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Discontinuous Galerkin with Weakly Over-Penalized Techniques for Reissner–Mindlin Plates

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Abstract In this article we introduce a new locking-free completely discontinuous formulation for Reissner–Mindlin plates that combines the discontinuous Galerkin methods with weakly over-penalized techniques. We establish a new discrete version of Helmholtz decomposition and some important residual estimates. Combining the residual estimates with enriching operators we derive an optimal a priori error estimate in the energy norm. We obtain robust a posteriori error estimators and prove their reliability and efficiency.

Keywords Reissner–Mindlin · Discontinuous Galerkin · A priori error estimates · A posteriori error estimates

1 Introduction

The weak formulation for the Reissner–Mindlin plate model reads: Given $g \in L^2(\Omega)$ and $\boldsymbol{f} \in L^2(\Omega; \mathbb{R}^2)$, seek $(\boldsymbol{\theta},\ w,\ \boldsymbol{\gamma}) \in H_0^1(\Omega; \mathbb{R}^2) \times H_0^1(\Omega) \times L^2(\Omega; \mathbb{R}^2)$ such that

$$
\begin{aligned}
&a(\boldsymbol{\theta}, \boldsymbol{\eta}) + (\boldsymbol{\gamma}, \boldsymbol{\eta})_\Omega = (\boldsymbol{f}, \boldsymbol{\eta})_\Omega \quad \text{for all } \boldsymbol{\eta} \in H_0^1(\Omega; \mathbb{R}^2)\\
&-(\boldsymbol{\gamma}, \nabla v)_\Omega = (g, v)_\Omega \quad \text{for all } v \in H_0^1(\Omega)\\
&t^2 \mu^{-1} (\boldsymbol{\gamma}, \boldsymbol{\phi})_\Omega - (\boldsymbol{\theta} - \nabla w, \boldsymbol{\phi})_\Omega = 0 \quad \text{for all } \boldsymbol{\phi} \in L^2(\Omega; \mathbb{R}^2).
\end{aligned} \tag{1}
$$

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Here, and throughout this paper, t is the plate thickness, Ω is a convex polygonal domain, and $e(\boldsymbol{\xi})$ is the symmetric part of the gradient of $\boldsymbol{\xi}$,

$$\mathcal{C}e(\boldsymbol{\xi}) = \frac{1}{3}\left[2\mu e(\boldsymbol{\xi}) + \frac{2\mu\lambda}{2\mu+\lambda}\operatorname{div}\boldsymbol{\xi}\, I\right]$$

where μ and λ are the Lamé coefficients and I is the identity 2×2 matrix, and

$$a(\boldsymbol{\theta},\boldsymbol{\eta}) = (e(\boldsymbol{\theta}), \mathcal{C}e(\boldsymbol{\eta}))_\Omega.$$

In this paper we introduce a new locking-free completely discontinuous formulation for (1) that combines the traditional discontinuous Galerkin methods with the weakly over-penalized symmetric interior penalty (WOPSIP) methods. For the first equation of the Reissner–Mindlin model we will apply the same formulation used, for example, in [1,2,7], while for the second equation we will introduce a type of WOPSIP method similar to that presented in [15]. With this approach the interior penalty term for the displacement will be over-penalized, but the penalty parameter can be any positive constant. However, for polynomials of degree $k = 2$, for which we have the required theoretical regularity available for the convex domain (see Theorem 8 and [3,4]), the over-penalization (the power of h) will be the same as that used in [7] and also in [35–37] for the biharmonic equation.

Locking-free formulations where completely discontinuous spaces are used for all the variables have been reported in [1,2] and [7] (see [34] for an overview of the first two articles). In the second article, polynomials of the same degree k were used for the displacement and rotation and $k-1$ was used for the shear stress, where k is an odd degree. For this formulation, an optimal rate of convergence in the energy norm was proved. In [1], for any $k \geq 2$ the formulation considers degree k for displacement and $k-1$ for rotation and shear stress. Using Helmholtz decomposition optimal rates of convergence in the energy norm and L^2 norm were proved. In the third article a new formulation was proposed, which does not introduce the shear as an unknown and does not need reduced integration (as [1]). Using degree k for displacement and $k-1$ for rotation, for any $k \geq 2$, optimal rates of convergence in the energy norm were proved and numerically confirmed. In this article we will apply degree k for displacement and $k-1$ for rotation and shear stress. Assuming Helmholtz decomposition we will prove optimal rates of convergence in the energy norm and prove the reliability and efficiency of the a posteriori error estimators.

Many other formulations for the Reissner–Mindlin model that combine (with or without the bubble function) nonconforming, conforming and fully discontinuous elements are available [1–3,5,6,16,17,21–23,25,28,32,38]. In [24], a general review of the finite elements methods for the Reissner–Mindlin model and related problems, such as dimensional reduction of the model, properties of the solution, regularity results and the locking problem, can be found. A description of the main approaches used to solve the Reissner–Mindlin model, including the Durán–Liberman element, MITC triangular families, Falk–Tu elements, linked interpolation methods, nonconforming Arnold and Falk and some rectangular elements, is provided. Other approaches, such as the discontinuous Galerkin methods and least-squares schemes, are also discussed in the above-cited paper.

The weakly over-penalized symmetric methods were introduced in [14] for second-order elliptic problems and extended to any higher-order polynomials in [15]. The main characteristic of WOPSIP methods is that the jumps across the element boundary are weakly over-penalized. Unfortunately, because of this, the resulting discrete system is ill-conditioned. However, in [15] (see also [12]), an adequate preconditioner, which reduces the condition number of the discrete problem to $O(h^{-2})$ (the same as a typical discretization) when odd-orders are considered, was constructed. This formulation is stable for any positive penalty parameter and has optimal errors in both the energy norm and L^2 norm. For $k = 1$, in [11] it was shown that WOPSIP is intrinsically parallel. Furthermore, a nonsymmetric version of the over-penalized method was introduced in [13] for the same class of problem covered by the symmetric version.

On combining discontinuous Galerkin (dG) methods with WOPSIP techniques the resultant discrete formulation is not consistent. This prevents us from obtaining the Galerkin orthogonality. Therefore, the traditional error analysis of dG methods can not be applied. Furthermore, since the consistency term is dependent on the shear stress we can obtain only suboptimal error estimates on applying the WOPSIP analysis techniques. To obtain optimal error estimates we will proceed with the analysis through the residual estimates, which are typical for a posteriori error analysis [18–20,33], together with enriching operators [8,9]. A similar approach has been previously used, for example, in [26] and [27] to analyze dG methods under minimal regularity. To succeed with this strategy we need to assume that the Helmholtz decomposition is valid. Fortunately, this is the case if $k = 2$ (at least) and if Ω is a convex polygon domain, basically.

We highlight that this new formulation of dG for Reissner–Mindlin have the following advantages: (a) more freedom in the choosing of the penalty parameters; (b) the formulation is simpler, in the sense that have less terms; (c) we obtain robust a posteriori error estimators and prove their reliability and efficiency; and (d) we required only reasonable and standard hypotheses on the domain. Moreover, the error analysis was designed in an unusual way and the Theorem 8, for $k = 2$, shows an error estimate in the energy norm which requires only the regularity provided theoretically for the solution in the case of a convex polygon domain (or smooth domain). In addition, the norms of the solution present on the right-hand side are uniformly bounded with respect to t.

The rest of this paper is organized as follows: In the next section we introduce the necessary notation and recall some definitions to deal with discontinuous Galerkin methods. In Sect. 3 we introduce the new discrete formulation which combines dG with WOPSIP techniques. Some residual estimates which are fundamental for error analysis are present in Sect. 4 together with a discrete version of Helmholtz decomposition. In Sect. 5 we describe the a priori error analysis in the energy norm and the final section is dedicated to the a posteriori error analysis.

2 Notation and Preliminaries

Let $\mathbb{T}$ be a shape-regular family of regular triangulations of $\Omega \subset \mathbb{R}^2$ into triangles T, where the T are open, convex and pairwise disjoint, such that

$$\overline{\Omega} = \bigcup_{T \in \mathcal{T}} \overline{T}.$$

On the regular triangulation $\mathcal{T} \in \mathbb{T}$, the piecewise constant function $h_{\mathcal{T}}$ is defined by

$$h_{\mathcal{T}|T} = h_T := \operatorname{diam}(T) \;\text{ on }\; T \in \mathcal{T}$$

and we denote by h the maximum of h_T for $T \in \mathcal{T}$. Let $\mathcal{E}$ be the set of all edges E of all the triangles in $\mathcal{T}$ and let us define the piecewise constant function $h_{\mathcal{E}}$ as

$$h_{\mathcal{E}|E} = h_E := \operatorname{diam}(E) \;\text{ on }\; E \in \mathcal{E}.$$

$\mathcal{E}(T)$ denotes the set of the three edges of T. The set $\mathcal{E}$ will be divided into two subsets, $\mathcal{E}(\Omega)$ and $\mathcal{E}(\partial\Omega)$, defined by

$$\mathcal{E}(\Omega) = \{E \in \mathcal{E} : E \subset \Omega\} \text{ and } \mathcal{E}(\partial\Omega) = \{E \in \mathcal{E} : E \subset \partial\Omega\}.$$

The shape-regularity of $\mathbb{T}$, provides some constant $0 < \gamma(\mathbb{T}) \leq 1$ such that $\forall \mathcal{T} \in \mathbb{T},\ \forall T \in \mathcal{T},\ \forall E \in \mathcal{E}(T)$

$$\gamma h_T \leq h_E \leq h_T.$$

The Sobolev space of real order (or index) s of real-valued functions defined on $\omega \subset \Omega$, will be labeled by $H^s(\omega)$. Its inner product, norm and semi-norm will be denoted by $(\cdot,\cdot)_{s,\omega}$, $\|\cdot\|_{s,\omega}$, and $|\cdot|_{s,\omega}$, respectively. In particular, we will write $\|\cdot\|_\omega$ and $(\cdot,\cdot)_\omega$ instead of $\|\cdot\|_{0,\omega}$ and $(\cdot,\cdot)_{0,\omega}$, respectively. Similarly, for any $E \in \mathcal{E}$ we will denote by $\langle\cdot,\cdot\rangle_E$ and $\|\cdot\|_E$ the inner product and the induced norm in the space $L^2(E)$, respectively. Also, we will denote by $H^s(\omega;\mathbb{R}^2) = H^s(\omega) \times H^s(\omega)$ the Sobolev space of vector functions for which, as in the case of the scalar function, $(\cdot,\cdot)_{s,\omega}$ will denote the inner product. Note that the same notation for the inner product also will be used occasionally for symmetric tensors. Let

$$H^s(\mathcal{T}) = \{v \in L^2(\Omega) : v_{|T} \in H^s(T),\ \text{for all } T \in \mathcal{T}\}$$

be the space of piecewise Sobolev H^s-functions. We denote its inner product, norm and semi-norm by $(\cdot,\cdot)_{s,h}$, $\|\cdot\|_{s,h}$ and $|\cdot|_{s,h}$, respectively. $H^s(\mathcal{T};\mathbb{R}^2) = H^s(\mathcal{T}) \times H^s(\mathcal{T})$ denotes the space of piecewise Sobolev H^s-vector functions.

We use the following differential operators: $\mathrm{Curl}(v) = (\partial v/\partial y, -\partial v/\partial x)$ for a scalar function v, and $rot(\boldsymbol{\eta}) = \partial\eta_2/\partial x - \partial\eta_1/\partial y$ for a vector function $\boldsymbol{\eta} = (\eta_1, \eta_2)$. We observe that any differential operator defined over a piecewise Sobolev space will be indicated by a subscript h.

For any $T \in \mathcal{T}$, let $\boldsymbol{\nu}_T = (\nu_1, \nu_2)$ be the outer unit normal to the boundary ∂T and let $\boldsymbol{\tau}_T = (-\nu_2, \nu_1)$ be the tangential vector. Let T^- and T^+ be two distinct elements of $\mathcal{T}$ sharing the edge $E = \overline{T^-} \bigcap \overline{T^+} \in \mathcal{E}(\Omega)$. We define the jump of $v \in H^1(\mathcal{T})$ by

$$[v] = v^-\boldsymbol{\nu}^- + v^+\boldsymbol{\nu}^+,$$

where $v^\pm := v|_{T^\pm}$ and $\boldsymbol{\nu}^\pm$ denote the outer unit normal $\boldsymbol{\nu}_{T^\pm}$ on $T^\pm$. For a vector function $\boldsymbol{\eta} \in H^1(\mathcal{T};\mathbb{R}^2)$, define

$$[\boldsymbol{\eta}] = \boldsymbol{\eta}^- \cdot \boldsymbol{\nu}^- + \boldsymbol{\eta}^+ \cdot \boldsymbol{\nu}^+ \text{ and } [\![\boldsymbol{\eta}]\!] = \boldsymbol{\eta}^- \odot \boldsymbol{\nu}^- + \boldsymbol{\eta}^+ \odot \boldsymbol{\nu}^+,$$

where $\boldsymbol{\eta} \odot \boldsymbol{\nu} = (\boldsymbol{\eta}\boldsymbol{\nu}^T + \boldsymbol{\nu}\boldsymbol{\eta}^T)/2$. Similarly, for a tensor $\epsilon \in H^1(\Omega;\mathbb{R}^{2\times 2})$ the jump on E is defined by

$$[\![\epsilon]\!] = \epsilon^-\boldsymbol{\nu}^- + \epsilon^+\boldsymbol{\nu}^+.$$

Note that the jump of a scalar function is a vector. For a vector function $\boldsymbol{\eta}$, the jump $[\boldsymbol{\eta}]$ is a scalar, while the jump $[\![\boldsymbol{\eta}]\!]$ is a symmetric matrix, and for a tensor the jump is a vector. The average of a tensor, scalar function or vector function χ is defined by $\{\chi\} = \frac{1}{2}(\chi^- + \chi^+)$. On a boundary edge, we define the average $\{\chi\}$ as the trace of χ, while we consider $[\phi]$ to be $\phi\boldsymbol{\nu}$, $[\boldsymbol{\eta}]$ to be $\boldsymbol{\eta}\cdot\boldsymbol{\nu}$, $[\![\boldsymbol{\eta}]\!]$ to be $\boldsymbol{\eta} \odot \boldsymbol{\nu}$ and $[\![\epsilon]\!]$ to be $\epsilon\boldsymbol{\nu}$.

Occasionally, we shall use the jump on E in relation to the tangent vector, in this case denoted by $[v]_{\boldsymbol{\tau}}$, that is, $[v]_{\boldsymbol{\tau}} = v^-\boldsymbol{\tau}^- + v^+\boldsymbol{\tau}^+$ (*idem* for a vector function).

For a positive integer k, $\mathcal{P}_k(T)$ will denote the linear space of polynomials on T with a total degree of less than or equal to k, and $\mathcal{P}_k(T;\mathbb{R}^2) := \mathcal{P}_k(T) \times \mathcal{P}_k(T)$. The discrete space for the displacement will be

$$\mathcal{P}_k(\mathcal{T}) = \left\{v \in L^2(\Omega) : \ \forall T \in \mathcal{T},\ v_{|T} \in \mathcal{P}_k(T)\right\},$$

and for the rotation and shear stress it will be

$$\mathcal{P}_{k-1}(\mathcal{T};\mathbb{R}^2) = \{\boldsymbol{\eta} \in L^2(\Omega;\mathbb{R}^2);\ \ \forall T \in \mathcal{T},\ \boldsymbol{\eta}_{|T} \in \mathcal{P}_{k-1}(T;\mathbb{R}^2)\}$$

for any $k \geq 2$.

Let π_W denote the natural projection onto $\mathcal{P}_k(\mathcal{T})$ (see [1] for definition of π_W). For $w \in H^{k+1}(\Omega)$ let $w^I = \pi_W w$ be the interpolant of w. It then follows that $w^I \in \mathcal{P}_k(\mathcal{T}) \cap H^1(\Omega)$ and that for $0 \leq q \leq k+1$, there exists a constant c such that

$$\|w - w^I\|_{q,h} \leq ch^{k+1-q}\,\|w\|_{k+1,\Omega} \quad \text{for all } w \in H^{k+1}(\Omega). \tag{2}$$

The rotated Brezzi–Douglas–Marini space of degree $k-1$, i.e., the space of all piecewise polynomial vector fields of degree $k-1$ subject to interelement continuity of the tangential components, will be denoted by $\mathbf{BDM}^R_{k-1}$. Let $\boldsymbol{\pi}_{\boldsymbol{\Theta}}$ be the natural projection operator of $H^1(\Omega;\mathbb{R}^2)$ into $\mathbf{BDM}^R_{k-1} \subset \mathcal{P}_{k-1}(\mathcal{T};\mathbb{R}^2)$. For $\boldsymbol{\theta} \in H^k(\Omega;\mathbb{R}^2)$ we define its interpolant $\boldsymbol{\theta}^I$ by $\boldsymbol{\theta}^I := \boldsymbol{\pi}_{\boldsymbol{\Theta}}\boldsymbol{\theta}$. With this choice, for $0 \leq s \leq \ell$, and $1 \leq \ell \leq k$ we have

$$\|\boldsymbol{\theta} - \boldsymbol{\theta}^I\|_{s,h} \leq ch^{\ell-s}\|\boldsymbol{\theta}\|_{\ell,\Omega} \quad \text{for all } \boldsymbol{\theta} \in H^\ell(\Omega;\mathbb{R}^2). \tag{3}$$

Defining $\boldsymbol{\gamma}^I = t^{-2}(\boldsymbol{\theta}^I - \nabla w^I)$, it follows from the commutative property $\boldsymbol{\pi}_{\boldsymbol{\Theta}}\nabla w = \nabla\pi_W w$ that

$$\boldsymbol{\pi}_{\boldsymbol{\Theta}}\boldsymbol{\gamma} = t^{-2}\boldsymbol{\pi}_{\boldsymbol{\Theta}}(\boldsymbol{\theta} - \nabla w) = t^{-2}(\boldsymbol{\pi}_{\boldsymbol{\Theta}}\boldsymbol{\theta} - \nabla\pi_W w) = t^{-2}(\boldsymbol{\theta}^I - \nabla w^I) = \boldsymbol{\gamma}^I.$$

Thus $\boldsymbol{\gamma}^I$ interpolates $\boldsymbol{\gamma}$ and for $0 \leq s \leq \ell$ and $1 \leq \ell \leq k$ we have

$$\|\boldsymbol{\gamma} - \boldsymbol{\gamma}^I\|_{s,h} \leq ch^{\ell-s}\|\boldsymbol{\gamma}\|_{\ell,\Omega} \quad \text{for all } \boldsymbol{\gamma} \in H^\ell(\Omega;\mathbb{R}^2). \tag{4}$$

To develop our dG with WOPSIP for the Reissner–Mindlin model, we need to define the following auxiliary norms

$$\begin{aligned}
\|v\|_h^2 &:= \sum_{T\in\mathcal{T}} \|\nabla_h v\|_T^2 + \sum_{E\in\mathcal{E}} \frac{\sigma_2}{h_E^\rho}\|\boldsymbol{\Pi}^{k-1}[v]\|_E^2 \quad \text{for all } v \in H^1(\mathcal{T});\\
\|\boldsymbol{\eta}\|_h^2 &:= \sum_{T\in\mathcal{T}} \|e_h(\boldsymbol{\eta})\|_T^2 + \sum_{E\in\mathcal{E}} \frac{\sigma_1}{h_E}\|[\![\boldsymbol{\eta}]\!]\|_E^2 \quad \text{for all } \boldsymbol{\eta} \in H^1(\mathcal{T};\mathbb{R}^2);\\
\|\boldsymbol{\eta}, v, \boldsymbol{\phi}\|_h^2 &:= \|\boldsymbol{\eta}\|_h^2 + \|v\|_h^2 + t^2\|\boldsymbol{\phi}\|_{0,h}^2 \text{ for all } (\boldsymbol{\eta}, v, \boldsymbol{\phi}) \in H^1(\mathcal{T};\mathbb{R}^2)\\
&\qquad \times H^1(\mathcal{T}) \times L^2(\mathcal{T};\mathbb{R}^2).
\end{aligned}$$

Here, and throughout this paper, ρ, σ_1 and σ_2 are positive constants that will be defined below. The operator $\boldsymbol{\Pi}^{k-1}$ is the orthogonal projections from $L^2(E;\mathbb{R}^2)$ onto $\mathcal{P}_{k-1}(E;\mathbb{R}^2)$ where $\mathcal{P}_{k-1}(E)$ is the space of polynomials of degree less than or equal to $k-1$ on E.

3 Combined Formulation of dG and WOPSIP

The new formulation for the Reissner–Mindlin model that combines WOPSIP and dG uses the following form on $\left(H^{1+\kappa}(\mathcal{T};\mathbb{R}^2)\times H^1(\mathcal{T})\times L^2(\mathcal{T};\mathbb{R}^2)\right)^2$ with $\kappa > 1/2$, namely

$$\begin{aligned}\mathcal{A}_h(\boldsymbol{\xi},u,\boldsymbol{\zeta};\boldsymbol{\eta},v,\boldsymbol{\phi}) = \mathcal{B}_h(\boldsymbol{\xi},\boldsymbol{\eta}) &+ \mathcal{J}(u,v)\\ &+ \sum_{T\in\mathcal{T}}\big((\boldsymbol{\zeta},\boldsymbol{\eta}-\nabla_h v)_T\\ &-(\boldsymbol{\xi}-\nabla_h u,\boldsymbol{\phi})_T + t^2\mu^{-1}(\boldsymbol{\zeta},\boldsymbol{\phi})_T\big)\end{aligned} \tag{5}$$

where

$$\mathcal{B}_h(\boldsymbol{\xi},\boldsymbol{\eta}) = a_h(\boldsymbol{\xi},\boldsymbol{\eta}) - \sum_{E\in\mathcal{E}}\langle\{\mathcal{C}e_h(\boldsymbol{\xi})\},[\![\boldsymbol{\eta}]\!]\rangle_E - \delta\sum_{E\in\mathcal{E}}\langle\{\mathcal{C}e_h(\boldsymbol{\eta})\},[\![\boldsymbol{\xi}]\!]\rangle_E + J(\boldsymbol{\xi},\boldsymbol{\eta}),$$

$$a_h(\boldsymbol{\xi},\boldsymbol{\eta}) = \sum_{T\in\mathcal{T}}(\mathcal{C}e_h(\boldsymbol{\xi}),e_h(\boldsymbol{\eta}))_T,\quad J(\boldsymbol{\xi},\boldsymbol{\eta}) = \sum_{E\in\mathcal{E}}\frac{\sigma_1}{h_E}\langle[\![\boldsymbol{\xi}]\!],[\![\boldsymbol{\eta}]\!]\rangle_E$$

and

$$\mathcal{J}(u,v) = \sum_{E\in\mathcal{E}}\frac{\sigma_2}{h_E^{\rho}}\langle\boldsymbol{\Pi}^{k-1}[u],\boldsymbol{\Pi}^{k-1}[v]\rangle_E.$$

Moreover, σ_1 and σ_2 are the penalty parameters and $\rho > 1$ (which is dependent on k) will be defined below. The parameter δ is the symmetric/nonsymmetric bilinear form parameter with $-1 \le \delta \le 1$. This gives the following energy norms

$$|||\boldsymbol{\eta},v,\boldsymbol{\phi}|||^2 = \|e_h(\boldsymbol{\eta})\|_{0,h}^2 + t^2\|\boldsymbol{\phi}\|_{0,h}^2 + J(\boldsymbol{\eta},\boldsymbol{\eta}) + \mathcal{J}(v,v) + \sum_{E\in\mathcal{E}}\frac{h_E}{\sigma_1}\|\{\mathcal{C}e_h(\boldsymbol{\eta})\}\|_E^2;$$

$$|||\boldsymbol{\eta},v,\boldsymbol{\phi}|||_*^2 = |||\boldsymbol{\eta},v,\boldsymbol{\phi}|||^2 + t^{-2}\|\boldsymbol{\eta}-\nabla_h v\|_{0,h}^2$$

for all $(\boldsymbol{\eta},v,\boldsymbol{\phi})\in H^{1+\kappa}(\mathcal{T};\mathbb{R}^2)\times H^1(\mathcal{T})\times L^2(\mathcal{T};\mathbb{R}^2)$.

The weakly over-penalized interior penalty associated with the discontinuous Galerkin (dGWOPIP) method for the Reissner–Mindlin model reads: Seek $(\boldsymbol{\theta}_h,\ w_h,\boldsymbol{\gamma}_h)\in\mathcal{P}_{k-1}(\mathcal{T};\mathbb{R}^2)\times\mathcal{P}_k(\mathcal{T})\times\mathcal{P}_{k-1}(\mathcal{T};\mathbb{R}^2)$ such that

$$\begin{aligned}&\mathcal{A}_h(\boldsymbol{\theta}_h,w_h,\boldsymbol{\gamma}_h;\boldsymbol{\eta},v,\boldsymbol{\phi}) = (g,v)_\Omega + (\boldsymbol{f},\boldsymbol{\eta})_\Omega\\ &\quad\text{for all }(\boldsymbol{\eta},v,\boldsymbol{\phi})\in\mathcal{P}_{k-1}(\mathcal{T};\mathbb{R}^2)\times\mathcal{P}_k(\mathcal{T})\times\mathcal{P}_{k-1}(\mathcal{T};\mathbb{R}^2).\end{aligned} \tag{6}$$

This formulation differs from those of [2] and [1] as follows: (a) the dGWOPIP formulation does not have the terms $\langle\{\boldsymbol{\gamma}_h\},[v]\rangle_{\mathcal{E}}$ and $\langle\{\boldsymbol{\phi}\},[w_h]\rangle_{\mathcal{E}}$ as in [2] and [1]; (b) the dGWOPIP formulation does not need reduced integration while in [2] it is needed; (c) the dGWOPIP formulation over-penalizes the jump of the displacement (even for $k = 2$ the penalization is different); and (d) the dGWOPIP formulation involves the projection of the jump while in [1] the projection is not present.

Lemma 1 *Let $\mathcal{T}$ be a shape-regular partition, then there exists a positive constant c independent of h and t, such that for all $((\boldsymbol{\xi},u,\boldsymbol{\zeta}),(\boldsymbol{\eta},v,\boldsymbol{\phi}))\in(H^{1+\kappa}(\mathcal{T};\mathbb{R}^2)\times H^1(\mathcal{T})\times L^2(\mathcal{T},\mathbb{R}^2))^2$ satisfies*

$$|\mathcal{A}_h(\boldsymbol{\xi},u,\boldsymbol{\zeta};\boldsymbol{\eta},v,\boldsymbol{\phi})| \le c|||\boldsymbol{\xi},u,\boldsymbol{\zeta}|||_*|||\boldsymbol{\eta},v,\boldsymbol{\phi}|||_*.$$

Lemma 2 *Let $\mathcal{T}$ be a shape-regular partition and assume that the Lamé coefficients are uniformly bounded. Then, there exists a positive constant $\tilde{\sigma}_1$, such that, if $\sigma_1 > \tilde{\sigma}_1$, there exists a positive constant ς independent of h and t such that,*

$$\varsigma |||\boldsymbol{\eta}, v, \boldsymbol{\phi}|||^2 \leq \mathcal{A}_h(\boldsymbol{\eta}, v, \boldsymbol{\phi}; \boldsymbol{\eta}, v, \boldsymbol{\phi})$$

for all $(\boldsymbol{\eta}, v, \boldsymbol{\phi}) \in \mathcal{P}_{k-1}(\mathcal{T}; \mathbb{R}^2) \times \mathcal{P}_k(\mathcal{T}) \times \mathcal{P}_{k-1}(\mathcal{T}; \mathbb{R}^2)$ and for any choice of $\sigma_2 > \tilde{\sigma}_2 > 0$ where $\tilde{\sigma}_2$ is arbitrary but fixed.

Proof Let Λ_0, Λ_1 be positive constants such that

$$\Lambda_0 \|e_h(\boldsymbol{\eta})\|_{0,h}^2 \leq |a_h(\boldsymbol{\eta}, \boldsymbol{\eta})| \leq \Lambda_1 \|e_h(\boldsymbol{\eta})\|_{0,h}^2. \tag{7}$$

Then we have

$$\begin{aligned}\mathcal{A}_h(\boldsymbol{\eta}, v, \boldsymbol{\phi}; \boldsymbol{\eta}, v, \boldsymbol{\phi}) - \varsigma |||\boldsymbol{\eta}, v, \boldsymbol{\phi}|||^2 &\geq (\Lambda_0 - \varsigma)\|e_h(\boldsymbol{\eta})\|_{0,h}^2 \\ &+(\mu^{-1} - \varsigma)t^2\|\boldsymbol{\phi}\|_{0,h}^2 + (1-\varsigma)\left(\mathcal{J}(\boldsymbol{\eta}, \boldsymbol{\eta}) + \mathcal{J}(v, v)\right) \\ &-(1+\delta)\sum_{E\in\mathcal{E}} \langle \{\mathcal{C}e_h(\boldsymbol{\eta})\}, [\![\boldsymbol{\eta}]\!]\rangle_E - \varsigma \sum_{E\in\mathcal{E}} \frac{h_E}{\sigma_1}\|\{\mathcal{C}e_h(\boldsymbol{\eta})\}\|_E^2.\end{aligned}$$

For any positive constant ϱ the Cauchy–Schwarz inequality and arithmetic-geometric inequality show that

$$-\langle \{\mathcal{C}e_h(\boldsymbol{\eta})\}, [\![\boldsymbol{\eta}]\!]\rangle_E \geq -\frac{\varrho}{2}\frac{h_E}{\sigma_1}\|\{\mathcal{C}e_h(\boldsymbol{\eta})\}\|_E^2 - \frac{1}{2\varrho}\frac{\sigma_1}{h_E}\|[\![\boldsymbol{\eta}]\!]\|_E^2.$$

An inverse inequality implies that

$$h_E \|\{\mathcal{C}e_h(\boldsymbol{\eta})\}\|_E^2 \leq c\|e_h(\boldsymbol{\eta})\|_T^2. \tag{8}$$

With this we obtain

$$\begin{aligned}\mathcal{A}_h(\boldsymbol{\eta}, v, \boldsymbol{\phi}; \boldsymbol{\eta}, v, \boldsymbol{\phi}) - \varsigma |||\boldsymbol{\eta}, v, \boldsymbol{\phi}|||^2 &\geq \left(\mu^{-1} - \varsigma\right) t^2 \|\boldsymbol{\phi}\|_{0,h}^2 + (1-\varsigma)\mathcal{J}(v, v) \\ &+\left(\Lambda_0 - \varsigma\left(1 + \frac{c}{\sigma_1}\right) - (1+\delta)\frac{\varrho c}{2\sigma_1}\right)\|e_h(\boldsymbol{\eta})\|_{0,h}^2 \\ &+\left(1 - \varsigma - \frac{(1+\delta)}{2\varrho}\right)\mathcal{J}(\boldsymbol{\eta}, \boldsymbol{\eta}).\end{aligned}$$

If $\delta \neq -1$ we first choose ϱ such that $1 - \frac{(1+\delta)}{2\varrho} > 0$. In the following we choose $\tilde{\sigma}_1$ such that $\Lambda_0 - (1+\delta)\frac{\varrho c}{2\tilde{\sigma}_1} > 0$. The assumption follows with $\varsigma > 0$ be such that

$$\varsigma < \min\left\{1,\ \mu^{-1},\ 1 - \frac{(1+\delta)}{2\varrho},\ \frac{\Lambda_0 - (1+\delta)\frac{\varrho c}{2\tilde{\sigma}_1}}{1 + \frac{c}{\tilde{\sigma}_1}}\right\}.$$

On the other hand, if $\delta = -1$ the assumption follows for any choice of $\sigma_1 > \tilde{\sigma}_1 > 0$, with $\tilde{\sigma}_1$ arbitrary but fixed, if $\varsigma > 0$ be such that

$$\varsigma < \min\left\{1,\ \mu^{-1},\ \frac{\Lambda_0}{1 + \frac{c}{\tilde{\sigma}_1}}\right\}.$$

□

Note that if $\delta \neq -1$, any choice of $\varrho > 1$ implies that $1 - \frac{(1+\delta)}{2\varrho} > 0$, for all $\delta \in (-1, 1]$. Thereby, if $\tilde{\sigma}_1 > \frac{(1+\delta)c\varrho}{2\Lambda_0}$, where c is given by (8) and Λ_0 by (7), the assumption follows with the suitable ς.

4 Residual Estimates

This section provides some residual estimates that will be necessary in Sect. 5 to prove the a priori error estimates. In order to achieve sharp residual estimates, we will also prove a discrete Helmholtz decomposition.

The first theorem gives various preliminary residual estimates for some quantities of interest. The proof of this result follows directly from that of Theorem 5.

Theorem 3 *Let $g_h \in \mathcal{P}_k(\mathcal{T})$, $\boldsymbol{f}_h \in \mathcal{P}_{k-1}(\mathcal{T};\mathbb{R}^2)$ and $\boldsymbol{\phi}, \boldsymbol{\eta} \in \mathcal{P}_{k-1}(\mathcal{T};\mathbb{R}^2)$ be arbitrary. Then, it holds for all $T \in \mathcal{T}$ and for all $E \in \mathcal{E}(\Omega)$ that*

$$
\begin{aligned}
&h_T\|\boldsymbol{f}_h + \mathbf{div}_h\mathcal{C}e_h(\boldsymbol{\eta}) - \boldsymbol{\phi}\|_T \lesssim \|e(\boldsymbol{\theta}) - e_h(\boldsymbol{\eta})\|_T + h_T\|\boldsymbol{\gamma} - \boldsymbol{\phi}\|_T \\
&\quad + \|\boldsymbol{f}_T - \boldsymbol{f}_h\|_{H^{-1}(T)}, \\
&h_T\|g_h - \mathrm{div}_h(\boldsymbol{\phi})\|_T \lesssim \|\boldsymbol{\gamma} - \boldsymbol{\phi}\|_T + \|g_T - g_h\|_{H^{-1}(T)}, \\
&h_E^{1/2}\|[\![\mathcal{C}e_h(\boldsymbol{\eta})]\!]\|_E \lesssim \|e(\boldsymbol{\theta}) - e_h(\boldsymbol{\eta})\|_{\omega_E} + h_E\|\boldsymbol{\gamma} - \boldsymbol{\phi}\|_{\omega_E} + \|\boldsymbol{f}_E - \boldsymbol{f}_h\|_{H^{-1}(\omega_E)}, \\
&h_E^{1/2}\|[\boldsymbol{\phi}]\|_E \lesssim \|\boldsymbol{\gamma} - \boldsymbol{\phi}\|_{\omega_E} + \|g_E - g_h\|_{H^{-1}(\omega_E)}.
\end{aligned}
$$

Here, and throughout this paper, $g_T = g_{|T}$, $g_E = g_{|\omega_E}$ (idem for $\boldsymbol{f}$) and ω_E is the patch of two triangles sharing the face E. Moreover, an inequality $a \lesssim b$ replaces $a \le Cb$ with a multiplicative (t, h_T, h_E)-independent constant C.

If we apply Theorem 3 directly, the our a priori error estimate (see Theorem 8 below) will be optimal with respect to h but will not be optimal with respect to t because the norm $\|\boldsymbol{\gamma}\|_{k-1}$, which is not bounded as t tends to zero, will appear on the right-hand side. The Helmholtz decomposition provides a remedy for this. As in [1], we assume that $\boldsymbol{\gamma}$ has a Helmholtz decomposition in the form

$$
\boldsymbol{\gamma} = \nabla\alpha + \mathrm{Curl}(\beta) \ \text{ with } \alpha \in H^k(\Omega) \cap H_0^1(\Omega) \text{ and } \beta \in H^k(\Omega)/\mathbb{R}. \tag{9}
$$

In addition we will assume that

$$
\|\alpha\|_{k,\Omega} + \|\beta\|_{k,\Omega} \lesssim \|\boldsymbol{\gamma}\|_{k-1,\Omega}, \ \text{ and } \|\alpha\|_{k,\Omega} + \|\beta\|_{k-1,\Omega} \lesssim \|\boldsymbol{\gamma}\|_{H^{k-2}(\mathrm{div})}, \tag{10}
$$

where $H^{k-2}(\mathrm{div})$ is the space of vectors in $H^{k-2}(\Omega;\mathbb{R}^2)$ that have divergence in $H^{k-2}(\Omega)$. We note that this result holds for $k = 2$ (at least) if Ω is a convex polygon and if we have H^k regularity for the Poisson problem $\Delta\alpha = \mathrm{div}(\boldsymbol{\gamma})$.

In order to obtain a result similar to that of Theorem 3 using Helmholtz decomposition, we first need to prove the follow discrete version of Helmholtz decomposition. This consists of splitting any piecewise polynomial $\boldsymbol{\phi} \in \mathcal{P}_{k-1}(\mathcal{T};\mathbb{R}^2)$ into two parts, where one is the gradient of $z \in \mathcal{P}_k(\mathcal{T})$ and the other is the curl of $r \in \mathcal{P}_k(\mathcal{T})$. To stabilize this split we assume that $\|z\|_{1,h} + \|r\|_{1,h} \lesssim \|\boldsymbol{\phi}\|_{0,h}$. Another version of discrete Helmholtz decomposition can be found in [31, Lemma 5.2].

Lemma 4 *Any piecewise polynomial $\boldsymbol{\phi} \in \mathcal{P}_{k-1}(\mathcal{T};\mathbb{R}^2)$ can be written as $\nabla_h z + \mathrm{Curl}_h(r)$ where $z \in \mathcal{P}_k(\mathcal{T})$ and $r \in \mathcal{P}_k(\mathcal{T})$.*

Proof It is suffice to prove the assumption for a generic $T \in \mathcal{T}$. Given $\boldsymbol{\phi} \in \mathcal{P}_{k-1}(T;\mathbb{R}^2)$ suppose that $\boldsymbol{\phi} = (x_1^a x_2^b, 0)$, where $a, b \in \mathbb{N}$ are such that $a + b = k - 1$ (the maximal degree of $\mathcal{P}_{k-1}(T;\mathbb{R}^2)$). If $b = 0$ set $z = \frac{1}{a+1}x_1^{a+1}$ and $r = 0$. If $a = 0$ set $z = 0$ and $r = \frac{1}{b+1}x_2^{b+1}$. In all other cases set $z_1 = \frac{1}{a+1}x_1^{a+1}x_2^b$ and $r_1 = \frac{b}{(a+1)(a+2)}x_1^{a+2}x_2^{b-1}$ and observe the following.

If $a = k - 2$ then $b = 1$ and the assumption reads

$$\nabla z_1 + \mathrm{Curl}(r_1) = \left(x_1^a x_2^b,\ \frac{b}{a+1} x_1^{a+1} x_2^{b-1}\right) + \left(0,\ -\frac{b}{a+1} x_1^{a+1} x_2^{b-1}\right) = \boldsymbol{\phi}$$

with a solution $z = z_1$ and $r = r_1$. Otherwise, it holds that

$$\begin{aligned}\nabla z_1 + \mathrm{Curl}(r_1) &= \left(x_1^a x_2^b,\ \frac{b}{a+1} x_1^{a+1} x_2^{b-1}\right)\\ &\quad + \left(\frac{b(b-1)}{(a+1)(a+2)} x_1^{a+2} x_2^{b-2},\ -\frac{b}{a+1} x_1^{a+1} x_2^{b-1}\right)\\ &= \boldsymbol{\phi} + \left(\frac{b(b-1)}{(a+1)(a+2)} (x_1^{a+2} x_2^{b-2},\ 0) =: \boldsymbol{\phi} + \boldsymbol{\phi}_1\right).\end{aligned}$$

If $a = k - 3$ set $z_2 = -c_1 \frac{1}{a+3} x_1^{a+3} x_2^{b-2}$ and $r_2 = 0$, where $c_1 = \frac{b(b-1)}{(a+1)(a+2)}$. This allows the solution $z = z_1 + z_2$ and $r = r_1$ because

$$\nabla z_2 + \mathrm{Curl}(r_2) = \left(-c_1 (x_1^{a+2} x_2^{b-2},\ 0) - c_1 (0,\ 0)\right) = -\boldsymbol{\phi}_1.$$

Otherwise, set $z_3 = z_2$ and $r_3 = -c_1 \frac{b-2}{(a+3)(a+4)} x_1^{a+4} x_2^{b-3}$.

If $a = k - 4$ then $b = 3$ and

$$\begin{aligned}\nabla z_3 + \mathrm{Curl}(r_3) &= -c_1 \left(x_1^{a+2} x_2^{b-2},\ \frac{b-2}{a+3} x_1^{a+3} x_2^{b-3}\right)\\ &\quad + c_1 \left(0,\ \frac{b-2}{a+3} x_1^{a+3} x_2^{b-3}\right) = -\boldsymbol{\phi}_1.\end{aligned}$$

In this case the process finishes with the solution $z = z_1 + z_3$ and $r = r_1 + r_3$. Otherwise,

$$\begin{aligned}\nabla z_3 + \mathrm{Curl}(r_3) &= -c_1 \left(x_1^{a+2} x_2^{b-2},\ c_2 x_1^{a+3} x_2^{b-3}\right)\\ &\quad + c_1 \left(-\frac{c_2(b-3)}{(a+4)} x_1^{a+4} x_2^{b-4},\ c_2 x_1^{a+3} x_2^{b-3}\right)\\ &= -\boldsymbol{\phi}_1 \left(-c_1 c_2 \frac{(b-3)}{(a+4)} (x_1^{a+4} x_2^{b-4},\ 0) =: -\boldsymbol{\phi}_1 + \boldsymbol{\phi}_2\right),\end{aligned}$$

where $c_2 = \frac{b-2}{a+3}$. It is easily observed that on continuing the process, after j ($j \leq k - 1$) steps, we found the solution.

If the first component of $\boldsymbol{\phi}$ is a sum of parcels we perform this process for each one of them. To complete, if $\boldsymbol{\phi}$ has two components different from zero, then we apply this process to each component. □

Clearly, the decomposition of Lemma 4 is not unique, for example, by adding any constant to z and/or r the result will continue to be a solution. Exploring this freedom, we will require for any $\boldsymbol{\phi} = \nabla_h z + \mathrm{Curl}_h(r) \in \mathcal{P}_{k-1}(\mathcal{T}, \mathbb{R}^2)$ that $\int_T z\,dx = \int_T \alpha\,dx$ and $\int_T r\,dx = \int_T \beta\,dx\ \forall\ T \in \mathcal{T}$, where α and β are given by (9).

In the following down we will apply Lemma 4 only to the subspace of $\mathcal{P}_{k-1}(\mathcal{T}, \mathbb{R}^2)$ which consists of all elements $\boldsymbol{\phi} \in \mathcal{P}_{k-1}(\mathcal{T}, \mathbb{R}^2)$ such that $z,\ r \in H^1(\Omega) \cap \mathcal{P}_k(\mathcal{T})$, where z and r are the counterpart of $\boldsymbol{\phi}$, that is, $\boldsymbol{\phi} = \nabla z + \mathrm{Curl}(r)$. Using this subspace our results for Theorems 5 and 7 become clearer, and the proof of error estimates for the solution of the dGWOPIP given in Theorem 8 is not affected.

Under the Helmholtz decomposition hypothesis the next theorem provides residual estimates that improve those of Theorem 3. Note that the proof of the next theorem is based on the idea described in [26, Lemma 2.2].

Theorem 5 *Let $\boldsymbol{\eta}, \boldsymbol{\phi} \in \mathcal{P}_{k-1}(\mathcal{T};\mathbb{R}^2)$ be arbitrary but such that $\boldsymbol{\phi} = \nabla z + \mathrm{Curl}(r)$ with $z, r \in \mathcal{P}_k(\mathcal{T}) \cap H^1(\Omega)$. Then, for $g_h \in \mathcal{P}_k(\mathcal{T})$ and $\boldsymbol{f}_h \in \mathcal{P}_{k-1}(\mathcal{T};\mathbb{R}^2)$ it holds for all $T \in \mathcal{T}$ and for all $E \in \mathcal{E}(\Omega)$ that*

$$h_T\|\boldsymbol{f}_h + \mathbf{div}_h\mathcal{C}e_h(\boldsymbol{\eta}) - \boldsymbol{\phi}\|_T \lesssim \|e(\boldsymbol{\theta}) - e_h(\boldsymbol{\eta})\|_T + \|\alpha - z\|_T + \|\beta - r\|_T + \|\boldsymbol{f}_T - \boldsymbol{f}_h\|_{H^{-1}(T)}, \tag{11a}$$

$$h_T\|g_h - \mathrm{div}_h(\boldsymbol{\phi})\|_T \lesssim \|\nabla(\alpha - z)\|_T + \|g_T - g_h\|_{H^{-1}(T)}, \tag{11b}$$

$$h_E^{1/2}\|[\![\mathcal{C}e_h(\boldsymbol{\eta})]\!]\|_E \lesssim \|e(\boldsymbol{\theta}) - e_h(\boldsymbol{\eta})\|_{\omega_E} + \|\alpha - z\|_{\omega_E} + \|\beta - r\|_{\omega_E} + \|\boldsymbol{f}_E - \boldsymbol{f}_h\|_{H^{-1}(\omega_E)}, \tag{11c}$$

$$h_E^{1/2}\|[\boldsymbol{\phi}]\|_E \lesssim \|\nabla(\alpha - z)\|_{\omega_E} + \|g_E - g_h\|_{H^{-1}(\omega_E)}. \tag{11d}$$

Proof Let $b_T \in H_0^1(T)$ be the bubble function that takes the value of one at the barycenter of T. Then,

$$\|b_T(\boldsymbol{f}_h + \mathbf{div}_h(\mathcal{C}e_h(\boldsymbol{\eta})) - \boldsymbol{\phi})\|_T \le \|\boldsymbol{f}_h + \mathbf{div}_h(\mathcal{C}e_h(\boldsymbol{\eta})) - \boldsymbol{\phi}\|_T. \tag{12}$$

As we are dealing with a finite dimension there exists a positive constant c such that

$$c\|\boldsymbol{f}_h + \mathbf{div}_h(\mathcal{C}e_h(\boldsymbol{\eta})) - \boldsymbol{\phi}\|_T^2 \le \|b_T^{1/2}(\boldsymbol{f}_h + \mathbf{div}_h(\mathcal{C}e_h(\boldsymbol{\eta})) - \boldsymbol{\phi})\|_T^2.$$

Note that $\boldsymbol{\vartheta} := b_T(\boldsymbol{f}_h + \mathbf{div}_h(\mathcal{C}e_h(\boldsymbol{\eta})) - \boldsymbol{\phi}) \in H_0^1(T;\mathbb{R}^2)$. Hence

$$\begin{aligned} c\int_T (\boldsymbol{f}_h + \mathbf{div}_h(\mathcal{C}e_h(\boldsymbol{\eta})) - \boldsymbol{\phi})^2\,dx &\le \int_T \boldsymbol{\vartheta}\cdot(\boldsymbol{f}_h + \mathbf{div}_h(\mathcal{C}e_h(\boldsymbol{\eta})) - \boldsymbol{\phi})\,dx \\ = \int_T \boldsymbol{f}_T\cdot\boldsymbol{\vartheta}\,dx + \int_T \boldsymbol{\vartheta}\cdot(\mathbf{div}_h(\mathcal{C}e_h(\boldsymbol{\eta})) - \boldsymbol{\phi})\,dx \\ + \int_T (\boldsymbol{f}_h - \boldsymbol{f}_T)\cdot\boldsymbol{\vartheta}\,dx =: \Upsilon_1 + \Upsilon_2 + \Upsilon_3. \end{aligned}$$

Let $\tilde{\boldsymbol{\vartheta}}$ be the extension of $\boldsymbol{\vartheta}$ by zero outside of T. Then

$$\begin{aligned} \Upsilon_1 + \Upsilon_2 &= \int_T \boldsymbol{f}_T\cdot\tilde{\boldsymbol{\vartheta}}\,dx + \int_T \boldsymbol{\vartheta}\cdot(\mathbf{div}_h(\mathcal{C}e_h(\boldsymbol{\eta})) - \boldsymbol{\phi})\,dx \\ &= \int_\Omega \left(\mathcal{C}e(\boldsymbol{\theta}) : e_h(\tilde{\boldsymbol{\vartheta}}) + \boldsymbol{\gamma}\cdot\tilde{\boldsymbol{\vartheta}}\right)dx - \int_T (\mathcal{C}e_h(\boldsymbol{\eta}) : e_h(\boldsymbol{\vartheta}) + \boldsymbol{\phi}\cdot\boldsymbol{\vartheta})\,dx \\ &= \int_T (\mathcal{C}e(\boldsymbol{\theta}) - \mathcal{C}e_h(\boldsymbol{\eta})) : e_h(\boldsymbol{\vartheta})\,dx + \int_T (\boldsymbol{\gamma} - \boldsymbol{\phi})\cdot\boldsymbol{\vartheta}\,dx, \end{aligned}$$

where we use (1). Using the Helmholtz decomposition (9), integration by parts and the properties of the bubble function we obtain

$$\int_T (\boldsymbol{\gamma} - \boldsymbol{\phi})\cdot\boldsymbol{\vartheta}\,dx = -\int_T (\alpha - z)div(\boldsymbol{\vartheta})\,dx + \int_T (\beta - r)rot(\boldsymbol{\vartheta})\,dx.$$

From Cauchy–Schwarz inequality and inverse inequality

$$\begin{aligned} \Upsilon_1 + \Upsilon_2 &\lesssim \|\mathcal{C}e(\boldsymbol{\theta}) - \mathcal{C}e_h(\boldsymbol{\eta})\|_T\|e_h(\boldsymbol{\vartheta})\|_T + \|div(\boldsymbol{\vartheta})\|_T\|\alpha - z\|_T \\ &+ \|rot(\boldsymbol{\vartheta})\|_T\|\beta - r\|_T \lesssim h_T^{-1}\left(\|e(\boldsymbol{\theta}) - e_h(\boldsymbol{\eta})\|_T + \|\alpha - z\|_T + \|\beta - r\|_T\right)\|\boldsymbol{\vartheta}\|_T. \end{aligned}$$

In the same way we obtain

$$\Upsilon_3 \lesssim \|\boldsymbol{f}_h - \boldsymbol{f}_T\|_{H^{-1}(T)}\|\boldsymbol{\vartheta}\|_{H^1(T)} \lesssim h_T^{-1}\|\boldsymbol{f}_h - \boldsymbol{f}_T\|_{H^{-1}(T)}\|\boldsymbol{\vartheta}\|_T.$$

Combining this and using (12) we complete the proof of inequality (11a).

For the second inequality, proceeding in a similar way, we obtain

$$c\int_T (g_h - \mathrm{div}_h(\boldsymbol{\phi}))^2\,dx \le \|g_h - g_T\|_{H^{-1}(T)}\|\vartheta\|_{H^1(T)} + \int_T (\boldsymbol{\gamma} - \boldsymbol{\phi})\cdot\nabla\vartheta\,dx.$$

Using Helmholtz decomposition, integration by parts (term with Curl) and the properties of the bubble function, together with inverse inequality and Cauchy–Schwarz inequality, we obtain

$$\begin{aligned}\int_T (\boldsymbol{\gamma} - \boldsymbol{\phi})\cdot\nabla\vartheta\,dx &= \int_T \nabla(\alpha - z)\cdot\nabla\vartheta\,dx + \int_T (\beta - r)rot(\nabla\vartheta)\,dx\\ &\quad - \int_{\partial T}(\beta - r)\nabla\vartheta\cdot\boldsymbol{\tau}\,dx \lesssim \|\nabla(\alpha - z)\|_T\|\nabla\vartheta\|_T\\ &\lesssim h_T^{-1}\|\nabla(\alpha - z)\|_T\|\vartheta\|_T,\end{aligned}$$

because $\nabla\vartheta\cdot\boldsymbol{\tau} = 0$ on ∂T. The combination of the previous arguments concludes the proof of inequality (11b).

To prove the third inequality, let $b_E \in H_0^1(\omega_E)$ be the edge-bubble function that takes the value of one at the barycenter of the edge E. Let Ψ be the extension of $[\![\mathcal{C}e_h(\boldsymbol{\eta})]\!]$ to ω_E by constants along lines orthogonal to the edge E and set $\boldsymbol{\vartheta} = b_E\Psi$. Then,

$$\|\boldsymbol{\vartheta}\|_{\omega_E} \lesssim \|h_E^{1/2}[\![\mathcal{C}e_h(\boldsymbol{\eta})]\!]\|_E \tag{13}$$

and

$$\begin{aligned}\|[\![\mathcal{C}e_h(\boldsymbol{\eta})]\!]\|_E^2 &\lesssim \|b_E^{1/2}[\![\mathcal{C}e_h(\boldsymbol{\eta})]\!]\|_E^2 = \int_E \boldsymbol{\vartheta}\cdot[\![\mathcal{C}e_h(\boldsymbol{\eta})]\!]\,ds\\ &= \sum_{T\in\omega_E}\int_T \mathbf{div}_h(\mathcal{C}e_h(\boldsymbol{\eta}))\cdot\boldsymbol{\vartheta}\,dx + \sum_{T\in\omega_E}\int_T \mathcal{C}e_h(\boldsymbol{\eta}) : e_h(\boldsymbol{\vartheta})\,dx\\ &\quad \pm \sum_{T\in\omega_E}\int_T \boldsymbol{\phi}\cdot\boldsymbol{\vartheta}\,dx \pm \sum_{T\in\omega_E}\int_T \boldsymbol{f}_h\cdot\boldsymbol{\vartheta}\,dx \pm \sum_{T\in\omega_E}\int_T \boldsymbol{f}_E\cdot\boldsymbol{\vartheta}\,dx\\ &= \sum_{T\in\omega_E}\int_T \left(\boldsymbol{f}_h + \mathbf{div}_h(\mathcal{C}e_h(\boldsymbol{\eta})) - \boldsymbol{\phi}\right)\cdot\boldsymbol{\vartheta}\,dx + \sum_{T\in\omega_E}\int_T (\boldsymbol{\phi} - \boldsymbol{\gamma})\cdot\boldsymbol{\vartheta}\,dx\\ &\quad + \sum_{T\in\omega_E}\int_T (\mathcal{C}e_h(\boldsymbol{\eta}) - \mathcal{C}e(\boldsymbol{\theta})) : e_h(\boldsymbol{\vartheta})\,dx + \sum_{T\in\omega_E}\int_T (\boldsymbol{f}_E - \boldsymbol{f}_h)\cdot\boldsymbol{\vartheta}\,dx.\end{aligned}$$

Applying Cauchy–Schwarz inequality

$$\begin{aligned}\|[\![\mathcal{C}e_h(\boldsymbol{\eta})]\!]\|_E^2 &\lesssim \|\boldsymbol{f}_E - \boldsymbol{f}_h\|_{H^{-1}(\omega_E)}\|\boldsymbol{\vartheta}\|_{H^1(\omega_E)} + \sum_{T\in\omega_E}\int_T (\boldsymbol{\phi} - \boldsymbol{\gamma})\cdot\boldsymbol{\vartheta}\,dx\\ &\quad + \sum_{T\in\omega_E}\left(\|\boldsymbol{f}_h + \mathbf{div}_h(\mathcal{C}e_h(\boldsymbol{\eta})) - \boldsymbol{\phi}\|_T\|\boldsymbol{\vartheta}\|_T + \|\mathcal{C}e_h(\boldsymbol{\eta}) - \mathcal{C}e(\boldsymbol{\theta})\|_T\|e_h(\boldsymbol{\vartheta})\|_T\right).\end{aligned} \tag{14}$$

For the last term of the first line, using the Helmholtz decomposition and integration by parts we have

$$\Upsilon_4 := \sum_{T\in\omega_E}\int_T (\boldsymbol{\gamma}-\boldsymbol{\phi})\cdot\boldsymbol{\vartheta}\,dx = -\sum_{T\in\omega_E}\int_T div(\boldsymbol{\vartheta})(\alpha - z)\,dx$$
$$+\sum_{T\in\omega_E}\int_T rot(\boldsymbol{\vartheta})(\beta - r)\,dx + \int_E \{\alpha - z\}[\boldsymbol{\vartheta}]\,ds + \int_E \{\beta - r\}[\boldsymbol{\vartheta}]_{\boldsymbol{\tau}}\,ds.$$

From Cauchy–Schwarz inequality and inverse inequality (as $[\boldsymbol{\vartheta}]=0$ and $[\boldsymbol{\vartheta}]_{\boldsymbol{\tau}}=0$)

$$|\Upsilon_4| \lesssim \sum_{T\in\omega_E}\left(\|div(\boldsymbol{\vartheta})\|_T\|\alpha - z\|_T + \|rot(\boldsymbol{\vartheta})\|_T\|\beta - r\|_T\right)$$
$$\lesssim \sum_{T\in\omega_E} h_T^{-1}\left(\|\alpha - z\|_T + \|\beta - r\|_T\right)\|\boldsymbol{\vartheta}\|_T.$$

The inequality (11c) follows from (13), (14), inverse inequality and (11a).

For the last inequality, let Ψ be the extension of $[\boldsymbol{\phi}]$ to ω_E by constants along lines orthogonal to the edge E. Let $b_E \in H_0^1(\omega_E)$ be the edge-bubble function that takes the value of one at the barycenter of the edge E. Defining $\vartheta = b_E\Psi$ we have that

$$h_E^{-1/2}\|\vartheta\|_{\omega_E} \lesssim \|[\boldsymbol{\phi}]\|_E \tag{15}$$

and

$$\|[\boldsymbol{\phi}]\|_E^2 \lesssim \|b_E^{1/2}[\boldsymbol{\phi}]\|_E^2 = \int_E \vartheta[\boldsymbol{\phi}]\,ds = \sum_{T\in\omega_E}\int_T \mathrm{div}_h(\boldsymbol{\phi})\vartheta\,dx$$
$$+\sum_{T\in\omega_E}\int_T \boldsymbol{\phi}\cdot\nabla\vartheta\,dx \pm \sum_{T\in\omega_E}\int_T g_h\vartheta\,dx \pm \sum_{T\in\omega_E}\int_T g_E\vartheta\,dx$$
$$=\sum_{T\in\omega_E}\int_T (-g_h + \mathrm{div}_h(\boldsymbol{\phi}))\,\vartheta\,dx + \sum_{T\in\omega_E}\int_T (g_h - g_E)\vartheta\,dx$$
$$+\sum_{T\in\omega_E}\int_T (\boldsymbol{\phi}-\boldsymbol{\gamma})\cdot\nabla\vartheta\,dx.$$

Applying Cauchy–Schwarz inequality

$$\|[\boldsymbol{\phi}]\|_E^2 \lesssim +\sum_{T\in\omega_E}\int_T (\boldsymbol{\phi}-\boldsymbol{\gamma})\cdot\nabla\vartheta\,dx$$
$$+\sum_{T\in\omega_E}\left(\|-g_h + \mathrm{div}_h(\boldsymbol{\phi})\|_T\|\vartheta\|_T + \|g_h - g_E\|_{H^{-1}(\omega_E)}\|\vartheta\|_{H^1(\omega_E)}\right). \tag{16}$$

Helmholtz decomposition, integration by parts (in the term with Curl) and the properties of the bubble function, together with inverse inequality and Cauchy–Schwarz inequality, show that

$$\sum_{T\in\omega_E}\int_T(\boldsymbol{\gamma}-\boldsymbol{\phi})\cdot\nabla\vartheta\,dx=\int_E\{\beta-r\}[\nabla\vartheta]_{\boldsymbol{\tau}}\,ds$$
$$+\sum_{T\in\omega_E}\left(\int_T\nabla(\alpha-z)\cdot\nabla\vartheta\,dx-\int_T(\beta-r)rot(\nabla\vartheta)\,dx\right)$$
$$\lesssim\sum_{T\in\omega_E}\|\nabla(\alpha-z)\|_T\|\nabla\vartheta\|_T\lesssim\sum_{T\in\omega_E}h_T^{-1}\|\nabla(\alpha-z)\|_T\|\vartheta\|_T.$$

We complete the proof of inequality (11d) combining the last inequality with (16) and using (15), inverse inequality and (11b). □

5 A Priori Error Analysis

In this section we use the residual estimates to derive an optimal error estimate in the energy norm. Initially, we recall some definitions and results. We begin with the following lemma, which was proved in [15].

Lemma 6 ([15, Lemma 2.2]) *Any $v\in H^1(\mathcal{T})$ satisfies*

$$\sum_{E\in\mathcal{E}}h_E^{-1}\|[v]\|_E^2\lesssim\|v\|_h^2.$$

The following enriching operators use averaging techniques (see [8] and [9] for details): $\mathsf{E}_\mathsf{h}:\mathcal{P}_{k-1}(\mathcal{T};\mathbb{R}^2)\to\mathcal{P}_{k-1}(\mathcal{T};\mathbb{R}^2)\cap H_0^1(\Omega,\mathbb{R}^2)$ such that

$$\left(\sum_{T\in\mathcal{T}}h_E^{-2}\|\mathsf{E}_\mathsf{h}\boldsymbol{\eta}-\boldsymbol{\eta}\|_T^2\right)^{1/2}+\|\nabla_h(\mathsf{E}_\mathsf{h}\boldsymbol{\eta}-\boldsymbol{\eta})\|_\Omega\lesssim\|\boldsymbol{\eta}\|_h \tag{17}$$

and $\mathsf{E}_\mathsf{h}:\mathcal{P}_k(\mathcal{T})\to\mathcal{P}_k(\mathcal{T})\cap H_0^1(\Omega)$ such that

$$\left(\sum_{T\in\mathcal{T}}h_E^{-2}\|\mathsf{E}_\mathsf{h}v-v\|_T^2\right)^{1/2}+\|\nabla_h(\mathsf{E}_\mathsf{h}v-v)\|_\Omega\lesssim\|v\|_h. \tag{18}$$

The previous inequality (17) follows from the enriching operator properties and from discrete Korn's inequality (see [10] and [2]), while (18) follows from the enriching operator properties and from Lemma 6 (recall that $\rho>1$).

We recall now the following definitions of oscillation for a scalar function and for a vector function

$$Osc(g)=\left(\sum_{E\in\mathcal{E}}\|g_E-Pg\|_{H^{-1}(\omega_E)}^2\right)^{1/2}$$

and

$$Osc(\boldsymbol{f})=\left(\sum_{E\in\mathcal{E}}\|\boldsymbol{f}_E-\mathbf{P}\boldsymbol{f}\|_{H^{-1}(\omega_E)}^2\right)^{1/2},$$

where $P:L^2(\Omega)\to\mathcal{P}_k(\mathcal{T})$ is the L^2 orthogonal projection onto $\mathcal{P}_k(\mathcal{T})$ and $\mathbf{P}:L^2(\Omega;\mathbb{R}^2)\to\mathcal{P}_{k-1}(\mathcal{T};\mathbb{R}^2)$ is the L^2 orthogonal projection onto $\mathcal{P}_{k-1}(\mathcal{T};\mathbb{R}^2)$. That is,

$$\int_\Omega(Pg-g)v\,dx=0\ \ \forall\ v\in\mathcal{P}_k(\mathcal{T})\ \text{(analogous for }\mathbf{P}\boldsymbol{f}).$$

As proved in [26], if $\boldsymbol{f} \in L^p(\Omega;\mathbb{R}^2)$ for $p > 1$ we have that

$$Osc(\boldsymbol{f}) \lesssim h^{1-2(1/2-1/q)}\|\boldsymbol{f} - \mathbf{P}\boldsymbol{f}\|_{L^p(\Omega)}, \tag{19}$$

where p and q are such that $1/p + 1/q = 1$. In the same way, it is possible to obtain

$$Osc(g) \lesssim h^{1-2(1/2-1/q)}\|g - Pg\|_{L^p(\Omega)} \tag{20}$$

if $g \in L^p(\Omega)$ for $p > 1$.

In the analysis that follows we will divide the error into two parts, where one is related to the interpolation error and the other is related to the nonconforming error and the consistency error (as in Strang's second lemma). Using enriching operators and the residual estimates we bound the consistency/nonconforming error by one factor similar to an interpolation error plus one factor (the terms with the coefficient $h^{\rho-1}$ in Theorem 7) which will be controlled by over-penalization. To ensure the convergence we need to set ρ appropriately, this means over-penalizing the jump of the displacement in (5). With this strategy we can prove an optimal a priori error bound (see Theorem 8). The motivation comes from [27, Lemma 2.1], where this error decomposition is explicit. We observe that the terms Υ_a and Υ_b defined below are related to the interpolation and consistency/nonconforming error part, respectively. Unfortunately, the analysis of Υ_a is more complex here because the condition N3 necessary for [27, Lemma 2.1] was not established.

Theorem 7 *Let $(\boldsymbol{\theta}, w, \boldsymbol{\gamma})$ be the solution of (1), and let $(\boldsymbol{\theta}_h, w_h, \boldsymbol{\gamma}_h)$ be the solution of the dGWOPIP formulation (6). Assume further that the Helmholtz decomposition (9) is valid. Then we have*

$$\begin{aligned}
|||\boldsymbol{\theta} - \boldsymbol{\theta}_h, w - w_h, \boldsymbol{\gamma} - \boldsymbol{\gamma}_h|||^2 &\lesssim Osc^2(g) + Osc^2(\boldsymbol{f}) + h^{\rho-1}\|\boldsymbol{\gamma}\|^2_\Omega \\
&+ \inf_{\boldsymbol{\phi} \in \mathcal{P}_{k-1}(\mathcal{T};\mathbb{R}^2)} \Bigg\{ \sum_{T\in\mathcal{T}} \Big(\|\alpha - z\|_T^2 + \|\beta - r\|_T^2 + (h_T^{\rho-1} + t^2)\|\boldsymbol{\gamma} - \boldsymbol{\phi}\|_T^2 \\
&+ \|\nabla(\alpha - z)\|_T^2\Big)\Bigg\} + \inf_{\substack{\boldsymbol{\eta} \in \mathcal{P}_{k-1}(\mathcal{T};\mathbb{R}^2)\\ v \in \mathcal{P}_k(\mathcal{T})}} \Bigg\{ t^{-2}\|\boldsymbol{\theta} - \nabla w - (\boldsymbol{\eta} - \nabla_h v)\|^2_{0,h} \\
&+ \|\boldsymbol{\theta} - \boldsymbol{\eta}\|_h^2 + \sum_{E\in\mathcal{E}} \frac{h_E}{\sigma_1}\|\{\mathcal{C}\epsilon_h(\boldsymbol{\theta} - \boldsymbol{\eta})\}\|_E^2 + \mathcal{J}(w - v, w - v)\Bigg\},
\end{aligned}$$

where $\boldsymbol{\phi} = \nabla_h z + \mathrm{Curl}(r)$ with $z, r \in \mathcal{P}_k(\mathcal{T}) \cap H^1(\Omega)$.

Proof **Step 0:** Let $\tilde{\boldsymbol{\eta}} = \boldsymbol{\theta}_h - \boldsymbol{\eta}$, $\tilde{v} = w_h - v$ and $\tilde{\boldsymbol{\phi}} = \boldsymbol{\gamma}_h - \boldsymbol{\phi}$ where $\boldsymbol{\theta}_h$, w_h and $\boldsymbol{\gamma}_h$ are the solution for the dGWOPIP formulation (6), $\boldsymbol{\eta}$ and v are arbitrary in $\mathcal{P}_{k-1}(\mathcal{T};\mathbb{R}^2)$ and $\mathcal{P}_k(\mathcal{T})$, respectively, and $\boldsymbol{\phi}$ is arbitrary in $\mathcal{P}_{k-1}(\mathcal{T};\mathbb{R}^2)$ but such that its counterpart $z, r \in \mathcal{P}_k(\mathcal{T}) \cap H^1(\Omega)$. The coercivity of the bilinear form given by Lemma 2 and (6) implies that

$$\begin{aligned}
&|||\tilde{\boldsymbol{\eta}}, \tilde{v}, \tilde{\boldsymbol{\phi}}|||^2 \lesssim \mathcal{A}_h(\tilde{\boldsymbol{\eta}}, \tilde{v}, \tilde{\boldsymbol{\phi}}; \tilde{\boldsymbol{\eta}}, \tilde{v}, \tilde{\boldsymbol{\phi}}) = \mathcal{A}_h(\boldsymbol{\theta}_h, w_h, \boldsymbol{\gamma}_h; \tilde{\boldsymbol{\eta}}, \tilde{v}, \tilde{\boldsymbol{\phi}}) \\
&\quad -\mathcal{A}_h(\boldsymbol{\eta}, v, \boldsymbol{\phi}; \tilde{\boldsymbol{\eta}}, \tilde{v}, \tilde{\boldsymbol{\phi}}) = (\boldsymbol{f}, \tilde{\boldsymbol{\eta}}) + (g, \tilde{v}) - \mathcal{A}_h(\boldsymbol{\eta}, v, \boldsymbol{\phi}; \tilde{\boldsymbol{\eta}}, \tilde{v}, \tilde{\boldsymbol{\phi}}) \\
&= (\boldsymbol{f}, \tilde{\boldsymbol{\eta}} - \mathsf{E_h}\tilde{\boldsymbol{\eta}}) + (g, \tilde{v} - \mathsf{E_h}\tilde{v}) - \Big(\sum_{T\in\mathcal{T}} (\boldsymbol{\phi}, \tilde{\boldsymbol{\eta}} - \mathsf{E_h}\tilde{\boldsymbol{\eta}} - \nabla_h(\tilde{v} - \mathsf{E_h}\tilde{v}))_T + \mathcal{J}(v, \tilde{v}) \\
&\quad +\mathcal{B}_h(\boldsymbol{\eta}, \tilde{\boldsymbol{\eta}} - \mathsf{E_h}\tilde{\boldsymbol{\eta}})\Big) + (\boldsymbol{f}, \mathsf{E_h}\tilde{\boldsymbol{\eta}}) + (g, \mathsf{E_h}\tilde{v}) - \mathcal{B}_h(\boldsymbol{\eta}, \mathsf{E_h}\tilde{\boldsymbol{\eta}}) \\
&\quad - \sum_{T\in\mathcal{T}} \Big((\boldsymbol{\phi}, \mathsf{E_h}\tilde{\boldsymbol{\eta}} - \nabla_h(\mathsf{E_h}\tilde{v}))_T - (\boldsymbol{\eta} - \nabla_h v, \tilde{\boldsymbol{\phi}})_T + t^2\mu^{-1}(\boldsymbol{\phi}, \tilde{\boldsymbol{\phi}})_T \Big).
\end{aligned} \tag{21}$$

Step 1: Proof of

$$\Upsilon_a \lesssim \left(\sum_{T\in\mathcal{T}} \left(\|e(\boldsymbol{\theta}) - e_h(\boldsymbol{\eta})\|_T^2 + \|\alpha - z\|_T^2 + \|\beta - r\|_T^2 \right) \right)^{1/2} \|\tilde{\boldsymbol{\eta}}\|_h$$
$$+ \left(\sum_{T\in\mathcal{T}} \|\nabla(\alpha - z)\|_T^2 \right)^{1/2} \|\tilde{v}\|_h + (J(\boldsymbol{\theta} - \boldsymbol{\eta}, \boldsymbol{\theta} - \boldsymbol{\eta}))^{1/2} \|\tilde{\boldsymbol{\eta}}\|_h$$
$$+ \left(\sum_{T\in\mathcal{T}} t^2 \|\boldsymbol{\gamma} - \boldsymbol{\phi}\|_T^2 + t^{-2} \|\boldsymbol{\theta} - \nabla w - (\boldsymbol{\eta} - \nabla_h v)\|_T^2 \right)^{1/2} t \|\tilde{\boldsymbol{\phi}}\|_{0,h}, \qquad (22)$$

where

$$\Upsilon_a := -\sum_{T\in\mathcal{T}} \left((\boldsymbol{\phi}, \mathsf{E_h}\tilde{\boldsymbol{\eta}} - \nabla(\mathsf{E_h}\tilde{v}))_T - (\boldsymbol{\eta} - \nabla_h v, \tilde{\boldsymbol{\phi}})_T + t^2\mu^{-1}(\boldsymbol{\phi}, \tilde{\boldsymbol{\phi}})_T \right)$$
$$-\mathcal{B}_h(\boldsymbol{\eta}, \mathsf{E_h}\tilde{\boldsymbol{\eta}}) + (\boldsymbol{f}, \mathsf{E_h}\tilde{\boldsymbol{\eta}}) + (g, \mathsf{E_h}\tilde{v}).$$

For the analysis of Υ_a observe that $\mathsf{E_h}\tilde{\boldsymbol{\eta}} \in H_0^1(\Omega;\mathbb{R}^2) \cap \mathcal{P}_{k-1}(\mathcal{T};\mathbb{R}^2)$ and $\mathsf{E_h}\tilde{v} \in H_0^1(\Omega) \cap \mathcal{P}_k(\mathcal{T})$. Hence (1) and the definition of $\mathcal{B}_h(\cdot,\cdot)$ lead to

$$\Upsilon_a = \sum_{T\in\mathcal{T}} ((\mathcal{C}e(\boldsymbol{\theta}) - \mathcal{C}e_h(\boldsymbol{\eta}), e_h(\mathsf{E_h}\tilde{\boldsymbol{\eta}}))_T + \mu(\boldsymbol{\gamma} - \boldsymbol{\phi}, \mathsf{E_h}\tilde{\boldsymbol{\eta}})_T$$
$$-\mu(\boldsymbol{\gamma} - \boldsymbol{\phi}, \nabla\mathsf{E_h}\tilde{v})_T) + \sum_{E\in\mathcal{E}} \delta\langle[\![\boldsymbol{\eta}]\!], \{\mathcal{C}e_h(\mathsf{E_h}\tilde{\boldsymbol{\eta}})\}\rangle_E$$
$$+ \sum_{T\in\mathcal{T}} \left((\boldsymbol{\eta} - \nabla_h v, \tilde{\boldsymbol{\phi}})_T - t^2\mu^{-1}(\boldsymbol{\phi}, \tilde{\boldsymbol{\phi}})_T \right) =: \Upsilon_1 + \cdots + \Upsilon_6.$$

The Cauchy–Schwarz inequality implies that

$$\Upsilon_1 = \sum_{T\in\mathcal{T}} (\mathcal{C}e(\boldsymbol{\theta}) - \mathcal{C}e_h(\boldsymbol{\eta}), e_h(\mathsf{E_h}\tilde{\boldsymbol{\eta}}))_T$$
$$\lesssim \sum_{T\in\mathcal{T}} \|e(\boldsymbol{\theta}) - e_h(\boldsymbol{\eta})\|_T \|e_h(\mathsf{E_h}\tilde{\boldsymbol{\eta}})\|_T.$$

From Helmholtz decomposition (9) and integration by parts we obtain

$$\Upsilon_2 = -\mu \sum_{T\in\mathcal{T}} \left(\int_T (\alpha - z) div(\mathsf{E_h}\tilde{\boldsymbol{\eta}})\, dx - \int_{\partial T} (\alpha - z) \mathsf{E_h}\tilde{\boldsymbol{\eta}} \cdot \boldsymbol{\nu}\, ds \right)$$
$$-\mu \sum_{T\in\mathcal{T}} \left(\int_T (\beta - r) rot(\mathsf{E_h}\tilde{\boldsymbol{\eta}})\, dx - \int_{\partial T} (\beta - r) \mathsf{E_h}\tilde{\boldsymbol{\eta}} \cdot \boldsymbol{\tau}\, ds \right)$$
$$= -\mu \sum_{T\in\mathcal{T}} \int_T (\alpha - z) div(\mathsf{E_h}\tilde{\boldsymbol{\eta}})\, dx + \mu \sum_{E\in\mathcal{E}} \int_E \{\alpha - z\}[\mathsf{E_h}\tilde{\boldsymbol{\eta}}]\, ds$$
$$-\mu \sum_{T\in\mathcal{T}} \int_T (\beta - r) rot(\mathsf{E_h}\tilde{\boldsymbol{\eta}})\, dx - \mu \sum_{E\in\mathcal{E}} \int_E \{\beta - r\}[\mathsf{E_h}\tilde{\boldsymbol{\eta}}]_{\boldsymbol{\tau}}\, ds.$$

Since $\mathsf{E_h}\tilde{\boldsymbol{\eta}} \in H_0^1(\Omega;\mathbb{R}^2) \cap \mathcal{P}_{k-1}(\mathcal{T};\mathbb{R}^2)$ the Cauchy–Schwarz inequality leads to

$$|\Upsilon_2| \lesssim \sum_{T\in\mathcal{T}} (\|\alpha - z\|_T \|div(\mathsf{E_h}\tilde{\boldsymbol{\eta}})\|_T + \|\beta - r\|_T \|rot(\mathsf{E_h}\tilde{\boldsymbol{\eta}})\|) .$$

For the third term we proceed as in the case of the second, but this time we integrate by parts only the term with $\mathrm{Curl}(\beta - r)$. This leads to

$$\begin{aligned}\Upsilon_3 &= \mu \sum_{T\in\mathcal{T}} \left(\int_T \nabla(\alpha - z)\cdot\nabla(\mathsf{E}_\mathsf{h}\tilde{v})\,dx - \int_T (\beta - r) rot(\nabla\mathsf{E}_\mathsf{h}\tilde{v})\,dx \right) \\ &\quad + \mu \sum_{E\in\mathcal{E}} \int_E \{\beta - r\}[\nabla\mathsf{E}_\mathsf{h}\tilde{v}]_{\boldsymbol{\tau}}\,ds = \mu \sum_{T\in\mathcal{T}} \int_T \nabla(\alpha - z)\cdot\nabla(\mathsf{E}_\mathsf{h}\tilde{v})\,dx.\end{aligned}$$

Subsequently, it follows from Cauchy–Schwarz inequality that

$$|\Upsilon_3| \lesssim \sum_{T\in\mathcal{T}} \|\nabla(\alpha - z)\|_T \|\nabla\mathsf{E}_\mathsf{h}\tilde{v}\|_T.$$

Applying Cauchy–Schwarz inequality, the fourth term leads to

$$\begin{aligned}\Upsilon_4 &\lesssim \sum_{E\in\mathcal{E}} \delta \left\| \sqrt{\frac{\sigma_1}{h_E}} [\![\boldsymbol{\eta}]\!] \right\|_E \left\| \sqrt{\frac{h_E}{\sigma_1}} \{\mathcal{C}e_h(\mathsf{E}_\mathsf{h}\tilde{\boldsymbol{\eta}})\} \right\|_E \\ &\lesssim (J(\boldsymbol{\eta},\boldsymbol{\eta}))^{1/2} \left(\sum_{T\in\mathcal{T}} \|e_h(\mathsf{E}_\mathsf{h}\tilde{\boldsymbol{\eta}})\|_T^2 \right)^{1/2}.\end{aligned}$$

For the last two terms, since $\tilde{\boldsymbol{\phi}} \in L^2(\Omega;\mathbb{R}^2)$, from (1) and Cauchy–Schwarz inequality, it follows that

$$\begin{aligned}\Upsilon_5 + \Upsilon_6 &= \sum_{T\in\mathcal{T}} \left(t^2\mu^{-1}(\boldsymbol{\gamma},\tilde{\boldsymbol{\phi}})_T - (\boldsymbol{\theta} - \nabla w, \tilde{\boldsymbol{\phi}})_T + (\boldsymbol{\eta} - \nabla_h v, \tilde{\boldsymbol{\phi}})_T - t^2\mu^{-1}(\boldsymbol{\phi},\tilde{\boldsymbol{\phi}})_T \right) \\ &\lesssim \sum_{T\in\mathcal{T}} \left(t\|\boldsymbol{\gamma} - \boldsymbol{\phi}\|_T t\|\tilde{\boldsymbol{\phi}}\|_T + t^{-1}\|\boldsymbol{\eta} - \nabla_h v - (\boldsymbol{\theta} - \nabla w)\|_T t\|\tilde{\boldsymbol{\phi}}\|_T \right) \\ &\lesssim \left(\sum_{T\in\mathcal{T}} \left(t^2\|\boldsymbol{\gamma} - \boldsymbol{\phi}\|_T^2 + t^{-2}\|\boldsymbol{\eta} - \nabla_h v - (\boldsymbol{\theta} - \nabla w)\|_T^2 \right) \right)^{1/2} t\|\tilde{\boldsymbol{\phi}}\|_{0,h}.\end{aligned}$$

The combination of these bounds shows that

$$\begin{aligned}\Upsilon_a &\lesssim \sum_{T\in\mathcal{T}} \left(\|e(\boldsymbol{\theta}) - e_h(\boldsymbol{\eta})\|_T \|e_h(\mathsf{E}_\mathsf{h}\tilde{\boldsymbol{\eta}})\|_T + (\|\alpha - z\|_T + \|\beta - r\|_T)\, |\mathsf{E}_\mathsf{h}\tilde{\boldsymbol{\eta}}|_{1,T} \right) \\ &\quad + \left(\sum_{T\in\mathcal{T}} \left(t^2\|\boldsymbol{\gamma} - \boldsymbol{\phi}\|_T^2 + t^{-2}\|\boldsymbol{\eta} - \nabla_h v - (\boldsymbol{\theta} - \nabla w)\|_T^2 \right) \right)^{1/2} t\|\tilde{\boldsymbol{\phi}}\|_{0,h} \\ &\quad + \sum_{T\in\mathcal{T}} \|\nabla(\alpha - z)\|_T \|\nabla\mathsf{E}_\mathsf{h}\tilde{v}\|_T + (J(\boldsymbol{\eta},\boldsymbol{\eta}))^{1/2} \left(\sum_{T\in\mathcal{T}} \|e_h(\mathsf{E}_\mathsf{h}\tilde{\boldsymbol{\eta}})\|_T^2 \right)^{1/2}.\end{aligned}$$

Applying the properties of the enriching operators (17) and (18) we arrive at (22)

Step 2: Proof of

$$\Upsilon_b \lesssim \left(\sum_{T\in\mathcal{T}} \left(\|e(\boldsymbol{\theta}) - e_h(\boldsymbol{\eta})\|_T^2 + \|\alpha - z\|_T^2 + \|\beta - r\|_T^2 \right) \right)^{1/2} \|\tilde{\boldsymbol{\eta}}\|_h$$
$$+ \left(\sum_{T\in\mathcal{T}} (\|\nabla(\alpha - z)\|_T^2 + h_T^{\rho-1}(\|\boldsymbol{\gamma} - \boldsymbol{\phi}\|_T^2 + \|\boldsymbol{\gamma}\|_T^2)) \right)^{1/2} \|\tilde{v}\|_h$$
$$+ \left(\mathcal{J}(w - v, w - v) + Osc^2(g) \right)^{1/2} \|\tilde{v}\|_h$$
$$+ \left(\boldsymbol{J}(\boldsymbol{\theta} - \boldsymbol{\eta}, \boldsymbol{\theta} - \boldsymbol{\eta}) + Osc^2(\boldsymbol{f}) \right)^{1/2} \|\tilde{\boldsymbol{\eta}}\|_h, \tag{23}$$

where

$$\Upsilon_b := (\boldsymbol{f}, \tilde{\boldsymbol{\eta}} - \mathsf{E}_\mathsf{h}\tilde{\boldsymbol{\eta}}) + (g, \tilde{v} - \mathsf{E}_\mathsf{h}\tilde{v}) - \mathcal{B}_h(\boldsymbol{\eta}, \tilde{\boldsymbol{\eta}} - \mathsf{E}_\mathsf{h}\tilde{\boldsymbol{\eta}})$$
$$- \sum_{T\in\mathcal{T}} (\boldsymbol{\phi}, \tilde{\boldsymbol{\eta}} - \mathsf{E}_\mathsf{h}\tilde{\boldsymbol{\eta}} - \nabla_h(\tilde{v} - \mathsf{E}_\mathsf{h}\tilde{v}))_T - \mathcal{J}(v, \tilde{v}).$$

To facilitate the handling we use the definition of the bilinear form $\mathcal{B}_h(\cdot,\cdot)$ to write all terms of Υ_b, that is,

$$\Upsilon_b = (\boldsymbol{f}, \tilde{\boldsymbol{\eta}} - \mathsf{E}_\mathsf{h}\tilde{\boldsymbol{\eta}})_\Omega + (g, \tilde{v} - \mathsf{E}_\mathsf{h}\tilde{v})_\Omega - \sum_{T\in\mathcal{T}} (\mathcal{C}e_h(\boldsymbol{\eta}), e_h(\tilde{\boldsymbol{\eta}} - \mathsf{E}_\mathsf{h}\tilde{\boldsymbol{\eta}}))_T$$
$$- \sum_{T\in\mathcal{T}} (\boldsymbol{\phi}, \tilde{\boldsymbol{\eta}} - \mathsf{E}_\mathsf{h}\tilde{\boldsymbol{\eta}})_T + \sum_{T\in\mathcal{T}} (\boldsymbol{\phi}, \nabla_h(\tilde{v} - \mathsf{E}_\mathsf{h}\tilde{v}))_T$$
$$+ \sum_{E\in\mathcal{E}} \langle \{\mathcal{C}e_h(\boldsymbol{\eta})\}, [\![\tilde{\boldsymbol{\eta}} - \mathsf{E}_\mathsf{h}\tilde{\boldsymbol{\eta}}]\!] \rangle_E + \delta \sum_{E\in\mathcal{E}} \langle \{\mathcal{C}e_h(\tilde{\boldsymbol{\eta}} - \mathsf{E}_\mathsf{h}\tilde{\boldsymbol{\eta}})\}, [\![\boldsymbol{\eta}]\!] \rangle_E$$
$$- \boldsymbol{J}(\boldsymbol{\eta}, \tilde{\boldsymbol{\eta}}) - \mathcal{J}(v, \tilde{v}) =: \Upsilon_1 + \cdots + \Upsilon_9.$$

We start by integrating by parts the first term of the bilinear form $\mathcal{B}_h(\boldsymbol{\eta}, \tilde{\boldsymbol{\eta}} - \mathsf{E}_\mathsf{h}\tilde{\boldsymbol{\eta}})$ in order to obtain

$$- \sum_{T\in\mathcal{T}} (\mathcal{C}e_h(\boldsymbol{\eta}), e_h(\tilde{\boldsymbol{\eta}} - \mathsf{E}_\mathsf{h}\tilde{\boldsymbol{\eta}}))_T = \sum_{T\in\mathcal{T}} (\mathbf{div}_h(\mathcal{C}e_h(\boldsymbol{\eta})), \tilde{\boldsymbol{\eta}} - \mathsf{E}_\mathsf{h}\tilde{\boldsymbol{\eta}})_T$$
$$- \sum_{E\in\mathcal{E}} \langle \{\mathcal{C}e_h(\boldsymbol{\eta})\}, [\![\tilde{\boldsymbol{\eta}} - \mathsf{E}_\mathsf{h}\tilde{\boldsymbol{\eta}}]\!] \rangle_E - \sum_{E\in\mathcal{E}} \langle \{\tilde{\boldsymbol{\eta}} - \mathsf{E}_\mathsf{h}\tilde{\boldsymbol{\eta}}\}, [\![\mathcal{C}e_h(\boldsymbol{\eta})]\!] \rangle_E. \tag{24}$$

Based on (24), inverse inequality, Cauchy–Schwarz inequality and the definition of orthogonal projection $\mathbf{P}$ we obtain

$$\Upsilon_1 + \Upsilon_3 + \Upsilon_4 + \Upsilon_6 = \sum_{T\in\mathcal{T}} \left((\mathbf{P}\boldsymbol{f} + \mathbf{div}_h(\mathcal{C}e_h(\boldsymbol{\eta})) - \boldsymbol{\phi}), \tilde{\boldsymbol{\eta}} - \mathsf{E}_\mathsf{h}\tilde{\boldsymbol{\eta}} \right)_T$$
$$- \sum_{E\in\mathcal{E}} \langle \{\tilde{\boldsymbol{\eta}} - \mathsf{E}_\mathsf{h}\tilde{\boldsymbol{\eta}}\}, [\![\mathcal{C}e_h(\boldsymbol{\eta})]\!] \rangle_E$$
$$\lesssim \left(\sum_{T\in\mathcal{T}} h_T^2 \|\mathbf{P}\boldsymbol{f} + \mathbf{div}_h(\mathcal{C}e_h(\boldsymbol{\eta})) - \boldsymbol{\phi}\|_T^2 \right)^{1/2} \left(\sum_{T\in\mathcal{T}} h_T^{-2} \|\tilde{\boldsymbol{\eta}} - \mathsf{E}_\mathsf{h}\tilde{\boldsymbol{\eta}}\|_T^2 \right)^{1/2}$$
$$+ \left(\sum_{T\in\mathcal{T}} h_E^{-2} \|\tilde{\boldsymbol{\eta}} - \mathsf{E}_\mathsf{h}\tilde{\boldsymbol{\eta}}\|_T^2 \right)^{1/2} \left(\sum_{E\in\mathcal{E}} h_E \|[\![\mathcal{C}e_h(\boldsymbol{\eta})]\!]\|_E^2 \right)^{1/2}.$$

Integrating by parts the term Υ_5 and from the definition of orthogonal projection P we have

$$\Upsilon_2 + \Upsilon_5 = \sum_{T\in\mathcal{T}} \int_T (Pg - \mathrm{div}_h(\boldsymbol{\phi}))(\tilde{v} - \mathsf{E}_\mathsf{h}\tilde{v})\, dx$$
$$+ \sum_{E\in\mathcal{E}} \int_E \left(\{\boldsymbol{\phi}\} \cdot [\tilde{v} - \mathsf{E}_\mathsf{h}\tilde{v}] + \{\tilde{v} - \mathsf{E}_\mathsf{h}\tilde{v}\}[\boldsymbol{\phi}]\right) ds =: S_1 + S_2.$$

For the first term of S_2, since $\boldsymbol{\phi} \in \mathcal{P}_{k-1}(\mathcal{T}, \mathbb{R}^2)$, we obtain from Cauchy–Schwarz inequality and inverse inequality that

$$\int_E \{\boldsymbol{\phi}\} \cdot [\tilde{v} - \mathsf{E}_\mathsf{h}\tilde{v}]\, ds = \int_E \{\boldsymbol{\phi}\} \cdot \boldsymbol{\Pi}^{k-1}[\tilde{v} - \mathsf{E}_\mathsf{h}\tilde{v}]\, ds$$
$$\lesssim \sqrt{\frac{h_E^{\rho-1}}{\sigma_2}}\, \|\boldsymbol{\phi}\|_T \left\| \sqrt{\frac{\sigma_2}{h_E^{\rho}}}\, \boldsymbol{\Pi}^{k-1}[\tilde{v} - \mathsf{E}_\mathsf{h}\tilde{v}] \right\|_E .$$

From triangle inequality and Cauchy–Schwarz inequality we obtain

$$S_2 \lesssim \left(\sum_{T\in\mathcal{T}} \frac{h_E^{\rho-1}}{\sigma_2} \left(\|\boldsymbol{\gamma} - \boldsymbol{\phi}\|_T^2 + \|\boldsymbol{\gamma}\|_T^2 \right) \right)^{1/2} \mathcal{J}(\tilde{v}, \tilde{v})^{1/2}$$
$$+ \sum_{E\in\mathcal{E}} h_E^{-1/2} \|\{\tilde{v} - \mathsf{E}_\mathsf{h}\tilde{v}\}\|_E h_E^{1/2} \|[\boldsymbol{\phi}]\|_E.$$

On combining these results and applying Cauchy–Schwarz inequality and inverse inequality we obtain

$$\Upsilon_2 + \Upsilon_5 \lesssim \left(\sum_{T\in\mathcal{T}} h_E^2 \|Pg - \mathrm{div}_h(\boldsymbol{\phi})\|_T^2 \right)^{1/2} \left(\sum_{T\in\mathcal{T}} h_E^{-2} \|\tilde{v} - \mathsf{E}_\mathsf{h}\tilde{v}\|_T^2 \right)^{1/2}$$
$$+ \left(\sum_{T\in\mathcal{T}} \frac{h_E^{\rho-1}}{\sigma_2} \left(\|\boldsymbol{\gamma} - \boldsymbol{\phi}\|_T^2 + \|\boldsymbol{\gamma}\|_T^2 \right) \right)^{1/2} \mathcal{J}(\tilde{v}, \tilde{v})^{1/2}$$
$$+ \left(\sum_{E\in\mathcal{E}} h_E \|[\boldsymbol{\phi}]\|_E^2 \right)^{1/2} \left(\sum_{T\in\mathcal{T}} h_E^{-2} \|\tilde{v} - \mathsf{E}_\mathsf{h}\tilde{v}\|_T^2 \right)^{1/2} .$$

For the last three terms we once again apply Cauchy–Schwarz inequality and inverse inequality and consider that $\boldsymbol{\theta} \in H^1(\Omega, \mathbb{R}^2)$ and $w \in H^1(\Omega)$ to obtain

$$\Upsilon_7 + \Upsilon_8 + \Upsilon_9 \lesssim \delta \sum_{E\in\mathcal{E}} \sqrt{\frac{h_T}{\sigma_1}} \|\{\mathcal{C}e_h(\tilde{\boldsymbol{\eta}} - \mathsf{E}_\mathsf{h}\tilde{\boldsymbol{\eta}})\}\|_E \sqrt{\frac{\sigma_1}{h_T}} \|[\![\boldsymbol{\eta}]\!]\|_E$$
$$+ \left(J(\boldsymbol{\eta} - \boldsymbol{\theta}, \boldsymbol{\eta} - \boldsymbol{\theta})\right)^{1/2} \left(J(\tilde{\boldsymbol{\eta}}, \tilde{\boldsymbol{\eta}})\right)^{1/2} + \left(\mathcal{J}(w - v, w - v)\right)^{1/2} \left(\mathcal{J}(\tilde{v}, \tilde{v})\right)^{1/2}$$
$$\lesssim \left(\delta \left(\sum_{T\in\mathcal{T}} \|e_h(\tilde{\boldsymbol{\eta}} - \mathsf{E}_\mathsf{h}\tilde{\boldsymbol{\eta}})\|_T^2 \right)^{1/2} + \left(J(\tilde{\boldsymbol{\eta}}, \tilde{\boldsymbol{\eta}})\right)^{1/2} \right) \left(J(\boldsymbol{\eta} - \boldsymbol{\theta}, \boldsymbol{\eta} - \boldsymbol{\theta})\right)^{1/2}$$
$$+ \left(\mathcal{J}(w - v, w - v)\right)^{1/2} \left(\mathcal{J}(\tilde{v}, \tilde{v})\right)^{1/2} .$$

By combining all of these inequalities and using the enriching operator properties (17) and (18), together with the Theorem 5, we prove (23).

Step 3: We combine the previous steps to finish the proof. Firstly, observe that $\|\boldsymbol{\eta}\|_h \leq \|\boldsymbol{\eta}, v, \boldsymbol{\phi}\|_h$, $\|v\|_h \leq \|\boldsymbol{\eta}, v, \boldsymbol{\phi}\|_h$ and $t\|\boldsymbol{\phi}\|_{0,h} \leq \|\boldsymbol{\eta}, v, \boldsymbol{\phi}\|_h$. Also, there exists positive constants $\tilde{c}_1$ and $\tilde{c}_2$ such that (finite dimension)

$$\tilde{c}_1|||\boldsymbol{\eta}, v, \boldsymbol{\phi}||| \leq \|\boldsymbol{\eta}, v, \boldsymbol{\phi}\|_h \leq \tilde{c}_2|||\boldsymbol{\eta}, v, \boldsymbol{\phi}|||. \tag{25}$$

Finally, from (21)–(23) and (25) we have

$$\begin{aligned}|||\tilde{\boldsymbol{\eta}}, \tilde{v}, \tilde{\boldsymbol{\phi}}|||^2 &\lesssim \sum_{T\in\mathcal{T}} \left(\|e(\boldsymbol{\theta}) - e_h(\boldsymbol{\eta})\|_T^2 + t^{-2}\|\boldsymbol{\theta} - \nabla w - (\boldsymbol{\eta} - \nabla_h v)\|_T^2\right)\\ &+ \sum_{T\in\mathcal{T}} \Big(\|\nabla(\alpha - z)\|_T^2 + (t^2 + h_T^{\rho-1})\|\boldsymbol{\gamma} - \boldsymbol{\phi}\|_T^2 + h_T^{\rho-1}\|\boldsymbol{\gamma}\|_T^2 + \|\alpha - z\|_T^2\\ &+ \|\beta - r\|_T^2\Big) + \mathcal{J}(w - v, w - v) + J(\boldsymbol{\theta} - \boldsymbol{\eta}, \boldsymbol{\theta} - \boldsymbol{\eta}) + Osc^2(g) + Osc^2(\boldsymbol{f}).\end{aligned}$$

From triangle inequality we complete the proof. □

Once again, as in [1], let $\alpha^I \in \mathcal{P}_k(\mathcal{T}) \cap H_0^1(\Omega)$ and $\beta^I \in \mathcal{P}_k(\mathcal{T}) \cap H_0^1(\Omega)/\mathbb{R}$ be the interpolants of α and β, respectively, which satisfies the following estimates

$$\begin{aligned}&\|\alpha - \alpha^I\|_\Omega + h|\alpha - \alpha^I|_{1,\Omega} \lesssim h^\ell|\alpha|_{\ell,\Omega} \quad \ell = 1, \ldots, k,\\ &\|\beta - \beta^I\|_\Omega + h|\beta - \beta^I|_{1,\Omega} \lesssim h^\ell|\beta|_{\ell,\Omega} \quad \ell = 1, \ldots, k.\end{aligned} \tag{26}$$

The choice of $\tilde{\boldsymbol{\gamma}}^I = \nabla\alpha^I + \mathrm{Curl}(\beta^I)$ proves with (10) that

$$\|\boldsymbol{\gamma} - \tilde{\boldsymbol{\gamma}}^I\|_\Omega \lesssim h^{\ell-1}\left(|\alpha|_{\ell,\Omega} + |\beta|_{\ell,\Omega}\right) \lesssim h^{\ell-1}\|\boldsymbol{\gamma}\|_{\ell-1,\Omega} \quad \ell = 1, \ldots, k. \tag{27}$$

Exploring the infima on the right-hand side of Theorem 7 we can prove the following convergence result.

Theorem 8 *Let $(\boldsymbol{\theta}, w, \boldsymbol{\gamma})$ be the solution of (1), and let $(\boldsymbol{\theta}_h, w_h, \boldsymbol{\gamma}_h)$ be the solution of the dGWOPIP formulation (6). Assume that the solution $(\boldsymbol{\theta}, w, \boldsymbol{\gamma}) \in H^k(\Omega;\mathbb{R}^2) \times H^k(\Omega) \times H^{k-1}(\Omega;\mathbb{R}^2)$, $\boldsymbol{f} \in H^{k-2}(\Omega;\mathbb{R}^2)$ and $g \in H^{k-2}(\Omega)$ for $k \geq 2$. Moreover, assume that the Helmholtz decomposition (9) is valid. Then, if $\rho = 2k - 1$ we have the following error estimate*

$$\begin{aligned}|||\boldsymbol{\theta} - \boldsymbol{\theta}_h, w - w_h, \boldsymbol{\gamma} - \boldsymbol{\gamma}_h||| &\lesssim h^{k-1}\left(\|\boldsymbol{f}\|_{k-2,\Omega} + \|g\|_{k-2,\Omega}\right)\\ &+ h^{k-1}\left(\|\boldsymbol{\gamma}\|_{k-2,\Omega} + t\|\boldsymbol{\gamma}\|_{k-1,\Omega} + \|\boldsymbol{\gamma}\|_\Omega + \|\boldsymbol{\gamma}\|_{H^{k-2}(\mathrm{div})} + \|\boldsymbol{\theta}\|_{k,\Omega}\right).\end{aligned} \tag{28}$$

Proof First note that if $v = w^I$ and $\boldsymbol{\eta} = \boldsymbol{\theta}^I$ we have that $t^{-2}(\boldsymbol{\eta} - \nabla v) = \boldsymbol{\gamma}^I$. Since $w^I \in H^1(\Omega) \cap \mathcal{P}_k(\mathcal{T})$ we obtain from trace inequality and interpolation estimates (3) and (4) that

$$\begin{aligned}&\inf_{\substack{\boldsymbol{\eta}\in\mathcal{P}_{k-1}(\mathcal{T};\mathbb{R}^2)\\ v\in\mathcal{P}_k(\mathcal{T})}} \Big\{\|\boldsymbol{\theta} - \boldsymbol{\eta}\|_h^2 + t^{-2}\|\boldsymbol{\theta} - \nabla w - (\boldsymbol{\eta} - \nabla_h v)\|_{0,h}^2\\ &\quad + \sum_{E\in\mathcal{E}} \frac{h_E}{\sigma_1}\|\{\mathcal{C}e_h(\boldsymbol{\theta} - \boldsymbol{\eta})\}\|_E^2 + \mathcal{J}(w - v, w - v)\Big\}\\ &\lesssim \|e(\boldsymbol{\theta}) - e_h(\boldsymbol{\theta}^I)\|_{0,h}^2 + t^2\|\boldsymbol{\gamma} - \boldsymbol{\gamma}^I\|_{0,h}^2 + \sum_{E\in\mathcal{E}} \frac{\sigma_1}{h_E}\|[\![\boldsymbol{\theta} - \boldsymbol{\theta}^I]\!]\|_E^2\\ &\quad + \sum_{T\in\mathcal{T}} h_T\left(h_T^{-1}\left\|e_h(\boldsymbol{\theta} - \boldsymbol{\theta}^I)\right\|_T^2 + h_T\left|e_h(\boldsymbol{\theta} - \boldsymbol{\theta}^I)\right|_{1,T}^2\right)\\ &\lesssim h^{2k-2}\left(\|\boldsymbol{\theta}\|_{k,\Omega}^2 + t^2\|\boldsymbol{\gamma}\|_{k-1,\Omega}^2\right).\end{aligned}$$

Recalling that $\alpha^I \in H_0^1(\Omega) \cap \mathcal{P}_k(\mathcal{T})$ and $\beta^I \in H^1(\Omega) \cap \mathcal{P}_k(\mathcal{T})/\mathbb{R}$ we obtain from the definition of $\tilde{\boldsymbol{\gamma}}^I$ (note that $\tilde{\boldsymbol{\gamma}}^I$ has counterpart in $H^1(\Omega) \cap \mathcal{P}_k(\mathcal{T})$) and interpolation estimates (26) and (27) that

$$\begin{aligned}
\Upsilon := & \inf_{\boldsymbol{\phi} \in \mathcal{P}_{k-1}(\mathcal{T};\mathbb{R}^2)} \sum_{T\in\mathcal{T}} \big(\|\alpha - z\|_T^2 + \|\beta - r\|_T^2 + \|\nabla(\alpha - z)\|_T^2 \\
& + \left(h_T^{\rho-1} + t^2\right) \|\boldsymbol{\gamma} - \boldsymbol{\phi}\|_T^2 \Big) \lesssim \sum_{T\in\mathcal{T}} \Big(\|\alpha - \alpha^I\|_T^2 + \|\beta - \beta^I\|_T^2 + \|\nabla(\alpha - \alpha^I)\|_T^2 \\
& + h_T^{\rho-1} \|\boldsymbol{\gamma} - \tilde{\boldsymbol{\gamma}}^I\|_T^2 + t^2 \|\boldsymbol{\gamma} - \tilde{\boldsymbol{\gamma}}^I\|_T^2 \Big) \lesssim \sum_{T\in\mathcal{T}} \Big(h_T^{2k-2} \left(|\alpha|_{k-1,T}^2 + |\beta|_{k-1,T}^2 \right) \\
& + h_T^{2k-2} |\alpha|_{k,T}^2 + h_T^{\rho-1} \|\boldsymbol{\gamma}\|_T^2 + h_T^{2k-2} t^2 \|\boldsymbol{\gamma}\|_{k-1,T}^2 \Big).
\end{aligned}$$

From (10) and choosing $\rho = 2k - 1$ we have

$$\Upsilon \lesssim h^{2k-2} \left(\|\boldsymbol{\gamma}\|_{k-2,\Omega}^2 + t^2 \|\boldsymbol{\gamma}\|_{k-1,\Omega}^2 + \|\boldsymbol{\gamma}\|_{H^{k-2}(\mathrm{div})}^2 + \|\boldsymbol{\gamma}\|_\Omega^2 \right).$$

Combining this result we have from Theorem 7 that

$$\begin{aligned}
&|||\boldsymbol{\theta} - \boldsymbol{\theta}_h, w - w_h, \boldsymbol{\gamma} - \boldsymbol{\gamma}_h|||^2 \lesssim Osc(\boldsymbol{f}) + Osc(g) \\
&\quad + h^{2k-2} \left(\|\boldsymbol{\gamma}\|_{k-2,\Omega}^2 + t^2 \|\boldsymbol{\gamma}\|_{k-1,\Omega}^2 + \|\boldsymbol{\gamma}\|_{H^{k-2}(\mathrm{div})}^2 + \|\boldsymbol{\gamma}\|_\Omega^2 + \|\boldsymbol{\theta}\|_{k,\Omega}^2 \right).
\end{aligned}$$

The result follows from (19) and (20). □

We note that for $k = 2$ the regularity required for the solution of (1) is $\boldsymbol{\theta} \in H^2(\Omega; \mathbb{R}^2)$, $w \in H^2(\Omega)$ and $\boldsymbol{\gamma} \in H^1(\Omega; \mathbb{R}^2)$. This regularity always holds if Ω is a convex polygon domain or a smooth bounded domain for $\boldsymbol{f} \in L^2(\Omega; \mathbb{R}^2)$ and $g \in L^2(\Omega)$. Furthermore, in this case ($k = 2$), the right-hand side, which is dependent on the solution, reads: $t\|\boldsymbol{\gamma}\|_{1,\Omega} + \|\boldsymbol{\gamma}\|_{H(\mathrm{div})} + \|\boldsymbol{\gamma}\|_\Omega + \|\boldsymbol{\theta}\|_{2,\Omega}$, which remains bounded as t tends to zero.

As this result was proved under the assumption that the Helmholtz decomposition holds for $\boldsymbol{\gamma}$, we highlight that the Helmholtz decomposition always hold if Ω is a convex polygon domain. Thus, our estimates (28) will hold for $k = 2$ (at least).

6 A Posteriori Error Analysis

We apply in this section the recent results regarding the a posteriori error control theory obtained in [29,30] for the dGWOPIP formulation (6). This allows robust a posteriori error estimators to be obtained and their reliability and efficiency to be proved.

Since the condition (H1)–(H3) of [29] holds for the formulation (6), we can use Theorem 3.2 of [29]. Moreover, as the dGWOPIP formulation is similar to that reported in Section 4 of [30], we will proceed with the a posteriori analysis in the same way as in [30].

Throughout this section we will use the same notation used in [29] and [30]. Let $\tilde{w}_h \in H_0^1(\Omega)$ and $\tilde{\boldsymbol{\theta}}_h \in H_0^1(\Omega, \mathbb{R}^2)$ be arbitrary. Setting $\tilde{\boldsymbol{\gamma}}_h' := \boldsymbol{\gamma}_h' = \tilde{\beta}^{-2}(\boldsymbol{\theta}_h - \nabla_h w_h)$ we find that the residual $\tilde{\boldsymbol{r}}_h := \tilde{\beta}^2 \tilde{\boldsymbol{\gamma}}_h' - (\tilde{\boldsymbol{\theta}}_h - \nabla \tilde{w}_h) = \boldsymbol{\theta}_h - \tilde{\boldsymbol{\theta}}_h - \nabla_h(w_h - \tilde{w}_h)$ and that $\tilde{\eta}_R := \|\frac{1}{\tilde{\alpha}}(\boldsymbol{\gamma}_h - \tilde{\alpha}^2(\tilde{\boldsymbol{\theta}}_h - \nabla \tilde{w}_h) - \tilde{\boldsymbol{\gamma}}_h')\|_\Omega \lesssim \|\boldsymbol{\theta}_h - \tilde{\boldsymbol{\theta}}_h - \nabla_h(w_h - \tilde{w}_h)\|_\Omega$, where for a given positive function $\tilde{\alpha} \in L^\infty(\Omega)$, with $\|\tilde{\alpha}\|_{L^\infty(\Omega)} \le \frac{\mu}{t^2}$, we define $\tilde{\beta} \in L^\infty(\Omega)$ satisfying $\frac{1}{\tilde{\beta}^2} = \frac{\mu}{t^2} - \tilde{\alpha}^2$.

Since the dGWOPIP formulation does not make use of a reduction integration operator we have $R_h = I$ which results in $\mu_h(\boldsymbol{\gamma}_h) = 0$.

Lemma 9 *It holds that*

$$
\begin{aligned}
\tilde{\eta}(\tilde{\boldsymbol{r}_h}) \lesssim & \left(\sum_{T\in\mathcal{T}} \min\left(\frac{1}{h_T^2}, \frac{1}{t^2}\right)\|\boldsymbol{\theta}_h - \tilde{\boldsymbol{\theta}}_h\|_T^2\right)^{1/2} + \left(\sum_{E\in\mathcal{E}} \frac{1}{h_E}\|[\nabla_h w_h]_{\boldsymbol{\tau}}\|_E^2\right)^{1/2} \\
& + \left(\sum_{E\in\mathcal{E}} \min\left(\frac{h_E}{t^2}, \frac{1}{t}\right)\|[\nabla_h w_h - \boldsymbol{\theta}_h]_{\boldsymbol{\tau}}\|_E^2\right)^{1/2} + \|\boldsymbol{\theta}_h - \tilde{\boldsymbol{\theta}}_h - \nabla_h(w_h - \tilde{w}_h)\|_\Omega \\
& + \left(\sum_{T\in\mathcal{T}} \min\left(\frac{h_T}{t}, 1\right)^2 \|rot(\boldsymbol{\theta}_h - \tilde{\boldsymbol{\theta}}_h)\|_T^2\right)^{1/2},
\end{aligned}
$$

where

$$
\begin{aligned}
\tilde{\eta}(\tilde{\boldsymbol{r}_h}) = & \sup_{0\neq p\in \check{H}^1(\Omega)} \frac{(\tilde{\boldsymbol{r}_h}, \mathrm{Curl}(Jp))_\Omega}{\|p\|_\Omega + \|t\nabla p\|_\Omega} + \left(\sum_{E\in\mathcal{E}} \min\left(\frac{h_E}{t^2}, \frac{1}{t}\right)\|[\tilde{\boldsymbol{r}_h}]_{\boldsymbol{\tau}}\|_E^2\right)^{1/2} \\
& + \left(\sum_{T\in\mathcal{T}} \min\left(\frac{h_T}{t}, 1\right)^2 \|rot(\tilde{\boldsymbol{r}_h})\|_T^2\right)^{1/2} + \|\tilde{\alpha}\tilde{\boldsymbol{r}_h}\|_\Omega.
\end{aligned}
$$

Proof The proof follows from integrating by parts the term $(\nabla w_h, \mathrm{Curl}(Jp))_\Omega$, the properties of Clément-type interpolation operator J and repeating the same lines described in [30, Lemma 4.1]. □

Aiming to apply [29, Theorem 3.2], we define the energy norm of the error by

$$
\begin{aligned}
\eth(\boldsymbol{\theta} - \boldsymbol{\theta}_h, w - w_h, \boldsymbol{\gamma} - \boldsymbol{\gamma}_h)^2 = & |||\boldsymbol{\theta} - \boldsymbol{\theta}_h|||_\Theta^2 + |||w - w_h|||_W^2 + t^2\|\boldsymbol{\gamma} - \boldsymbol{\gamma}_h\|_\Omega^2 \\
& + \|\boldsymbol{\gamma} - \boldsymbol{\gamma}_h\|_{H^{-1}(div)}^2 + \tilde{\mathcal{J}}(w - w_h, w - w_h),
\end{aligned} \tag{29}
$$

where

$$
\begin{aligned}
|||\boldsymbol{\eta}|||_\Theta^2 &= \|\boldsymbol{\eta}\|_{1,h}^2 + \sum_{E\in\mathcal{E}} \frac{1}{h_E}\|[\![\boldsymbol{\eta}]\!]\|_E^2 \ \forall\ \boldsymbol{\eta} \in H^2(\mathcal{T};\mathbb{R}^2), \\
|||v|||_W^2 &= \|v\|_{1,h}^2 + \sum_{E\in\mathcal{E}} \frac{1}{h_E}\|[v]\|_E^2 \ \forall\ v \in H^1(\mathcal{T}), \\
\tilde{\mathcal{J}}(v, v) &= \sum_{E\in\mathcal{E}} \frac{\sigma_2}{h_E^{2k-1}}\langle [v], [v]\rangle_E.
\end{aligned}
$$

We express the volume and edge terms η_K and η_E as

$$
\begin{aligned}
\eta_T &= h_T\|g - \mathrm{div}(\boldsymbol{\gamma}_h)\|_T + h_T\|\boldsymbol{f} + \mathrm{div}\,\mathcal{C}e(\boldsymbol{\theta}_h) - \boldsymbol{\gamma}_h\|_T, \\
\eta_E &= h_E^{1/2}\|[\boldsymbol{\gamma}_h]\|_E + h_E^{1/2}\|[\![\mathcal{C}e(\boldsymbol{\theta}_h)]\!]\|_E.
\end{aligned}
$$

Then we define the a posteriori error estimator η_h by

$$
\begin{aligned}
\eta_h^2 = & \sum_{T\in\mathcal{T}} \eta_T^2 + \sum_{E\in\mathcal{E}} \eta_E^2 + \sum_{E\in\mathcal{E}} h_E^{-1}\left(\|[w_h]\|_E^2 + \|[\![\boldsymbol{\theta}_h]\!]\|_E^2\right) \\
& + \sum_{E\in\mathcal{E}} \min\left(\frac{1}{t}, \frac{h_E}{t^2}\right)\|[\boldsymbol{\theta}_h - \nabla w_h]_{\boldsymbol{\tau}}\|_E^2 + \tilde{\mathcal{J}}(w_h, w_h) + \sum_{E\in\mathcal{E}} \frac{1}{h_E}\|[\nabla_h w_h]_{\boldsymbol{\tau}}\|_E^2.
\end{aligned}
$$

The proof of the next theorem is basically the same as in [30, Theorem 4.1].

Theorem 10 (The reliability of the estimator) *It holds that*

$$\eth(\boldsymbol{\theta} - \boldsymbol{\theta}_h, w - w_h, \boldsymbol{\gamma} - \boldsymbol{\gamma}_h) \lesssim \eta_h.$$

Finally, the next result shows the efficiency of the estimator.

Theorem 11 (The efficiency of the estimator) *It holds that*

$$\eta_h \lesssim \eth(\boldsymbol{\theta} - \boldsymbol{\theta}_h, w - w_h, \boldsymbol{\gamma} - \boldsymbol{\gamma}_h) + Osc(g) + Osc(\boldsymbol{f}).$$

Proof We only need to prove the efficiency for the last term of η_h. The others terms have been previously verified in [29, Theorem 4.5].

Let $b_E \in H_0^1(\omega_E)$ be the edge-bubble function that takes the value of one at the barycenter of the edge E. Let Ψ be the extension of $[\nabla_h w_h]_{\boldsymbol{\tau}}$ to ω_E by constants along lines orthogonal to the edge E and set $\vartheta = b_E \Psi$. Then,

$$\begin{aligned}\|[\nabla_h w_h]_{\boldsymbol{\tau}}\|_E^2 \lesssim \langle [\nabla_h w_h]_{\boldsymbol{\tau}}, \vartheta\rangle_E &= -\sum_{T\in\omega_E}\int_T \nabla_h w_h \cdot \mathrm{Curl}(\vartheta)\, dx \\ &= \sum_{T\in\omega_E}\int_T w_h div(\mathrm{Curl}(\vartheta))\, dx - \sum_{E\in\omega_E}\int_E w_h\, \mathrm{Curl}(\vartheta)\cdot \boldsymbol{\nu}\, ds \\ &= \int_E [w_h]\cdot \mathrm{Curl}(\vartheta)\, ds\end{aligned}$$

where in the last equality we consider that $\mathrm{Curl}(\vartheta) \in H(div)$ and $\nabla\vartheta \cdot \boldsymbol{\tau} = 0$ on $\partial\omega_E$. Applying Cauchy-Schwarz inequality and inverse inequality we obtain

$$\|[\nabla_h w_h]_{\boldsymbol{\tau}}\|_E^2 \lesssim \langle h_E^{-\rho/2}[w_h], h_E^{\rho/2}\,\mathrm{Curl}(\vartheta)\rangle_E \lesssim \|h_E^{-\rho/2}[w_h]\|_E h_E^{\rho/2-1}\|\vartheta\|_E.$$

From the definition of ϑ it follows that

$$\|[\nabla_h w_h]_{\boldsymbol{\tau}}\|_E \lesssim \|h_E^{-\rho/2}[w_h]\|_E h_E^{\rho/2-1},$$

and then setting $\rho = 2k - 1$ we have

$$\sum_{E\in\mathcal{E}}\frac{1}{h_E}\|[\nabla_h w_h]_{\boldsymbol{\tau}}\|_E^2 \lesssim \sum_{E\in\mathcal{E}}\frac{1}{h_E}\|h_E^{-\rho/2}[w_h]\|_E^2 h_E^{\rho-2} \lesssim \tilde{\mathcal{J}}(w_h, w_h)h^{2k-4}.$$

As $k \geq 2$, we have that $h^{2k-4} < 1$ for any choice of $k \geq 2$. Therefore

$$\sum_{E\in\mathcal{E}}\frac{1}{h_E}\|[\nabla_h w_h]_{\boldsymbol{\tau}}\|_E^2 \lesssim \tilde{\mathcal{J}}(w_h, w_h).$$

Since $w \in H^1(\Omega)$ we have

$$\begin{aligned}\sum_{E\in\mathcal{E}}\frac{1}{h_E}\|[\nabla_h w_h]_{\boldsymbol{\tau}}\|_E^2 &\lesssim \tilde{\mathcal{J}}(w_h, w_h) = \tilde{\mathcal{J}}(w - w_h, w - w_h) \\ &\lesssim \eth(\boldsymbol{\theta} - \boldsymbol{\theta}_h, w - w_h, \boldsymbol{\gamma} - \boldsymbol{\gamma}_h),\end{aligned}$$

which ends the proof. □

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