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Medius analysis and comparison results for first-order finite element methods in linear elasticity

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This paper enfolds a medius analysis for first-order nonconforming finite element methods (FEMs) in linear elasticity named after Crouzeix–Raviart and Kouhia–Stenberg, which are robust with respect to the incompressible limit as the Lamé parameter λ tends to infinity. The new result is a best-approximation error estimate for the stress error in L^2 up to data-oscillation terms. Even for very coarse shape-regular triangulations, two comparison results assert that the errors of the nonconforming FEM are equivalent to those of the conforming first-order FEM. The explicit role of the parameter λ in those equivalence constants leads to an advertisement of the robust and quasi-optimal Kouhia–Stenberg FEM, in particular for nonconvex polygons. The proofs are based on conforming companions, a new discrete Helmholtz decomposition and a new discrete-plus-continuous Korn inequality for Kouhia–Stenberg finite element functions. Numerical evidence strongly supports the robustness of the nonconforming FEMs with respect to incompressibility locking and with respect to singularities, and underlines that the dependence of the equivalence constants on λ in the comparison of conforming and nonconforming FEMs cannot be improved. This work therefore advertises the Kouhia–Stenberg FEM as a first-order robust discretization in linear elasticity in the presence of Neumann boundary conditions.

Keywords: linear elasticity; nonconforming finite elements; Kouhia–Stenberg; comparison; locking; discrete Korn inequality; Stokes equations.

1. Introduction

The textbook *a priori* error analysis of nonconforming finite element methods (FEMs) considers an inconsistency term with the normal derivative of the exact solution along edges and so requires $H^{3/2+\varepsilon}$ regularity of the exact solution for some positive ε. This regularity request fails to hold for certain mixed boundary value problems in linear elasticity and leaves the impression that nonconforming FEMs may be more sensitive for 'near singularities' than conforming FEM (Braess, 2007, p.111 and the web supplement). The medius analysis of Gudi (2010) and Carstensen *et al.* (2012b) does not rely on elliptic regularity at all and proves quasi-optimality for the linear elastic model problem of this paper in the sense that the total error is dominated by the approximation error. The medius analysis extends to non-constant coefficients λ and μ and higher space dimensions, while the more involved precise analysis of

singular functions in the case of nonconvex polygons appears to be limited to the simple linear elastic model problem at hand.

For a polygonal, bounded Lipschitz domain $\Omega \subset \mathbb{R}^2$ with closed Dirichlet boundary Γ_D of positive length and (relatively open) Neumann boundary $\Gamma_N := \partial\Omega \setminus \Gamma_D$ with outer unit normal ν, the strong formulation of the Navier–Lamé equations for volume forces $f \in L^2(\Omega;\mathbb{R}^2)$ and applied tractions $g \in L^2(\Gamma_N;\mathbb{R}^2)$ and homogeneous boundary conditions reads (in compact notation)

$$\begin{aligned} \operatorname{div} \mathbb{C}\varepsilon(u) &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \Gamma_D, \\ \mathbb{C}\varepsilon(u)\nu &= g \quad \text{on } \Gamma_N. \end{aligned}$$

The fourth-order elasticity tensor acts as $\mathbb{C}A := 2\mu A + \lambda\ \operatorname{tr}(A)1_{2\times 2}$ for positive Lamé parameters μ and λ, and for any general input variable $A \in \mathbb{R}^{2\times 2}$, and the linear Green strain is $\varepsilon(u) := (Du + Du^T)/2$.

The conforming first-order FEM of Fig. 1(a) (named after Courant (CFEM)) converges, but suffers from locking in the incompressible limit as $\lambda \to \infty$ (Falk, 2006; Braess, 2007; Brenner & Scott, 2008). This means for large values of λ that the L^2 error of the stresses shows the expected convergence rate for a very large number of degrees of freedom only. To overcome this phenomenon, finite element spaces should have good approximation properties for nearly incompressible materials. One possibility is the choice of a higher polynomial degree of the ansatz space ($\geqslant 4$) or the use of mixed methods. However, the lowest-order conforming mixed method of Arnold & Winther (2002) still has 30 degrees of freedom per triangle. Alternative approaches are the first-order nonconforming methods of Crouzeix and Raviart (Brenner & Sung, 1992) or of Kouhia & Stenberg (1995), which do not show such a locking phenomenon and are therefore of great interest. This paper enfolds a medius error analysis for the nonconforming FEM of Kouhia and Stenberg (KS-NCFEM) of Fig. 1(b) in the sense that mathematical arguments from an *a posteriori* error analysis lead to *a priori* error estimates. The notion of medius analysis was introduced in Gudi (2010) and leads to results which rely on no extra regularity of the weak solution and hold for arbitrarily coarse meshes with certain minimal conditions (a)–(d) of Section 2.3. In this respect, the error analysis of this paper is a refinement of the error analysis in Kouhia & Stenberg (1995). The main result of this analysis is the best-approximation property of the discrete stress $\sigma_{KS} := \mathbb{C}\varepsilon_{NC}(u_{KS})$ (ε_{NC} and D_{NC} are the piecewise analogues of ε and D, respectively) with respect to the exact stress $\sigma := \mathbb{C}\varepsilon(u)$ for the exact and discrete solutions $u \in H^1(\Omega;\mathbb{R}^2)$ and $u_{KS} \in \mathrm{KS}(\mathcal{T})$; i.e.,

$$\|\sigma - \sigma_{KS}\|_{L^2(\Omega)} \lesssim \min_{v_{KS}\in \mathrm{KS}(\mathcal{T})} \|\sigma - \mathbb{C}\varepsilon_{NC}(v_{KS})\|_{L^2(\Omega)} + \operatorname{osc}(f_2,\mathcal{T}) + \operatorname{osc}(g_2,\mathcal{E}(\Gamma_N)).$$

The definitions of the data oscillations $\operatorname{osc}(f_2,\mathcal{T})$ and $\operatorname{osc}(g_2,\mathcal{E}(\Gamma_N))$ and the precise definition of the Kouhia–Stenberg FEM space $\mathrm{KS}(\mathcal{T})$ follow in Section 2. The notation $A \lesssim B$ abbreviates an inequality $A \leqslant CB$ with some mesh-size- and λ-independent generic constant $0 < C < \infty$. The constant may

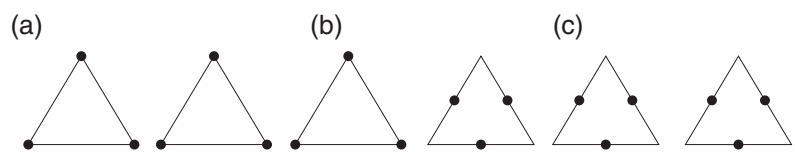

FIG. 1. Three first-order FEMs for linear elasticity: (a) CFEM; (b) KS-NCFEM; (c) CR-NCFEM.

depend on the constant $\alpha > 0$ in conditions (a)–(d) of Section 2.3 and on μ. Since the multiplicative constant (hidden behind $\lesssim$) does not depend on λ, the aforementioned error estimate also holds in the incompressible limit $\lambda \to \infty$. In other words, quasi-optimal convergence follows for the KS-NCFEM in the Stokes problem as well.

The proof relies on a new discrete Helmholtz decomposition (Theorem 3.1), a new discrete-plus-continuous Korn inequality (Theorem 4.1) and the conforming cubic companion of the nonconforming discrete solution from Lemma 3.3. This conforming companion $J_3 v_{\mathrm{CR}}$ fulfils for all Crouzeix–Raviart functions v_{CR} the integral mean properties

$$\int_T (v_{\mathrm{CR}} - J_3 v_{\mathrm{CR}})\,\mathrm{d}x = 0 \quad \text{and} \quad \int_T D_{\mathrm{NC}}(v_{\mathrm{CR}} - J_3 v_{\mathrm{CR}})\,\mathrm{d}x = 0 \quad \text{for all } T \in \mathcal{T},$$

and some stability and approximation properties.

The nonconforming FEM of Crouzeix and Raviart (CR-NCFEM) (Brenner & Sung, 1992) of Fig. 1(c) only allows a discretization of the pure Dirichlet problem $\Gamma_{\mathrm{D}} = \partial\Omega$, in which the (nonphysical) stress $\tilde{\sigma} := \tilde{\mathbb{C}} Du := \mu Du + (\lambda + \mu)\ \mathrm{div}(u) 1_{2\times 2}$ appears with its approximation $\tilde{\sigma}_{\mathrm{CR}} := \tilde{\mathbb{C}} D_{\mathrm{NC}} u_{\mathrm{CR}}$ in the Crouzeix–Raviart FEM. The best-approximation result of this paper reads

$$\|\sigma - \mathbb{C}\varepsilon_{\mathrm{NC}}(u_{\mathrm{CR}})\|_{L^2(\Omega)} \lesssim \|\tilde{\sigma} - \tilde{\sigma}_{\mathrm{CR}}\|_{L^2(\Omega)} \lesssim \|\tilde{\sigma} - \Pi_0 \tilde{\sigma}\|_{L^2(\Omega)} + \mathrm{osc}(f, \mathcal{T}).$$

Recent comparison results (Braess, 2009; Carstensen *et al.*, 2012b) lead to equivalences of approximation classes for the Poisson model problem. The best-approximation results and further analysis of this paper lead to comparison results between the three considered FEMs of Fig. 1 with explicit dependence on the Lamé parameter λ in the equivalence constants. For the conforming discrete solution u_{C} and the discrete stress $\sigma_{\mathrm{C}} := \mathbb{C}\varepsilon(u_{\mathrm{C}})$, the comparison between KS-NCFEM and CFEM reads

$$\begin{aligned} \lambda^{-1}\|\sigma - \sigma_{\mathrm{C}}\|_{L^2(\Omega)} &\lesssim \|\sigma - \sigma_{\mathrm{KS}}\|_{L^2(\Omega)} \\ &\lesssim \|\sigma - \sigma_{\mathrm{C}}\|_{L^2(\Omega)} + \mathrm{osc}(f_2, \mathcal{T}) + \mathrm{osc}(g_2, \mathcal{E}(\Gamma_{\mathrm{N}})). \end{aligned} \tag{1.1}$$

A detailed investigation of the gap in the dependence on λ, which is in fact sharp, is included in Section 6. For the pure Dirichlet problem $\Gamma_{\mathrm{D}} = \partial\Omega$, the solutions of CR-NCFEM and KS-NCFEM (with $\tilde{\sigma}_{\mathrm{KS}} := \tilde{\mathbb{C}} D_{\mathrm{NC}} u_{\mathrm{KS}}$) exist and can be compared by

$$\|\sigma - \sigma_{\mathrm{CR}}\|_{L^2(\Omega)} \lesssim \|\sigma - \sigma_{\mathrm{KS}}\|_{L^2(\Omega)} + \mathrm{osc}(f, \mathcal{T})$$

and

$$\|\tilde{\sigma} - \tilde{\sigma}_{\mathrm{KS}}\|_{L^2(\Omega)} \lesssim \|\tilde{\sigma} - \tilde{\sigma}_{\mathrm{CR}}\|_{L^2(\Omega)} + \mathrm{osc}(f_2, \mathcal{T}).$$

The paper focuses on the two-dimensional case; the generalization to higher dimensions is straightforward for CR-NCFEM and CFEM. The generalization of KS-NCFEM to three dimensions applies two nonconforming and one conforming FEM to the three components or two conforming and one nonconforming; the mathematical justification will be established in the near future (Hu & Schedensack).

Within the scope of low-order methods, despite the equivalence results of this paper, the explicit dependence on the Lamé parameter λ strongly suggests the use of nonconforming discretizations for nearly incompressible materials. If Neumann boundary conditions are present, this advertises the use of KS-NCFEM which, therefore, is apparently far too underrated in the engineering community despite striking numerical examples in Kouhia & Stenberg (1995) and Carstensen & Funken (2001a). It may appear strange to employ some scheme which depends on the choice of the coordinate system, but (in the

presence of Neumann boundary conditions) the KS-NCFEM is the only known robustly quasi-optimal first-order scheme.

The outline of this paper is as follows. Section 2 introduces the precise notation and states the main results, which imply the error estimates of this introduction. Section 3 presents some preliminary results which include the definition of the conforming companion and the new discrete Helmholtz decomposition. Sections 4 and 5 prove the main results including the new discrete-plus-continuous Korn inequality. Section 6 concludes the paper with numerical illustrations and provides striking numerical evidence for the equivalence of the three first-order methods and the claimed dependence on the equivalence constant as $\lambda \to \infty$.

Throughout this paper, standard notation on Lebesgue and Sobolev spaces and their norms is employed and further notation can be found in the following table for convenient reading.

$A \lesssim B$	$A \leqslant CB$ with a mesh-size-independent constant C
$v_k, v(k)$	The kth component of $v \in \mathbb{R}^2$
$A(k,j)$	The component kj of $A \in \mathbb{R}^{2\times 2}$
$A(k)$	The kth row $(A(k,1), A(k,2))$ of $A \in \mathbb{R}^{2\times 2}$
$a \cdot b$	$= \sum_{j=1}^{2} a(j)b(j)$ for $a, b \in \mathbb{R}^2$
$A : B$	$= \sum_{j,k=1,2} A(j,k)B(j,k)$ for $A, B \in \mathbb{R}^{2\times 2}$
$1_{2\times 2}$	Unit matrix in $\mathbb{R}^{2\times 2}$
$\mathbb{S}$	Set of symmetric matrices; $\mathbb{S} := \{A \in \mathbb{R}^{2\times 2} \mid A = A^{\mathrm{T}}\}$
$\varepsilon(u)$	Green strain $(Du + (Du)^{\mathrm{T}})/2$
$\mathbb{C}$	Elasticity tensor; $\mathbb{C}A = 2\mu A + \lambda \operatorname{tr}(A) 1_{2\times 2}$ for $A \in \mathbb{R}^{2\times 2}$
$\tilde{\mathbb{C}}$	Modified elasticity tensor; $\tilde{\mathbb{C}}A = \mu A + (\mu + \lambda) \operatorname{tr}(A) 1_{2\times 2}$ for $A \in \mathbb{R}^{2\times 2}$
$C_{\mathrm{D}}(\Omega)$ (respectively, $C_{\mathrm{N}}(\Omega)$)	Space of continuous functions with homogeneous boundary conditions on Γ_{D} (respectively, Γ_{N})
V	$V := \{v \in H^1(\Omega; \mathbb{R}^2) \mid v|_{\Gamma_{\mathrm{D}}} = 0\}$
Dv, ∇w, $\operatorname{div} v$	Derivative (of a vector-valued function $v \in V$), gradient of a scalar-valued function $w \in H^1(\Omega)$, divergence of v
Curl	$\operatorname{Curl} v = (\partial v/\partial x_2, -\partial v/\partial x_1) \in L^2(\Omega; \mathbb{R}^2)$ for $v \in H^1(\Omega)$, $\operatorname{Curl} w = (\operatorname{Curl} w(1); \operatorname{Curl} w(2)) \in L^2(\Omega; \mathbb{R}^{2\times 2})$ for $w \in V$
$\mathcal{T}, \mathcal{N}, \mathcal{E}$	Shape-regular triangulation with the set of vertices $\mathcal{N}$ and set of edges $\mathcal{E}$; Section 2.2
$\mathcal{N}(\omega)$	Set of vertices in ω, $\mathcal{N}(\omega) := \mathcal{N} \cap \omega$
$\mathcal{E}(\omega)$	$= \{E \in \mathcal{E} \mid E \subseteq \bar{\omega}, E \not\subseteq \partial\omega\}$
$\lvert T\rvert$, $\lvert E\rvert$	Area of a triangle T, length of an edge E
$\operatorname{mid}(E)$	Midpoint of an edge $E \in \mathcal{E}$
$P_k(\mathcal{T}; \mathbb{R}^m)$	Set of piecewise polynomials of degree $\leqslant k$; Section 2.2
Π_0	$\Pi_0 : L^2(\Omega; \mathbb{R}^m) \to P_0(\mathcal{T}; \mathbb{R}^m)$, L^2 projection on piecewise constants; Section 2.2
$\Pi_{\mathcal{E}}$	L^2 projection onto $\mathcal{E}$-piecewise constants; Section 2.2
$h_{\mathcal{T}}$	Piecewise constant mesh size, $h_{\mathcal{T}}|_T := \operatorname{diam}(T)$ for all T
$[v]_E$	Jump along an edge E; Section 2.2
$\operatorname{osc}(f,T)$, $\operatorname{osc}(f,\mathcal{T})$	Oscillations of f, $\operatorname{osc}(f,T) := \lVert h_{\mathcal{T}}(f - \Pi_0 f)\rVert_{L^2(T)}$, $\operatorname{osc}(f,\mathcal{T}) := \lVert h_{\mathcal{T}}(f - \Pi_0 f)\rVert_{L^2(\Omega)}$
$\operatorname{osc}(g, \mathcal{E}(\Gamma_{\mathrm{N}}))$	Edge oscillations; Section 2.2

D_{NC}, ∇_{NC}, $\mathrm{div}_{\mathrm{NC}}$, $\mathrm{Curl}_{\mathrm{NC}}$	Piecewise versions of D, ∇, div, Curl
$V_{\mathrm{C}}(\mathcal{T})$	Conforming finite element space; Section 2.3
$\mathrm{CR}^1_{\mathrm{D}}(\mathcal{T})$	Nonconforming Crouzeix–Raviart space; Section 2.3
$V_{\mathrm{CR}}(\mathcal{T})$	$V_{\mathrm{CR}}(\mathcal{T}) := \mathrm{CR}^1_{\mathrm{D}}(\mathcal{T}) \times \mathrm{CR}^1_{\mathrm{D}}(\mathcal{T})$
$\mathrm{KS}(\mathcal{T})$	Finite element space of KS-NCFEM; $\mathrm{KS}(\mathcal{T}) = (P_1(\mathcal{T}) \cap C_{\mathrm{D}}(\Omega)) \times \mathrm{CR}^1_{\mathrm{D}}(\mathcal{T})$; Section 2.3
$\mathrm{KS}^*(\mathcal{T})$	$\mathrm{KS}^*(\mathcal{T}) = \mathrm{CR}^1_{\mathrm{N}}(\mathcal{T}) \times (P_1(\mathcal{T}) \cap C_{\mathrm{N}}(\Omega))$; Section 3
$I_{\mathrm{NC}} : V \to V_{\mathrm{CR}}(\mathcal{T})$	Nonconforming interpolation operator with $(I_{\mathrm{NC}}v)(\mathrm{mid}(E)) = \fint_E v\,\mathrm{d}s$ for all $E \in \mathcal{E} \setminus \mathcal{E}(\Gamma_{\mathrm{D}})$
$(\bullet,\bullet)_{\mathbb{C}^{-1}}$	$(\sigma,\tau)_{\mathbb{C}^{-1}} := \int_\Omega \sigma : \mathbb{C}^{-1}\tau\,\mathrm{d}x$ for $\sigma, \tau \in L^2(\Omega;\mathbb{S})$

2. Notation and main results

This section defines the linear elastic model problem and all the considered FEMs, and states the main results.

2.1 *Linear elasticity*

Recall that the elastic body occupies the bounded Lipschitz domain Ω with boundary $\partial\Omega = \Gamma_{\mathrm{D}} \cup \Gamma_{\mathrm{N}}$. We assume that Γ_{D} consists of finitely many parts which lie on the outer boundary of Ω (on the unbounded connectivity component of $\mathbb{R}^2 \setminus \Omega$). The weak formulation based on the Green strain seeks $u \in V := \{v \in H^1(\Omega;\mathbb{R}^2) \mid v|_{\Gamma_{\mathrm{D}}} = 0\}$ such that

$$a(v,u) := \int_\Omega \varepsilon(v) : \mathbb{C}\varepsilon(u)\,\mathrm{d}x = \int_\Omega f \cdot v\,\mathrm{d}x + \int_{\Gamma_{\mathrm{N}}} g \cdot v\,\mathrm{d}s \quad \text{for all } v \in V. \tag{2.1}$$

For the pure Dirichlet problem, i.e., $\Gamma_{\mathrm{D}} = \partial\Omega$, an integration by parts and the commutation of the derivatives for $C_0^\infty(\Omega;\mathbb{R}^2)$ functions show that

$$\int_\Omega \varepsilon(\bullet) : \mathbb{C}\varepsilon(\bullet)\,\mathrm{d}x = \int_\Omega D\bullet : \tilde{\mathbb{C}} D\bullet\,\mathrm{d}x$$

on $V \times C_0^\infty(\Omega;\mathbb{R}^2)$. The denseness of $C_0^\infty(\Omega;\mathbb{R}^2)$ in V implies that the two bilinear forms are identical on $V \times V$. Thus, for the pure Dirichlet problem, the equivalent weak formulation based on the full gradient seeks $u \in H^1_0(\Omega;\mathbb{R}^2)$ with

$$\int_\Omega Dv : \tilde{\mathbb{C}} Du\,\mathrm{d}x = \int_\Omega f \cdot v\,\mathrm{d}x \quad \text{for all } v \in H^1_0(\Omega;\mathbb{R}^2). \tag{2.2}$$

Define the energy norm $|||\bullet||| := \sqrt{a(\bullet,\bullet)}$ in V and the scalar product

$$(\sigma,\tau)_{\mathbb{C}^{-1}} := \int_\Omega \sigma : \mathbb{C}^{-1}\tau\,\mathrm{d}x \quad \text{for all } \sigma,\ \tau \in L^2(\Omega;\mathbb{S}).$$

2.2 *Triangulations*

Let $\mathcal{T}$ denote some shape-regular triangulation of a polygonal, bounded Lipschitz domain Ω into triangles, i.e., $\bar{\Omega} = \bigcup \mathcal{T}$ and any two distinct triangles are either disjoint or share exactly one common

edge or one vertex. Let $\mathcal{E}$ denote the set of edges of $\mathcal{T}$ and $\mathcal{N}$ denote the set of vertices. Define for $\omega \subset \mathbb{R}^2$ the sets $\mathcal{N}(\omega) := \mathcal{N} \cap \omega$ and $\mathcal{E}(\omega) := \{E \in \mathcal{E} \mid \mathrm{int}(E) \subset \omega\}$ for the relative interior $\mathrm{int}(E)$ of an edge $E \in \mathcal{E}$. We assume that the boundary edges $\mathcal{E}(\partial\Omega)$ match the boundary conditions in the sense that the boundary conditions change only at nodes $\Gamma_{\mathrm{D}} \cap \bar{\Gamma}_{\mathrm{N}} \subseteq \mathcal{N}$. Let

$$\begin{aligned} P_k(T;\mathbb{R}^m) &:= \{v_k : T \to \mathbb{R}^m \mid \forall j = 1,\dots,m, \text{ the component} \\ &\qquad v_k(j) \text{ of } v_k \text{ is a polynomial of total degree} \leqslant k\}, \\ P_k(\mathcal{T};\mathbb{R}^m) &:= \{v_k : \Omega \to \mathbb{R}^m \mid \forall T \in \mathcal{T}, v_k|_T \in P_k(T;\mathbb{R}^m)\} \end{aligned}$$

denote the set of polynomials and the set of piecewise polynomials, respectively; $\Pi_0 : L^2(\Omega;\mathbb{R}^m) \to P_0(\mathcal{T};\mathbb{R}^m)$ denotes the L^2 projection onto $\mathcal{T}$-piecewise constant functions or vectors, i.e., $(\Pi_0 f)|_T = \fint_T f \,\mathrm{d}x := \int_T f \,\mathrm{d}x/|T|$ for all $T \in \mathcal{T}$ with area $|T|$ and all $f \in L^2(\Omega;\mathbb{R}^m)$. The operator $\Pi_{\mathcal{E}}$ denotes the L^2 projection onto $\mathcal{E}$-piecewise constant functions or vectors, i.e., $\Pi_{\mathcal{E}} g|_E = \fint_E g \,\mathrm{d}s := \int_E g \,\mathrm{d}s/|E|$ for all edges $E \in \mathcal{E}$ of length $|E|$. The volume oscillations read

$$\mathrm{osc}(f,T) := \|h_{\mathcal{T}}(f - \Pi_0 f)\|_{L^2(T)} \quad \text{and} \quad \mathrm{osc}(f,\mathcal{T}) := \|h_{\mathcal{T}}(f - \Pi_0 f)\|_{L^2(\Omega)},$$

while the edge oscillations read

$$\mathrm{osc}(g,\mathcal{E}(\Gamma_{\mathrm{N}})) := \sqrt{\sum_{E \in \mathcal{E}(\Gamma_{\mathrm{N}})} \|h_{\mathcal{T}}^{1/2}(g - \Pi_{\mathcal{E}} g)\|_{L^2(E)}^2}$$

with piecewise constant mesh size $h_{\mathcal{T}} \in P_0(\mathcal{T})$ with $h_{\mathcal{T}}|_T := \mathrm{diam}(T)$ for all $T \in \mathcal{T}$. The jump along an interior edge $E \in \mathcal{E}(\Omega)$ with adjacent triangles T_+ and T_-, i.e., $E = T_+ \cap T_-$, is defined by $[v]_E := v|_{T_+} - v|_{T_-}$. Given the homogeneous Dirichlet boundary conditions, the jump along boundary edges $E \in \mathcal{E}(\Gamma_{\mathrm{D}})$ reads $[v]_E := v|_{T_+}$ for the triangle $T_+ \in \mathcal{T}$ with $E \subset T_+$.

For piecewise affine functions $v_{\mathrm{NC}} \in P_1(\mathcal{T};\mathbb{R}^2)$ the $\mathcal{T}$-piecewise gradient $D_{\mathrm{NC}} v_{\mathrm{NC}}$ with $(D_{\mathrm{NC}} v_{\mathrm{NC}})|_T = D(v_{\mathrm{NC}}|_T)$ for all $T \in \mathcal{T}$ and, accordingly, $\varepsilon_{\mathrm{NC}}(v_{\mathrm{NC}})$ and $\mathrm{div}_{\mathrm{NC}}(v_{\mathrm{NC}})$, exists and $D_{\mathrm{NC}} v_{\mathrm{NC}} \in P_0(\mathcal{T};\mathbb{R}^{2\times 2})$ and $\varepsilon_{\mathrm{NC}}(v_{\mathrm{NC}}) \in P_0(\mathcal{T};\mathbb{S})$ and $\mathrm{div}_{\mathrm{NC}}(v_{\mathrm{NC}}) \in P_0(\mathcal{T})$.

2.3 *Discrete spaces*

CFEM. The Courant finite element space reads

$$V_{\mathrm{C}}(\mathcal{T}) := (P_1(\mathcal{T}) \cap C_{\mathrm{D}}(\Omega)) \times (P_1(\mathcal{T}) \cap C_{\mathrm{D}}(\Omega)).$$

The corresponding (unique) Galerkin approximation $u_{\mathrm{C}} \in V_{\mathrm{C}}(\mathcal{T})$ satisfies

$$\int_\Omega \varepsilon(v_{\mathrm{C}}) : \mathbb{C}\varepsilon(u_{\mathrm{C}}) \,\mathrm{d}x = \int_\Omega f \cdot v_{\mathrm{C}} \,\mathrm{d}x + \int_{\Gamma_{\mathrm{N}}} g \cdot v_{\mathrm{C}} \,\mathrm{d}s \quad \text{for all } v_{\mathrm{C}} \in V_{\mathrm{C}}(\mathcal{T}).$$

CR-NCFEM. Define the P_1 nonconforming space as

$$\mathrm{CR}^1(\mathcal{T}) := \{v_{\mathrm{CR}} \in P_1(\mathcal{T}) \mid v_{\mathrm{CR}} \text{ is continuous at midpoints of interior edges}\}.$$

The nonconforming Crouzeix–Raviart space reads

$$\mathrm{CR}^1_{\mathrm{D}}(\mathcal{T}) := \{v_{\mathrm{CR}} \in \mathrm{CR}^1(\mathcal{T}) \mid v_{\mathrm{CR}} \text{ vanishes at midpoints of edges } E \in \mathcal{E}(\Gamma_{\mathrm{D}})\}.$$

For the discretization of the pure Dirichlet problem $\Gamma_D = \partial\Omega$ of linear elasticity, define the space

$$V_{CR}(\mathcal{T}) := \mathrm{CR}^1_D(\mathcal{T}) \times \mathrm{CR}^1_D(\mathcal{T}).$$

Since the kernel of $\varepsilon_{NC} : V_{CR}(\mathcal{T}) \to P_0(\mathcal{T}; \mathbb{R}^{2\times 2})$ is in general not trivial, the weak formulation based on the full gradient is in use for the discretization and seeks $u_{CR} \in V_{CR}(\mathcal{T})$ with

$$\int_\Omega D_{NC} v_{CR} : \tilde{\mathbb{C}} D_{NC} u_{CR} \, dx = \int_\Omega f \cdot v_{CR} \, dx \quad \text{for all } v_{CR} \in V_{CR}(\mathcal{T}). \tag{2.3}$$

Here, the piecewise gradient D_{NC} replaces the weak differential operator.
KS-NCFEM. The Kouhia–Stenberg approximation $u_{KS} \in \mathrm{KS}(\mathcal{T}) := (P_1(\mathcal{T}) \cap C_D(\Omega)) \times \mathrm{CR}^1_D(\mathcal{T})$ satisfies

$$\int_\Omega \varepsilon_{NC}(v_{KS}) : \mathbb{C}\varepsilon_{NC}(u_{KS}) \, dx = \int_\Omega f \cdot v_{KS} \, dx + \int_{\Gamma_N} g \cdot v_{KS} \, ds \quad \text{for all } v_{KS} \in \mathrm{KS}(\mathcal{T}). \tag{2.4}$$

The following conditions (a)–(d) are given for completeness and replace the assumptions on the sufficiently small mesh size of $\mathcal{T}$ and the assumptions (AD) and (AN) of Kouhia & Stenberg (1995). These conditions are, for example, fulfilled if at least one vertex of each triangle lies in the interior of the domain. The existence of $\alpha > 0$ which fulfils conditions (a)–(d) ensures that the inf–sup condition and a discrete Korn inequality hold.

(a) For all $E \in \mathcal{E}(\Omega)$ with $\mathcal{N}(E) \subseteq \mathcal{N}(\partial\Omega)$, it holds that $|\nu_E(2)| \geqslant \alpha$.

(b) If $\Gamma_N \neq \emptyset$, it holds that $\mathcal{N} \cap \Gamma_N \neq \emptyset$ or there exists at least one $E \in \mathcal{E}(\Gamma_N)$ with $|\nu_E(2)| \geqslant \alpha$.

(c) In the case that the two vertices of an interior edge $E \in \mathcal{E}(\Omega)$ belong to the boundary, i.e., space $\mathcal{N}(E) \subseteq \mathcal{N}(\partial\Omega)$, and $|\nu_E(1)| < \alpha$, consider $z \in \mathcal{N}(E)$. For the nodal patch $\omega_z := \mathrm{int}(\bigcup\{T \in \mathcal{T} \mid z \in T\})$ let $\omega_1, \omega_2 \subseteq \omega_z$ denote the two connected sets, which decompose the nodal patch in the upper and lower part (i.e., $\bar{\omega}_1 \cap \bar{\omega}_2 = E$ and $\omega_1 \cup \omega_2 \cup \mathrm{int}(E) = \omega_z$). Then there exist edges $E_k \in \mathcal{E}(\partial\omega_k) \cap \mathcal{E}(\Gamma_D)$ for $k = 1, 2$ with $|\nu_{E_k}(1)| > \alpha$ (see Fig. 2(c)).

(d) If the entire Dirichlet boundary is nearly horizontal, i.e., for all $E \in \mathcal{E}(\Gamma_D)$ it holds that $|\nu_E(1)| < \alpha$, then there exist two adjacent edges on the Dirichlet boundary, i.e., there exist $E, F \in \mathcal{E}(\Gamma_D)$ with $E \neq F$ and $E \cap F \neq \emptyset$.

The generic multiplicative constants hidden in the notation $\lesssim$ are allowed to depend on α. Figure 2 illustrates conditions (a)–(d).

2.4 *Main results*

The main results below imply the statements of the introduction in Section 1.

THEOREM 2.1 (Best approximation of KS-NCFEM) The exact and the discrete stress $\sigma = \mathbb{C}\varepsilon(u)$ and $\sigma_{KS} = \mathbb{C}\varepsilon_{NC}(u_{KS})$ for the exact and discrete solutions $u \in V$ and $u_{KS} \in \mathrm{KS}(\mathcal{T})$, respectively, satisfy

$$\|\sigma - \sigma_{KS}\|_{L^2(\Omega)} \lesssim \min_{v_{KS} \in \mathrm{KS}(\mathcal{T})} \|\sigma - \mathbb{C}\varepsilon_{NC}(v_{KS})\|_{L^2(\Omega)} + \mathrm{osc}(f_2, \mathcal{T}) + \mathrm{osc}(g_2, \mathcal{E}(\Gamma_N)).$$

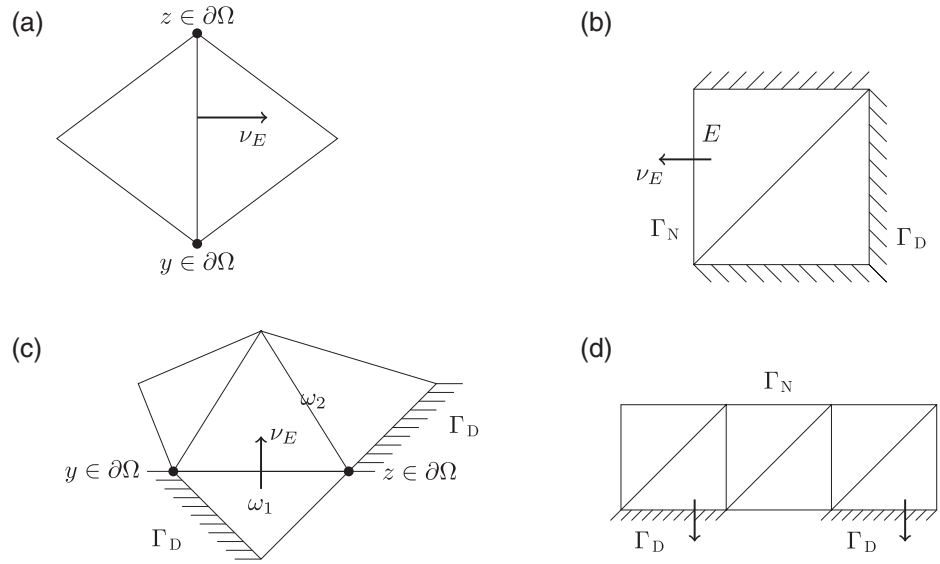

FIG. 2. Illustrations of conditions (a)–(d): (a) a situation excluded by condition (a); (b) triangulation which is excluded by condition (b); (c) a possible patch, which fulfils condition (c); (d) a situation excluded by condition (d).

REMARK 2.2 As one key ingredient for the proof of Theorem 2.1, the error estimate of Theorem 4.4 estimates the stress error of KS-NCFEM by some best-approximation error of the stress and the derivative in the piecewise constant functions plus some data approximation terms.

THEOREM 2.3 (Best approximation of CR-NCFEM) Let $\tilde{\sigma} := \tilde{\mathbb{C}}Du$ for the exact solution $u \in H^1_0(\Omega;\mathbb{R}^2)$. For the pure Dirichlet problem $\Gamma_D = \partial\Omega$, the discrete stress $\tilde{\sigma}_{CR} := \tilde{\mathbb{C}}D_{NC}u_{CR}$ for the discrete solution $u_{CR} \in V_{CR}(\mathcal{T})$ of (2.3) satisfies

$$\|\tilde{\sigma} - \tilde{\sigma}_{CR}\|_{L^2(\Omega)} \lesssim \|\tilde{\sigma} - \Pi_0\tilde{\sigma}\|_{L^2(\Omega)} + \operatorname{osc}(f,\mathcal{T}).$$

THEOREM 2.4 (Comparison of CFEM, CR-NCFEM and KS-NCFEM) The exact stress $\sigma = \mathbb{C}\varepsilon(u)$ and the discrete stresses $\sigma_C = \mathbb{C}\varepsilon(u_C)$ and $\sigma_{KS} = \mathbb{C}\varepsilon_{NC}(u_{KS})$ satisfy

$$\begin{aligned}\lambda^{-1}\|\sigma - \sigma_C\|_{L^2(\Omega)} &\lesssim \|\sigma - \sigma_{KS}\|_{L^2(\Omega)}\\ &\lesssim \|\sigma - \sigma_C\|_{L^2(\Omega)} + \operatorname{osc}(f_2,\mathcal{T}) + \operatorname{osc}(g_2,\mathcal{E}(\Gamma_N)).\end{aligned}$$

For the pure Dirichlet problem $\Gamma_D = \partial\Omega$, the discrete stress $\sigma_{CR} = \mathbb{C}\varepsilon_{NC}(u_{CR})$ satisfies

$$\|\sigma - \sigma_{CR}\|_{L^2(\Omega)} \lesssim \|\sigma - \sigma_{KS}\|_{L^2(\Omega)} + \operatorname{osc}(f,\mathcal{T}).$$

In addition, the stress error of KS-NCFEM is comparable with the error of the nonsymmetric approximation $\tilde{\sigma} := \tilde{\mathbb{C}}Du$ from (2.2) through

$$\|\sigma - \sigma_{KS}\|_{L^2(\Omega)} \lesssim \|\tilde{\sigma} - \tilde{\sigma}_{CR}\|_{L^2(\Omega)} + \operatorname{osc}(f_2,\mathcal{T}).$$

3. Preliminary results

The following discrete Helmholtz decomposition and some properties of a conforming companion are required below; cf. Falk & Morley (1990) for a first decomposition of this type. To this end, define

$$\begin{aligned}\mathrm{CR}^\star(\mathcal{T}) &:= \{v_{\mathrm{CR}} \in \mathrm{CR}(\mathcal{T}) \mid v_{\mathrm{CR}}(\mathrm{mid}(E_1)) = v_{\mathrm{CR}}(\mathrm{mid}(E_2)) \\ &\qquad \text{for } E_1, E_2 \text{ edges of the same connectivity component of } \Gamma_{\mathrm{N}}\},\\ V_{\mathrm{C}}^\star(\mathcal{T}) &:= \{v_{\mathrm{C}} \in P_1(\mathcal{T}) \cap H^1(\Omega) \mid v_{\mathrm{C}} \text{ is constant along} \\ &\qquad \text{each connectivity component of } \Gamma_{\mathrm{N}}\}.\end{aligned}$$

Recall that the boundary conditions match the triangulation $\mathcal{T}$ of the possibly multiply connected planar Lipschitz domain Ω with $\Gamma_{\mathrm{D}} \subseteq \Gamma_0$ for the outer boundary Γ_0 of Ω (Γ_0 is defined as the boundary of the unbounded connectivity component of $\mathbb{R}^2 \setminus \Omega$).

THEOREM 3.1 (Discrete Helmholtz decomposition) Let $\mathrm{KS}^*(\mathcal{T}) := \mathrm{CR}^\star(\mathcal{T}) \times V_{\mathrm{C}}^\star(\mathcal{T})$. Then it holds that

$$P_0(\mathcal{T};\mathbb{S}) = \mathbb{C}\varepsilon_{\mathrm{NC}}(\mathrm{KS}(\mathcal{T})) \oplus (\mathrm{Curl}_{\mathrm{NC}}(\mathrm{KS}^*(\mathcal{T})) \cap P_0(\mathcal{T};\mathbb{S}))$$

and the sum is orthogonal with respect to the scalar product $(\bullet,\bullet)_{\mathbb{C}^{-1}} := \int_\Omega \bullet : \mathbb{C}^{-1} \bullet \, \mathrm{d}x$.

REMARK 3.2 For any $v_{\mathrm{KS}}^* \in \mathrm{KS}^*(\mathcal{T})$ the assertion $\mathrm{Curl}_{\mathrm{NC}} v_{\mathrm{KS}}^* \in P_0(\mathcal{T};\mathbb{S})$ is equivalent to $\mathrm{div}_{\mathrm{NC}} v_{\mathrm{KS}}^* = 0$.

Proof of Theorem 3.1. For $\alpha_{\mathrm{KS}} \in \mathrm{KS}(\mathcal{T})$ and $\beta_{\mathrm{KS}^*} \in \mathrm{KS}^*(\mathcal{T})$ with $\mathrm{Curl}_{\mathrm{NC}}(\beta_{\mathrm{KS}^*}) \in P_0(\mathcal{T};\mathbb{S})$ let $\alpha_{\mathrm{C}} \in P_1(\mathcal{T}) \cap C_{\mathrm{D}}(\Omega)$, $\alpha_{\mathrm{CR}} \in \mathrm{CR}^1_{\mathrm{D}}(\mathcal{T})$, $\beta_{\mathrm{CR}} \in \mathrm{CR}^\star(\mathcal{T})$ and $\beta_{\mathrm{C}} \in V_{\mathrm{C}}^\star(\mathcal{T})$ with $\alpha_{\mathrm{KS}} = (\alpha_{\mathrm{C}}, \alpha_{\mathrm{CR}})$ and $\beta_{\mathrm{KS}^*} = (\beta_{\mathrm{CR}}, \beta_{\mathrm{C}})$. Then α_{KS} and β_{KS^*} satisfy

$$\begin{aligned}(\mathbb{C}\varepsilon(\alpha_{\mathrm{KS}}), \mathrm{Curl}_{\mathrm{NC}}\, \beta_{\mathrm{KS}^*})_{\mathbb{C}^{-1}} &= \int_\Omega \varepsilon_{\mathrm{NC}}(\alpha_{\mathrm{KS}}) : \mathrm{Curl}_{\mathrm{NC}}\, \beta_{\mathrm{KS}^*}\, \mathrm{d}x \\ &= \int_\Omega D_{\mathrm{NC}}(\alpha_{\mathrm{KS}}) : \mathrm{Curl}_{\mathrm{NC}}\, \beta_{\mathrm{KS}^*}\, \mathrm{d}x \\ &= \int_\Omega (\nabla \alpha_{\mathrm{C}} \cdot \mathrm{Curl}_{\mathrm{NC}}\, \beta_{\mathrm{CR}} + \nabla_{\mathrm{NC}} \alpha_{\mathrm{CR}} \cdot \mathrm{Curl}\, \beta_{\mathrm{C}})\, \mathrm{d}x.\end{aligned}$$

This and the L^2 orthogonalities

$$\nabla(P_1(\mathcal{T}) \cap C_{\mathrm{D}}(\Omega)) \perp_{L^2} \mathrm{Curl}_{\mathrm{NC}}(\mathrm{CR}^\star(\mathcal{T})), \quad \nabla_{\mathrm{NC}}\, \mathrm{CR}^1_{\mathrm{D}}(\mathcal{T}) \perp_{L^2} \mathrm{Curl}(V_{\mathrm{C}}^\star(\mathcal{T})) \tag{3.1}$$

imply the orthogonality (with respect to the scalar product $(\bullet,\bullet)_{\mathbb{C}^{-1}}$)

$$\mathbb{C}\varepsilon_{\mathrm{NC}}(\mathrm{KS}(\mathcal{T})) \perp (\mathrm{Curl}_{\mathrm{NC}}(\mathrm{KS}^*(\mathcal{T})) \cap P_0(\mathcal{T};\mathbb{S})).$$

Given $\sigma_h \in P_0(\mathcal{T};\mathbb{S})$, let $\alpha_{\mathrm{KS}} \in \mathrm{KS}(\mathcal{T})$ solve

$$\int_\Omega \varepsilon_{\mathrm{NC}}(v_{\mathrm{KS}}) : \mathbb{C}\varepsilon_{\mathrm{NC}}(\alpha_{\mathrm{KS}})\, \mathrm{d}x = \int_\Omega \varepsilon_{\mathrm{NC}}(v_{\mathrm{KS}}) : \sigma_h \, \mathrm{d}x \quad \text{for all } v_{\mathrm{KS}} \in \mathrm{KS}(\mathcal{T}).$$

The jth row $\tau_h(j) := (\tau_h(j,1), \tau_h(j,2)) \in P_0(\mathcal{T};\mathbb{R}^2)$ of $\tau_h := \sigma_h - \mathbb{C}\varepsilon_{\mathrm{NC}}(\alpha_{\mathrm{KS}}) \in P_0(\mathcal{T};\mathbb{S})$ is piecewise constant for $j = 1, 2$. The discrete Helmholtz decomposition for Crouzeix–Raviart and conforming P_1

functions (Arnold & Falk, 1989) remains true for mixed boundary conditions and interchanged discrete spaces as

$$P_0(\mathcal{T};\mathbb{R}^2) = \nabla_{\mathrm{NC}}\,\mathrm{CR}^1_{\mathrm{D}}(\mathcal{T}) \oplus \mathrm{Curl}\,V^\star_{\mathrm{C}}(\mathcal{T}),$$
$$P_0(\mathcal{T};\mathbb{R}^2) = \nabla(P_1(\mathcal{T}) \cap C_{\mathrm{D}}(\Omega)) \oplus \mathrm{Curl}_{\mathrm{NC}}\,\mathrm{CR}^\star(\mathcal{T}).$$

(This can be proved, for example, by the orthogonalities (3.1) and a dimension argument.) This guarantees the existence of $p_{\mathrm{C}} \in P_1(\mathcal{T}) \cap C_{\mathrm{D}}(\Omega)$, $p_{\mathrm{CR}} \in \mathrm{CR}^\star(\mathcal{T})$, $q_{\mathrm{CR}} \in \mathrm{CR}^1_{\mathrm{D}}(\mathcal{T})$ and $q_{\mathrm{C}} \in V^\star_{\mathrm{C}}(\mathcal{T})$ with

$$\tau_h(1) = \nabla p_{\mathrm{C}} + \mathrm{Curl}_{\mathrm{NC}}\,p_{\mathrm{CR}} \quad \text{and} \quad \tau_h(2) = \nabla_{\mathrm{NC}} q_{\mathrm{CR}} + \mathrm{Curl}\;q_{\mathrm{C}}.$$

(Here, ∇p_{C}, $\mathrm{Curl}_{\mathrm{NC}}\,p_{\mathrm{CR}}$, $\nabla_{\mathrm{NC}} q_{\mathrm{CR}}$ and $\mathrm{Curl}\,q_{\mathrm{C}}$ are understood as row vectors.) Since τ_h is orthogonal to $\mathbb{C}\varepsilon_{\mathrm{NC}}(\mathrm{KS}(\mathcal{T}))$ with respect to $(\bullet,\bullet)_{\mathbb{C}^{-1}}$ and since $\tau_h \in P_0(\mathcal{T};\mathbb{S})$, the functions $v_{\mathrm{C}} \in P_1(\mathcal{T}) \cap C_{\mathrm{D}}(\Omega)$ and $v_{\mathrm{CR}} \in \mathrm{CR}^1_{\mathrm{D}}(\mathcal{T})$ satisfy

$$\int_\Omega (\nabla v_{\mathrm{C}}; \nabla_{\mathrm{NC}} v_{\mathrm{CR}}) : (\tau_h(1); \tau_h(2))\,\mathrm{d}x = \int_\Omega \varepsilon_{\mathrm{NC}}(v_{\mathrm{C}}, v_{\mathrm{CR}}) : \tau_h\,\mathrm{d}x = 0.$$

This implies the L^2 orthogonalities

$$\tau_h(1) \perp_{L^2} \nabla(P_1(\mathcal{T}) \cap C_{\mathrm{D}}(\Omega)) \quad \text{and} \quad \tau_h(2) \perp_{L^2} \nabla_{\mathrm{NC}}\,\mathrm{CR}^1_{\mathrm{D}}(\mathcal{T}).$$

This and the orthogonalities (3.1) lead to

$$\|\nabla p_{\mathrm{C}}\|^2_{L^2(\Omega)} = \int_\Omega \nabla p_{\mathrm{C}} \cdot (\nabla p_{\mathrm{C}} - \tau_h(1))\,\mathrm{d}x = 0.$$

Analogous arguments prove $\|\nabla_{\mathrm{NC}} q_{\mathrm{CR}}\|^2_{L^2(\Omega)} = 0$. Hence,

$$\tau_h = \mathrm{Curl}_{\mathrm{NC}}(p_{\mathrm{CR}}, q_{\mathrm{C}}) \in \mathrm{Curl}_{\mathrm{NC}}(\mathrm{KS}^*(\mathcal{T})).$$

□

LEMMA 3.3 There exists an operator $J_3 : \mathrm{CR}^1_{\mathrm{D}}(\mathcal{T}) \to (P_3(\mathcal{T}) \cap C_{\mathrm{D}}(\Omega))$ with the conservation properties

$$\int_T (v_{\mathrm{CR}} - J_3 v_{\mathrm{CR}})\,\mathrm{d}x = 0 \quad \text{for all } T \in \mathcal{T}, \tag{3.2a}$$
$$\int_E (v_{\mathrm{CR}} - J_3 v_{\mathrm{CR}})\,\mathrm{d}x = 0 \quad \text{for all } E \in \mathcal{E}, \tag{3.2b}$$

and the approximation and stability properties

$$\begin{aligned}\|h_{\mathcal{T}}^{-1}(v_{\mathrm{CR}} - J_3 v_{\mathrm{CR}})\|_{L^2(\Omega)} &\approx \|\nabla_{\mathrm{NC}}(v_{\mathrm{CR}} - J_3 v_{\mathrm{CR}})\|_{L^2(\Omega)} \\ &\approx \min_{\varphi \in H^1(\Omega) \cap C_{\mathrm{D}}(\Omega)} \|\nabla_{\mathrm{NC}}(v_{\mathrm{CR}} - \varphi)\|_{L^2(\Omega)} \\ &\leqslant \|\nabla_{\mathrm{NC}} v_{\mathrm{CR}}\|_{L^2(\Omega)}.\end{aligned} \tag{3.3}$$

REMARK 3.4 The conservation property along edges (3.2b) and an integration by parts reveal the conservation property of the gradients $\Pi_0 \nabla J_3 = \nabla_{\mathrm{NC}}$ in the sense that

$$\int_T \nabla J_3 v_{\mathrm{CR}} \, \mathrm{d}x = \int_T \nabla_{\mathrm{NC}} v_{\mathrm{CR}} \, \mathrm{d}x \quad \text{for all } T \in \mathcal{T} \text{ and all } v_{\mathrm{CR}} \in \mathrm{CR}^1_{\mathrm{D}}(\mathcal{T}).$$

Proof of Lemma 3.3. The design is based on three successive steps.
Step 1. The operator $J_1 : \mathrm{CR}^1_{\mathrm{D}}(\mathcal{T}) \to P_1(\mathcal{T}) \cap C_{\mathrm{D}}(\Omega)$ acts on any function $v_{\mathrm{CR}} \in \mathrm{CR}^1_{\mathrm{D}}(\mathcal{T})$ by averaging the function values at each node $z \in \mathcal{N}(\Omega \cup \Gamma_{\mathrm{N}})$:

$$J_1 v_{\mathrm{CR}}(z) = |\mathcal{T}(z)|^{-1} \sum_{T \in \mathcal{T}(z)} v_{\mathrm{CR}}|_T(z) \quad \text{for all } z \in \mathcal{N}(\Omega \cup \Gamma_{\mathrm{N}}) \tag{3.4}$$

with $\mathcal{T}(z) := \{T \in \mathcal{T} \mid z \in T\}$. (This operator is also known as the enriching operator in the context of fast solvers; see Brenner, 1996.) The arguments of Carstensen *et al.* (2012a, Theorem 5.1) prove the approximation property

$$\begin{aligned} \|h_{\mathcal{T}}^{-1}(v_{\mathrm{CR}} - J_1 v_{\mathrm{CR}})\|_{L^2(\Omega)} &\lesssim \sqrt{\sum_{E \in \mathcal{E}(\Omega \cup \Gamma_{\mathrm{D}})} |E|^{-1} \|[v_{\mathrm{CR}}]_E\|^2_{L^2(E)}} \\ &\lesssim \sqrt{\sum_{E \in \mathcal{E}(\Omega \cup \Gamma_{\mathrm{D}})} |E| \, \|[\nabla_{\mathrm{NC}} v_{\mathrm{CR}} \cdot \tau_E]_E\|^2_{L^2(E)}} \\ &\lesssim \min_{v \in H^1(\Omega) \cap C_{\mathrm{D}}(\Omega)} \|\nabla_{\mathrm{NC}}(v_{\mathrm{CR}} - v)\|_{L^2(\Omega)}. \end{aligned} \tag{3.5}$$

This and an inverse estimate imply the stability property

$$\|\nabla_{\mathrm{NC}}(v_{\mathrm{CR}} - J_1 v_{\mathrm{CR}})\|_{L^2(\Omega)} \lesssim \min_{v \in H^1(\Omega) \cap C_{\mathrm{D}}(\Omega)} \|\nabla_{\mathrm{NC}}(v_{\mathrm{CR}} - v)\|_{L^2(\Omega)}. \tag{3.6}$$

Step 2. Given any edge $E = \mathrm{conv}\{a, b\} \in \mathcal{E}(\Omega \cup \Gamma_{\mathrm{N}})$ with nodal P_1 conforming basis functions $\varphi_a, \varphi_b \in P_1(\mathcal{T}) \cap C(\Omega)$ (defined by $\varphi_a(a) = \varphi_b(b) = 1, \varphi_a(z) = 0$ for $z \in \mathcal{N} \setminus \{a\}$ and $\varphi_b(z) = 0$ for $z \in \mathcal{N} \setminus \{b\}$), the quadratic edge-bubble function

$$\flat_E := 6 \, \varphi_a \varphi_b$$

has the support $\tilde{\omega}_E$ and satisfies $\fint_E \flat_E \, \mathrm{d}s = 1$. For any function $v_{\mathrm{CR}} \in \mathrm{CR}^1_{\mathrm{D}}(\mathcal{T})$ the operator $J_2 : \mathrm{CR}^1_{\mathrm{D}}(\mathcal{T}) \to P_2(\mathcal{T}) \cap C_{\mathrm{D}}(\Omega)$ acts as

$$J_2 v_{\mathrm{CR}} := J_1 v_{\mathrm{CR}} + \sum_{E \in \mathcal{E}(\Omega \cup \Gamma_{\mathrm{N}})} \left(\fint_E (v_{\mathrm{CR}} - J_1 v_{\mathrm{CR}}) \, \mathrm{d}s \right) \flat_E.$$

An immediate consequence of this choice is

$$\fint_E J_2 v_{\mathrm{CR}} \, \mathrm{d}s = \fint_E v_{\mathrm{CR}} \, \mathrm{d}s \quad \text{for all } E \in \mathcal{E}.$$

An integration by parts shows, for the vertex $P_E \in \mathcal{N}(T) \setminus E$ opposite to $E \in \mathcal{E}(T)$ in the triangle T, the trace identity

$$\fint_E (v_{\mathrm{CR}} - J_1 v_{\mathrm{CR}})\,\mathrm{d}s = \fint_T (v_{\mathrm{CR}} - J_1 v_{\mathrm{CR}})\,\mathrm{d}x + \frac{1}{2}\fint_T (x - P_E)\cdot \nabla_{\mathrm{NC}}(v_{\mathrm{CR}} - J_1 v_{\mathrm{CR}})\,\mathrm{d}x.$$

The scaling $\|\flat_E\|_{L^2(\Omega)} \lesssim h_T$ shows

$$\begin{aligned}\left\| h_T^{-1} \sum_{E\in\mathcal{E}(T)} \left(\fint_E (v_{\mathrm{CR}} - J_1 v_{\mathrm{CR}})\,\mathrm{d}s\right)\flat_E \right\|_{L^2(T)} &\lesssim \sum_{E\in\mathcal{E}(T)} \left|\fint_E (v_{\mathrm{CR}} - J_1 v_{\mathrm{CR}})\,\mathrm{d}s\right| \\ &\lesssim h_T^{-1}\|v_{\mathrm{CR}} - J_1 v_{\mathrm{CR}}\|_{L^2(T)} + \|\nabla_{\mathrm{NC}}(v_{\mathrm{CR}} - J_1 v_{\mathrm{CR}})\|_{L^2(T)}.\end{aligned}$$

This together with (3.5) and (3.6) yield

$$\|h_{\mathcal{T}}^{-1}(v_{\mathrm{CR}} - J_2 v_{\mathrm{CR}})\|_{L^2(\Omega)} \lesssim \min_{v\in H^1(\Omega)\cap C_{\mathrm{D}}(\Omega)} \|\nabla_{\mathrm{NC}}(v_{\mathrm{CR}} - v)\|_{L^2(\Omega)}. \tag{3.7}$$

The stability property of J_2 follows with an inverse estimate:

$$\begin{aligned}\|\nabla_{\mathrm{NC}}(v_{\mathrm{CR}} - J_2 v_{\mathrm{CR}})\|_{L^2(\Omega)} &\lesssim \|h_{\mathcal{T}}^{-1}(v_{\mathrm{CR}} - J_2 v_{\mathrm{CR}})\|_{L^2(\Omega)} \\ &\lesssim \min_{v\in H^1(\Omega)\cap C_{\mathrm{D}}(\Omega)} \|\nabla_{\mathrm{NC}}(v_{\mathrm{CR}} - v)\|_{L^2(\Omega)}.\end{aligned} \tag{3.8}$$

Step 3. On any triangle $T = \mathrm{conv}\{a, b, c\}$ with nodal basis functions $\varphi_a, \varphi_b, \varphi_c$, the cubic volume bubble function reads

$$\flat_T := 60\varphi_a\varphi_b\varphi_c \in H_0^1(T)$$

and enjoys the scaling $\|\nabla\flat_T\|_{L^2(\Omega)} \approx 1$. Define

$$J_3 v_{\mathrm{CR}} := J_2 v_{\mathrm{CR}} + \sum_{T\in\mathcal{T}} \left(\fint_T (v_{\mathrm{CR}} - J_2 v_{\mathrm{CR}})\,\mathrm{d}x\right)\flat_T.$$

Then J_3 fulfils the conservation properties (3.2) and

$$\begin{aligned}\left\| \sum_{T\in\mathcal{T}} \left(\fint_T (v_{\mathrm{CR}} - J_2 v_{\mathrm{CR}})\,\mathrm{d}x\right)\nabla\flat_T \right\|_{L^2(\Omega)}^2 &\approx \sum_{T\in\mathcal{T}} \left|\left(\fint_T (v_{\mathrm{CR}} - J_2 v_{\mathrm{CR}})\,\mathrm{d}x\right)\right|^2 \\ &\lesssim \|h_{\mathcal{T}}^{-1}(v_{\mathrm{CR}} - J_2 v_{\mathrm{CR}})\|_{L^2(\Omega)}^2.\end{aligned}$$

This together with (3.7) and (3.8) imply

$$\begin{aligned}\|D_{\mathrm{NC}}(v_{\mathrm{CR}} - J_3 v_{\mathrm{CR}})\|_{L^2(\Omega)} &\leqslant \|D_{\mathrm{NC}}(v_{\mathrm{CR}} - J_2 v_{\mathrm{CR}})\|_{L^2(\Omega)} + \left\| \sum_{T\in\mathcal{T}} \left(\fint_T (v_{\mathrm{CR}} - J_2 v_{\mathrm{CR}})\,\mathrm{d}x\right)\nabla\flat_T \right\|_{L^2(\Omega)} \\ &\lesssim \min_{v\in H^1(\Omega)\cap C_{\mathrm{D}}(\Omega)} \|\nabla_{\mathrm{NC}}(v_{\mathrm{CR}} - v)\|_{L^2(\Omega)}.\end{aligned}$$

This and a Poincaré inequality lead to (3.3). □

LEMMA 3.5 Any $v_{\mathrm{KS}} \in \mathrm{KS}(\mathcal{T})$ satisfies

$$\int_\Omega (\Pi_0\sigma - \sigma_{\mathrm{KS}}) : \varepsilon_{\mathrm{NC}}(v_{\mathrm{KS}})\,\mathrm{d}x \lesssim (\|(\sigma - \Pi_0\sigma)(2)\|_{L^2(\Omega)} + \mathrm{osc}(f_2, \mathcal{T}) + \mathrm{osc}(g_2, \mathcal{E}(\Gamma_{\mathrm{N}})))\|\nabla_{\mathrm{NC}} v_{\mathrm{KS}}(2)\|_{L^2(\Omega)}.$$

Proof. Given any $v_{\mathrm{KS}} \in \mathrm{KS}(\mathcal{T})$, let $v_{\mathrm{C}} \in P_1(\mathcal{T}) \cap C_{\mathrm{D}}(\Omega)$ and $v_{\mathrm{CR}} \in \mathrm{CR}^1_{\mathrm{D}}(\mathcal{T})$ be the components of v_{KS}, i.e., $v_{\mathrm{KS}} = (v_{\mathrm{C}}, v_{\mathrm{CR}})$. Lemma 3.3 guarantees the existence of $J_3 v_{\mathrm{CR}} \in P_3(\mathcal{T}) \cap H^1(\Omega) \cap C_{\mathrm{D}}(\Omega)$ with

$$\int_T (v_{\mathrm{CR}} - J_3 v_{\mathrm{CR}})\,\mathrm{d}x = 0 = \int_T \nabla_{\mathrm{NC}}(v_{\mathrm{CR}} - J_3 v_{\mathrm{CR}})\,\mathrm{d}x \quad \text{for all } T \in \mathcal{T}$$
$$\text{and} \quad \|\nabla_{\mathrm{NC}}(v_{\mathrm{CR}} - J_3 v_{\mathrm{CR}})\|_{L^2(\Omega)} \lesssim \|\nabla_{\mathrm{NC}} v_{\mathrm{CR}}\|_{L^2(\Omega)}. \tag{3.9}$$

Since $\Pi_0\sigma$ is piecewise constant, the integral mean property (3.9) implies

$$\int_\Omega \Pi_0\sigma : \varepsilon_{\mathrm{NC}}(0, v_{\mathrm{CR}})\,\mathrm{d}x = \int_\Omega \Pi_0\sigma : \varepsilon_{\mathrm{NC}}(0, J_3 v_{\mathrm{CR}})\,\mathrm{d}x = \int_\Omega (\Pi_0\sigma - \sigma) : D(0, J_3 v_{\mathrm{CR}})\,\mathrm{d}x + \int_\Omega \sigma : \varepsilon(0, J_3 v_{\mathrm{CR}})\,\mathrm{d}x.$$

Since σ is the stress of the exact solution and $J_3 v_{\mathrm{CR}} \in H^1(\Omega) \cap C_{\mathrm{D}}(\Omega)$, the Cauchy–Schwarz inequality implies

$$\int_\Omega (\Pi_0\sigma - \sigma) : D(0, J_3 v_{\mathrm{CR}})\,\mathrm{d}x + \int_\Omega \sigma : \varepsilon(0, J_3 v_{\mathrm{CR}})\,\mathrm{d}x \leqslant \|(\sigma - \Pi_0\sigma)(2)\|_{L^2(\Omega)}\|\nabla J_3 v_{\mathrm{CR}}\|_{L^2(\Omega)} + \int_\Omega f_2 J_3 v_{\mathrm{CR}}\,\mathrm{d}x + \int_{\Gamma_{\mathrm{N}}} g_2 J_3 v_{\mathrm{CR}}\,\mathrm{d}s.$$

The triangle inequality and the stability property (3.9) show

$$\|\nabla J_3 v_{\mathrm{CR}}\|_{L^2(\Omega)} \leqslant \|\nabla J_3 v_{\mathrm{CR}} - \nabla_{\mathrm{NC}} v_{\mathrm{CR}}\|_{L^2(\Omega)} + \|\nabla_{\mathrm{NC}} v_{\mathrm{CR}}\|_{L^2(\Omega)} \lesssim \|\nabla_{\mathrm{NC}} v_{\mathrm{CR}}\|_{L^2(\Omega)}.$$

The combination of the above inequalities yields

$$\int_\Omega \Pi_0\sigma : \varepsilon_{\mathrm{NC}}(0, v_{\mathrm{CR}})\,\mathrm{d}x \lesssim \|(\sigma - \Pi_0\sigma)(2)\|_{L^2(\Omega)}\|\nabla_{\mathrm{NC}} v_{\mathrm{CR}}\|_{L^2(\Omega)} + \int_\Omega f_2 J_3 v_{\mathrm{CR}}\,\mathrm{d}x + \int_{\Gamma_{\mathrm{N}}} g_2 J_3 v_{\mathrm{CR}}\,\mathrm{d}s.$$

Since σ and σ_{KS} are the stresses of the exact and the discrete solution, respectively, it follows that

$$\int_\Omega (\Pi_0\sigma - \sigma_{\mathrm{KS}}) : \varepsilon_{\mathrm{NC}}(v_{\mathrm{KS}})\,\mathrm{d}x = \int_\Omega (\Pi_0\sigma - \sigma_{\mathrm{KS}}) : \varepsilon_{\mathrm{NC}}(0, v_{\mathrm{CR}})\,\mathrm{d}x = \int_\Omega \Pi_0\sigma : \varepsilon_{\mathrm{NC}}(0, v_{\mathrm{CR}})\,\mathrm{d}x - \int_\Omega f_2 v_{\mathrm{CR}}\,\mathrm{d}x - \int_{\Gamma_{\mathrm{N}}} g_2 v_{\mathrm{CR}}\,\mathrm{d}s.$$

The combination of the previous displayed formulas proves

$$\int_\Omega (\Pi_0\sigma - \sigma_{\mathrm{KS}}) : \varepsilon_{\mathrm{NC}}(v_{\mathrm{KS}})\,\mathrm{d}x \lesssim \|(\sigma - \Pi_0\sigma)(2)\|_{L^2(\Omega)} \|\nabla_{\mathrm{NC}} v_{\mathrm{CR}}\|_{L^2(\Omega)} + \int_\Omega f_2(J_3 v_{\mathrm{CR}} - v_{\mathrm{CR}})\,\mathrm{d}x + \int_{\Gamma_{\mathrm{N}}} g_2(J_3 v_{\mathrm{CR}} - v_{\mathrm{CR}})\,\mathrm{d}s. \tag{3.10}$$

Since the integral mean of $J_3 v_{\mathrm{CR}} - v_{\mathrm{CR}}$ vanishes on triangles, the trace inequality (Brenner & Scott, 2008, p.282) followed by a Poincaré inequality yields, for $E \in \mathcal{E}(\Gamma_{\mathrm{N}})$ and $T \in \mathcal{T}$ with $E \in \mathcal{E}(T)$,

$$\begin{aligned}\|h_{\mathcal{T}}^{-1/2}(J_3 v_{\mathrm{CR}} - v_{\mathrm{CR}})\|_{L^2(E)} &\lesssim \|h_{\mathcal{T}}^{-1}(J_3 v_{\mathrm{CR}} - v_{\mathrm{CR}})\|_{L^2(T)} + \|\nabla_{\mathrm{NC}}(J_3 v_{\mathrm{CR}} - v_{\mathrm{CR}})\|_{L^2(T)} \\ &\leqslant \|\nabla_{\mathrm{NC}}(J_3 v_{\mathrm{CR}} - v_{\mathrm{CR}})\|_{L^2(T)}.\end{aligned}$$

Since the integral mean of $J_3 v_{\mathrm{CR}} - v_{\mathrm{CR}}$ vanishes on edges, this leads to

$$\int_E g_2\,(J_3 v_{\mathrm{CR}} - v_{\mathrm{CR}})\,\mathrm{d}s \lesssim \|h_{\mathcal{T}}^{1/2}(g_2 - \Pi_{\mathcal{E}} g_2)\|_{L^2(E)} \|\nabla_{\mathrm{NC}}(J_3 v_{\mathrm{CR}} - v_{\mathrm{CR}})\|_{L^2(T)}.$$

Since the integral mean of $J_3 v_{\mathrm{CR}} - v_{\mathrm{CR}}$ vanishes on triangles, (3.10) implies

$$\begin{aligned}&\int_\Omega (\Pi_0\sigma - \sigma_{\mathrm{KS}}) : \varepsilon_{\mathrm{NC}}(v_{\mathrm{KS}})\,\mathrm{d}x \\ &\quad\lesssim (\|(\sigma - \Pi_0\sigma)(2)\|_{L^2(\Omega)} + \operatorname{osc}(f_2, \mathcal{T}) + \operatorname{osc}(g_2, \mathcal{E}(\Gamma_{\mathrm{N}})))\|\nabla_{\mathrm{NC}} v_{\mathrm{CR}}\|_{L^2(\Omega)}.\end{aligned}$$

This concludes the proof. □

The nonconforming interpolation operator $I_{\mathrm{NC}} : V \to V_{\mathrm{CR}}(\mathcal{T})$ is defined by $(I_{\mathrm{NC}} v)(\mathrm{mid}(E)) = \fint_E v\,\mathrm{d}s$ for all $E \in \mathcal{E} \setminus \mathcal{E}(\Gamma_{\mathrm{D}})$ and fulfils the integral mean property $D_{\mathrm{NC}} I_{\mathrm{NC}} = \Pi_0 D$ in the sense that

$$D_{\mathrm{NC}} I_{\mathrm{NC}} v|_T = \fint_T Dv\,\mathrm{d}x \quad \text{for all } T \in \mathcal{T} \text{ and all } v \in V. \tag{3.11}$$

LEMMA 3.6 Any $\beta_{\mathrm{KS}^*} \in \mathrm{KS}^*(\mathcal{T})$ with $\mathrm{Curl}_{\mathrm{NC}}\, \beta_{\mathrm{KS}^*} \in P_0(\mathcal{T}; \mathbb{S})$ satisfies

$$\int_\Omega \varepsilon_{\mathrm{NC}}(I_{\mathrm{NC}} u - u_{\mathrm{KS}}) : \mathrm{Curl}_{\mathrm{NC}}\, \beta_{\mathrm{KS}^*}\, \mathrm{d}x \lesssim \|(Du - \Pi_0 Du)(1)\|_{L^2(\Omega)} \|\, \mathrm{Curl}_{\mathrm{NC}}\, \beta_{\mathrm{KS}^*}(1)\|_{L^2(\Omega)}.$$

Proof. According to the definition, $\beta_{\mathrm{KS}^*}(1) \in \mathrm{CR}^\star(\mathcal{T})$ and $\beta_{\mathrm{KS}^*}(2) \in V_{\mathrm{C}}^\star(\mathcal{T})$. The orthogonalities (3.1) and $\mathrm{Curl}_{\mathrm{NC}}\, \beta_{\mathrm{KS}^*} \in \mathbb{S}$ show, for any $\phi_{\mathrm{C}} \in P_1(\mathcal{T}) \cap C_{\mathrm{D}}(\Omega)$, that

$$\int_\Omega \varepsilon_{\mathrm{NC}}(I_{\mathrm{NC}} u - u_{\mathrm{KS}}) : \mathrm{Curl}_{\mathrm{NC}}\, \beta_{\mathrm{KS}^*}\, \mathrm{d}x = \int_\Omega (\nabla_{\mathrm{NC}} I_{\mathrm{NC}} u(1) - \nabla \phi_{\mathrm{C}}) \cdot \mathrm{Curl}_{\mathrm{NC}}\, \beta_{\mathrm{KS}^*}(1)\, \mathrm{d}x.$$

Since $\phi_{\mathrm{C}} \in P_1(\mathcal{T}) \cap C_{\mathrm{D}}(\Omega)$ is arbitrary, this implies

$$\begin{aligned} &\int_\Omega \varepsilon_{\mathrm{NC}}(I_{\mathrm{NC}} u - u_{\mathrm{KS}}) : \mathrm{Curl}_{\mathrm{NC}}\, \beta_{\mathrm{KS}^*}\, \mathrm{d}x \\ &\quad \leqslant \min_{\phi_{\mathrm{C}} \in P_1(\mathcal{T}) \cap C_{\mathrm{D}}(\Omega)} \|\nabla_{\mathrm{NC}} I_{\mathrm{NC}} u(1) - \nabla \phi_{\mathrm{C}}\|_{L^2(\Omega)} \|\, \mathrm{Curl}_{\mathrm{NC}}\, \beta_{\mathrm{KS}^*}(1)\|_{L^2(\Omega)}. \end{aligned} \tag{3.12}$$

The integral mean property (3.11) of I_{NC} and Carstensen *et al.* (2012a, Theorem 5.1) show

$$\begin{aligned} \min_{\phi_{\mathrm{C}} \in P_1(\mathcal{T}) \cap C_{\mathrm{D}}(\Omega)} \|\nabla_{\mathrm{NC}} I_{\mathrm{NC}} u(1) - \nabla \phi_{\mathrm{C}}\|_{L^2(\Omega)} &\leqslant \|\nabla u(1) - \nabla_{\mathrm{NC}} I_{\mathrm{NC}} u(1)\|_{L^2(\Omega)} \\ &= \|\nabla u(1) - \Pi_0 \nabla u(1)\|_{L^2(\Omega)}. \end{aligned} \tag{3.13}$$

The combination of (3.12) and (3.13) concludes the proof. □

4. Proof of Theorem 2.1

The main step in the proof of Theorem 2.1 is the error estimate

$$\|\sigma - \sigma_{\mathrm{KS}}\|_{L^2(\Omega)} \lesssim \|\sigma - \Pi_0 \sigma\|_{L^2(\Omega)} + \|(Du - \Pi_0 Du)(1)\|_{L^2(\Omega)} + \mathrm{osc}(f_2, \mathcal{T}) + \mathrm{osc}(g_2, \mathcal{E}(\Gamma_{\mathrm{N}}))$$

from Theorem 4.4. The discrete-plus-continuous Korn inequality from Theorem 4.1 allows control of the nonsymmetric term $\|Du - \Pi_0 Du\|_{L^2(\Omega)}$ in terms of the symmetric stress error $\|\sigma - \mathbb{C}\varepsilon_{\mathrm{NC}}(v_{\mathrm{KS}})\|_{L^2(\Omega)}$. This proves Theorem 2.1.

The remaining parts of this section prove first Theorem 4.1 and then Theorem 4.4.

Theorem 4.1 generalizes the discrete Korn inequality from Kouhia & Stenberg (1995) in that the underlying function space is $V + \mathrm{KS}(\mathcal{T})$ and not just $\mathrm{KS}(\mathcal{T})$. Then Carstensen & Funken (2001b, Remark 4.1.v) gives the general warning that the Korn inequality in the form of Theorem 4.1 is only stated but not proved completely in Bao & Barrett (1998).

THEOREM 4.1 (Discrete-plus-continuous Korn inequality) For a triangulation $\mathcal{T}$ which fulfils conditions (c),(d) of Section 2.3, any $v_{\mathrm{NC}} \in V + \mathrm{KS}(\mathcal{T})$ satisfies

$$\|D_{\mathrm{NC}} v_{\mathrm{NC}}\|_{L^2(\Omega)} \lesssim \|\varepsilon_{\mathrm{NC}}(v_{\mathrm{NC}})\|_{L^2(\Omega)}.$$

REMARK 4.2 The discrete-plus-continuous Korn inequality could be proved for slightly weaker conditions than conditions (c),(d) from Section 2.3 as in the situation of Fig. 2(d). In those situations, the proof of Theorem 4.1 considers some larger neighbourhoods of the patches. In the situation of Fig. 2(d), it is not guaranteed that those patches do not become arbitrarily large under some refinement strategies

and so the constant from the discrete Korn inequality is not uniformly bounded. For the ease of this presentation and the sake of clarity, the slightly stronger versions (c),(d) are assumed.

The proof of Theorem 4.1 considers a set of vertices $\mathcal{Z} \subseteq \mathcal{N}$ defined by $z \in \mathcal{Z}$ if and only if at least one of the following conditions (i)–(iii) is fulfilled with $\alpha > 0$ from conditions (c), (d) of Section 2.3:

(i) The node $z \in \mathcal{N}(\Omega)$ is an interior node.

(ii) The node $z \in \mathcal{N}(\partial\Omega)$ is a boundary node with $|v_E(1)| > \alpha$ for all $E \in \mathcal{E}(\omega_z)$ for the nodal patch $\omega_z := \operatorname{int}(\bigcup\{T \in \mathcal{T} \mid z \in T\})$, and if $|\{E \in \mathcal{E}(\Gamma_D \cap \bar{\omega}_z) \mid |v_E(1)| < \alpha\}| = 1$, then $|\{E \in \mathcal{E}(\Gamma_D \cap \bar{\omega}_z) \mid |v_E(1)| > \alpha\}| > 0$.

(iii) The node $z \in \mathcal{N}(\partial\Omega)$ is a boundary node and there exists an edge $E \in \mathcal{E}(\omega_z)$ with $\mathcal{N}(E) \subseteq \mathcal{N}(\partial\Omega)$ and $|v_E(1)| < \alpha$, which decomposes the patch ω_z in the two domains ω_1, ω_2 (i.e., ω_1, ω_2 connected with $\bar{\omega}_1 \cap \bar{\omega}_2 = E$ and $\omega_1 \cup \omega_2 \cup \operatorname{int}(E) = \omega_z$). For each of the two domains ω_1 and ω_2 there exist $E_1 \in \mathcal{E}(\partial\omega_1) \cap \mathcal{E}(\Gamma_D)$ and $E_2 \in \mathcal{E}(\partial\omega_2) \cap \mathcal{E}(\Gamma_D)$ on the Dirichlet boundary with $|v_{E_1}(1)| > \alpha$ and $|v_{E_2}(1)| > \alpha$, as depicted in Fig. 2(c).

Recall that the generic multiplicative constants hidden in the notation $\lesssim$ may depend on α.

The set $\mathcal{Z}$ contains all interior nodes and some nodes on the boundary, for which some local discrete Korn inequality holds on the nodal patches. The proof of Theorem 4.1 uses the fact that, under conditions (c), (d) of Section 2.3, the set $\mathcal{Z}$ is large enough to prove the theorem even if that set is empty and the mesh is very coarse (without any interior nodes). The first step of this proof is the subsequent lemma.

LEMMA 4.3 (Characterization of rigid body motions) Let $\mathcal{T}$ be a triangulation which fulfils conditions (c),(d) of Section 2.3 and define $\mathcal{Z}$ as above. Then any $v_{KS} \in \mathrm{KS}(\mathcal{T})$ with $\varepsilon_{NC}(v_{KS}|_{\omega_z}) = 0$ on the nodal patch ω_z for $z \in \mathcal{Z}$ is continuous on ω_z. For $E \in \mathcal{E}(\Omega)$ with $|v_E(1)| \geqslant \alpha$, any $v_{KS} \in \mathrm{KS}(\mathcal{T})$ with $\varepsilon_{NC}(v_{KS}|_{\omega_E}) = 0$ on the edge patch $\omega_E := \operatorname{int}(\bigcup\{T \in \mathcal{T} \mid E \in \mathcal{E}(T)\})$ is continuous on ω_E.

Proof. The critical situation concerns horizontal edges as depicted in Fig. 3(a). For interior nodes the rigid body motions are fixed through two midpoints of those horizontal edges (see Fig. 3(c)). For nodes on the boundary, condition (iii) guarantees that the rigid body motions are fixed by the boundary conditions. In the case of the edge patches, such critical situations are excluded. □

Proof of Theorem 4.1. Define $\tilde{\Gamma}_D := \bigcup\{E \in \mathcal{E}(\Gamma_D) \mid |v_E(1)| > \alpha \text{ or } \exists F \in \mathcal{E}(\Gamma_D) \text{ with } F \neq E \text{ and } F \cap E \neq \emptyset\}$. The point of departure is the discrete Korn inequality for piecewise H^1 functions (Brenner, 2004, Equation (1.19)),

$$\|D_{NC} v_{NC}\|_{L^2(\Omega)} \lesssim \|\varepsilon_{NC}(v_{NC})\|_{L^2(\Omega)} + \|v_{NC}\|_{L^2(\tilde{\Gamma}_D)} + \sqrt{\sum_{E \in \mathcal{E}(\Omega)} |E|^{-1} \|[v_{NC}]_E\|^2_{L^2(E)}}.$$

For any vertex $z \in \mathcal{Z}$ set $\mathcal{T}(z) := \{T \in \mathcal{T} \mid z \in T\}$, the set of all triangles with vertex z, and define $\mathcal{E}_z := \{E \in \mathcal{E}(\Omega \cup \tilde{\Gamma}_D) \mid z \in E \text{ and if } E \in \mathcal{E}(\tilde{\Gamma}_D) \text{ and } |v_E(1)| < \alpha, \text{ then } |\mathcal{E}(\tilde{\Gamma}_D)| > 1\}$ and let $\omega_z := \operatorname{int}(\bigcup \mathcal{T}(z))$ be the nodal patch. On $\mathrm{KS}(\mathcal{T}(z)) := \{v_{KS} \in P_1(\mathcal{T}(z), \mathbb{R}^2) \mid \exists w_{KS} \in \mathrm{KS}(\mathcal{T}) \text{ s.t. } v_{KS} = w_{KS}|_{\omega_z}\}$ the maps

$$\begin{aligned} \rho_1(v_{KS}) &:= \sqrt{\sum_{E \in \mathcal{E}_z} |E|^{-1} \|[v_{KS}]_E\|^2_{L^2(E)}} \\ \text{and} \quad \rho_2(v_{KS}) &:= \inf_{v \in V(\omega_z)} \|\varepsilon_{NC}(v_{KS} - v)\|_{L^2(\omega_z)} \end{aligned} \tag{4.1}$$

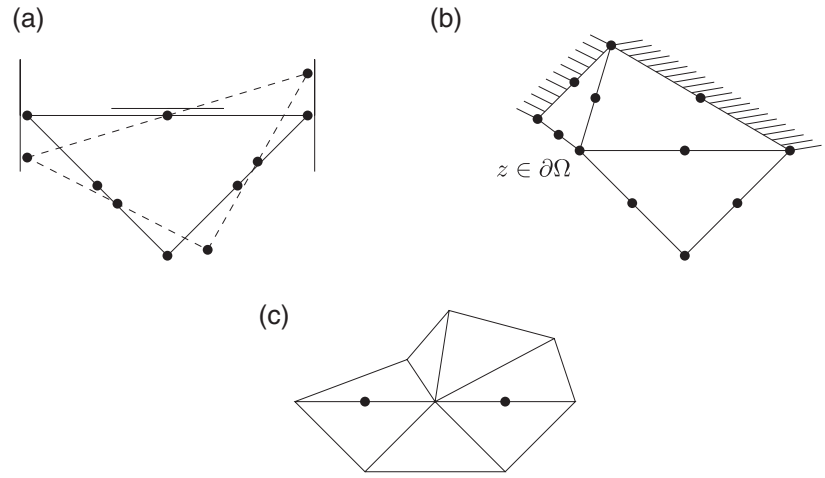

FIG. 3. Illustration of critical situations for Lemma 4.3: (a) an excluded infinitesimal rigid body motion; (b) situation of Fig. 3(a) embedded in a triangulation, excluded by condition (c); (c) interior patch.

define two seminorms, where

$$V(\omega_z) := \{w \in L^2(\omega_z;\mathbb{R}^2) \mid \exists v \in V \text{ with } w = v|_{\omega_z}\}.$$

The triangle inequality implies that infimizing sequences $v_n \in V(\omega_z)$ in (4.1) are bounded in $H^1(\omega_z;\mathbb{R}^2)$. Since $V(\omega_z)$ is a closed subspace of the reflexive space $H^1(\omega_z;\mathbb{R}^2)$, there exist a subsequence v_{n_k} and a function $v_\infty \in V(\omega_z)$ with $v_{n_k} \rightharpoonup v_\infty$. This and the weak lower semicontinuity of the norm $\|\varepsilon(\bullet)\|_{L^2(\omega_z)}$ on $V(\omega_z)$ imply that the infimum is in fact a minimum.

If $\rho_2(v_{\mathrm{KS}}) = 0$ for $v_{\mathrm{KS}} \in \mathrm{KS}(\mathcal{T}(z))$, then there exists some $v \in V(\omega_z)$ with $\varepsilon_{\mathrm{NC}}(v_{\mathrm{KS}}) = \varepsilon(v)$. Therefore, $w_{\mathrm{KS}} := v - v_{\mathrm{KS}} \in P_1(\mathcal{T}(z);\mathbb{R}^2)$ is a piecewise rigid body motion. This implies

$$v \in P_1(\mathcal{T}(z);\mathbb{R}^2) \cap C(\omega_z;\mathbb{R}^2) \subset \mathrm{KS}(\mathcal{T}(z))$$

and therefore $w_{\mathrm{KS}} \in \mathrm{KS}(\mathcal{T}(z))$. Lemma 4.3 implies that $w_{\mathrm{KS}} \in C(\omega_z;\mathbb{R}^2)$ is continuous. Hence, $v_{\mathrm{KS}} = v - w_{\mathrm{KS}} \in C(\omega_z;\mathbb{R}^2)$ and $v_{\mathrm{KS}}|_E \equiv 0$ for $E \in \mathcal{E}_z \cup \tilde{\Gamma}_{\mathrm{D}}$ and therefore $\rho_1(v_{\mathrm{KS}}) = 0$. Since ρ_1 and ρ_2 are seminorms on the finite-dimensional space $\mathrm{KS}(\mathcal{T}(z))$, there exists a constant $C(\mathcal{T}(z))$, such that $\rho_1 \leqslant C(\mathcal{T}(z))\rho_2$. A scaling argument shows that the constant $C(\mathcal{T}(z))$ is independent of the mesh size and depends on the minimal angle in $\mathcal{T}(z)$ and on $\alpha > 0$ from conditions (c),(d) of Section 2.3 only.

For $E \in \mathcal{E}(\Omega \cup \tilde{\Gamma}_{\mathrm{D}})$ with $|\nu_E(1)| \geqslant \alpha$, a similar argument shows the inequality $\rho_1 \lesssim \rho_2$ for the two seminorms (of $v_{\mathrm{KS}} \in \mathrm{KS}(\mathcal{T}(z))$)

$$\rho_1(v_{\mathrm{KS}}) := |E|^{-1/2}\|[v_{\mathrm{KS}}]_E\|_{L^2(E)}$$

and

$$\rho_2(v_{\mathrm{KS}}) := \inf_{v \in V(\omega_E)} \|\varepsilon_{\mathrm{NC}}(v_{\mathrm{KS}} - v)\|_{L^2(\omega_E)}.$$

Note that for all $E \in \mathcal{E}(\Omega \cup \tilde{\Gamma}_{\mathrm{D}})$ with $|\nu_E(1)| < \alpha$, conditions (c),(d) guarantee the existence of a node $z \in \mathcal{Z}$ with $E \in \mathcal{E}_z$. Since the length of edges $E \in \mathcal{E}(\Gamma_{\mathrm{D}})$ on the Dirichlet boundary is bounded

($|E| \lesssim 1$), the sum over all vertices $z \in \mathcal{Z}$ and the bounded overlap of the patches show

$$\begin{aligned}
&\|v_{\mathrm{KS}}\|_{L^2(\tilde{\Gamma}_{\mathrm{D}})} + \sqrt{\sum_{E\in\mathcal{E}(\Omega)} |E|^{-1}\|[v_{\mathrm{KS}}]_E\|^2_{L^2(E)}} \\
&\quad\lesssim \sqrt{\sum_{E\in\mathcal{E}(\Omega\cup\tilde{\Gamma}_{\mathrm{D}})} |E|^{-1}\|[v_{\mathrm{KS}}]_E\|^2_{L^2(E)}} \\
&\quad\lesssim \sqrt{\sum_{\substack{E\in\mathcal{E}(\Omega\cup\tilde{\Gamma}_{\mathrm{D}})\\ |v_E(1)|\geqslant\alpha}} |E|^{-1}\|[v_{\mathrm{KS}}]_E\|^2_{L^2(E)} + \sum_{z\in\mathcal{Z}}\sum_{E\in\mathcal{E}_z} |E|^{-1}\|[v_{\mathrm{KS}}]_E\|^2_{L^2(E)}} \\
&\quad\lesssim \inf_{v\in V} \|\varepsilon_{\mathrm{NC}}(v_{\mathrm{KS}} - v)\|_{L^2(\Omega)}. \qquad (4.2)
\end{aligned}$$

For $v_{\mathrm{NC}} \in V + \mathrm{KS}(\mathcal{T})$ and $v \in V$ and $v_{\mathrm{KS}} \in \mathrm{KS}(\mathcal{T})$ with $v_{\mathrm{NC}} = v + v_{\mathrm{KS}}$, it holds that $[v_{\mathrm{NC}}]_E = [v_{\mathrm{KS}}]_E$ and $v_{\mathrm{NC}}|_{\Gamma_{\mathrm{D}}} = v_{\mathrm{KS}}|_{\Gamma_{\mathrm{D}}}$. Inequality (4.2) implies

$$\begin{aligned}
\|v_{\mathrm{NC}}\|_{L^2(\tilde{\Gamma}_{\mathrm{D}})} + \sqrt{\sum_{E\in\mathcal{E}(\Omega)} |E|^{-1}\|[v_{\mathrm{NC}}]_E\|^2_{L^2(E)}} &\lesssim \inf_{w\in V} \|\varepsilon_{\mathrm{NC}}(v_{\mathrm{KS}} - w)\|_{L^2(\Omega)} \\
&\leqslant \|\varepsilon_{\mathrm{NC}}(v_{\mathrm{NC}})\|_{L^2(\Omega)}.
\end{aligned}$$

□

The remaining part of this section proves Theorem 4.4.

THEOREM 4.4 It holds that

$$\begin{aligned}
\|\sigma - \sigma_{\mathrm{KS}}\|_{L^2(\Omega)} \lesssim \|\sigma - \Pi_0\sigma\|_{L^2(\Omega)} + \|(Du - \Pi_0 Du)(1)\|_{L^2(\Omega)} \\
+ \operatorname{osc}(f_2, \mathcal{T}) + \operatorname{osc}(g_2, \mathcal{E}(\Gamma_{\mathrm{N}})).
\end{aligned}$$

REMARK 4.5 It remains an open question whether one can neglect the term $\|(Du - \Pi_0 Du)(1)\|_{L^2(\Omega)}$ in the upper bound in Theorem 4.4; it is not clear how to control this term by the stress error.

The inf–sup condition from Theorem 4.6 plays an important role for the independence from λ in the proof of Theorem 4.4.

THEOREM 4.6 (Inf–sup condition, Kouhia & Stenberg, 1995) Let $\mathcal{T}$ satisfy conditions (a),(b) from Section 2.3. Then it holds that

$$\|p_0\|_{L^2(\Omega)} \lesssim \sup_{v_{\mathrm{KS}}\in\mathrm{KS}(\mathcal{T})\setminus\{0\}} \frac{\int_\Omega p_0 \operatorname{div}_{\mathrm{NC}} v_{\mathrm{KS}}\,\mathrm{d}x}{\|D_{\mathrm{NC}} v_{\mathrm{KS}}\|_{L^2(\Omega)}} \qquad (4.3)$$

for all $p_0 \in P_0(\mathcal{T})$ if $\Gamma_{\mathrm{N}} \neq \emptyset$ and for all $p_0 \in P_0(\mathcal{T})$ with $\int_\Omega p_0\,\mathrm{d}x = 0$ if $\Gamma_{\mathrm{N}} = \emptyset$.

Proof. The first paper (Kouhia & Stenberg, 1995) on this nonconforming FEM aims at an asymptotic result for sufficiently fine mesh sizes and therefore reasonably ignores the possibly pathological cases on coarse meshes. Following the arguments of Kouhia & Stenberg (1995, pp. 208–210), one can verify that condition (a) is stronger than Kouhia & Stenberg (1995, condition (AD), p. 198) but avoids the

modification of the domain necessary in Kouhia & Stenberg (1995, Step 4 of the proof). In fact, the proof in Kouhia & Stenberg (1995) reduces the discrete stability to that on the continuous level but changing the mesh results in changing the domain. One possible criticism is that the change of the continuous inf–sup constant with respect to the change of the domain is neglected without a detailed discussion in Kouhia & Stenberg (1995). Conditions (a),(b) of Section 2.3 are sufficient to argue on the original domain in a way analogous to Kouhia & Stenberg (1995, pp. 208–210). Since there are no additional ideas in the proof, further details of this technicality are omitted. □

Proof of Theorem 4.4. The triangle inequality implies that it suffices to consider the difference $\|\Pi_0\sigma - \sigma_{\mathrm{KS}}\|_{L^2(\Omega)}$. The L^2-orthogonal decomposition in the isochoric and deviatoric part reads

$$\|\Pi_0\sigma - \sigma_{\mathrm{KS}}\|^2_{L^2(\Omega)} = \|\operatorname{dev}(\Pi_0\sigma - \sigma_{\mathrm{KS}})\|^2_{L^2(\Omega)} + \tfrac{1}{4}\|\operatorname{tr}(\Pi_0\sigma - \sigma_{\mathrm{KS}})1_{2\times 2}\|^2_{L^2(\Omega)}.$$

For $\Gamma_{\mathrm{N}} = \emptyset$ the homogeneous boundary conditions of u and u_{KS} allow an integration by parts for the second term. The continuity condition $\int_E [u_{\mathrm{KS}}]_E \,\mathrm{d}s = 0$ for all $E \in \mathcal{E}(\Omega)$ leads to

$$\int_\Omega \operatorname{tr}(\Pi_0\sigma)\,\mathrm{d}x = 0 = \int_\Omega \operatorname{tr}(\sigma_{\mathrm{KS}})\,\mathrm{d}x, \tag{4.4}$$

i.e., $\operatorname{tr}(\Pi_0\sigma - \sigma_{\mathrm{KS}}) \in P_0(\mathcal{T})/\mathbb{R}$. Theorem 4.6 guarantees, for $\Gamma_{\mathrm{N}} = \emptyset$ and $\Gamma_{\mathrm{N}} \neq \emptyset$, the existence of $v_{\mathrm{KS}} \in \mathrm{KS}(\mathcal{T})$ with $\|D_{\mathrm{NC}} v_{\mathrm{KS}}\|_{L^2(\Omega)} = 1$ and

$$\begin{aligned}\|\operatorname{tr}(\Pi_0\sigma - \sigma_{\mathrm{KS}})\|_{L^2(\Omega)} &\lesssim \int_\Omega \operatorname{tr}(\Pi_0\sigma - \sigma_{\mathrm{KS}})\ \operatorname{div}_{\mathrm{NC}} v_{\mathrm{KS}}\,\mathrm{d}x \\ &= \int_\Omega (\Pi_0\sigma - \sigma_{\mathrm{KS}}) : D_{\mathrm{NC}} v_{\mathrm{KS}}\,\mathrm{d}x - \int_\Omega \operatorname{dev}(\Pi_0\sigma - \sigma_{\mathrm{KS}}) : D_{\mathrm{NC}} v_{\mathrm{KS}}\,\mathrm{d}x.\end{aligned}$$

The application of Lemma 3.5 to the first term of the right-hand side yields

$$\begin{aligned}\|\operatorname{tr}(\Pi_0\sigma - \sigma_{\mathrm{KS}})\|_{L^2(\Omega)} \lesssim{}& \|(\sigma - \Pi_0\sigma)(2)\|_{L^2(\Omega)} + \operatorname{osc}(f_2, \mathcal{T}) + \operatorname{osc}(g_2, \mathcal{E}(\Gamma_{\mathrm{N}})) \\ &+ \|\operatorname{dev}(\Pi_0\sigma - \sigma_{\mathrm{KS}})\|_{L^2(\Omega)}.\end{aligned}$$

The analysis of $\|\operatorname{dev}(\Pi_0\sigma - \sigma_{\mathrm{KS}})\|_{L^2(\Omega)}$ remains. Algebraic manipulation shows $\operatorname{dev}\mathbb{C}A : \operatorname{dev}\mathbb{C}A \lesssim A : \mathbb{C}A$ for all $A \in \mathbb{R}^{2\times 2}$. Applied to the above situation this reads

$$\|\operatorname{dev}(\Pi_0\sigma - \sigma_{\mathrm{KS}})\|^2_{L^2(\Omega)} \lesssim \int_\Omega (\Pi_0\sigma - \sigma_{\mathrm{KS}}) : \varepsilon_{\mathrm{NC}}(I_{\mathrm{NC}}u - u_{\mathrm{KS}})\,\mathrm{d}x. \tag{4.5}$$

The point is that $\mathbb{C}\operatorname{dev} A$ does not depend on λ. Theorem 3.1 guarantees the existence of $\alpha_{\mathrm{KS}} \in \mathrm{KS}(\mathcal{T})$ and $\beta_{\mathrm{KS}^*} \in \mathrm{KS}^*(\mathcal{T})$ with the property that $\operatorname{Curl}_{\mathrm{NC}} \beta_{\mathrm{KS}^*} \in P_0(\mathcal{T}; \mathbb{S})$ and $\Pi_0\sigma - \sigma_{\mathrm{KS}} = \mathbb{C}\varepsilon_{\mathrm{NC}}(\alpha_{\mathrm{KS}}) +$

$\operatorname{Curl}_{\mathrm{NC}} \beta_{\mathrm{KS}^*}$. Lemmas 3.5 and 3.6 yield

$$\begin{aligned}
&\int_\Omega (\Pi_0\sigma - \sigma_{\mathrm{KS}}) : \varepsilon_{\mathrm{NC}}(I_{\mathrm{NC}}u - u_{\mathrm{KS}})\,\mathrm{d}x \\
&\quad = \int_\Omega \varepsilon_{\mathrm{NC}}(\alpha_{\mathrm{KS}}) : (\Pi_0\sigma - \sigma_{\mathrm{KS}})\,\mathrm{d}x + \int_\Omega \operatorname{Curl}_{\mathrm{NC}} \beta_{\mathrm{KS}^*} : \varepsilon_{\mathrm{NC}}(I_{\mathrm{NC}}u - u_{\mathrm{KS}})\,\mathrm{d}x \\
&\quad \lesssim (\|(\Pi_0\sigma - \sigma)(2)\|_{L^2(\Omega)} + \operatorname{osc}(f_2, \mathcal{T}) + \operatorname{osc}(g_2, \mathcal{E}(\Gamma_{\mathrm{N}})))\|\nabla_{\mathrm{NC}}\alpha_{\mathrm{KS}}(2)\|_{L^2(\Omega)} \\
&\qquad + \|(Du - \Pi_0 Du)(1)\|_{L^2(\Omega)}\|\operatorname{Curl}_{\mathrm{NC}} \beta_{\mathrm{KS}^*}(1)\|_{L^2(\Omega)}.
\end{aligned} \tag{4.6}$$

A similar argument as for the decomposition of $\Pi_0\sigma - \sigma_{\mathrm{KS}}$ in the isochoric and the deviatoric part at the beginning of the proof bounds the term $\|\operatorname{Curl}_{\mathrm{NC}} \beta_{\mathrm{KS}^*}(1)\|^2_{L^2(\Omega)}$ by $(\operatorname{Curl}_{\mathrm{NC}} \beta_{\mathrm{KS}^*}, \mathbb{C}^{-1} \operatorname{Curl}_{\mathrm{NC}} \beta_{\mathrm{KS}^*})_{\mathbb{C}^{-1}}$. For this purpose $\operatorname{Curl}_{\mathrm{NC}} \beta_{\mathrm{KS}^*}$ is L^2-orthogonally decomposed in the isochoric and the deviatoric part, i.e.,

$$\|\operatorname{Curl}_{\mathrm{NC}} \beta_{\mathrm{KS}^*}\|^2_{L^2(\Omega)} = \|\operatorname{dev}(\operatorname{Curl}_{\mathrm{NC}} \beta_{\mathrm{KS}^*})\|^2_{L^2(\Omega)} + \tfrac{1}{4}\|\operatorname{tr}(\operatorname{Curl}_{\mathrm{NC}} \beta_{\mathrm{KS}^*})1_{2\times 2}\|^2_{L^2(\Omega)}. \tag{4.7}$$

For $\Gamma_{\mathrm{N}} = \emptyset$ the function α_{KS} satisfies

$$\begin{aligned}
\int_\Omega \operatorname{tr}(\mathbb{C}\varepsilon_{\mathrm{NC}}(\alpha_{\mathrm{KS}}))\,\mathrm{d}x &= (2\mu + 2\lambda)\int_\Omega \operatorname{div}_{\mathrm{NC}}(\alpha_{\mathrm{KS}})\,\mathrm{d}x \\
&= (2\mu + 2\lambda)\sum_{E\in\mathcal{E}}\int_E [\alpha_{\mathrm{KS}}]_E\,\nu_E\,\mathrm{d}s = 0.
\end{aligned}$$

With (4.4), it follows that $\int_\Omega \operatorname{tr}(\operatorname{Curl}_{\mathrm{NC}} \beta_{\mathrm{KS}^*})\,\mathrm{d}x = 0$. The inf–sup condition for Kouhia–Stenberg functions, Theorem 4.6, guarantees, for $\Gamma_{\mathrm{N}} = \emptyset$ and $\Gamma_{\mathrm{N}} \neq \emptyset$, the existence of $v_{\mathrm{KS}} \in \mathrm{KS}(\mathcal{T})$ with $\|D_{\mathrm{NC}} v_{\mathrm{KS}}\|_{L^2(\Omega)} = 1$ and

$$\|\operatorname{tr}(\operatorname{Curl}_{\mathrm{NC}} \beta_{\mathrm{KS}^*})\|_{L^2(\Omega)} \lesssim \int_\Omega \operatorname{tr}(\operatorname{Curl}_{\mathrm{NC}} \beta_{\mathrm{KS}^*})\operatorname{div}_{\mathrm{NC}} v_{\mathrm{KS}}\,\mathrm{d}x.$$

For $\beta_{\mathrm{KS}^*} = (\beta_{\mathrm{CR}}, \beta_{\mathrm{C}})$ with $\beta_{\mathrm{CR}} \in \mathrm{CR}^1_{\mathrm{N}}(\mathcal{T})$ and $\beta_{\mathrm{C}} \in P_1(\mathcal{T}) \cap C_{\mathrm{N}}(\Omega)$ and $v_{\mathrm{KS}} = (v_{\mathrm{C}}, v_{\mathrm{CR}})$ with $v_{\mathrm{C}} \in P_1(\mathcal{T}) \cap C_{\mathrm{D}}(\Omega)$ and $v_{\mathrm{CR}} \in \mathrm{CR}^1_{\mathrm{D}}(\mathcal{T})$, it follows that

$$\begin{aligned}
\|\operatorname{tr}(\operatorname{Curl}_{\mathrm{NC}} \beta_{\mathrm{KS}^*})\|_{L^2(\Omega)} &\lesssim \int_\Omega (\operatorname{Curl}_{\mathrm{NC}} \beta_{\mathrm{KS}^*} - \operatorname{dev}\operatorname{Curl}_{\mathrm{NC}} \beta_{\mathrm{KS}^*}) : D_{\mathrm{NC}} v_{\mathrm{KS}}\,\mathrm{d}x \\
&\leqslant \|\operatorname{dev}\operatorname{Curl}_{\mathrm{NC}} \beta\|_{L^2(\Omega)}\|D_{\mathrm{NC}} v_{\mathrm{KS}}\|_{L^2(\Omega)} \\
&\quad + \int_\Omega \operatorname{Curl}_{\mathrm{NC}} \beta_{\mathrm{CR}} \cdot \nabla v_{\mathrm{C}}\,\mathrm{d}x + \int_\Omega \operatorname{Curl} \beta_{\mathrm{C}} \cdot \nabla_{\mathrm{NC}} v_{\mathrm{CR}}\,\mathrm{d}x.
\end{aligned} \tag{4.8}$$

Since $\nabla v_{\mathrm{C}}\tau_E$ vanishes on Γ_{D}, an integration by parts leads to

$$\int_\Omega \operatorname{Curl}_{\mathrm{NC}} \beta_{\mathrm{CR}} \cdot \nabla v_{\mathrm{C}}\,\mathrm{d}x = \sum_{E\in\mathcal{E}(\Omega)}\int_E [\beta_{\mathrm{CR}}]_E\,\mathrm{d}s \nabla v_{\mathrm{C}} \cdot \tau_E + \sum_{E\in\mathcal{E}(\Gamma_{\mathrm{N}})}\int_E \beta_{\mathrm{CR}}\,\mathrm{d}s \nabla v_{\mathrm{C}} \cdot \tau_E = 0.$$

Since $\nabla\beta_{\mathrm{C}} \cdot \tau_E$ vanishes on $E \in \mathcal{E}(\Gamma_{\mathrm{N}})$,

$$\int_\Omega \operatorname{Curl} \beta_{\mathrm{C}} \cdot \nabla_{\mathrm{NC}} v_{\mathrm{CR}} \, \mathrm{d}x = \sum_{E\in\mathcal{E}(\Gamma_{\mathrm{D}})} \int_E v_{\mathrm{CR}} \, \mathrm{d}s(\nabla\beta_{\mathrm{C}} \cdot \tau_E) + \sum_{E\in\mathcal{E}(\Omega)} \int_E [v_{\mathrm{CR}}]_E \, \mathrm{d}s(\nabla\beta_{\mathrm{C}} \cdot \tau_E) = 0.$$

Together with (4.7) and (4.8), it follows that

$$\| \operatorname{tr}(\operatorname{Curl}_{\mathrm{NC}} \beta_{\mathrm{KS}^*})\|_{L^2(\Omega)} + \| \operatorname{Curl}_{\mathrm{NC}} \beta_{\mathrm{KS}^*}\|_{L^2(\Omega)} \lesssim \| \operatorname{dev} \operatorname{Curl}_{\mathrm{NC}} \beta_{\mathrm{KS}^*}\|_{L^2(\Omega)}.$$

Since $\operatorname{dev} \mathbb{C}A : \operatorname{dev} \mathbb{C}A \lesssim A : \mathbb{C}A$ for all $A \in \mathbb{R}^{2\times 2}$, it follows, as above, that

$$\| \operatorname{Curl}_{\mathrm{NC}} \beta_{\mathrm{KS}^*}\|^2_{L^2(\Omega)} \lesssim (\operatorname{Curl}_{\mathrm{NC}} \beta_{\mathrm{KS}^*}, \operatorname{Curl}_{\mathrm{NC}} \beta_{\mathrm{KS}^*})_{\mathbb{C}^{-1}}.$$

Theorem 4.1 implies

$$\|\nabla_{\mathrm{NC}}\alpha_{\mathrm{KS}}(2)\|^2_{L^2(\Omega)} \lesssim \|\varepsilon_{\mathrm{NC}}(\alpha_{\mathrm{KS}})\|^2_{L^2(\Omega)} \lesssim (\mathbb{C}\varepsilon_{\mathrm{NC}}(\alpha_{\mathrm{KS}}), \mathbb{C}\varepsilon_{\mathrm{NC}}(\alpha_{\mathrm{KS}}))_{C^{-1}}.$$

The orthogonality of the decomposition $\Pi_0\sigma - \sigma_{\mathrm{KS}} = \mathbb{C}\varepsilon_{\mathrm{NC}}(\alpha_{\mathrm{KS}}) + \operatorname{Curl}_{\mathrm{NC}} \beta_{\mathrm{KS}^*}$ with respect to $(\bullet, \bullet)_{\mathbb{C}^{-1}}$, together with the above estimate, implies

$$\|\nabla_{\mathrm{NC}}\alpha_{\mathrm{KS}}(2)\|_{L^2(\Omega)} + \| \operatorname{Curl}_{\mathrm{NC}} \beta_{\mathrm{KS}^*}\|_{L^2(\Omega)} \lesssim \left(\int_\Omega (\Pi_0\sigma - \sigma_{\mathrm{KS}}) : \varepsilon_{\mathrm{NC}}(I_{\mathrm{NC}}u - u_{\mathrm{KS}}) \, \mathrm{d}x \right)^{1/2}.$$

Inequality (4.6) proves

$$\begin{aligned}\left(\int_\Omega (\Pi_0\sigma - \sigma_{\mathrm{KS}}) : \varepsilon_{\mathrm{NC}}(I_{\mathrm{NC}}u - u_{\mathrm{KS}}) \, \mathrm{d}x \right)^{1/2} &\lesssim \|(\Pi_0\sigma - \sigma)(2)\|_{L^2(\Omega)} + \operatorname{osc}(f_2, \mathcal{T}) + \operatorname{osc}(g_2, \mathcal{E}(\Gamma_{\mathrm{N}})) \\ &\quad + \|(Du - \Pi_0 Du)(1)\|_{L^2(\Omega)}.\end{aligned}$$

This and (4.5) conclude the proof of Theorem 4.4. □

5. Proof of Theorems 2.3 and 2.4

The first part of this section proves Theorem 2.3, while the second proves Theorem 2.4.

The proof of Theorem 2.3 is based on the following lemma. It corresponds to Lemma 3.5 for Crouzeix–Raviart functions.

LEMMA 5.1 Any $v_{\mathrm{CR}} \in V_{\mathrm{CR}}(\mathcal{T})$ satisfies

$$\int_\Omega (\tilde\sigma - \tilde\sigma_{\mathrm{CR}}) : D_{\mathrm{NC}} v_{\mathrm{CR}} \, \mathrm{d}x \lesssim (\|\tilde\sigma - \Pi_0\tilde\sigma\|_{L^2(\Omega)} + \operatorname{osc}(f, \mathcal{T}))\|D_{\mathrm{NC}} v_{\mathrm{CR}}\|_{L^2(\Omega)}.$$

Proof. Lemma 3.3 implies, with a piecewise Poincaré inequality for J_3 (applied componentwise),

$$\begin{aligned}\int_\Omega (\tilde\sigma - \tilde\sigma_{\mathrm{CR}}) : D_{\mathrm{NC}} v_{\mathrm{CR}}\,\mathrm{d}x &= \int_\Omega (\tilde{\mathbb{C}}(D_{\mathrm{NC}} I_{\mathrm{NC}} u) - \tilde\sigma_{\mathrm{CR}}) : D_{\mathrm{NC}} v_{\mathrm{CR}}\,\mathrm{d}x \\ &= \int_\Omega f \cdot (J_3 v_{\mathrm{CR}} - v_{\mathrm{CR}})\,\mathrm{d}x + \int_\Omega (\tilde{\mathbb{C}}(D_{\mathrm{NC}} I_{\mathrm{NC}} u) - \tilde\sigma) : DJ_3 v_{\mathrm{CR}}\,\mathrm{d}x \\ &\leqslant \|h_{\mathcal{T}}(f - \Pi_0 f)\|_{L^2(\Omega)} \|(J_3 v_{\mathrm{CR}} - v_{\mathrm{CR}})/h_{\mathcal{T}}\|_{L^2(\Omega)} \\ &\quad + \|\Pi_0\tilde\sigma - \tilde\sigma\|_{L^2(\Omega)} \|DJ_3 v_{\mathrm{CR}}\|_{L^2(\Omega)} \\ &\lesssim \mathrm{osc}(f, \mathcal{T}) \|D_{\mathrm{NC}} v_{\mathrm{CR}}\|_{L^2(\Omega)} + \|\Pi_0\tilde\sigma - \tilde\sigma\|_{L^2(\Omega)} \|D_{\mathrm{NC}} v_{\mathrm{CR}}\|_{L^2(\Omega)}.\end{aligned} \tag{5.1}$$

□

Proof of Theorem 2.3. The point of departure is an inequality of Carstensen & Rabus (2012, Lemma 3.8),

$$\|\tilde\sigma - \tilde\sigma_{\mathrm{CR}}\|^2_{L^2(\Omega)} \lesssim \int_\Omega (\tilde\sigma - \tilde\sigma_{\mathrm{CR}}) : (Du - D_{\mathrm{NC}} u_{\mathrm{CR}})\,\mathrm{d}x + \|h_{\mathcal{T}} f\|^2_{L^2(\Omega)}. \tag{5.2}$$

Define the bubble function $b_T := (\varphi_T, \varphi_T) \in P_3(\mathcal{T}; \mathbb{R}^2)$ with φ_T as in the proof of Lemma 3.5. The property $\int_T b_T\,\mathrm{d}x \approx |T|$ implies

$$\begin{aligned}\|h_{\mathcal{T}} f\|_{L^2(T)} &\leqslant \mathrm{osc}(f, T) + \|h_{\mathcal{T}} \Pi_0 f\|_{L^(T)} \\ &\approx \mathrm{osc}(f, T) + \left| \int_T b_T \cdot \Pi_0 f\,\mathrm{d}x \right|.\end{aligned}$$

The scaling $\|b_T\|_{L^2(T)} \approx h_{\mathcal{T}}|_T$ and an integration by parts show

$$\begin{aligned}\left| \int_T b_T \cdot \Pi_0 f\,\mathrm{d}x \right| &\leqslant \|b_T\|_{L^2(T)} \|f - \Pi_0 f\|_{L^2(T)} + \left| \int_T b_T \cdot f\,\mathrm{d}x \right| \\ &\lesssim \mathrm{osc}(f, T) + \left| \int_T Db_T : (\sigma - \Pi_0\sigma)\,\mathrm{d}x \right|.\end{aligned}$$

Since $\|Db_T\|_{L^2(T)} \approx 1$ and $(A + A^{\mathrm{T}}) : (A + A^{\mathrm{T}}) \leqslant 4A : A$, it follows that

$$\begin{aligned}\left| \int_T Db_T : (\sigma - \Pi_0\sigma)\,\mathrm{d}x \right| &\lesssim \|\sigma - \Pi_0\sigma\|_{L^2(T)} \\ &\leqslant \|\tilde\sigma - \Pi_0\tilde\sigma\|_{L^2(T)}.\end{aligned}$$

Altogether,

$$\|h_{\mathcal{T}} f\|_{L^2(T)} \leqslant \mathrm{osc}(f, T) + \|\tilde\sigma - \Pi_0\tilde\sigma\|_{L^2(T)}.$$

This, (5.2) and Lemma 5.1 imply

$$\begin{aligned}\|\tilde{\sigma} - \tilde{\sigma}_{\mathrm{CR}}\|_{L^2(\Omega)}^2 &\lesssim \int_\Omega (\tilde{\sigma} - \Pi_0\tilde{\sigma}) : (Du - \Pi_0 Du)\,\mathrm{d}x + \int_\Omega (\tilde{\sigma} - \tilde{\sigma}_{\mathrm{CR}}) : D_{\mathrm{NC}}(I_{\mathrm{NC}}u - u_{\mathrm{CR}})\,\mathrm{d}x \\ &\quad + \mathrm{osc}^2(f,\mathcal{T}) + \|\tilde{\sigma} - \Pi_0\tilde{\sigma}\|_{L^2(\Omega)}^2 \\ &\lesssim \|\tilde{\sigma} - \Pi_0\tilde{\sigma}\|_{L^2(\Omega)}^2 + \mathrm{osc}^2(f,\mathcal{T}) \\ &\quad + (\|\tilde{\sigma} - \Pi_0\tilde{\sigma}\|_{L^2(\Omega)} + \mathrm{osc}(f,\mathcal{T}))\|D_{\mathrm{NC}}(I_{\mathrm{NC}}u - u_{\mathrm{CR}})\|_{L^2(\Omega)},\end{aligned}$$

where the last inequality follows from $\|Du - \Pi_0 Du\|_{L^2(\Omega)} \leqslant \|\tilde{\sigma} - \Pi_0\tilde{\sigma}\|_{L^2(\Omega)}$. The Young inequality $2\,a\,b \leqslant \alpha\,a^2 + \alpha^{-1}\,b^2$ for $\alpha > 0$ implies

$$\begin{aligned}&(\|\tilde{\sigma} - \Pi_0\tilde{\sigma}\|_{L^2(\Omega)} + \mathrm{osc}(f,\mathcal{T}))\|D_{\mathrm{NC}}(I_{\mathrm{NC}}u - u_{\mathrm{CR}})\|_{L^2(\Omega)} \\ &\quad \leqslant 1/(4\alpha)(\|\tilde{\sigma} - \Pi_0\tilde{\sigma}\|_{L^2(\Omega)} + \mathrm{osc}(f,\mathcal{T}))^2 + \alpha\|D_{\mathrm{NC}}(I_{\mathrm{NC}}u - u_{\mathrm{CR}})\|_{L^2(\Omega)}^2.\end{aligned}$$

For sufficiently small α the last term is absorbed. It follows that

$$\|\tilde{\sigma} - \tilde{\sigma}_{\mathrm{CR}}\|_{L^2(\Omega)} \lesssim \|\tilde{\sigma} - \Pi_0\tilde{\sigma}\|_{L^2(\Omega)} + \mathrm{osc}(f,\mathcal{T}).$$

□

The remaining parts of this section are devoted to the proof of Theorem 2.4, which is based on the following proposition.

PROPOSITION 5.2 For $u_{\mathrm{KS}} \in \mathrm{KS}(\mathcal{T})$ and $1 \lesssim \lambda$ it holds that

$$\min_{v_{\mathrm{C}} \in V_{\mathrm{C}}(\mathcal{T})} |||u_{\mathrm{KS}} - v_{\mathrm{C}}|||_{\mathrm{NC}} \lesssim \lambda^{1/2} \min_{v \in V} |||u_{\mathrm{KS}} - v|||_{\mathrm{NC}}.$$

Proof. The arguments of Carstensen *et al.* (2012a, Theorem 5.1) prove the crucial point, namely

$$\min_{v_{\mathrm{C}} \in V_{\mathrm{C}}(\mathcal{T})} \|D_{\mathrm{NC}}(v_{\mathrm{KS}} - v_{\mathrm{C}})\|_{L^2(\Omega)} \approx \min_{v \in V} \|D_{\mathrm{NC}}(v_{\mathrm{KS}} - v)\|_{L^2(\Omega)}.$$

(This is proved for scalar functions and the pure Dirichlet problem in Carstensen *et al.*, 2012a, but the local arguments in the proof are still valid for the weaker boundary conditions and for two components.) The estimate

$$|||v_{\mathrm{KS}}|||_{\mathrm{NC}} \lesssim \lambda^{1/2}\|\varepsilon_{\mathrm{NC}}(v_{\mathrm{KS}})\|_{L^2(\Omega)} \leqslant \lambda^{1/2}\|D_{\mathrm{NC}}v_{\mathrm{KS}}\|_{L^2(\Omega)}$$

and

$$\|D_{\mathrm{NC}}(v_{\mathrm{KS}} - v)\|_{L^2(\Omega)} \leqslant |||v_{\mathrm{KS}} - v|||_{\mathrm{NC}}$$

conclude the proof of the proposition. □

Proof of Theorem 2.4. The proof follows in three steps.

Step 1. The inclusion $V_{\mathrm{C}}(\mathcal{T}) \subset \mathrm{KS}(\mathcal{T})$ and Galerkin orthogonality show, together with Proposition 5.2,

$$\begin{aligned}|\!|\!| u_{\mathrm{KS}} - u_{\mathrm{C}} |\!|\!|_{\mathrm{NC}} &\leqslant \min_{v_{\mathrm{C}} \in V_{\mathrm{C}}(\mathcal{T})} |\!|\!| u_{\mathrm{KS}} - v_{\mathrm{C}} |\!|\!|_{\mathrm{NC}} \\ &\lesssim \lambda^{1/2} \min_{v \in V} |\!|\!| u_{\mathrm{KS}} - v |\!|\!| \leqslant \lambda^{1/2} |\!|\!| u - u_{\mathrm{KS}} |\!|\!|.\end{aligned}$$

This implies the following inequality for the energy norm:

$$|\!|\!| u - u_{\mathrm{C}} |\!|\!| \leqslant |\!|\!| u - u_{\mathrm{KS}} |\!|\!|_{\mathrm{NC}} + |\!|\!| u_{\mathrm{C}} - u_{\mathrm{KS}} |\!|\!|_{\mathrm{NC}} \lesssim (1 + \lambda^{1/2}) |\!|\!| u - u_{\mathrm{KS}} |\!|\!|_{\mathrm{NC}}.$$

Since $|\mathbb{C}A|^2 \lesssim \lambda\, (A : \mathbb{C}A)$, it follows that

$$\|\sigma - \sigma_{\mathrm{C}}\|_{L^2(\Omega)} \lesssim \lambda^{1/2} |\!|\!| u - u_{\mathrm{C}} |\!|\!| \lesssim \lambda |\!|\!| u - u_{\mathrm{KS}} |\!|\!|_{\mathrm{NC}} \lesssim \lambda \|\sigma - \sigma_{\mathrm{KS}}\|_{L^2(\Omega)}.$$

Step 2. The inequalities

$$\begin{aligned}\|\sigma - \sigma_{\mathrm{KS}}\|_{L^2(\Omega)} &\lesssim \|\sigma - \sigma_{\mathrm{C}}\|_{L^2(\Omega)} + \operatorname{osc}(f_2, \mathcal{T}) + \operatorname{osc}(g_2, \mathcal{E}(\Gamma_{\mathrm{N}})), \\ \|\sigma - \sigma_{\mathrm{KS}}\|_{L^2(\Omega)} &\lesssim \|\tilde{\sigma} - \tilde{\sigma}_{\mathrm{CR}}\|_{L^2(\Omega)} + \operatorname{osc}(f_2, \mathcal{T}) \quad (\text{if } \Gamma_{\mathrm{D}} = \partial\Omega)\end{aligned}$$

are direct consequences of Theorems 2.1 and 4.4.
Step 3. The inequality $(A + A^{\mathrm{T}}) : (A + A^{\mathrm{T}}) \leqslant 4\, A : A$ implies

$$(\sigma - \sigma_{\mathrm{CR}}) : (\sigma - \sigma_{\mathrm{CR}}) \lesssim (\tilde{\sigma} - \tilde{\sigma}_{\mathrm{CR}}) : (\tilde{\sigma} - \tilde{\sigma}_{\mathrm{CR}}).$$

From Theorem 4.1 it follows, for $\tilde{\sigma}_{\mathrm{KS}} := \tilde{\mathbb{C}} D_{\mathrm{NC}} u_{\mathrm{KS}}$, that

$$\begin{aligned}\|\tilde{\sigma} - \tilde{\sigma}_{\mathrm{KS}}\|^2_{L^2(\Omega)} &= \mu^2 \|D_{\mathrm{NC}}(u - u_{\mathrm{KS}})\|^2_{L^2(\Omega)} \\ &\quad + (2\mu(\lambda + \mu) + (\lambda + \mu)^2) \|\operatorname{div}_{\mathrm{NC}}(u - u_{\mathrm{KS}})\|^2_{L^2(\Omega)} \\ &\lesssim 4\mu^2 \|\varepsilon_{\mathrm{NC}}(u - u_{\mathrm{KS}})\|^2_{L^2(\Omega)} + (4\mu\lambda + \lambda^2) \|\operatorname{div}_{\mathrm{NC}}(u - u_{\mathrm{KS}})\|^2_{L^2(\Omega)} \\ &= \|\sigma - \sigma_{\mathrm{KS}}\|^2_{L^2(\Omega)}.\end{aligned}$$

Altogether,

$$\begin{aligned}\|\sigma - \sigma_{\mathrm{CR}}\|_{L^2(\Omega)} &\lesssim \|\tilde{\sigma} - \tilde{\sigma}_{\mathrm{CR}}\|_{L^2(\Omega)} \\ &\lesssim \|\tilde{\sigma} - \tilde{\sigma}_{\mathrm{KS}}\|_{L^2(\Omega)} + \operatorname{osc}(f, \mathcal{T}) \\ &\lesssim \|\sigma - \sigma_{\mathrm{KS}}\|_{L^2(\Omega)} + \operatorname{osc}(f, \mathcal{T}).\end{aligned}$$

This concludes the proof of Theorem 2.4. □

6. Numerical investigations

This section provides numerical evidence that the claimed equivalence of σ_{CR} and σ_{KS} is independent of the parameter λ for the pure Dirichlet problem in linear elasticity and that the dependence of the equivalence constants in (1.1) on $\lambda = 1.6 \times 10^k$ for $k = 6, 7, 8, 9$ cannot be improved.

6.1 *Preliminaries*

Throughout this section, the elastic modulus is $E = 10^5$ and the Poisson ratio varies between $\nu = 0.4, 0.49, 0.499, 0.4999$ with corresponding values of $\mu = E/(2(1+\nu))$ and $\lambda = E\nu/((1+\nu)(1-2\nu)) = 1.6 \times 10^k$ for $k = 6, 7, 8, 9$. The initial triangulations $\mathcal{T}_0$ of all four numerical examples are depicted in Figs 4 and 5. The discrete problems are solved on a sequence of triangulations $\mathcal{T}_\ell$ obtained by successive red-refinements; a red-refinement of a triangle subdivides each triangle into four congruent subtriangles via straight lines through the edges' midpoints, as depicted in Fig. 4(a).

Since the error is known only in the first example, the averaging error estimator defined in Carstensen & Funken (2001a, Equation (2.17)) serves as an error indicator. Although the proofs of efficiency and reliability from Carstensen & Funken (2001a) provide no information about the efficiency and reliability constants, there is numerical evidence that the averaging error estimator often yields results very close to the exact error (Carstensen & Funken, 2001a). The first example confirms this observation and so partly justifies the use of this error estimator for the further examples. Let $|T|$ denote the area of a triangle $T \in \mathcal{T}$ and τ_E the tangent of an edge $E \in \mathcal{E}$. The residual error estimators

$$\eta_{\mathrm{C}}(u_{\mathrm{C}}) := \left(\sum_{T\in\mathcal{T}} \left(|T| \|f\|^2_{L^2(T)} + |T|^{1/2} \sum_{E\in\mathcal{E}(T)\setminus\mathcal{E}(\Gamma_{\mathrm{D}})} \|[\sigma_{\mathrm{C}}]_E \nu_E\|^2_{L^2(E)} \right) \right)^{1/2},$$

$$\eta_{\mathrm{CR}}(u_{\mathrm{CR}}) := \left(\sum_{T\in\mathcal{T}} \left(|T| \|f\|^2_{L^2(T)} + |T|^{1/2} \sum_{E\in\mathcal{E}(T)\setminus\mathcal{E}(\Gamma_{\mathrm{N}})} \|[D_{\mathrm{NC}} u_{\mathrm{CR}}]_E \tau_E\|^2_{L^2(E)} \right) \right)^{1/2},$$

$$\eta_{\mathrm{KS}}(u_{\mathrm{KS}}) := \left(\sum_{T\in\mathcal{T}} \left(|T| \|f\|^2_{L^2(T)} + |T|^{1/2} \sum_{E\in\mathcal{E}(T)\setminus\mathcal{E}(\Gamma_{\mathrm{N}})} \|[D_{\mathrm{NC}} u_{\mathrm{KS}}]_E \tau_E\|^2_{L^2(E)} \right.\right.$$
$$\left.\left. + |T|^{1/2} \sum_{E\in\mathcal{E}(T)\setminus\mathcal{E}(\Gamma_{\mathrm{D}})} \|(1,0)\cdot([\sigma_{\mathrm{KS}}]_E \nu_E)\|^2_{L^2(E)} \right) \right)^{1/2}$$

for CFEM, CR-NCFEM and KS-NCFEM, respectively, are reliable and efficient (Carstensen & Funken, 2001a; Carstensen & Rabus, 2012). In contrast to Carstensen & Funken (2001a), the normal jump of the second component of the stress is omitted for KS-NCFEM in the spirit of Dari *et al.* (1995).

A close investigation of the dependency on the parameter λ for $\nu = 0.4$, 0.49, 0.499 and 0.4999 in the comparison result (1.1) considers the quotients

$$q(\nu,\ell) := \|\sigma_\nu - \sigma_{\mathrm{C}}^{\ell,\nu}\|_{L^2(\Omega)} / \|\sigma_\nu - \sigma_{\mathrm{KS}}^{\ell,\nu}\|_{L^2(\Omega)} \quad \text{for } \ell = 1, \ldots, 9. \tag{6.1}$$

Here and in Sections 6.2 and 6.4, σ_ν denotes the exact stress for the Poisson ratio ν, and $\sigma_{\mathrm{C}}^{\ell,\nu}$ and $\sigma_{\mathrm{KS}}^{\ell,\nu}$ denote, respectively, the discrete stresses of CFEM and KS-NCFEM for the Poisson ratio ν and the ℓth red-refinement $\mathcal{T}_\ell := \mathrm{red}^{(\ell)}(\mathcal{T}_0)$ of $\mathcal{T}_0$. (For the experiment from Section 6.4 the quotients are approximated by the corresponding values of the averaging error estimator.)

6.2 *Academic example*

Under homogeneous pure Dirichlet boundary conditions, the unit square $\Omega = (0,1)^2$ is loaded with the applied force

$$f(x,y) = \begin{pmatrix} -2\mu\pi^3 \cos(\pi y)\sin(\pi y)(2\cos(2\pi x) - 1) \\ 2\mu\pi^3 \cos(\pi x)\sin(\pi x)(2\cos(2\pi y) - 1) \end{pmatrix}$$

(written as a function of the coordinates x and y) so that (2.1) leads to the exact smooth solution

$$u(x,y) = \begin{pmatrix} \pi \cos(\pi y)\sin^2(\pi x)\sin(\pi y) \\ -\pi \cos(\pi x)\sin^2(\pi y)\sin(\pi x) \end{pmatrix}.$$

Given the initial mesh $\mathcal{T}_0$ of Fig. 4(b) with one interior node and eight interior edges, the three FEMs with the number of degrees of freedom lead on each triangulation $\mathcal{T}_\ell$ to the discrete stresses $\sigma_C, \sigma_{CR}, \sigma_{KS}$; on level zero, for instance, ndof $= 2$ for CFEM, ndof $= 16$ for CR-NCFEM and ndof $= 9$ for KS-NCFEM. The convergence history plot of Fig. 6 displays various errors and error estimators versus the number of degrees of freedom for the Poisson ratios $\nu = 0.4, 0.49, 0.499, 0.4999$ (from dark to light; in the color picture in the online edition of this paper the values correspond to red, blue, green, cyan) for the three FEMs.

The graphs of the averaging error estimators and the exact error of CR-NCFEM and KS-NCFEM for all values of ν lie on top of each other and the values of the residual error estimator for KS-NCFEM and also the values of the residual error estimator for CR-NCFEM behave in the same way.

For the initial triangulation $\mathcal{T}_0$ of Fig. 4(b) with two degrees of freedom in CFEM, the averaging error estimator strongly underestimates and is omitted. Apart from that case, the values of the averaging error estimator are very close to the exact error. This example therefore serves as an empirical validation of the averaging error estimator in the following examples where it is expected to indicate the (unknown) errors to high accuracy.

Equivalent convergence rates are observed for all three FEMs with a strong dependency on λ for CFEM, while the errors in KS-NCFEM and CR-NCFEM are of similar size. Table 1 displays the quotients (6.1) and reveals a linear dependency on λ. This is clear numerical evidence that the dependence on λ in the first estimate of (1.1) and in Theorem 2.4 is sharp.

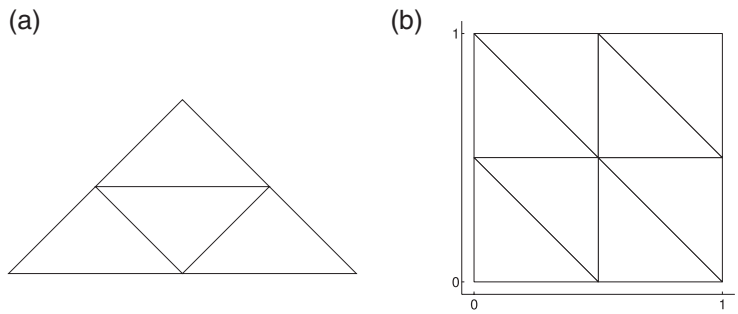

FIG. 4. (a) Red-refined triangle. (b) Initial triangulation $\mathcal{T}_0$ on the unit square from Section 6.2.

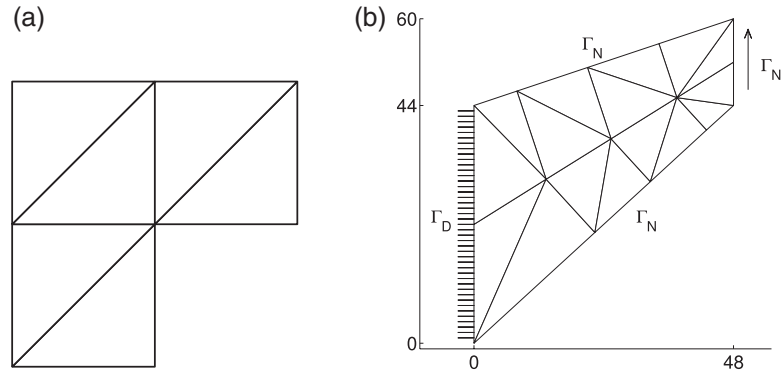

FIG. 5. (a) Initial triangulation $\mathcal{T}_0$ on the L-shaped domain in Sections 6.4 and 6.5. (b) Cook's membrane and initial triangulation $\mathcal{T}_0$ in Section 6.3.

TABLE 1 *Quotient* $q(\nu, \ell)$ *from* (6.1) *for* CFEM *and* KS-NCFEM *in Section* 6.2

ν	$\ell = 1$	2	3	4	5	6	7	8	9
0.4	0.8461	1.588	2.411	2.947	3.165	3.229	3.246	3.250	3.251
0.49	0.6717	3.327	9.667	17.95	24.84	28.46	29.71	30.05	30.14
0.499	0.6498	4.053	17.61	56.26	127.5	207.0	264.0	289.3	297.3
0.4999	0.6476	4.150	19.52	78.29	277.7	778.8	1556	2301	2755

6.3 *Cook's membrane benchmark*

This benchmark in linear elasticity concerns the domain Ω of Fig. 5(b) with vertices $(0, 0)$, $(48, 44)$, $(48, 60)$, $(0, 44)$ and the Dirichlet boundary $\Gamma_D := \text{conv}\{(0, 0), (0, 44)\}$ and $\Gamma_N := \partial\Omega \setminus \Gamma_D$. The applied forces are $f \equiv 0$ in Ω and $g(x) = (0, 1)$ if $x_1 = 48$ on the right vertical edge of $\partial\Omega$ while $g \equiv 0$ on the remaining two parts of Γ_N. The Neumann boundary of the problem excludes CR-NCFEM. The estimated errors of CFEM and KS-NCFEM are plotted against the number of degrees of freedom in Fig. 7. For $\nu = 0.49, 0.499, 0.4999$ the values of the averaging error estimator for KS-NCFEM lie on top of each other and the values of the residual error estimator for KS-NCFEM behave in the same way.

The locking behaviour of CFEM and the robustness of KS-NCFEM (with respect to λ) is clearly visible in the sense that the preasymptotic range for CFEM is so big that it covers the full range of our computational capabilities with the effect that, for $\nu = 0.4999$, none of the computational values are better than the initial stress approximation (relative to the L^2 norm).

Note that the jump in the boundary conditions at the vertex $(0, 44)$ causes a solution $u \notin H^2(\Omega; \mathbb{R}^2)$ in agreement with the reduced convergence rates (under uniform mesh refinement) and, hence, the conditions of Kouhia & Stenberg (1995) are violated.

6.4 L-*shaped domain without locking*

This example shows that the equivalence constant in the second inequality of (1.1) cannot be replaced by any negative power of λ. The underlying domain of this example is the L-shaped domain $\Omega := (-1, 1)^2 \setminus [0, 1] \times [-1, 0]$ with $\Gamma_D = \partial\Omega$ and the initial mesh $\mathcal{T}_0$ of Fig. 5(a). The piecewise constant

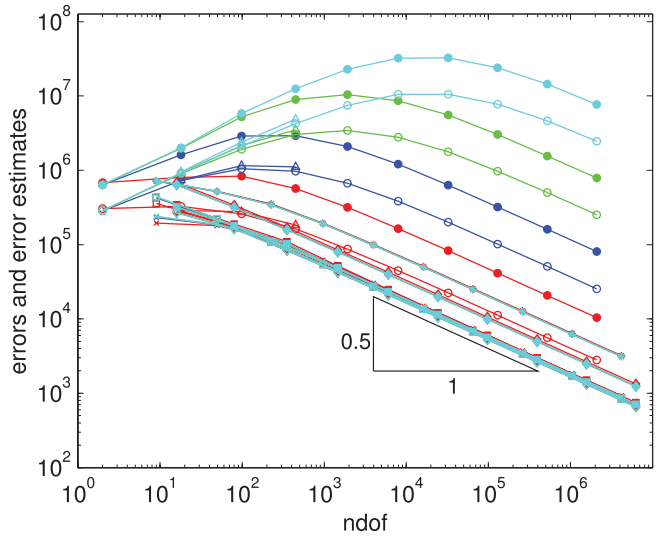

FIG. 6. Estimated errors of CFEM (residual estimator (•), averaging estimator (△) and exact error (○)), CR-NCFEM (residual estimator (♦), averaging estimator (■) and exact error (◊)) and KS-NCFEM (residual estimator (∗), averaging estimator (×) and exact error (□)) for $\nu = 0.4, 0.49, 0.499, 0.4999$ (from dark to light; in the color picture in the online edition of this paper the values correspond to red, blue, green, cyan) on uniform red-refined meshes for the unit square from Section 6.2.

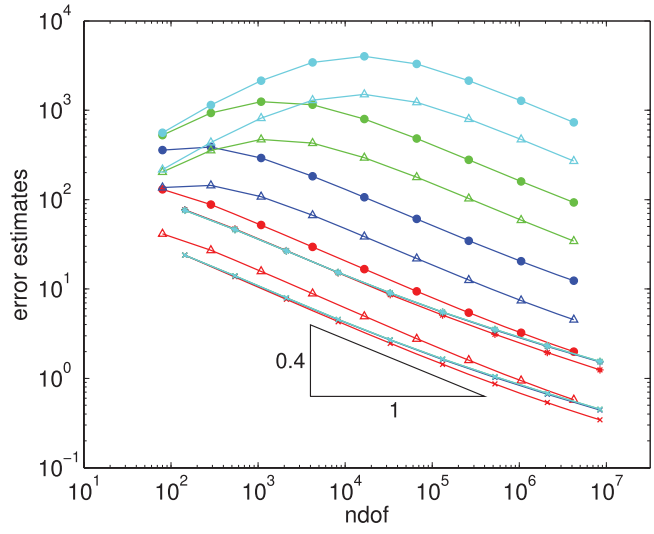

FIG. 7. Estimated errors of CFEM (residual estimator (•) and averaging estimator (△)) and KS-NCFEM (residual estimator (∗) and averaging estimator (×)) for $\nu = 0.4, 0.49, 0.499, 0.4999$ (from dark to light; in the color picture in the online edition of this paper the values correspond to red, blue, green, cyan) on uniform red-refined meshes for Cook's membrane from Section 6.3.

volume force f reads

$$f(x,y) := \begin{cases} (0,-1) & \text{if } x \leqslant 0 \text{ and } y \geqslant 0, \\ (1,-1) & \text{if } x, y \geqslant 0, \\ (0,0) & \text{if } x, y \leqslant 0. \end{cases}$$

Figure 8 displays the averaging and residual error estimators for a sequence of red-refined triangulations against the number of degrees of freedom. For $\nu = 0.49, 0.499, 0.4999$ the values of the averaging error estimator lie on top of each other for all three FEMs and the values of the residual error estimator behave in the same way. In Table 2, the quotients from (6.1) are approximated by the corresponding values of the averaging estimator. The values of these quotients are all of the same order of magnitude; this indicates no dependency on λ in the second inequality of (1.1).

Since f is a gradient, we do not expect and do not observe the locking behaviour while λ increases over several orders of magnitude.

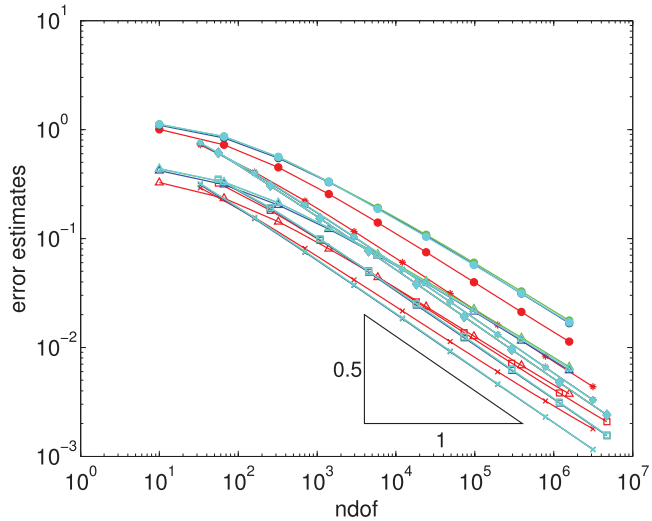

FIG. 8. Estimated errors of CFEM (residual estimator (•) and averaging estimator (△)), CR-NCFEM (residual estimator (♦) and averaging estimator (□)) and KS-NCFEM (residual estimator (∗) and averaging estimator (×)) for $\nu = 0.4, 0.49, 0.499, 0.4999$ (from dark to light; in the color picture in the online edition of this paper the values correspond to red, blue, green, cyan) on the L-shaped domain from Section 6.4.

TABLE 2 *Approximated quotient* $q(\nu, \ell)$ *from* (6.1) *for* CFEM *and* KS-NCFEM *in Section* 6.4

ν	$\ell = 1$	2	3	4	5	6	7	8
0.4	1.123	1.502	1.762	1.931	2.037	2.097	2.123	2.118
0.49	1.348	2.057	2.705	3.291	3.807	4.252	4.641	4.989
0.499	1.371	2.130	2.790	3.342	3.869	4.393	4.882	5.334
0.4999	1.373	2.138	2.803	3.336	3.783	4.211	4.666	5.127

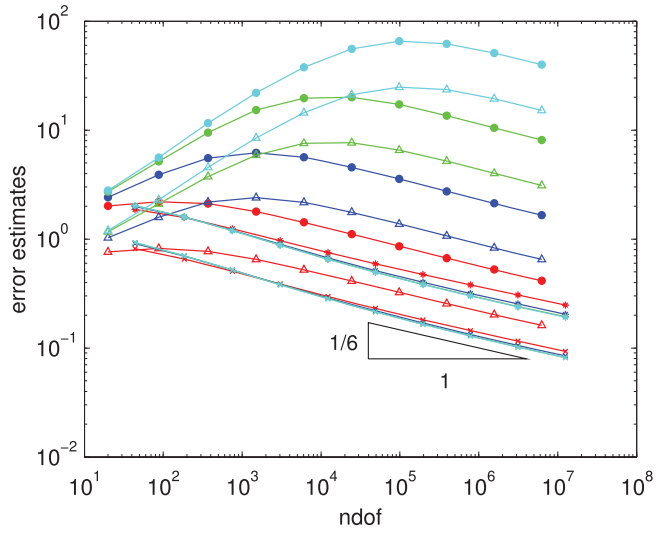

FIG. 9. Estimated error for CFEM (residual estimator (•) and averaging estimator (△)) and KS-NCFEM (residual estimator (∗) and averaging estimator (×)) for $\nu = 0.4, 0.49, 0.499, 0.4999$ (from dark to light; in the color picture in the online edition of this paper the values correspond to red, blue, green, cyan) on the L-shaped domain from Section 6.5.

6.5 L-*shaped domain with Neumann boundary conditions*

This example confirms our theoretical findings in the case of a nonempty Neumann boundary. The boundary conditions change type at the re-entrant corner point. This means that one cannot expect a regularity of $H^{3/2+\varepsilon}$ for some positive ε. The empirical convergence rate 1/6 of Fig. 9 in terms of the number of degrees of freedom clearly indicates that $u \notin H^{3/2}(\Omega;\mathbb{R}^2)$. This situation excludes even a mathematical justification via a straightforward though technical generalization of the error analysis from Kouhia & Stenberg (1995).

The domain Ω and the initial triangulation $\mathcal{T}_0$ is as in Section 6.4, while the volume force f reads

$$f(x,y) := \begin{cases} (0,0) & \text{if } x \leqslant 0 \text{ and } y \geqslant 0, \\ (0,1) & \text{if } x, y \geqslant 0, \\ (1,0) & \text{if } x, y \leqslant 0. \end{cases}$$

The boundary is divided into the Neumann boundary $\Gamma_N := \{(x_1,x_2) \in \partial\Omega \mid x_1 > 0\}$ with applied tractions $g \equiv 0$ and the Dirichlet boundary $\Gamma_D = \partial\Omega \setminus \Gamma_N$. Figure 9 displays the estimated errors in terms of the number of degrees of freedom. For $\nu = 0.499$ and 0.4999, the values of the averaging error estimator for KS-NCFEM lie on top of each other and the values of the residual error estimator for KS-NCFEM behave in the same way. The equivalence of KS-NCFEM and CFEM up to a multiplicative factor which scales linearly in λ is visible also for this singular problem. The numerical experiments provide striking empirical evidence for robustness with respect to the locking behaviour and to possible singularities and mark the superiority of the somehow bizarre but simple and well-justified KS-NCFEM.

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