sum_{E\in\mathcal{E}(z)} \frac{n\,|E|^{1/(n-1)}}{(n!)^{n/(n-1)}}\left\|\varphi_z^{1/p'}[\sigma_h\cdot\nu_E]_E\right\|_{L^{p'}(E)}^{p'}\right)^{1/p'}\\
&\quad\times\left(\sum_{F\in\mathcal{F}(z)} |F|^{p-pn/(p'(n-1))}\left|\fint_F (v - v_{\omega_z^\star})ds\right|^p\right)^{1/p}.\end{aligned}$$

For any $F \in \mathcal{F}$, the side patch $\omega_F^\star$ consists of one or two neighboring elements $T^\star = \operatorname{conv}\{F, \operatorname{mid}(T)\} \in \mathcal{T}^\star$, where $\operatorname{mid}(T)$ is the midpoint of the simplex $T \in \mathcal{T}$ with $T^\star \subseteq T$. The trace identity for any $T^\star = \operatorname{conv}\{F, \operatorname{mid}(T)\}$, reads, for any $w \in W^{1,p}(\omega_z^\star)$,

$$\frac{1}{|F|}\int_F w ds = \frac{1}{|T^\star|}\int_{T^\star} w dx + \frac{1}{n\,|T^\star|}\int_{T^\star}(x - \operatorname{mid}(\mathcal{T}))\cdot\nabla w dx.$$

Their weighted summation leads to

$$\frac{|\omega_F^\star|}{|F|}\int_F w ds = \int_{\omega_F^\star} w dx + \frac{1}{n}\int_{\omega_F^\star}(x - \operatorname{mid}(\mathcal{T})(x))\cdot\nabla w dx. \tag{4.3}$$

A Hölder inequality $X \cdot Y \leqslant \|X\|_{p'}\|Y\|_p$ with $X := \left(C_{\mathrm{PF}}(p,\omega_z^\star)\,|\omega_F^\star|^{1/p'-1},\right.$ $\left.\left\|\bullet - \operatorname{mid}(\mathcal{T})\right\|_{L^{p'}(\omega_F^\star)}/(n\,|\omega_F^\star|)\right)$ and $Y := \left(\|w\|_{L^p(\omega_F^\star)}/C_{\mathrm{PF}}(p,\omega_z^\star), \|\nabla w\|_{L^p(\omega_F^\star)}\right)$ in $\mathbb{R}^2$ proves

$$\begin{aligned}\left|\fint_F w ds\right|^p &\leqslant \left(\|w\|_{L^p(\omega_F^\star)}\,|\omega_F^\star|^{1/p'-1} + \frac{1}{n\,|\omega_F^\star|}\left\|\bullet - \operatorname{mid}(\mathcal{T})\right\|_{L^{p'}(\omega_F^\star)}\|\nabla w\|_{L^p(\omega_F^\star)}\right)^p\\
&\leqslant \left(C_{\mathrm{PF}}(p,\omega_z^\star)^{p'}\,|\omega_F^\star|^{1-p'} + \frac{\left\|\bullet - \operatorname{mid}(\mathcal{T})\right\|_{L^{p'}(\omega_F^\star)}^{p'}}{n^{p'}\,|\omega_F^\star|^{p'}}\right)^{p/p'}\\
&\quad\times\left(\frac{\|w\|_{L^p(\omega_F^\star)}^p}{C_{\mathrm{PF}}(p,\omega_z^\star)^p} + \|\nabla w\|_{L^p(\omega_F^\star)}^p\right).\end{aligned}$$

The sum over all $F \in \mathcal{F}(z)$ and a Poincaré or Friedrichs inequality for $w = v - v_{\omega_z^\star}$ lead to

$$\begin{aligned}&\sum_{F\in\mathcal{F}(z)} |F|^{p-pn/(p'(n-1))}\left|\fint_F (v - v_{\omega_z^\star})ds\right|^p\\
&\leqslant 2\|\nabla v\|_{L^p(\omega_z^\star)}^p \max_{F\in\mathcal{F}(z)} |F|^{p-pn/(p'(n-1))}\\
&\quad\times\left(C_{\mathrm{PF}}(p,\omega_z^\star)^{p'}\,|\omega_F^\star|^{1-p'} + \frac{\left\|\bullet - \operatorname{mid}(\mathcal{T})\right\|_{L^{p'}(\omega_F^\star)}^{p'}}{n^{p'}\,|\omega_F^\star|^{p'}}\right)^{p/p'}.\end{aligned}$$

The previous estimates prove that $\mathbb{III}$ is bounded by

$$\begin{aligned}&\|\nabla v\|_{L^p(\omega_z^\star)}\left(\sum_{E\in\mathcal{E}(z)}|E|^{1/(n-1)}\,\|[\sigma_h\cdot\nu_E]_E\|^{p'}_{L^{p'}(E)}\right)^{1/p'}\\&\quad\times\frac{n^{1/p'}2^{1/p}}{(n!)^{n/((n-1)p')}}\max_{F\in\mathcal{F}(z)}|F|^{1-n/(p'(n-1))}\\&\quad\times\left(C_{\mathrm{PF}}(p,\omega_z^\star)^{p'}\,|\omega_F^\star|^{1-p'}+\frac{\|\bullet-\mathrm{mid}(\mathcal{T})\|^{p'}_{L^{p'}(\omega_F^\star)}}{n^{p'}\,|\omega_F^\star|^{p'}}\right)^{1/p'}.\end{aligned}$$

A Hölder inequality concludes the proof. ∎

5. A new proof of the explicit Veeser–Verfürth upper bounds. This section shows how to retain the explicit upper bounds from [VV09] with even improved constants in benchmark examples. To express our results in the notation from [VV09], consider $\mu(z):=\mathrm{diam}(\omega_z)\|\varphi_z^{1/p'}f\|_{L^{p'}(\omega_z)}$ and

$$\mu(\mathcal{E}(z)):=\mathrm{diam}(\omega_z)\left(\sum_{E\in\mathcal{E}(z)}(h_E^\perp)^{1-p'}\|\varphi_z^{1/p'}[\sigma_h\cdot\nu]_E\|^{p'}_{L^{p'}(E)}\right)^{1/p'}$$

with $h_E^\perp:=\int_{\omega_E}\varphi_z dx/\int_E\varphi_z dx=\gamma_n|\omega_E|/|E|$ with $\gamma_2=2/3$ and $\gamma_3=3/4$. These are the two main contributions for the explicit Veeser–Verfürth upper bound

$$\|\!|\mathrm{Res}|\!\|_\star\leqslant\left(\sum_{z\in\mathcal{N}}\left(c_p(\omega_z)\mu(z)+c_p(\sigma_z)\mu(\mathcal{E}(z))\right)^{p'}\right)^{1/p'}\tag{5.1}$$

with constants $c_p(\omega_z)$ and $c_p(\sigma_z)$ to compare with.

PROPOSITION 5.1. *It holds that*

$$\|\!|Res|\!\|_\star\leqslant C_p\|h_\mathcal{T}(f-f^\star)\|_{L^{p'}(\Omega)}+\left(\sum_{z\in\mathcal{N}}\left(c_1(z)m_1(z)\mu(z)+c_2(z)m_2(z)\mu(\mathcal{E}(z))\right)^{p'}\right)^{1/p'}$$

with the multipliers $1/2\leqslant m_1(z):=\mathrm{diam}(\omega_z^\star)/\mathrm{diam}(\omega_z)<1$ *and*

$$m_2(z):=\max_{E\in\mathcal{E}(z)\setminus\mathcal{E}(\Gamma_D)}\left(|E|^{1/(n-1)}(h_E^\perp)^{p'-1}/\mathrm{diam}(\omega_z)^{p'}\right)^{1/p'}.$$

Proof. Follow the proof of section 4 with $f_{\omega_z}=0$ for every $z\in\mathcal{N}$ to obtain

$$\|\!|\mathrm{div}(\sigma_h^\star-\sigma_h)|\!\|_\star^{p'}\leqslant\sum_{z\in\mathcal{N}}\left(c_1(z)\eta(z)+c_2(z)\eta(\mathcal{E}(z))\right)^{p'}$$

with $\eta(z):=\mathrm{diam}(\omega_z^\star)\|\varphi_z^{1/p'}f\|_{L^{p'}(\omega_z)}$ and

$$\eta(\mathcal{E}(z)):=\left(\sum_{E\in\mathcal{E}(z)}|E|^{1/(n-1)}\,\|\varphi_z^{1/p'}[\sigma_h\cdot\nu_E]_E\|^{p'}_{L^{p'}(E)}\right)^{1/p'}.$$

Then, it is easy to check that $\eta(z)\leqslant m_1(z)\mu(z)$ and $\eta(\mathcal{E}(z))\leqslant m_2(z)\mu(\mathcal{E}(z))$. ∎

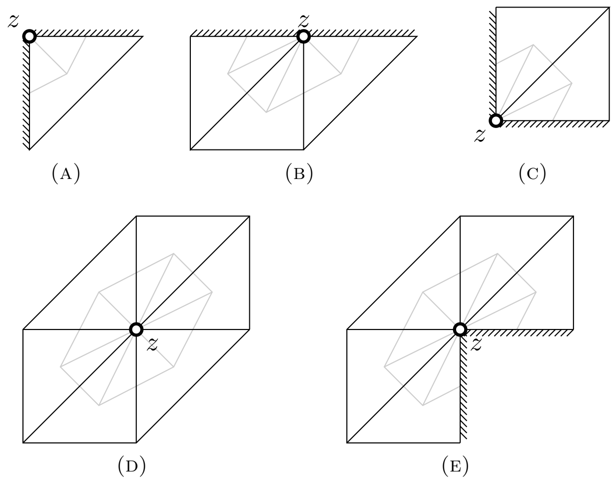

FIG. 3. *Standard node patches* (A) *to* (E) *(up to rotations and reflections) from the uniform triangulation of the unit square and their dual node patches (light gray). Boundary edges along* $\partial\Omega$ *in patches* (A) *to* (C) *are shaded.*

This section concludes with a few comments on and comparisons of the new upper bounds and the Veeser–Verfürth upper bounds.

Remark 5.2 ($m_2(z) \leqslant 0.5774$ for $n = 2 = p$). Elementary geometry for regular triangulations into triangles allows for $|\omega_E| \leqslant |E|\ \mathrm{diam}(\omega_z)/2$ and hence for

$$m_2(z) \leqslant \max_{E\in\mathcal{E}(z)\setminus\mathcal{E}(\Gamma_D)} (|E|/(3\,\mathrm{diam}(\omega_z)))^{1/2} \leqslant 0.5774.$$

Remark 5.3 (anisotropic meshes for $n = 2 = p$). The proofs in this paper employ the isotropic Poincaré or Friedrichs inequalities for the usual (and that means isotropic) Sobolev norm. Hence it cannot be expected that the resulting error estimator is robust with respect to anisotropic meshes like those of the cross refinement of uniform axi-parallel rectangles in two dimensions with a mesh-size d and h in the first and second components, respectively, for $h \ll d$ and large aspect ratio $\kappa := d/h$. The geometry leads to $m_2(z) \lesssim \kappa^{-1/2}$, which tends to zero as the aspect ratio κ tends to ∞. One possible interpretation is that the novel upper bound is more robust than the Veeser–Verfürth bound in the edge-jump term.

Example 5.4 (two-dimensional L-shaped domain example). This example illustrates that the new estimate is indeed comparable to or even sharper than the reliability constants of [CF00, VV09].

Uniform cross refinements of the L-shaped domain $\Omega = (-1, 1)^2\setminus([0, 1] \times [-1, 0])$ with right-isosceles triangles of area $|T| = h^2/2$ and a diagonal parallel to the main diagonal result in five different patches depicted in Figure 3. Table 1 lists computed values of $c_1(z)$, $c_2(z)$, the multipliers $m_1(z)$, $m_2(z)$, and their products for patches (A)–(E) from Figure 3 as well as $c_2(\omega_z)$ and $c_2(\sigma_z)$ from [VV09]. The Poincaré and Friedrichs constants from section 3 for the patches (A)–(D) read $C_{\mathrm{PF}}(2, \omega_z) = \mathrm{diam}(\omega_z)/\pi$, while for patch (E) we use $C_{\mathrm{PF}}(2, \omega_z) = \sqrt{2}\,\mathrm{diam}(\omega_z)/\pi$. Furthermore, we use $C_2(\mathcal{T}) = 1/j_{1,1}$ and $C_2(\mathcal{E}(\Gamma_N)) = 0.96478$.

The competition with [VV09] leads to the comparison of $m_1(z)c_1(z) \leqslant 0.17272$ in this paper with $c_2(\omega_z) \geqslant 1/\pi = 0.31831$ from [VV09] and $m_2(z)c_2(z) \leqslant 0.6c_2(\sigma_z)$ with

TABLE 1. *Constants for standard patches* (A) *to* (E) *(with Dirichlet boundary edges) from Figure* 3.

Patch	$c_1(z)$	$c_2(z)$	$m_1(z)$	$m_2(z)$	$m_1(z)c_1(z)$	$m_2(z)c_2(z)$	$c_2(\omega_z)$	$c_2(\sigma_z)$
(A)	0.31831	–	0.50000	–	0.15915	–	0.77909	–
(B)	0.31831	1.03037	0.54263	0.36515	0.17272	0.37624	0.96636	1.63786
(C)	0.31831	0.71021	0.52705	0.57735	0.16776	0.41004	0.96995	2.40030
(D)	0.31831	1.23192	0.52705	0.28868	0.16776	0.35562	0.31831	0.64409
(E)	0.45016	1.69368	0.52705	0.28868	0.23725	0.48892	1.42544	2.23921

TABLE 2. *Constants for standard patches* (A)–(C) *and* (E) *from Figure* 3 *for Neumann boundary edges.*

Patch	$c_1(z)$	$c_2(z)$	$m_1(z)$	$m_2(z)$	$m_1(z)c_1(z)$	$m_2(z)c_2(z)$	$c_2(\omega_z)$	$c_2(\sigma_z)$
(A)	0.31831	0.90347	0.50000	0.40825	0.15915	0.36884	0.31831	0.79187
(B)	0.31831	1.49945	0.54263	0.36515	0.17272	0.54752	0.31831	0.68634
(C)	0.31831	1.06480	0.52705	0.57735	0.16776	0.61476	0.31831	1.21486
(E)	0.45016	2.42118	0.52705	0.28868	0.23725	0.69894	0.45016	0.83988

$c_2(\omega_z)$ and $c_2(\sigma_z)$ from [VV09]. In conclusion, the analysis leads to an improvement of [VV09] by at least 40%. For patches with Neumann boundary the numbers are displayed in Table 2. Also in this case, the improvement is still significant. A further comparison is possible with [CF99] and computer-based constants.

6. Comparison of guaranteed upper bounds. This section compares the three guaranteed upper bounds

$$\begin{aligned}\eta_{\mathrm{LW}} &:= C_p(\mathcal{T})\big\|h_{\mathcal{T}}(f-f^\star)\big\|_{L^{p'}(\Omega)} + C_p(\mathcal{E}(\Gamma_N))\big\|h_{\mathcal{T}}^{1/p'}(g-g^\star)\big\|_{L^{p'}(\Omega)}\\ &\quad + \big\|\sigma_h^\star-\sigma_h\big\|_{L^{p'}(\Omega)} && \text{from (2.1)},\\ \eta_{\mathrm{RCM}} &:= C_p(\mathcal{T})\big\|h_{\mathcal{T}}(f-f^\star)\big\|_{L^{p'}(\Omega)} + C_p(\mathcal{E}(\Gamma_N))\big\|h_{\mathcal{T}}^{1/p'}(g-g^\star)\big\|_{L^{p'}(\Omega)}\\ &\quad + \left(\sum_{z\in\mathcal{N}}\big(c_1(z)\eta(z)+c_2(z)\eta(\mathcal{E}(z))\big)^{p'}\right)^{1/p'} && \text{from (3.1)},\\ \eta_{\mathrm{RVV}} &:= \left(\sum_{z\in\mathcal{N}}\big(c_p(\omega_z)\mu(z)+c_p(\sigma_z)\mu(\mathcal{E}(z))\big)^{p'}\right)^{1/p'} && \text{from (5.1)}.\end{aligned}$$

While the last two estimates are fully explicit and easy to compute, the first one is a very sharp but is also a more expensive a posteriori error estimator.

Table 3 shows their values and efficiency indices compared to the exact energy error $|||u-u_h||| = |||\mathrm{Res}|||_\star$ for the L-shaped domain in some Poisson model problem with constant right-hand side $f\equiv 1$ and discrete flux $\sigma_h = \nabla u_h$ of the P_1-conforming finite element solution $u_h \in P_1(\mathcal{T})\cap C(\Omega)$ on the uniform cross refinements $\mathcal{T}$ discussed at the end of section 5. The exact solution is unknown; however, the Galerkin orthogonality allows us to compute the energy error by $|||u-u_h|||^2 = |||u|||^2 - |||u_h|||^2$ and with $|||u|||^2 = 0.214075802680976$ (calculated with higher-order finite element methods).

TABLE 3. *Exact energy error $\|\!|e|\!\|$ and guaranteed upper bounds η_{LW}, η_{RCM}, and η_{RVV} for the L-shaped domain example and uniform cross refinement.*

$n$dof	$\|\!\|e\|\!\|$	η_{LW}	$\eta_{\mathrm{LW}}/\|\!\|e\|\!\|$	η_{RCM}	$\eta_{\mathrm{RCM}}/\|\!\|e\|\!\|$	η_{RVV}	$\eta_{\mathrm{RVV}}/\|\!\|e\|\!\|$
5	2.84e−01	3.74e−01	1.32	9.33e−01	3.28	4.20e+00	14.8
33	1.58e−01	2.10e−01	1.33	6.19e−01	3.91	1.97e+00	12.5
161	8.62e−02	1.17e−01	1.36	3.66e−01	4.24	9.97e−01	11.6
705	4.76e−02	6.62e−02	1.39	2.09e−01	4.39	5.37e−01	11.3
2945	2.69e−02	3.83e−02	1.42	1.19e−01	4.44	3.06e−01	11.4
12033	1.56e−02	2.26e−02	1.45	6.93e−02	4.44	1.81e−01	11.6
48641	9.23e−03	1.36e−02	1.48	4.09e−02	4.43	1.10e−01	11.9
195585	5.57e−03	8.33e−03	1.50	2.46e−02	4.41	6.78e−02	12.2
784385	3.41e−03	5.15e−03	1.51	1.50e−02	4.40	4.21e−02	12.4

The comparison of the efficiency indices of all three estimators of Table 3 reveals some behavior like $\eta_{\mathrm{RVV}} \leqslant 2.5\,\eta_{\mathrm{RCM}}$, $\eta_{\mathrm{RCM}} \leqslant 3\,\eta_{\mathrm{LW}}$, and $\eta_{\mathrm{LW}} \leqslant 1.5|\!|\!|e|\!|\!|$. The comparison favors η_{RCM} on coarse meshes.

7. Fully explicit error control for two-dimensional nonconforming FEM.

This section discusses an application to the error in the nonconforming finite element method which involves the set of Crouzeix–Raviart functions

$$\mathrm{CR}^1(\mathcal{T}) := \{v \in P_1(\mathcal{T}) \,|\, v \text{ is continuous in } \mathrm{mid}(\mathcal{E})\},$$
$$\mathrm{CR}^1_0(\mathcal{T}) := \{v \in \mathrm{CR}^1(\mathcal{T}) \,|\, v(\mathrm{mid}(\mathcal{E}(\partial\Omega))) = 0\}.$$

The nonconforming FEM for the Poisson model problem with right-hand side $f \in L^2(\Omega)$ and Dirichlet data $u_D \in H^1(\partial\Omega)$ for a simply-connected, bounded polygonal Lipschitz domain $\Omega \subset \mathbb{R}^2$ seeks $u_{\mathrm{CR}} \in \mathrm{CR}^1(\mathcal{T})$ with $u_{\mathrm{CR}}(\mathrm{mid}(E)) = ⨍_E u_D ds$ for all $E \in \mathcal{E}(\partial\Omega)$ and

$$\int_\Omega \nabla_{\mathrm{NC}}\, u_{\mathrm{CR}} \cdot \nabla_{\mathrm{NC}}\, v_{\mathrm{CR}} dx = \int_\Omega f v_{\mathrm{CR}} dx \quad \forall\ v_{\mathrm{CR}} \in \mathrm{CR}^1_0(\mathcal{T})$$

where ∇_{NC} is the piecewise gradient. The error analysis in [CMx1] shows

$$|\!|\!|u - u_{\mathrm{CR}}|\!|\!| \leqslant \mu^2 + \sup\left\{\mathrm{Res}_{\mathrm{NC}}(v)^2 \,\middle|\, v \in H^1(\Omega;\mathbb{R}^k) \text{ with } \left\|\mathrm{Curl}\, v\right\|_{L^2(\Omega)} = 1\right\}$$

with $\mu := \left\|f_{\mathcal{T}}/2\,(\bullet - \mathrm{mid}(\mathcal{T}))\right\|_{L^2(\Omega)} + \mathrm{osc}(f,\mathcal{T})/j_{1,1}$ for $n = 2$ and the nonconforming residual

$$\mathrm{Res}_{\mathrm{NC}}(v) := \int_{\partial\Omega} \gamma_t(\nabla\, u_D) \cdot v ds - \int_\Omega \nabla_{\mathrm{NC}}\, u_{\mathrm{CR}} \cdot \mathrm{Curl}\, v dx \qquad \forall\ v \in V := H^1(\Omega;\mathbb{R}^k) \tag{7.1}$$

with $k = 1$ for $n = 2$ and $k = 3$ for $n = 3$ and Dirichlet data u_D along $\Gamma_D = \partial\Omega$. This form is in agreement with the definition of the $\mathrm{Curl}\, v \in L^2(\Omega;\mathbb{R}^n)$ for some $v \in H^1(\Omega;\mathbb{R}^k)$ defined by

$$\mathrm{Curl}\, v := (0,-1;1,0)\,\nabla\, v \text{ for } n = 2 \qquad \text{and} \qquad \mathrm{Curl}\, v := \nabla \times v \text{ for } n = 3.$$

TABLE 4. *Exact energy error $\|\|e\|\|_{NC}$ and guaranteed upper bounds $\eta_{\rm LW}$, $\eta_{\rm RCM}$, and $\eta_{\rm RVV}$ for the nonconforming Crouzeix–Raviart FEM in the L-shaped domain example with uniform cross refinement.*

$n$dof	$\|\|e\|\|$	$\eta_{\rm LW}$	$\eta_{\rm LW}/\|\|e\|\|$	$\eta_{\rm RCM}$	$\eta_{\rm RCM}/\|\|e\|\|$	$\eta_{\rm RVV}$	$\eta_{\rm RVV}/\|\|e\|\|$
28	2.85e−01	4.60e−01	1.62	1.32e+00	4.64	1.85e+00	6.48
128	1.89e−01	2.93e−01	1.55	8.89e−01	4.69	1.30e+00	6.87
544	1.23e−01	1.85e−01	1.51	5.76e−01	4.69	8.63e−01	7.03
2240	7.87e−02	1.17e−01	1.49	3.68e−01	4.68	5.59e−01	7.11
9088	5.01e−02	7.38e−02	1.47	2.34e−01	4.68	3.58e−01	7.15
36608	3.17e−02	4.65e−02	1.47	1.48e−01	4.68	2.28e−01	7.17
146944	2.01e−02	2.93e−02	1.46	9.38e−02	4.68	1.44e−01	7.19

The tangential component $\gamma_{\rm t}(v)$ of some vector $v \in \mathbb{R}^n$ with respect to some normal vector ν reads

$$\gamma_{\rm t}(v) := \begin{cases} v \cdot (-\nu(2), \nu(1)) & \text{if } n = 2, \\ v \times \nu & \text{if } n = 3. \end{cases}$$

For the two-dimensional case the residual (7.1) easily transforms into

(7.2)
$$\mathrm{Res}_{\rm NC}(v) = \int_{\partial\Omega} (\partial u_D/\partial s) v\,ds - \int_\Omega \mathrm{Curl}_{\rm NC}\, u_{\rm CR} \cdot \nabla v\,dx \qquad \forall\ v \in V := H^1(\Omega)$$

and equals the standard residual with $f \equiv 0$, Neumann data $g := \partial u_D/\partial s$, and discrete flux $\sigma_h := \mathrm{Curl}_{\rm NC}\, u_{\rm CR}$. Thus, Theorem 3.1 gives a computable upper bound for

$$\|\|\mathrm{Res}_{\rm NC}\|\|_\star := \sup\left\{\mathrm{Res}_{\rm NC}(v)\,\middle|\, v \in H^1(\Omega;\mathbb{R}^k) \text{ with } \left\|\,\mathrm{Curl}\, v\right\|_{L^2(\Omega)} = 1\right\}.$$

THEOREM 7.1. *It holds that*

$$\|\|Res_{NC}\|\|_\star^2 \leqslant \sum_{z\in\mathcal{N}} c(z)\eta(\mathcal{E}(z))^2 \quad \textit{with} \quad \eta(\mathcal{E}(z))^2$$
$$:= \sum_{E\in\mathcal{E}(z)} |E| \left\|\varphi_z^{1/2}[\mathrm{Curl}_{NC}\, u_{CR} \cdot \nu_E]_E\right\|_{L^2(E)}^2.$$

Proof. Apply Theorem 3.1 to the transformed residual (7.2). ∎

Table 4 shows the results for some model example, again on the L-shaped domain, with known exact solution $u(r,\varphi) = r^{2/3}\sin(2\varphi/3)$ in polar coordinates, right-hand side $f \equiv 0$, and nonhomogeneous boundary data u_D. The Luce–Wohlmuth error estimator $\eta_{\rm LW}$ in this example (with $f \equiv 0$, $g \equiv \partial u_D/\partial s$, and $\sigma_h = \mathrm{Curl}_{\rm NC}\, u_{\rm CR}$ in (2.1)) reads

$$\eta_{\rm LW} := C_2(\mathcal{E}(\Gamma_N))\left\|h_{\mathcal{T}}^{1/2}(\partial u_D/\partial s - (\partial u_D/\partial s)^\star)\right\|_{L^2(\Omega)} + \left\|\sigma_h^\star - \mathrm{Curl}_{\rm NC}\, u_{\rm CR}\right\|_{L^2(\Omega)}.$$

The improvement of $\eta_{\rm RCM}$ compared to $\eta_{\rm RVV}$ is less striking but still significant.

Remark 7.2. In three dimensions $\|\mathrm{D}\, v\|_{L^2(\Omega)} \neq \|\mathrm{Curl}\, v\|_{L^2(\Omega)}$ does not allow an immediate application of Theorem 3.1. The decomposition in three dimensions (even for a Lipschitz domain with connected boundary) involves a further stability constant which needs to be computed and the untransformed residual (7.1) requires novel equilibrated fluxes beyond the design in section 2.

8. Extension to higher-order approximations. This sections extends Theorem 3.1 to discrete stresses $\sigma_h \in P_k(\mathcal{T};\mathbb{R}^n)$ from higher-order approximations that satisfy property (2.2) as the higher-order finite element approximations from [HSV12, section 7.2.2]. Given $f^\star \in P_k(\mathcal{T}^\star)$ and $g^\star \in P_k(\mathcal{E}^\star(\Gamma_N))$ with $f - f^\star \perp P_k(T)$ in $L^2(T)$, $g - g^\star \perp P_k(E)$ in $L^2(E)$, and

$$\int_{\omega_z} f\varphi_z dx - \int_{\omega_z^\star} f^\star dx = 0 \quad \forall\ z \in \mathcal{N}, \tag{8.1}$$

there exists $\sigma_h^\star \in RT_k(\mathcal{T}^\star(z))$ such that for all $z \in \mathcal{N}$,

$$\begin{aligned}\sigma_h^\star|_{\omega_z^\star} &\in Q(\mathcal{T}^\star(z))\\ &:= \big\{\tau_h \in RT_k(\mathcal{T}^\star(z)) \,\big|\, \operatorname{div}\tau_h\\ &\qquad + f^\star = 0 \text{ in } \omega_z^\star \;\&\; \tau_h\cdot\nu = \sigma_h\cdot\nu \text{ along } \partial\omega_z^\star\backslash\partial\Omega \;\&\; \tau_h\cdot\nu = g^\star \text{ along } \partial\omega_z^\star\cap\Gamma_N\big\}.\end{aligned}$$

The guaranteed explicit upper bound consists of the contributions

$$\eta(z) := \operatorname{diam}(\omega_z^\star) \min_{f_{\omega_z}\in\mathbb{R}} \|f^\star + \operatorname{div}\sigma_h - f_{\omega_z}\|_{L^{p'}(\omega_z^\star)} \quad \text{and}$$
$$\eta(\mathcal{E}^\star(z))^{p'} := \sum_{F\in\mathcal{E}^\star(z)} |F|^{1/(n-1)} \|[\sigma_h\cdot\nu_F]_F\|_{L^{p'}(F)}^{p'}.$$

THEOREM 8.1. *Any $\sigma_h^\star \in RT_k(\mathcal{T}^\star)$ with $\sigma_h^\star|_{\omega_z^\star} \in Q(\mathcal{T}^\star(z))$ for all $z\in\mathcal{N}$ satisfies*

$$|\!|\!| \operatorname{div}(\sigma_h^\star - \sigma_h)|\!|\!|_\star^{p'} \leqslant \sum_{z\in\mathcal{N}} \left(c_1(z)\eta(z) + c_3(z)\eta(\mathcal{E}^\star(z))\right)^{p'}.$$

The constants are bounded by $c_1(z) \leqslant C_{\mathrm{PF}}(p,\omega_z^\star)/\operatorname{diam}(\omega_z^\star) \lesssim 1$ and, with $\mathcal{F}(z)$ as before,

$$\begin{aligned}c_3(z) &\leqslant \max_{F\in\mathcal{F}(z)} |F|^{\frac{1}{p}-\frac{1}{p'(n-1)}} / |\omega_F^\star|^{1/p}\\ &\qquad\times \left(C_{\mathrm{PF}}(p,\omega_z^\star)^p + C_{\mathrm{PF}}(p,\omega_z^\star)^{p/p'} p/n \,\|\bullet - \operatorname{mid}(\mathcal{T})\|_{L^\infty(\omega_z^\star)}\right)^{1/p} \lesssim 1.\end{aligned}$$

Proof. The proof follows that of Theorem 3.1 and starts with an integration by parts, i.e.,

$$\begin{aligned}\int_{\omega_z^\star} (\sigma_h^\star - \sigma_h)\cdot\nabla v dx &= \int_{\omega_z^\star} (\sigma_h^\star - \sigma_h)\cdot\nabla(v - v_{\omega_z^\star}) dx\\ &= \int_{\omega_z^\star} (v - v_{\omega_z^\star})(f^\star + \operatorname{div}\sigma_h) dx - \sum_{F\in\mathcal{E}^\star(z)} \int_F (v - v_{\omega_z^\star})[\sigma_h\cdot\nu_F]_F ds\\ &\quad + \int_{\partial\omega_z^\star\cap\Gamma_D} (v - v_{\omega_z^\star})(\sigma_h^\star - \sigma_h)\cdot\nu ds =: \mathbb{I} + \mathbb{III} + \mathbb{IIII}.\end{aligned}$$

The third term vanishes and the first term is estimated by

$$\int_{\omega_z^\star}(v-v_{\omega_z^\star})(f^\star+\operatorname{div}\sigma_h)dx \\ \leqslant C_{\mathrm{PF}}(p,\omega_z^\star)\min_{f_{\omega_z}\in\mathbb{R}}\left\|f^\star+\operatorname{div}\sigma_h-f_{\omega_z}\right\|_{L^{p'}(\omega_z^\star)}\left\|\nabla v\right\|_{L^p(\omega_z^\star)}.$$

The arguments about the estimation of quantity $\mathbb{III}$ have to be modified as follows and lead to the constant $c_3(z)$. Since $[\sigma_h\cdot\nu_F]_F=0$ on every edge $F\in\mathcal{E}^\star(z)\backslash\mathcal{F}$, the second term reduces to

$$\begin{aligned}\mathbb{III} &:= -\sum_{F\in\mathcal{E}^\star(z)}\int_F(v-v_{\omega_z^\star})[\sigma_h\cdot\nu_F]_F ds = -\sum_{F\in\mathcal{F}(z)}\int_{F\cap\bigcup\mathcal{E}(z)}[\sigma_h\cdot\nu_F]_F(v-v_{\omega_z^\star})ds\\ &\leqslant \sum_{F\in\mathcal{F}(z)}\left\|[\sigma_h\cdot\nu_F]_F\right\|_{L^p(F)}\left\|v-v_{\omega_z^\star}\right\|_{L^{p'}(F)}\\ &\leqslant\left(\sum_{F\in\mathcal{F}(z)}|F|^{1/(n-1)}\left\|[\sigma_h\cdot\nu_F]_F\right\|_{L^{p'}(F)}^{p'}\right)^{1/p'}\\ &\quad\times\left(\sum_{F\in\mathcal{F}(z)}|F|^{-p/(p'(n-1))}\left\|v-v_{\omega_z^\star}\right\|_{L^p(F)}^p\right)^{1/p}.\end{aligned}$$

The weighted trace identity with $w=\left|v-v_{\omega_z^\star}\right|^p$ leads to (4.3) and the sum over all $F\in\mathcal{F}(z)$ reads

$$\sum_{F\in\mathcal{F}(z)}\frac{|\omega_F^\star|}{|F|}\int_F w ds=\int_{\omega_z^\star}w dx+\frac{1}{n}\int_{\omega_z^\star}(x-\operatorname{mid}(\mathcal{T}))\cdot\nabla w dx.$$

The weak derivate of the modulus function and the chain rule show $|\nabla w|=p\left|v-v_{\omega_z^\star}\right|^{p-1}|\nabla v|$ and elementary calculations lead to

$$\left\|\nabla w\right\|_{L^1(\omega_z^\star)}\leqslant p\left\|\left|v-v_{\omega_z^\star}\right|^{p-1}\right\|_{L^{p'}(\omega_z^\star)}\left\|\nabla v\right\|_{L^p(\omega_z^\star)}\leqslant p\left\|v-v_{\omega_z^\star}\right\|_{L^p(\omega_z^\star)}^{p/p'}\left\|\nabla v\right\|_{L^p(\omega_z^\star)}.$$

The previous estimate and a Poincaré or Friedrichs inequality yield

$$\begin{aligned}&\sum_{F\in\mathcal{F}(z)}\frac{|\omega_F^\star|}{|F|}\left\|v-v_{\omega_z^\star}\right\|_{L^p(F)}^p\\ &\leqslant\left\|v-v_{\omega_z^\star}\right\|_{L^p(\omega_z^\star)}^p+1/n\left\|\bullet-\operatorname{mid}(\mathcal{T})\right\|_{L^\infty(\omega_z^\star)}\left\|\nabla w\right\|_{L^1(\omega_z^\star)}\\ &\leqslant\left(C_{\mathrm{PF}}(p,\omega_z^\star)^p+C_{\mathrm{PF}}(p,\omega_z^\star)^{p/p'}p/n\left\|\bullet-\operatorname{mid}(\mathcal{T})\right\|_{L^\infty(\omega_z^\star)}\right)\left\|\nabla v\right\|_{L^p(\omega_z^\star)}^p.\end{aligned}$$

This leads to

$$\begin{aligned}\sum_{F\in\mathcal{F}(z)}|F|^{-p/(p'(n-1))}\left\|v-v_{\omega_z^\star}\right\|_{L^p(F)}^p&\leqslant\max_{F\in\mathcal{F}(z)}|F|^{1-p/(p'(n-1))}/|\omega_F^\star|\\ &\times\left(C_{\mathrm{PF}}(p,\omega_z^\star)^p+C_{\mathrm{PF}}(p,\omega_z^\star)^{p/p'}p/n\left\|\bullet-\operatorname{mid}(\mathcal{T})\right\|_{L^\infty(\omega_z^\star)}\right)\left\|\nabla v\right\|_{L^p(\omega_z^\star)}^p.\end{aligned}$$

Hence, $\mathbb{III}$ is bounded by

$$\|\nabla v\|_{L^p(\omega_z^\star)} \eta(\mathcal{E}^\star(z)) \max_{F\in\mathcal{F}(z)} |F|^{1/p-1/(p'(n-1))} / |\omega_F^\star|^{1/p}$$
$$\times \left(C_{\mathrm{PF}}(p,\omega_z^\star)^p + C_{\mathrm{PF}}(p,\omega_z^\star)^{p/p'} \frac{p}{n} \|\bullet - \mathrm{mid}(\mathcal{T})\|_{L^\infty(\omega_z^\star)} \right)^{1/p}.$$

This concludes the proof. ∎

Remark 8.2. The term $\eta(z)$ clearly is efficient by a triangle inequality

$$\eta(z) \leqslant \mathrm{diam}(\omega_z^\star)\|f^\star - f\|_{L^{p'}(\omega_z^\star)} + \mathrm{diam}(\omega_z^\star)\|f + \mathrm{div}\,\sigma_h\|_{L^{p'}(\omega_z^\star)}.$$

Since $f - f^\star \perp P_k(T)$ in $L^2(T)$ for all $T \in \mathcal{T}$, the sum of $\mathrm{diam}(\omega_z^\star)\|f^\star - f\|_{L^{p'}(\omega_z^\star)}$ over all $z \in \mathcal{N}$ is of higher order for piecewise smooth f. The second term on the right-hand side amounts to $\|h_\mathcal{T}(f + \mathrm{div}\,\sigma_h)\|_{L^{p'}(\Omega)}$, which is a standard contribution in a posteriori error analysis and well-known to be efficient up to data oscillations.

Remark 8.3 (design of $f^\star$ for $k = 1$ in two dimensions). Let $K_j := \omega_z^\star \cap T$ denote the union of the two triangles in $\mathcal{T}^\star(T)$ with vertex P_j (e.g., $K_1 = T_1^\star \cup T_6^\star$ in Figure 4) and let φ_j for $j = 1,2,3$ denote the nodal basis functions on the triangle $T \in \mathcal{T}$. Since the affine functions $P_1(K_j)$ are linear independent, there exists exactly one $\psi_j^\star \in P_1(K_j)$ with

$$\int_{K_j} \varphi_k \psi_j^\star dx = \delta_{jk}.$$

Since $\varphi_1 + \varphi_2 + \varphi_3 = 1$, the sum over $k = 1,2,3$ in the previous identity shows $\int_{K_j} \psi_j^\star dx = 1$. Altogether, the extension of $\psi_j^\star$ by zero defines a function $\psi_j^\star \in P_1(\{K_1, K_2, K_3\})$ with

$$\int_T \varphi_k \psi_j^\star dx = \delta_{jk} = \int_{K_k} \psi_j dx.$$

Based on those duality relations, one designs a function $f^\star := \sum_{j=1}^3 (\int_T f\varphi_j dx)\psi_j^\star$ on the triangle T with $f^\star - f \perp P_1(T)$ in $L^2(T)$ for the given $f \in L^2(T)$ and the additional property that

$$\int_{K_j} f^\star dx = \int_T f^\star \varphi_j dx = \int_T f\varphi_j dx.$$

(The second identity follows from the aforementioned orthogonality.) The sum over all triangles in a patch of some node z guarantees $\int_{\omega_z} f\varphi_z dx = \int_{\omega_z^\star} f^\star dx$. In conclusion, the design of the function $f^\star$ follows for $k = 1$.

Remark 8.4 (design of $f^\star$ for $k = 2$ in two dimensions). Let $\lambda_1, \dots, \lambda_6$ denote a nodal basis of $P_2(T)$ dual to the point evaluations at the three vertices of the triangle P_1, P_2, P_3 and the three edge midpoints $\mathrm{mid}(E_1), \mathrm{mid}(E_2), \mathrm{mid}(E_3)$ of the three edges $E_1, \dots, E_3$ of the triangle in Figure 4. For every $T^\star \in \mathcal{T}^\star(T)$ let $\psi_{T^\star,k}^\star \in P_2(T^\star)$ denote the polynomial with

$$\int_{T^\star} \psi_{T^\star,k}^\star \lambda_j dx = \delta_{jk} \quad \text{for } j,k = 1,\dots,6$$

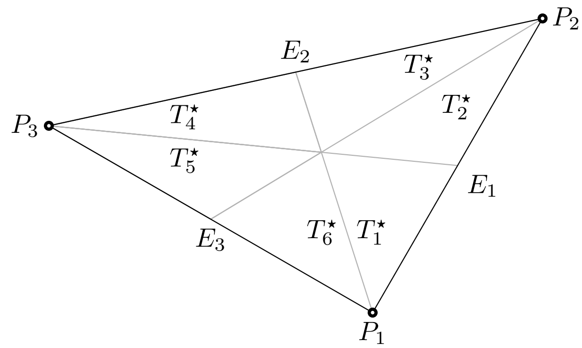

FIG. 4. *Enumeration of the vertices and edges in a triangle $T \in \mathcal{T}$ and its subtriangles $T^\star \in \mathcal{T}^\star(T)$.*

extended by zero to some $\psi^\star_{T^\star,k} \in P_2(\mathcal{T}^\star(T))$. The partition of unity property of the λ_j shows $\int_{T^\star} \psi^\star_{T^\star,k} dx = 1$. With the coefficients $\alpha_j := \int_T f\lambda_j dx$, $f^\star$ is defined via

$$2f^\star = \alpha_1(\psi^\star_{T_1^\star,1} + \psi^\star_{T_6^\star,1}) + \alpha_2(\psi^\star_{T_2^\star,2} + \psi^\star_{T_3^\star,2}) + \alpha_3(\psi^\star_{T_4^\star,3} + \psi^\star_{T_5^\star,3})$$
$$+ \alpha_4(\psi^\star_{T_1^\star,4} + \psi^\star_{T_2^\star,4}) + \alpha_5(\psi^\star_{T_3^\star,5} + \psi^\star_{T_4^\star,5}) + \alpha_6(\psi^\star_{T_5^\star,6} + \psi^\star_{T_6^\star,6}).$$

Then, it holds that

$$\int_T f^\star \lambda_1 dx = \alpha_1/2 \left(\int_{T_1^\star} \psi^\star_{T_1^\star,1}\lambda_1 dx + \int_{T_6^\star} \psi^\star_{T_6^\star,1}\lambda_1 dx \right) = \alpha_1 = \int_T f\lambda_1 dx.$$

Analog direct calculations show $f^\star - f \perp P_2(T)$ in $L^2(T)$. Since $\varphi_1 = \lambda_1 + (\lambda_4 + \lambda_6)/2$, it holds that

$$\int_{T_1^\star \cup T_6^\star} f^\star dx = \alpha_1 + (\alpha_4 + \alpha_6)/2 = \int_T f(\lambda_1 + (\lambda_4 + \lambda_6)/2)dx = \int_T f\varphi_1 dx.$$

Analogous formulas hold for $\omega^\star_{P_j} \cap T$ as well. Their sum proves (8.1).

Similar designs of $f^\star$ exist for higher polynomial degrees $k \geqslant 3$.

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