Numer. Math.
DOI 10.1007/s00211-015-0738-1

Numerische Mathematik

A natural nonconforming FEM for the Bingham flow problem is quasi-optimal

C. Carstensen[1,2] · B. D. Reddy[3] · M. Schedensack[4]

Received: 30 November 2013 / Revised: 10 May 2015

Abstract This paper introduces a novel three-field formulation for the Bingham flow problem and its two-dimensional version named after Mosolov together with low-order discretizations: a nonconforming for the classical formulation and a mixed finite element method for the three-field model. The two discretizations are equivalent and quasi-optimal in the sense that the H^1 error of the primal variable is bounded by the error of the L^2 best-approximation of the stress variable. This improves the predicted convergence rate by a log factor of the maximal mesh-size in comparison to the first-order conforming finite element method in a model scenario. Despite that numerical experiments lead to comparable results, the nonconforming scheme is proven to be quasi-optimal while this is not guaranteed for the conforming one.

Mathematics Subject Classification 65N30 · 76M10

✉ C. Carstensen
cc@math.hu-berlin.de

B. D. Reddy
daya.reddy@uct.ac.za

M. Schedensack
schedensack@ins.uni-bonn.de

1 Institut für Mathematik, Humboldt-Universität zu Berlin, Unter den Linden 6, 10099 Berlin, Germany

2 Department of Computational Science and Engineering, Yonsei University, Seoul, Korea

3 Department of Mathematics and Applied Mathematics, University of Cape Town, Rondebosch 7701, South Africa

4 Institut für Numerische Simulation, Universität Bonn, Wegelerstraße 6, 53115 Bonn, Germany

Published online: 20 June 2015

1 Introduction

After a short introduction in the modelling of a Bingham-flow, Sect. 1.2 discusses the main results of the paper which are quasi-optimal error estimates for a mixed and a nonconforming discretization.

1.1 Modelling of Bingham flow

The Bingham flow problem refers to the behaviour of a fluid modeled as rigid-viscous plastic. With the stress denoted by $\sigma : \Omega \to \mathbb{R}^{3\times 3}$, the fluid satisfies the conventional equation for momentum balance

$$\operatorname{div}\sigma + f = 0$$

with the given body force f. This stationary mathematical model assumes that the flows are sufficiently slow so that the inertial terms can be neglected. The velocity $u : \Omega \subseteq \mathbb{R}^3 \to \mathbb{R}^3$, the stress deviator $\sigma^D := \sigma - (1/3)\mathrm{tr}(\sigma)I_{3\times 3}$, the rate of deformation $\varepsilon(u) = \frac{1}{2}(\nabla u + (\nabla u)^T)$, the potential $\overline{W}$,

$$\overline{W}(\varepsilon(u)) = \frac{\mu}{2}|\varepsilon(u)|^2 + g|\varepsilon(u)|$$

with the (pointwise) Frobenius norm $|\bullet|$, the viscosity μ of the fluid and the yield limit $g > 0$ lead to the constitutive relation for the fluid

$$\sigma^D \in \partial\overline{W}(\varepsilon(u)). \tag{1.1}$$

From (1.1) we have, using also the definition of the subdifferential,

$$\overline{W}(\varepsilon(v)) - \overline{W}(\varepsilon(u)) - \sigma^D : (\varepsilon(v) - \varepsilon(u)) \geq 0$$

for all vector fields v. When $\varepsilon(u) \neq 0$, this gives

$$\sigma^D = \mu\varepsilon(u) + g\frac{\varepsilon(u)}{|\varepsilon(u)|}. \tag{1.2}$$

More generally we have

$$\varepsilon(u) = \begin{cases} \dfrac{1}{\mu}\left(1 - \dfrac{g}{|\sigma^D|}\right)\sigma^D & \text{if } |\sigma^D| > g, \\ 0 & \text{if } |\sigma^D| < g. \end{cases} \tag{1.3}$$

Thus, when the stress exceeds the threshold defined by the yield limit g, flow takes place in the same direction as σ^D. Below this threshold, the material is rigid. From

(1.3) it follows that the velocity field satisfies $\operatorname{tr}\varepsilon(u) = \operatorname{div} u = 0$. Thus, the constitutive relation captures the physically based assumption of incompressibility. This formulation is characteristic also of that for problems of elastoplasticity.

In compact notation, the Bingham flow problem with homogeneous Dirichlet boundary conditions seeks $u \in Z := \{w \in H_0^1(\Omega; \mathbb{R}^3) \mid \operatorname{div} w = 0\}$ with

$$E(u) = \min_{v\in Z} E(v) \tag{1.4}$$

with

$$\begin{aligned} E(v) &:= \int_\Omega \overline{W}(\varepsilon(v))\,dx - \int_\Omega f\cdot v\,dx \\ &= \frac{\mu}{2}\int_\Omega |\varepsilon(v)|^2\,dx + g\int_\Omega |\varepsilon(v)|\,dx - \int_\Omega f\cdot v\,dx. \end{aligned}$$

The Bingham flow problem has been studied mathematically in [9,12,16–18]; see also [13–15] for results pertaining to finite element approximations.

Of interest in this work are uni-directional flows such as, for example, the flow in a pipe. In this case there is a single nontrivial component of velocity u; furthermore, with the generator of the pipe coinciding with the x_3 axis, the domain of interest is then $\Omega \subset \mathbb{R}^2$, the cross-section of the pipe, assumed here to be bounded and polygonal. This is also called Mosolov's problem in [11] and leads to the potential function

$$W(F) = \frac{\mu}{2}|F|^2 + g|F| \quad \text{for all} \quad F \in \mathbb{R}^2.$$

The resulting variational problem seeks $u \in H_0^1(\Omega)$ with

$$E(u) = \min_{v\in H_0^1(\Omega)} E(v) \tag{1.5}$$

for the energy

$$\begin{aligned} E(v) &:= \int_\Omega W(\nabla v)\,dx - \int_\Omega f v\,dx \\ &= \frac{\mu}{2}\int_\Omega |\nabla v|^2\,dx + g\int_\Omega |\nabla v|\,dx - \int_\Omega f\,v\,dx. \end{aligned} \tag{1.6}$$

1.2 Conforming versus nonconforming discretization

For the first-order conforming finite element method (FEM), the presence of the non-differentiable function

$$j(\nabla v) := g\int_\Omega |\varepsilon(v)|\,dx = g\,\|\varepsilon(v)\|_{L^1(\Omega)}$$
$$\left(\text{resp. } j(\nabla v) := g\int_\Omega |\nabla v|\,dx = g\,\|\nabla v\|_{L^1(\Omega)}\right)$$

for the Bingham problem (resp. for the uni-directional flow problem) leads in general to a reduced convergence rate of $h^{1/2}$ in the H^1 error of u even for smooth solutions and for the particular prototype example of a circular domain with $f \equiv$ const to a convergence rate of $h\sqrt{\log(1/h)}$ [13,14]. Here and throughout the paper, $h_{\mathcal{T}}$ denotes the piecewise constant mesh-size function with $h_{\mathcal{T}}|_T := \mathrm{diam}(T)$ on $T \in \mathcal{T}$ and maximal mesh-size $h := \max h_{\mathcal{T}} := \|h_{\mathcal{T}}\|_{L^\infty(\Omega)}$ in the underlying regular triangulation $\mathcal{T}$ into triangles or tetrahedra in the FEM.

This paper introduces mixed variational inequality formulations that are new for the Bingham problem. The formulations are equivalent to (1.4) and (1.5), and there is a corresponding equivalence between the discrete formulations. Furthermore, the discrete formulations are equivalent to natural nonconforming discretizations of first order of (1.4) and (1.5). The striking main result of this paper is to prove a linear convergence of h for the flux error $\|p - p_h\|_{L^2(\Omega)}$ for smooth stress-type variable $\sigma \in \partial W(p)$ for $p = \nabla u$ and its approximation p_h. Moreover, there is quasi-optimal convergence up to an explicit data term with optimal convergence rate in the sense of

$$\left(\mu/\sqrt{2}\right) \|p - p_h\|_{L^2(\Omega)} \leq \min_{\tau_h \in Q(f,\mathcal{T})} \|\sigma - \tau_h\|_{L^2(\Omega)} + \|h_{\mathcal{T}} f\|_{L^2(\Omega)} / \sqrt{2} \tag{1.7}$$

for the piecewise constant flux approximation p_h and the lowest-order Raviart–Thomas space $\mathrm{RT}_0(\mathcal{T})$ and its natural subspace $Q(f, \mathcal{T})$ of all Raviart–Thomas functions with prescribed divergence $-\Pi_0 f$, the piecewise constant approximation of $-f$. Furthermore, a direct analysis of the nonconforming scheme with approximation p_{CR} of the flux proves the best-approximation result

$$\|p - p_{\mathrm{CR}}\| \lesssim \|\sigma - \Pi_0\sigma\|_{L^2(\Omega)} + \mathrm{osc}(f, \mathcal{T}) \tag{1.8}$$

for the best-approximation $\Pi_0\sigma$ of σ in the piecewise constant functions. The notation $A \lesssim B$ is equivalent to the statement that there exists a positive generic mesh-size independent constant $C > 0$ such that $A \leq CB$. Those surprising quasi-optimalities (1.7) and (1.8) have to contrasted with the analysis in [14, Eqn (6.48) on p 87] and [13, Eqn 4.14], which leads to a convergence $O(h^{1/2})$, which, in view of the numerical experiments in Sect. 6 below, appears suboptimal. The remedy via a Jensen inequality in this paper has been observed before in [11, p. 143, line 19].

Another difficulty is the higher regularity of the exact solutions with further limitations of the convergence rates. For a circular domain and a constant right-hand side, the exact solution $u \in H^2(\Omega)$ is known explicitly in closed form [13,14]. An explicit analysis with the closed form of u, although it certainly serves as a role model, leads to the convergence rate as $h\sqrt{\log(1/h)}$ for conforming FEM while the general best-approximation result (1.7) leads to h whenever $\sigma \in H^1(\Omega; \mathbb{R}^2)$ for the nonconforming FEM. In view of equivalence of nonconforming and conforming first-order FEM [6], this omission of an extra log factor is somehow surprising.

First order convergence rates are apparent in all the numerical experiments of this paper. The nonconforming first-order FEM and its quasi-optimal convergence has in fact been mentioned before in [11, Sec. 5.2] in a very abstract fashion: the word nonconforming is not even mentioned in [11]. Moreover, the authors of [11] do *not* recommend the scheme but suggest the numerical treatment of some dual formulation in order to avoid non-smoothness [11, p. 143, lines 23–28]. This paper provides a regularization of the nonconforming FEM, which can directly be applied. Moreover, the novel three-field formulation and its equivalence to the nonconforming FEM (cf. Theorem 3.1 below) open the door for future applications of nonstandard discretizations to non-Newtonian flow problems.

1.3 Outline of the paper

The remaining parts of this paper are organized as follows. Since the crucial points in the analysis for the 3D Bingham problem and for the uni-directional flow are the same, Sects. 2, 3 and 4 focus on Mosolov's problem, while Sect. 5 extends the results to the 3D problem. In more detail, Sect. 2 introduces the new three-field formulation for the Mosolov problem and shows its equivalence to (1.5). This implies existence of solutions of the mixed formulation. Section 3 introduces the discretizations of (1.5) and of the mixed formulation and shows their equivalence. This implies existence of discrete solutions. Section 4 is concerned with a priori analyses for the FEMs for the Mosolov problem and the derivation of a best-approximation result. Section 5 is devoted to the three dimensional Bingham problem. It introduces the three-field formulation and the variational inequality and their discretizations and proves the best approximation results for it. Section 6 concludes the paper with numerical experiments for Mosolov's problem.

1.4 General notation

Standard notation on Lebesgue and Sobolev spaces applies throughout the paper and $(\bullet, \bullet)_{L^2(\Omega)}$ denotes the L^2 scalar product and $\|\bullet\|_{L^2(\Omega)}$ the L^2-norm. The notation $\bullet$ abbreviates the identity mapping. The formula $A \lesssim B$ abbreviates that there exists a positive generic mesh-size independent constant $C > 0$ such that $A \leq CB$. The formula $A \approx B$ abbreviates $A \lesssim B \lesssim A$.

2 Three-field formulation

This section introduces the three-field formulation in Sect. 2.1 and proves its equivalence with (1.5) in Sect. 2.2. The existence of solutions to (1.5) is stated in Sect. 2.3.

2.1 Mathematical model

Define $X := L^2(\Omega) \times L^2(\Omega; \mathbb{R}^2) \equiv L^2(\Omega; \mathbb{R} \times \mathbb{R}^2)$, and define the bilinear forms $a : L^2(\Omega; \mathbb{R}^2) \times L^2(\Omega; \mathbb{R}^2) \to \mathbb{R}$ and $b : X \times H(\text{div}, \Omega) \to \mathbb{R}$ for $\varphi = (u, p)$, $\psi = (v, q) \in X$ and $\tau \in H(\text{div}, \Omega)$ by

$$
\begin{aligned}
a(p, q) &:= \mu(p, q)_{L^2(\Omega)},\\
b(\psi, \tau) &:= -(v, \text{div}\ \tau)_{L^2(\Omega)} - (q, \tau)_{L^2(\Omega)}.
\end{aligned}
$$

We define also the functionals

$$j(q) := g \int_\Omega |q|\,dx \quad \text{for all} \;\; q \in L^2(\Omega;\mathbb{R}^2),$$
$$F(v) = (f, v)_{L^2(\Omega)} \quad \text{for all} \;\; v \in L^2(\Omega).$$

The three-field formulation for the Bingham flow seeks $\varphi = (u, p) \in X$ and $\sigma \in H(\text{div}, \Omega)$ with

$$\begin{aligned} F(v-u) &\le a(p, q-p) + j(q) - j(p) + b(\psi - \varphi, \sigma), \\ b(\varphi, \tau) &= 0 \end{aligned} \tag{3FF}$$

for all $\psi = (v, q) \in X$ and all $\tau \in H(\text{div}, \Omega)$.

Remark 1 (*Non-uniqueness of* σ) The solutions to (3FF) for $f \equiv 0$ read $(0, 0, \sigma)$ for $\sigma \in H(\text{div}, \Omega)$ with

$$\text{div}\,\sigma = 0 \quad \text{and} \quad (q, \sigma)_{L^2(\Omega)} \le j(q) \quad \text{for all} \;\; q \in L^2(\Omega;\mathbb{R}^2). \tag{2.1}$$

Any $\alpha \in H^1_0(\Omega)$ satisfies $\text{Curl}\,\alpha := (-\partial\alpha/\partial y, \partial\alpha/\partial x) \in H(\text{div}, \Omega)$ with $\text{div}\,\text{Curl}\,\alpha = 0$. Therefore $\sigma := g\,\text{Curl}\,\alpha/\|\text{Curl}\,\alpha\|_{L^\infty(\Omega)}) \in H(\text{div}, \Omega)$ is a solution to (2.1) for any $\alpha \in H^1_0(\Omega)$ with $\text{Curl}\,\alpha \in L^\infty(\Omega)$. In particular, the stress $\sigma \in H(\text{div}, \Omega)$ is not unique.

2.2 Equivalence to the Bingham flow problem (1.5)

The quadratic growth, the strong convexity and the continuity of the energy E of (1.6) implies the unique existence of a minimizer of E in $H^1_0(\Omega)$ (see for example [14, Chapter 1, Lemma 4.1]). The minimization problem (1.5) of E is equivalent to the variational inequality

$$F(v-u) \le a(\nabla u, \nabla(v-u)) + j(\nabla v) - j(\nabla u) \tag{VI}$$

for all $v \in H^1_0(\Omega)$. The unique existence of a solution u to (VI) follows from its equivalence to the minimization of E in $H^1_0(\Omega)$.

Theorem 2.1 [Equivalence of (3FF) with (VI)] *Let* $u \in H^1_0(\Omega)$ *be the solution of* (VI) *and* $\tau \in H(\text{div}, \Omega)$ *with* $\tau \in \partial W(\nabla u)$ *and* $f + \text{div}\,\tau = 0$. *Then* $(u, \nabla u, \tau) \in X \times H(\text{div}, \Omega)$ *is a solution of* (3FF).

Conversely, if $(u, p, \sigma) \in X \times H(\text{div}, \Omega)$ *is a solution of* (3FF), *then* $u \in H^1_0(\Omega)$ *solves* (VI) *and* $p = \nabla u$ *and* $\sigma \in \partial W(\nabla u)$ *with* $f + \text{div}\,\sigma = 0$.

Proof Let $u \in H^1_0(\Omega)$ be the solution to (VI) and $\tau \in H(\text{div}, \Omega)$ satisfy $\tau \in \partial W(\nabla u)$ and $f + \text{div}\,\tau = 0$. Since $\tau \in \partial W(\nabla u) = \mu\nabla u + \partial j(\nabla u)$, the definition of the subderivative implies, for any $q \in L^2(\Omega)$, that

$$(q - \nabla u, \tau - \mu\nabla u)_{L^2(\Omega)} \le j(q) - j(\nabla u).$$

Set $p := \nabla u$. Since $f + \operatorname{div} \tau = 0$, any $v \in L^2(\Omega)$ satisfies

$$F(v-u) \leq j(q) - j(p) - (q-p, \tau - \mu p)_{L^2(\Omega)} - (v-u, \operatorname{div} \tau)_{L^2(\Omega)}.$$

This is the first formula of (3FF) while the second formula follows from $p = \nabla u$.

Suppose that $(u, p, \sigma) \in X \times H(\operatorname{div}, \Omega)$ denotes a solution to (3FF). The second formula of (3FF) implies that $u \in H_0^1(\Omega)$ with $\nabla u = p$. Given $\psi := (v, \nabla v)$ for any $v \in H_0^1(\Omega)$, any $\tau \in H(\operatorname{div}, \Omega)$ satisfies $b(\psi, \tau) = 0$ and, hence,

$$\begin{aligned} a(\nabla u, \nabla(v-u)) + j(\nabla v) - j(\nabla u) &= a(p, \nabla v - p) + j(\nabla v) - j(p) \\ &\geq F(v-u). \end{aligned}$$

Therefore u solves (VI). For $q = p$ in the first formula of (3FF), it follows that $f + \operatorname{div} \sigma = 0$ and, for $v = u$,

$$(q-p, \sigma - \mu p)_{L^2(\Omega)} \leq j(q) - j(p).$$

That is, $\sigma - \mu p \in \partial j(p)$, whence $\sigma \in \partial W(\nabla u)$. □

2.3 Existence of solutions

The following theorem guarantees the existence of $\tau \in H(\operatorname{div}, \Omega)$ with $\tau \in \partial W(\nabla u)$ and $f + \operatorname{div} \tau = 0$ for the solution $u \in H_0^1(\Omega)$ of (VI).

Theorem 2.2 (Euler–Lagrange equations for the Mosolov problem) *The solution $u \in H_0^1(\Omega)$ of (VI) satisfies the Euler–Lagrange equations in the sense that there exists $\sigma \in H(\operatorname{div}, \Omega)$ with $\sigma \in \partial W(\nabla u)$ and*

$$\big(\sigma, \nabla v\big)_{L^2(\Omega)} = F(v) \quad \textit{for all} \quad v \in H_0^1(\Omega).$$

Proof See for example [14, Chapter II, Theorem 6.3] and [17, Theorem 1.1]. □

Theorem 2.3 (Existence of solutions) *There exists (at least) one solution of problem* (3FF).

Proof This is a direct consequence of Theorem 2.1 and 2.2. □

Remark 2 (*Uniqueness of* $(u, p) \in X$) Since W is strictly convex, the solution $u_{\min} \in H_0^1(\Omega)$ to (1.5) is unique. This and Theorem 2.1 imply that the solution $(u, p) = (u_{\min}, \nabla u_{\min})$ to (3FF) is unique.

Remark 3 Molosov and Miasnikov [16–18] have obtained results on the existence of nuclei, that is, subsets of the domain that behave as rigid bodies, moving with constant velocity, and show that the nuclei are simply connected. Furthermore, the authors obtain results on the existence of stagnant zones, that is, nuclei or subsets of the domain that behave rigidly, and which in addition have part of their boundary coinciding with $\partial\Omega$. Since $u = 0$ on the boundary these stagnant zones have zero velocity.

3 Discrete problems

This section introduces notation on triangulations and basic finite element spaces in Sect. 3.1 and defines the P_1 nonconforming, a regularized P_1 nonconforming, and a mixed FEM in Sects. 3.2 and 3.4. Section 3.5 proves equivalence of the P_1 nonconforming discretization with that of the three-field formulation. This is exploited in Sect. 3.6 to prove the existence of discrete solutions.

3.1 Triangulations and finite element spaces

A shape-regular triangulation $\mathcal{T}$ of a polygonal bounded Lipschitz domain $\Omega \subseteq \mathbb{R}^n$ with $n = 2$ or $n = 3$ is a set of simplices (triangles if $n = 2$ and tetrahedra of $n = 3$) such that $\overline{\Omega} = \bigcup \mathcal{T}$ and any two distinct simplices are either disjoint or share exactly one common face, edge or vertex. Let $\mathcal{E}$ denote the set of edges for $n = 2$ and the set of faces for $n = 3$ of $\mathcal{T}$ and $\mathcal{N}$ the set of vertices. Let

$$
\begin{aligned}
P_k(T; \mathbb{R}^m) &:= \{v_k : T \to \mathbb{R}^m \mid \forall j = 1, \dots, m, \text{ the component} \\
&\qquad v_k(j) \text{ of } v_k \text{ is a polynomial of total degree} \le k\}, \\
P_k(\mathcal{T}; \mathbb{R}^m) &:= \{v_k : \Omega \to \mathbb{R}^m \mid \forall T \in \mathcal{T}, v_k|_T \in P_k(T; \mathbb{R}^m)\}
\end{aligned}
$$

denote the set of piecewise polynomials; The piecewise constant function $\operatorname{mid}(\mathcal{T}) \in P_0(\mathcal{T})$ is defined by $\operatorname{mid}(\mathcal{T})|_T = \operatorname{mid}(T)$ for a simplex $T \in \mathcal{T}$ with barycenter $\operatorname{mid}(T)$. For an edge or face $E \in \mathcal{E}$, $\operatorname{mid}(E)$ denotes the midpoint of E. The L^2-projection onto $\mathcal{T}$-piecewise constant functions or vectors $\Pi_0 : L^2(\Omega; \mathbb{R}^m) \to P_0(\mathcal{T}; \mathbb{R}^m)$ is defined by $(\Pi_0 f)|_T = \fint_T f\, dx := \int_T f\, dx / |T|$ for all $T \in \mathcal{T}$ with area $|T|$ for $n = 2$ and volume $|T|$ for $n = 3$ and all $f \in L^2(\Omega; \mathbb{R}^m)$. Let $h_{\mathcal{T}} \in P_0(\mathcal{T})$ denote the piecewise constant mesh-size with $h_{\mathcal{T}}|_T := \operatorname{diam}(T)$ for all $T \in \mathcal{T}$. The oscillations of f are defined by $\operatorname{osc}(f, \mathcal{T}) := \|h_{\mathcal{T}}(f - \Pi_0 f)\|_{L^2(\Omega)}$. The jump along an interior edge or face E with adjacent simplices T_+ and T_-, i.e., $E = T_+ \cap T_-$, is defined by $[v]_E := v|_{T_+} - v|_{T_-}$. The jump along boundary edges or faces $E \in \mathcal{E}(\Gamma_D)$ reads $[v]_E := v|_{T_+}$ for that simplex $T_+ \in \mathcal{T}$ with $E \subset T_+$ due to the homogeneous Dirichlet boundary conditions.

For piecewise affine functions $v_h \in P_1(\mathcal{T})$ the $\mathcal{T}$-piecewise gradient $\nabla_{\mathrm{NC}} v_h$ with $(\nabla_{\mathrm{NC}} v_h)|_T = \nabla(v_h|_T)$ for all $T \in \mathcal{T}$ and, accordingly, $\operatorname{div}_{\mathrm{NC}}(\tau_h)$ for $\tau_h \in P_1(\mathcal{T}; \mathbb{R}^2)$, exists and $\nabla_{\mathrm{NC}} v_h \in P_0(\mathcal{T}; \mathbb{R}^2)$ and $\operatorname{div}_{\mathrm{NC}}(v_h) \in P_0(\mathcal{T})$.

3.2 P_1 nonconforming discretization

The P_1 nonconforming finite element space [8], sometimes named after Crouzeix and Raviart, reads

$$
\begin{aligned}
\mathrm{CR}_0^1(\mathcal{T}) := \{v_{\mathrm{CR}} \in P_1(\mathcal{T}) \mid\ & v_{\mathrm{CR}} \text{ is continuous at midpoints of interior} \\
& \text{edges and vanishes at midpoints of boundary edges}\}
\end{aligned}
$$

and motivate the discrete energy

$$E_{\mathrm{NC}}(v_{\mathrm{CR}}) := \int_{\Omega} W(\nabla_{\mathrm{NC}} v_{\mathrm{CR}})\,dx - F(v_{\mathrm{CR}}).$$

The nonconforming discretization of (VI) seeks $u_{\mathrm{CR}} \in \mathrm{CR}_0^1(\mathcal{T})$ with

$$\begin{aligned} F(v_{\mathrm{CR}} - u_{\mathrm{CR}}) \le{}& a(\nabla_{\mathrm{NC}} u_{\mathrm{CR}}, \nabla_{\mathrm{NC}}(v_{\mathrm{CR}} - u_{\mathrm{CR}})) \\ &+ j(\nabla_{\mathrm{NC}} v_{\mathrm{CR}}) - j(\nabla_{\mathrm{NC}} u_{\mathrm{CR}}) \quad \text{for all} \quad v_{\mathrm{CR}} \in \mathrm{CR}_0^1(\mathcal{T}). \end{aligned} \tag{CRVI}$$

This is equivalent to the minimization of E_{NC} in $\mathrm{CR}_0^1(\mathcal{T})$.

Remark 4 The discretization in [11, Eqn (25)] of (1.5) suggests the space $M_h = \{p_h \in P_0(\mathcal{T};\mathbb{R}^2) \mid \forall \psi_h \in P_1(\mathcal{T}) \cap H^1(\Omega),\ (p_h, \nabla\psi_h)_{L^2(\Omega)} = 0\}$. The equivalence to the nonconforming problem (CRVI) can be shown by a discrete Helmholtz decomposition [1]

$$P_0(\mathcal{T};\mathbb{R}^2) = \nabla(P_1(\mathcal{T}) \cap C(\Omega)) \oplus \mathrm{Curl}_{\mathrm{NC}}(\mathrm{CR}_0^1(\mathcal{T}))$$

with $M_h = \mathrm{Curl}_{\mathrm{NC}}(\mathrm{CR}_0^1(\mathcal{T}))$ for a simply connected domain Ω.

3.3 Regularized problem

The regularization is one possibility to approximate the discrete solution. Given any $\varepsilon > 0$, define $W_\varepsilon \in C^1(\mathbb{R}^2)$ by

$$W_\varepsilon(F) := (\mu/2)|F|^2 + g\left(\sqrt{|F|^2 + \varepsilon^2} - \varepsilon\right).$$

The regularized problem seeks $u_{\varepsilon,\mathrm{CR}} \in \mathrm{CR}_0^1(\mathcal{T})$ with

$$E_{\varepsilon,\mathrm{NC}}(u_{\varepsilon,\mathrm{CR}}) = \min_{v_{\mathrm{CR}} \in \mathrm{CR}_0^1(\mathcal{T})} E_{\varepsilon,\mathrm{NC}}(v_{\mathrm{CR}}) \tag{3.1}$$

for the modified energy

$$E_{\varepsilon,\mathrm{NC}}(v_{\mathrm{CR}}) := \int_{\Omega} W_\varepsilon(\nabla_{\mathrm{NC}} v_{\mathrm{CR}})\,dx - F(v_{\mathrm{CR}}).$$

3.4 Discrete three-field formulation

The lowest-order Raviart–Thomas finite element space reads

$$\begin{aligned} \mathrm{RT}_0(\mathcal{T}) := \{q_h \in H(\operatorname{div},\Omega) \mid{}& \forall T \in \mathcal{T}\ \exists \alpha_T \in \mathbb{R}^2 \\ & \exists \beta_T \in \mathbb{R} \forall x \in T, q_h|_T(x) = \alpha_T + \beta_T x\}. \end{aligned}$$

The discrete three-field formulation seeks $\varphi_h = (u_h, p_h) \in P_0(\mathcal{T}) \times P_0(\mathcal{T};\mathbb{R}^2) \equiv P_0(\mathcal{T};\mathbb{R}\times\mathbb{R}^2)$ and $\sigma_h \in \mathrm{RT}_0(\mathcal{T})$ with

$$F(v_h - u_h) \le a(p_h, q_h - p_h) + j(q_h) - j(p_h) + b(\psi_h - \varphi_h, \sigma_h), \qquad \text{(d3FF)}$$
$$b(\varphi_h, \tau_h) = 0$$

for all $\psi_h = (v_h, q_h) \in P_0(\mathcal{T}; \mathbb{R} \times \mathbb{R}^2)$ and all $\tau_h \in \mathrm{RT}_0(\mathcal{T})$.

3.5 Equivalence of (d3FF) to (CRVI)

The following lemma provides a first link between the three-field formulation and the P_1 nonconforming finite element space.

Lemma 1 $(\ker(b|_{P_0(\mathcal{T};\mathbb{R}\times\mathbb{R}^2)\times\mathrm{RT}_0(\mathcal{T})}))$ *The kernel of b, restricted to* $P_0(\mathcal{T}; \mathbb{R}\times\mathbb{R}^2)\times \mathrm{RT}_0(\mathcal{T})$*, is given by*

$$\begin{aligned}
&\ker(b|_{P_0(\mathcal{T};\mathbb{R}\times\mathbb{R}^2)\times\mathrm{RT}_0(\mathcal{T})})\\
&\quad := \{\varphi = (v_h, q_h) \in P_0(\mathcal{T}; \mathbb{R} \times \mathbb{R}^2) \mid b(\varphi, \tau_h) = 0 \quad \textit{for all } \tau_h \in \mathrm{RT}_0(\mathcal{T})\}\\
&\quad = \{(\Pi_0 v_{\mathrm{CR}}, \nabla_{\mathrm{NC}} v_{\mathrm{CR}}) \mid v_{\mathrm{CR}} \in \mathrm{CR}_0^1(\mathcal{T})\}.
\end{aligned}$$

Proof Let $(v_h, q_h) \in P_0(\mathcal{T}; \mathbb{R} \times \mathbb{R}^2)$ with

$$(v_h, \operatorname{div} \tau_h)_{L^2(\Omega)} + (q_h, \tau_h)_{L^2(\Omega)} = 0 \quad \text{for all} \quad \tau_h \in \mathrm{RT}_0(\mathcal{T})$$

and define $v_{\mathrm{CR}} := v_h + q_h \cdot (\bullet - \operatorname{mid}(\mathcal{T}))$. Let $E \in \mathcal{E}$ and let, for an interior edge, $T_\pm \in \mathcal{T}$ with $E = T_+ \cap T_-$ and, for a boundary edge, $T_+ \in \mathcal{T}$ with $E \subset T_+$ and let $\phi_E \in \mathrm{RT}_0(\mathcal{T})$ denote the Raviart–Thomas basis function defined by $\phi_E|_{T_\pm} := \pm|E|(\bullet - P_{\pm,E})/(2|T_\pm|)$ for $P_{\pm,E} \in \mathcal{N} \cap T_\pm$ opposite to E and $\phi_E|_{\Omega \setminus T_\pm} \equiv 0$. In fact, $\phi_E \in \mathrm{RT}_0(\mathcal{T})$. An elementary calculation reveals $\phi_E \cdot \nu_E|_E = 1$ for the unit normal $\nu_E = \nu_{T_+}|_E$ pointing from T_+ to T_-. This implies

$$|E|[v_{\mathrm{CR}}]_E(\operatorname{mid}(E)) = \int_E [v_{\mathrm{CR}}]_E\, \phi_E \cdot \nu_E\, ds.$$

A piecewise integration by parts and the definition of v_{CR} imply

$$\begin{aligned}
\int_E [v_{\mathrm{CR}}]_E\, \phi_E \cdot \nu_E\, ds &= (v_{\mathrm{CR}}, \operatorname{div} \phi_E)_{L^2(\Omega)} + (\phi_E, \nabla_{\mathrm{NC}} v_{\mathrm{CR}})_{L^2(\Omega)}\\
&= (v_h, \operatorname{div} \phi_E)_{L^2(\Omega)} + (\phi_E, q_h)_{L^2(\Omega)} = 0.
\end{aligned}$$

Hence, $v_{\mathrm{CR}} \in \mathrm{CR}_0^1(\mathcal{T})$ and $v_h = \Pi_0 v_{\mathrm{CR}}$ and $q_h = \nabla_{\mathrm{NC}} v_{\mathrm{CR}}$.

Conversely, a piecewise integration by parts implies

$$\{(\Pi_0 v_{\mathrm{CR}}, \nabla_{\mathrm{NC}} v_{\mathrm{CR}}) \mid v_{\mathrm{CR}} \in \mathrm{CR}_0^1(\mathcal{T})\} \subseteq \ker(b|_{P_0(\mathcal{T};\mathbb{R}\times\mathbb{R}^2)\times\mathrm{RT}_0(\mathcal{T})}).$$

$\square$

Theorem 3.1 [Equivalence of (d3FF) and (CRVI)] *Let* $u_{\rm CR} \in {\rm CR}_0^1(\mathcal{T})$ *be the solution of* (CRVI) *for the piecewise constant right-hand side* $\Pi_0 f \in L^2(\Omega)$ *and define* $u_h := \Pi_0 u_{\rm CR}$ *and* $p_h := \nabla_{\rm NC} u_{\rm CR}$. *Let* $\sigma_{\rm CR} \in P_0(\mathcal{T};\mathbb{R}^2)$ *with* $\sigma_{\rm CR} \in \partial W(\nabla_{\rm NC} u_{\rm CR})$ *and*

$$(\sigma_{\rm CR}, \nabla_{\rm NC} v_{\rm CR})_{L^2(\Omega)} = (\Pi_0 f, v_{\rm CR})_{L^2(\Omega)} \quad \textit{for all} \quad v_{\rm CR} \in {\rm CR}_0^1(\mathcal{T}) \tag{3.2}$$

and define

$$\sigma_h := \sigma_{\rm CR} - \Pi_0 f(\bullet - \operatorname{mid}(\mathcal{T}))/2.$$

Then $(u_h, p_h, \sigma_h) \in (P_0(\mathcal{T}) \times P_0(\mathcal{T};\mathbb{R}^2)) \times {\rm RT}_0(\mathcal{T})$ *is a solution to* (d3FF).

Conversely, if $(u_h, p_h, \sigma_h) \in (P_0(\mathcal{T}) \times P_0(\mathcal{T};\mathbb{R}^2)) \times {\rm RT}_0(\mathcal{T})$ *is a solution to* (d3FF), *then* $u_{\rm CR} := u_h + p_h \cdot (\bullet - \operatorname{mid}(\mathcal{T})) \in {\rm CR}_0^1(\mathcal{T})$ *is a solution to* (CRVI) *for the piecewise constant right-hand side* $\Pi_0 f$.

Proof Let $E \in \mathcal{E}$ be an interior edge and $T_\pm \in \mathcal{T}$ with $E = T_+ \cap T_-$. Let $\psi_E \in {\rm CR}_0^1(\mathcal{T})$ denote the Crouzeix–Raviart basis function defined by $\psi_E(\operatorname{mid}(F)) = \delta_{EF}$ (with the Kronecker delta δ_{EF}). Since $(\bullet - \operatorname{mid}(T)) \cdot \nu_F$ is constant along $F \in \mathcal{E}$ for all $T \in \mathcal{T}$, a piecewise integration by parts implies

$$\begin{aligned} |E|\,[\sigma_h \cdot \nu_E]_E &= \int_E \psi_E [\sigma_h \cdot \nu_E]_E \, ds \\ &= \int_{T_+ \cup T_-} \sigma_h \cdot \nabla_{\rm NC} \psi_E \, dx + \int_{T_+ \cup T_-} \psi_E \operatorname{div}_{\rm NC} \sigma_h \, dx. \end{aligned}$$

Since $-\operatorname{div}_{\rm NC} \sigma_h = \Pi_0 f$ and $\Pi_0 \sigma_h = \sigma_{\rm CR}$, (3.2) implies

$$\begin{aligned} &\int_{T_+ \cup T_-} \sigma_h \cdot \nabla_{\rm NC} \psi_E \, dx + \int_{T_+ \cup T_-} \psi_E \operatorname{div}_{\rm NC} \sigma_h \, dx \\ &\quad = \big(\psi_E, \Pi_0 f\big)_{L^2(\Omega)} - \big(\psi_E, \Pi_0 f\big)_{L^2(\Omega)} = 0. \end{aligned}$$

This implies $\sigma_h \in H(\operatorname{div}, \Omega)$ and, hence, $\sigma_h \in {\rm RT}_0(\mathcal{T})$.

The definition of the subderivative implies for $\sigma_{\rm CR} \in \partial W(\nabla_{\rm NC} u_{\rm CR}) = \mu \nabla_{\rm NC} u_{\rm CR} + \partial j(\nabla_{\rm NC} u_{\rm CR})$ and all $q_h \in P_0(\mathcal{T};\mathbb{R}^2)$

$$(\sigma_{\rm CR} - \mu \nabla_{\rm NC} u_{\rm CR}, q_h - \nabla_{\rm NC} u_{\rm CR})_{L^2(\Omega)} \le j(q_h) - j(\nabla_{\rm NC} u_{\rm CR}).$$

Since $\Pi_0 \sigma_h = \sigma_{\rm CR}$, the definition of p_h implies

$$0 \le a(p_h, q_h - p_h) + j(q_h) - j(p_h) - (q_h - p_h, \sigma_h)_{L^2(\Omega)}.$$

Since $-\operatorname{div} \sigma_h = \Pi_0 f$, this implies

$$\begin{aligned} (\Pi_0 f, v_h - u_h)_{L^2(\Omega)} &\le a(p_h, q_h - p_h) + j(q_h) - j(p_h) \\ &\quad - (v_h - u_h, \operatorname{div} \sigma_h)_{L^2(\Omega)} - (q_h - p_h, \sigma_h)_{L^2(\Omega)}. \end{aligned}$$

This is the first formula of (d3FF). Since div $\tau_h \in P_0(\mathcal{T})$ for all $\tau_h \in \mathrm{RT}_0(\mathcal{T})$ the definitions of u_h and p_h imply

$$(u_h, \operatorname{div} \tau_h)_{L^2(\Omega)} = (u_{\mathrm{CR}}, \operatorname{div} \tau_h)_{L^2(\Omega)} = -(\tau_h, \nabla_{\mathrm{NC}} u_{\mathrm{CR}})_{L^2(\Omega)} = -(p_h, \tau_h)_{L^2(\Omega)}.$$

Hence, $(u_h, p_h, \sigma_h) \in (P_0(\mathcal{T}) \times P_0(\mathcal{T}; \mathbb{R}^2)) \times \mathrm{RT}_0(\mathcal{T})$ solves (d3FF).

Given a solution $(u_h, p_h, \sigma_h) \in (P_0(\mathcal{T}) \times P_0(\mathcal{T}; \mathbb{R}^2)) \times \mathrm{RT}_0(\mathcal{T})$ to (d3FF) and $u_{\mathrm{CR}} := u_h + p_h \cdot (\bullet - \operatorname{mid}(\mathcal{T}))$, Lemma 1 and the second formula of (d3FF) show the existence of $w_{\mathrm{CR}} \in \mathrm{CR}_0^1(\mathcal{T})$ with $u_h = \Pi_0 w_{\mathrm{CR}}$ and $p_h = \nabla_{\mathrm{NC}} w_{\mathrm{CR}}$. This implies $u_{\mathrm{CR}} = w_{\mathrm{CR}} \in \mathrm{CR}_0^1(\mathcal{T})$.

Let $v_{\mathrm{CR}} \in \mathrm{CR}_0^1(\mathcal{T})$ and define $v_h := \Pi_0 v_{\mathrm{CR}}$ and $q_h := \nabla_{\mathrm{NC}} v_{\mathrm{CR}}$. Then any $\tau_h \in \mathrm{RT}_0(\mathcal{T})$ satisfies

$$(v_h - u_h, \operatorname{div} \tau_h)_{L^2(\Omega)} = -(q_h - p_h, \tau_h)_{L^2(\Omega)}.$$

With those definitions, the first formula of (d3FF) becomes

$$\begin{aligned}&(\Pi_0 f, v_{\mathrm{CR}} - u_{\mathrm{CR}})_{L^2(\Omega)}\\ &\quad \le a(\nabla_{\mathrm{NC}} u_{\mathrm{CR}}, \nabla_{\mathrm{NC}}(v_{\mathrm{CR}} - u_{\mathrm{CR}})) + j(\nabla_{\mathrm{NC}} v_{\mathrm{CR}}) - j(\nabla_{\mathrm{NC}} u_{\mathrm{CR}}).\end{aligned}$$

This proves (CRVI). □

3.6 Existence of discrete solutions

The following theorem guarantees the existence of $\tau_{\mathrm{CR}} \in P_0(\mathcal{T}; \mathbb{R}^2)$ with $\tau_{\mathrm{CR}} \in \partial W(\nabla_{\mathrm{NC}} u_{\mathrm{CR}})$ and

$$(\tau_{\mathrm{CR}}, \nabla_{\mathrm{NC}} v_{\mathrm{CR}})_{L^2(\Omega)} = (f, v_{\mathrm{CR}})_{L^2(\Omega)} \quad \text{for all} \quad v_{\mathrm{CR}} \in \mathrm{CR}_0^1(\mathcal{T}). \tag{3.3}$$

Theorem 3.2 (Discrete Euler–Lagrange equations) *The solution $u_{\mathrm{CR}} \in \mathrm{CR}_0^1(\mathcal{T})$ of* (CRVI) *with the right-hand side f satisfies the Euler-Lagrange equations in the sense that there exists $\tau_{\mathrm{CR}} \in P_0(\mathcal{T}; \mathbb{R}^2) \cap \partial W(\nabla_{\mathrm{NC}} u_{\mathrm{CR}})$ with*

$$(\tau_{\mathrm{CR}}, \nabla_{\mathrm{NC}} v_{\mathrm{CR}})_{L^2(\Omega)} = (f, v_{\mathrm{CR}})_{L^2(\Omega)} \quad \textit{for all} \quad v_{\mathrm{CR}} \in \mathrm{CR}_0^1(\mathcal{T}).$$

Proof The proof follows that in [14,17], as cited for Theorem 2.2. □

Theorem 3.3 (Existence of discrete solutions) *There exists* (*at least*) *one solution of* (d3FF).

Proof This is a direct consequence of Theorems 3.1 and 3.2 and the equivalence of (CRVI) with the minimization of the convex functional E_{NC} over $\mathrm{CR}_0^1(\mathcal{T})$. Note that the choice of a piecewise constant ansatz space for u_h implies that, if $(u_h, q_h, \sigma_h) \in (P_0(\mathcal{T}) \times P_0(\mathcal{T}; \mathbb{R}^2)) \times \mathrm{RT}_0(\mathcal{T})$ is a solution of (d3FF) for the right-hand side $\Pi_0 f$, it also solves (d3FF) with right-hand side f. □

Remark 5 (*Uniqueness of* $(u_h, p_h) \in P_0(\mathcal{T}) \times P_0(\mathcal{T}; \mathbb{R}^2)$) Since W is convex, the solution $u_{\mathrm{CR}} \in \mathrm{CR}_0^1(\mathcal{T})$ of (CRVI) is unique. Together with Theorem 3.1, this proves uniqueness of $(u_h, p_h) = (\Pi_0 u_{\mathrm{CR}}, \nabla_{\mathrm{NC}} u_{\mathrm{CR}})$.

4 A priori analysis for (d3FF)

Section 4.1 states the main results of this paper. The proofs follow in Sects. 4.2, 4.3 and 4.4. Section 4.5 comments on the results and Sect. 4.6 deduces the convergence rate for the circular example with constant right-hand side and compares it with the predicted convergence rates for the conforming FEM.

4.1 Main results

Theorem 4.1 proves the a priori error estimate of (1.7) for the three-field formulation, while Theorem 4.2 carries out the direct medius analysis for the P_1 nonconforming FEM, that leads to (1.8). Theorem 4.3 states an a priori error estimate for the regularized discretization of Sect. 3.3. Recall

$$Q(f, \mathcal{T}) := \{\tau_{\mathrm{RT}} \in \mathrm{RT}_0(\mathcal{T}) \mid -\operatorname{div} \tau_{\mathrm{RT}} = \Pi_0 f\}.$$

Theorem 4.1 *Any solution* $(u, p, \sigma) \in X \times H(\operatorname{div}, \Omega)$ *to* (3FF) *and any discrete solution* $(u_h, p_h, \sigma_h) \in (P_0(\mathcal{T}) \times P_0(\mathcal{T}; \mathbb{R}^2)) \times \mathrm{RT}_0(\mathcal{T})$ *to* (d3FF) *satisfy*

$$\|\operatorname{div}(\sigma - \sigma_h)\|_{L^2(\Omega)} = \|f - \Pi_0 f\|_{L^2(\Omega)}, \tag{4.1}$$

$$\mu \|p - \Pi_0 p\|^2_{L^2(\Omega)}/2 + |j(p) - j(\Pi_0 p)| \le \|\sigma - \Pi_0 \sigma\|^2_{L^2(\Omega)}/(2\mu), \tag{4.2}$$

$$\begin{aligned} &\mu \|p - p_h\|^2_{L^2(\Omega)}/2 + |j(p) - j(\Pi_0 p)| \\ &\le \Big(\min_{\tau_h \in Q(f,\mathcal{T})} \|\sigma - \tau_h\|^2_{L^2(\Omega)} + \|h_{\mathcal{T}} f\|^2_{L^2(\Omega)}/2\Big)/\mu, \end{aligned} \tag{4.3}$$

$$\|u - u_h\|_{L^2(\Omega)} \lesssim \|u - \Pi_0 u\|_{L^2(\Omega)} + \|p - p_h\|_{L^2(\Omega)}. \tag{4.4}$$

Corollary 1 *The solution* $u \in H_0^1(\Omega)$ *to* (1.5) *and the solution* $u_{\mathrm{CR}} \in \mathrm{CR}_0^1(\mathcal{T})$ *to* (CRVI) *satisfy*

$$\begin{aligned} \mu^2 \|\nabla_{\mathrm{NC}}(u - u_{\mathrm{CR}})\|^2_{L^2(\Omega)} \le 4 &\min_{\tau_h \in Q(f,\mathcal{T})} \|\sigma - \tau_h\|^2_{L^2(\Omega)} \\ &+ 2\|h_{\mathcal{T}} f\|^2_{L^2(\Omega)} + 2\operatorname{osc}^2(f, \mathcal{T}). \end{aligned}$$

Proof Let $\widetilde{u}_{\mathrm{CR}} \in \mathrm{CR}_0^1(\mathcal{T})$ be the solution to (CRVI) with respect to the right-hand side $\Pi_0 f$. The sum of the variational inequalities (CRVI) applied to $\widetilde{u}_{\mathrm{CR}}$ and u_{CR} leads to

$$\mu \,\|\nabla_{\mathrm{NC}}(u_{\mathrm{CR}} - \widetilde{u}_{\mathrm{CR}})\|^2_{L^2(\Omega)} \leq (f - \Pi_0 f, u_{\mathrm{CR}} - \widetilde{u}_{\mathrm{CR}})_{L^2(\Omega)}$$
$$= (f - \Pi_0 f, u_{\mathrm{CR}} - \widetilde{u}_{\mathrm{CR}} - \Pi_0(u_{\mathrm{CR}} - \widetilde{u}_{\mathrm{CR}}))_{L^2(\Omega)}$$
$$\leq \mathrm{osc}(f, \mathcal{T})\, \|\nabla_{\mathrm{NC}}(u_{\mathrm{CR}} - \widetilde{u}_{\mathrm{CR}})\|_{L^2(\Omega)}\,.$$

The combination of this with Theorems 3.1 and 4.1 concludes the proof. □

The following theorem proves a best-approximation result with the best-approximation error of σ in the piecewise constant functions and implies (1.8).

Theorem 4.2 (Direct analysis of P_1 nonconforming FEM) *The solution $u \in H^1_0(\Omega)$ and the approximation $u_{\mathrm{CR}} \in \mathrm{CR}^1_0(\mathcal{T})$ satisfy*

$$\|\nabla_{\mathrm{NC}}(u - u_{\mathrm{CR}})\|_{L^2(\Omega)} \lesssim \|\sigma - \Pi_0\sigma\|_{L^2(\Omega)} + \mathrm{osc}(f, \mathcal{T}).$$

The following theorem proves an error estimate for the discrete solution of the regularized problem. This regularization is used in Sect. 6 for the practical treatment of the discrete problem.

Theorem 4.3 *The discrete solution $u_{\varepsilon,\mathrm{CR}} \in \mathrm{CR}^1_0(\mathcal{T})$ to (3.1) satisfies*

$$\mu \left\|\nabla_{\mathrm{NC}}(u - u_{\varepsilon,\mathrm{CR}})\right\|^2_{L^2(\Omega)} /4 \leq \varepsilon g|\Omega| + 2 \min_{\tau_h \in Q(f,\mathcal{T})} \|\sigma - \tau_h\|^2_{L^2(\Omega)} /\mu$$
$$+ \|h_{\mathcal{T}} f\|^2_{L^2(\Omega)} /\mu + \mathrm{osc}^2(f, \mathcal{T})/\mu. \qquad (4.5)$$

4.2 Proof of Theorem 4.1

This subsection proves the a priori error estimates of Theorem 4.1.

Proof of (4.1) The choice $q = p$ and $q_h = p_h$ implies

$$-\operatorname{div}\sigma = f \quad \text{almost everywhere and} \quad -\operatorname{div}\sigma_h = \Pi_0 f.$$

□

Proof of (4.2) The choice $v = u$ and $q = \Pi_0 p$ in the first formula of (3FF) leads to

$$(\sigma, \Pi_0 p - p)_{L^2(\Omega)} \leq \mu(p, \Pi_0 p - p)_{L^2(\Omega)} + j(\Pi_0 p) - j(p).$$

This yields

$$j(p) - j(\Pi_0 p) + \mu \,\|p - \Pi_0 p\|^2_{L^2(\Omega)} \leq (\sigma - \mu\Pi_0 p, p - \Pi_0 p)_{L^2(\Omega)}.$$

Since $\int_T (p - \Pi_0 p)\, dx = 0$ for all $T \in \mathcal{T}$,

$$(\sigma - \mu\Pi_0 p, p - \Pi_0 p)_{L^2(\Omega)} = (\sigma - \Pi_0\sigma, p - \Pi_0 p)_{L^2(\Omega)}$$
$$\leq \|\sigma - \Pi_0\sigma\|^2_{L^2(\Omega)} /(2\mu) + \mu \,\|p - \Pi_0 p\|^2_{L^2(\Omega)} /2.$$

Jensen's inequality [10] proves $j(\Pi_0 p) \le j(p)$. The combination with the previous two displayed inequalities concludes the proof. □

Proof of (4.3) The choice $v = u$ in problem (3FF) reads

$$(\sigma - \mu\, p, q - p)_{L^2(\Omega)} \le j(q) - j(p).$$

It follows for $q = p_h$ that

$$(\sigma - \mu\, p, p_h - p)_{L^2(\Omega)} \le j(p_h) - j(p).$$

The discrete problem (d3FF) implies for $v_h = u_h$ and $q_h = \Pi_0 p$ that

$$(\sigma_h - \mu\, p_h, \Pi_0 p - p_h)_{L^2(\Omega)} \le j(\Pi_0 p) - j(p_h).$$

The sum of the above inequalities and $(\sigma_h, \Pi_0 p - p_h)_{L^2(\Omega)} = (\Pi_0 \sigma_h, p - p_h)_{L^2(\Omega)}$ yield

$$\begin{aligned} \mu\, \|p - p_h\|^2_{L^2(\Omega)} + j(p) - j(\Pi_0 p) \le{}& (\sigma - \sigma_h, p - p_h)_{L^2(\Omega)} \\ &+ (\sigma_h - \Pi_0 \sigma_h, p - p_h)_{L^2(\Omega)}. \end{aligned} \tag{4.6}$$

The second formula of (3FF) implies for any $\tau_h \in \mathrm{RT}_0(\mathcal{T})$ that

$$\begin{aligned} (\tau_h, p - p_h)_{L^2(\Omega)} &= -b(\varphi - \varphi_h, \tau_h) - (u - u_h, \operatorname{div} \tau_h)_{L^2(\Omega)} \\ &= -(u - u_h, \operatorname{div} \tau_h)_{L^2(\Omega)}. \end{aligned}$$

Hence, any $\tau_h \in \mathrm{RT}_0(\mathcal{T})$ with $-\operatorname{div} \tau_h = \Pi_0 f$, written $\tau_h \in Q(f, \mathcal{T})$, satisfies

$$\begin{aligned} (\sigma - \sigma_h, p - p_h)_{L^2(\Omega)} &= (\sigma - \tau_h, p - p_h)_{L^2(\Omega)} \\ &\le \|\sigma - \tau_h\|^2_{L^2(\Omega)} / \mu + \mu\, \|p - p_h\|^2_{L^2(\Omega)} / 4. \end{aligned}$$

The second term in (4.6) reads

$$\begin{aligned} &(\operatorname{div} \sigma_h\, (\bullet - \operatorname{mid}(\mathcal{T}))/2, p - p_h)_{L^2(\Omega)} \\ &\quad = ((-\Pi_0 f)\, (\bullet - \operatorname{mid}(\mathcal{T}))/2, p - p_h)_{L^2(\Omega)} \\ &\quad \le \|h_{\mathcal{T}} f\|^2_{L^2(\Omega)} / (2\mu) + \mu\, \|p - p_h\|^2_{L^2(\Omega)} / 4. \end{aligned}$$

This results in

$$\begin{aligned} &\mu\, \|p - p_h\|^2_{L^2(\Omega)} / 2 + j(p) - j(\Pi_0 p) \\ &\quad \le \Big(\min_{\tau_h \in Q(f, \mathcal{T})} \|\sigma - \tau_h\|^2_{L^2(\Omega)} + \|h_{\mathcal{T}} f\|^2_{L^2(\Omega)} / 2 \Big) / \mu. \end{aligned}$$

Jensen's inequality [10] leads to $j(\Pi_0 p) \le j(p)$. This concludes the proof. □

Proof of (4.4) Define $e_h := \Pi_0 u - u_h \in P_0(\mathcal{T})$ and consider

$$\|u - u_h\|^2_{L^2(\Omega)} = \|u - \Pi_0 u\|^2_{L^2(\Omega)} + \|e_h\|^2_{L^2(\Omega)}\,.$$

The well-known inf–sup condition of the divergence in $\mathrm{RT}_0(\mathcal{T}) \times P_0(\mathcal{T})$ leads to the existence of $\tau_{\mathrm{RT}} \in \mathrm{RT}_0(\mathcal{T})$ with

$$\operatorname{div} \tau_{\mathrm{RT}} = e_h \quad \text{and} \quad \|\tau_{\mathrm{RT}}\|_{H(\operatorname{div},\Omega)} \lesssim \|e_h\|_{L^2(\Omega)}$$

The problem (3FF) leads to

$$\begin{aligned}\|e_h\|^2_{L^2(\Omega)} &= (u - u_h,\ \operatorname{div} \tau_{\mathrm{RT}})_{L^2(\Omega)}\\ &= -(p - p_h, \tau_{\mathrm{RT}})_{L^2(\Omega)}\\ &\lesssim \|p - p_h\|_{L^2(\Omega)}\ \|e_h\|_{L^2(\Omega)}\,.\end{aligned}$$

This concludes the proof. □

4.3 Proof of Theorem 4.2

Let $\sigma \in H(\operatorname{div}, \Omega)$ from Theorem 2.2. Then $\sigma(x) \in \partial W(\nabla u(x)) = \mu \nabla u(x) + \partial j(\nabla u(x))$ implies, for all $q \in L^2(\Omega)$, that

$$(\sigma - \mu\nabla u, q - \nabla u)_{L^2(\Omega)} \le j(q) - j(\nabla u).$$

Let $\sigma_{\mathrm{CR}} \in P_0(\mathcal{T}; \mathbb{R}^2)$ from Theorem 3.2. Then $\sigma_{\mathrm{CR}} \in \partial W(\nabla_{\mathrm{NC}} u_{\mathrm{CR}}) = \mu \nabla_{\mathrm{NC}} u_{\mathrm{CR}} + \partial j(\nabla_{\mathrm{NC}} u_{\mathrm{CR}})$ implies for $q_h \in P_0(\mathcal{T}; \mathbb{R}^2)$

$$(\sigma_{\mathrm{CR}} - \mu\nabla_{\mathrm{NC}} u_{\mathrm{CR}}, q_h - \nabla_{\mathrm{NC}} u_{\mathrm{CR}})_{L^2(\Omega)} \le j(q_h) - j(\nabla_{\mathrm{NC}} u_{\mathrm{CR}}).$$

Let $q = \nabla_{\mathrm{NC}} u_{\mathrm{CR}}$ and $q_h = \Pi_0 \nabla u$ in the previous inequalities. The sum of the two inequalities yields

$$\begin{aligned}\mu\,\|\nabla_{\mathrm{NC}}(u - u_{\mathrm{CR}})\|^2 \le{}& (\sigma - \sigma_{\mathrm{CR}}, \nabla_{\mathrm{NC}}(u - u_{\mathrm{CR}}))_{L^2(\Omega)}\\ &+ (\sigma_{\mathrm{CR}} - \mu\nabla_{\mathrm{NC}} u_{\mathrm{CR}}, \nabla u - \Pi_0 \nabla u)_{L^2(\Omega)}\\ &+ j(\Pi_0 \nabla u) - j(\nabla u).\end{aligned} \tag{4.7}$$

Since σ_{CR} and $\nabla_{\mathrm{NC}} u_{\mathrm{CR}}$ are piecewise constant, the second term vanishes. Jensen's inequality [10] leads to $j(\Pi_0 \nabla u) - j(\nabla u) \le 0$. Let $\mathcal{N}(\Omega) = \mathcal{N} \cap \Omega$ denote the set of the interior nodes and $\mathcal{T}(z) := \{T \in \mathcal{T} \mid z \in T\}$ the set of triangles that share the node z. Let $J_3 : \mathrm{CR}^1_0(\mathcal{T}) \to (P_3(\mathcal{T}) \cap H^1_0(\Omega))$ be defined as in [5, Subsection 2.4], [3,7] with the conservation property

$$⨍_T \nabla J_3 v_{\mathrm{CR}}\, dx = ⨍_T \nabla_{\mathrm{NC}} v_{\mathrm{CR}}\, dx \qquad \text{for all } T \in \mathcal{T} \tag{4.8}$$

and the approximation and stability property

$$\begin{aligned}\left\|h_{\mathcal{T}}^{-1}(v_{\mathrm{CR}}-J_3 v_{\mathrm{CR}})\right\|_{L^2(\Omega)} &\approx \|\nabla_{\mathrm{NC}}(v_{\mathrm{CR}}-J_3 v_{\mathrm{CR}})\|_{L^2(\Omega)}\\ &\approx \min_{\varphi\in H_0^1(\Omega)} \|\nabla_{\mathrm{NC}}(v_{\mathrm{CR}}-\varphi)\|_{L^2(\Omega)}\\ &\le \|\nabla_{\mathrm{NC}}(v_{\mathrm{CR}}-u)\|_{L^2(\Omega)}\end{aligned}$$

for all $v_{\mathrm{CR}} \in \mathrm{CR}_0^1(\mathcal{T})$. Let $I_{\mathrm{NC}} : H_0^1(\Omega) \to \mathrm{CR}_0^1(\mathcal{T})$ denote the nonconforming interpolation operator defined by

$$I_{\mathrm{NC}} v(\operatorname{mid}(E)) = \fint_E v\, ds$$

for all interior edges $E \in \mathcal{E}(\Omega)$ with the approximation property [4]

$$\begin{aligned}\|h_{\mathcal{T}}^{-1}(v-I_{\mathrm{NC}}v)\|_{L^2(\Omega)} &\lesssim \|\nabla_{\mathrm{NC}}(v-I_{\mathrm{NC}}v)\|_{L^2(\Omega)}\\ &= \min_{w_{\mathrm{CR}}\in \mathrm{CR}_0^1(\mathcal{T})} \|\nabla_{\mathrm{NC}}(v-w_{\mathrm{CR}})\|_{L^2(\Omega)}.\end{aligned}$$

A calculation reveals the integral mean property $\nabla_{\mathrm{NC}} I_{\mathrm{NC}} v = \Pi_0 \nabla v$ for all $T \in \mathcal{T}$ and $v \in H_0^1(\Omega)$. This and the conservation property (4.8) leads to

$$\begin{aligned}(\sigma-\sigma_{\mathrm{CR}}, \nabla_{\mathrm{NC}}(u-u_{\mathrm{CR}}))_{L^2(\Omega)} &= (\sigma, \nabla(u-J_3 u_{\mathrm{CR}}))_{L^2(\Omega)}\\ &\quad + (\sigma-\Pi_0\sigma, \nabla_{\mathrm{NC}}(J_3 u_{\mathrm{CR}}-u_{\mathrm{CR}}))_{L^2(\Omega)}\\ &\quad - (\sigma_{\mathrm{CR}}, \nabla_{\mathrm{NC}}(I_{\mathrm{NC}}u-u_{\mathrm{CR}}))_{L^2(\Omega)}.\end{aligned}$$

Theorems 2.2 and 3.2 imply

$$\begin{aligned}&(\sigma, \nabla(u-J_3 u_{\mathrm{CR}}))_{L^2(\Omega)} - (\sigma_{\mathrm{CR}}, \nabla_{\mathrm{NC}}(I_{\mathrm{NC}}u-u_{\mathrm{CR}}))_{L^2(\Omega)}\\ &\quad = (f, u-I_{\mathrm{NC}}u)_{L^2(\Omega)} + (f, u_{\mathrm{CR}}-J_3 u_{\mathrm{CR}})_{L^2(\Omega)}.\end{aligned}$$

The combination of the previous inequalities with the approximation properties of I_{NC} and J_3 yield

$$\begin{aligned}&(\sigma-\sigma_{\mathrm{CR}}, \nabla_{\mathrm{NC}}(u-u_{\mathrm{CR}}))_{L^2(\Omega)}\\ &\quad \lesssim (\|h_{\mathcal{T}} f\|_{L^2(\Omega)} + \|\sigma-\Pi_0\sigma\|_{L^2(\Omega)})\, \|\nabla_{\mathrm{NC}}(u-u_{\mathrm{CR}})\|_{L^2(\Omega)}\,.\end{aligned}$$

The efficiency of $\|h_{\mathcal{T}} f\|_{L^2(\Omega)}$, namely

$$\|h_{\mathcal{T}} f\|_{L^2(\Omega)} \lesssim \|\sigma-\Pi_0\sigma\|_{L^2(\Omega)} + \operatorname{osc}(f,\mathcal{T}),$$

follows from

$$\int_T f \flat_T \, dx = \int_T \sigma \cdot \nabla \flat_T \, dx = \int_T (\sigma - \Pi_0 \sigma) \cdot \nabla \flat_T \, dx$$

for the volume bubble function $\flat_T = \lambda_a \lambda_b \lambda_c$ with barycentric coordinates λ_a, λ_b, and λ_c for the triangle $T = \operatorname{conv}\{a, b, c\} \in \mathcal{T}$ as in the bubble function technique of [19, Chapter I]. The combination with (4.7) yields the assertion.

4.4 Proof of Theorem 4.3

Lemma 2 *Any $A, B \in \mathbb{R}^2$ satisfy*

$$\mu/2|A - B|^2 \leq W_\varepsilon(B) - W_\varepsilon(A) - DW_\varepsilon(A) \cdot (B - A).$$

Proof An elementary calculation reveals

$$\begin{aligned} &W_\varepsilon(B) - W_\varepsilon(A) - DW_\varepsilon(A)\,(B - A) \\ &\quad = (\mu/2)|A - B|^2 + g\left(\sqrt{|B|^2 + \varepsilon^2} - \sqrt{|A|^2 + \varepsilon^2} - A \cdot (B - A)/\sqrt{|A|^2 + \varepsilon^2}\right). \end{aligned}$$

The formula $2ab \leq a^2 + b^2$ together with the Cauchy inequality prove

$$(A \cdot B + \varepsilon^2)^2 \leq |A|^2|B|^2 + \varepsilon^2(|A|^2 + |B|^2) + \varepsilon^4 = (|A|^2 + \varepsilon^2)(|B|^2 + \varepsilon^2).$$

This yields

$$\begin{aligned} &\sqrt{|B|^2 + \varepsilon^2} - \sqrt{|A|^2 + \varepsilon^2} - A \cdot (B - A)/\sqrt{|A|^2 + \varepsilon^2} \\ &\quad = \sqrt{|B|^2 + \varepsilon^2} - (A \cdot B + \varepsilon^2)/\sqrt{|A|^2 + \varepsilon^2} \geq 0 \end{aligned}$$

This yields the assertion. □

Proof of Theorem 4.3 Since $\sqrt{a^2 + b^2} \leq a + b$ for $a, b > 0$, the functional j_ε defined by

$$j_\varepsilon(F) := g \int_\Omega \left(\sqrt{|F|^2 + \varepsilon^2} - \varepsilon\right) dx \quad \text{for all} \quad F \in L^2(\Omega; \mathbb{R}^2),$$

satisfies, for all $v_{\mathrm{NC}} \in H_0^1(\Omega) + \mathrm{CR}_0^1(\mathcal{T})$, that

$$j_\varepsilon(\nabla_{\mathrm{NC}} v_{\mathrm{NC}}) \leq j(\nabla_{\mathrm{NC}} v_{\mathrm{NC}}) \leq j_\varepsilon(\nabla_{\mathrm{NC}} v_{\mathrm{NC}}) + g\varepsilon|\Omega|.$$

This implies

$$E_{\varepsilon,\mathrm{NC}}(v_{\mathrm{NC}}) \leq E_{\mathrm{NC}}(v_{\mathrm{NC}}) \leq E_{\varepsilon,\mathrm{NC}}(v_{\mathrm{NC}}) + g\varepsilon|\Omega|. \tag{4.9}$$

Lemma 2 leads to

$$(\mu/2)\left\|\nabla_{\mathrm{NC}}(u_{\mathrm{CR}}-u_{\varepsilon,\mathrm{CR}})\right\|^2_{L^2(\Omega)} \le E_{\varepsilon,\mathrm{NC}}(u_{\mathrm{CR}})-E_{\varepsilon,\mathrm{NC}}(u_{\varepsilon,\mathrm{CR}})$$
$$-\left(DW_\varepsilon(\nabla_{\mathrm{NC}}u_{\varepsilon,\mathrm{CR}}),\nabla_{\mathrm{NC}}(u_{\mathrm{CR}}-u_{\varepsilon,\mathrm{CR}})\right)_{L^2(\Omega)}+F(u_{\mathrm{CR}}-u_{\varepsilon,\mathrm{CR}}).$$

The Euler–Lagrange Equations for the smooth W_ε and the previous inequalities lead to

$$\begin{aligned}(\mu/2)\left\|\nabla_{\mathrm{NC}}(u_{\mathrm{CR}}-u_{\varepsilon,\mathrm{CR}})\right\|^2_{L^2(\Omega)} &\le E_{\varepsilon,\mathrm{NC}}(u_{\mathrm{CR}})-E_{\varepsilon,\mathrm{NC}}(u_{\varepsilon,\mathrm{CR}})\\ &\le E_{\mathrm{NC}}(u_{\mathrm{CR}})-E_{\mathrm{NC}}(u_{\varepsilon,\mathrm{CR}})+g\varepsilon|\Omega|.\end{aligned}$$

Since u_{CR} minimizes E_{NC} in $\mathrm{CR}^1_0(\mathcal{T})$, this implies

$$(\mu/2)\left\|\nabla_{\mathrm{NC}}(u_{\mathrm{CR}}-u_{\varepsilon,\mathrm{CR}})\right\|^2_{L^2(\Omega)} \le g\varepsilon|\Omega|.$$

A triangle inequality and Corollary 1 conclude the proof. □

4.5 Comments

The a priori bounds (4.3)–(4.5) have the form of best-approximation estimates in terms of the stress as announced in (1.7) and (1.8) of the Sect. 1. Further information about the order of convergence relies on the stress regularity. It has been shown [12, Theorem 3.3.3] that $\nabla u \in C^0(\Omega,\mathbb{R}^2)$, specialized to the present problem of pipe flow. Moreover, if $\partial\Omega$ is smooth and $f \in L^2(\Omega)$ it holds $u \in H^2(\Omega)$ [2,14]. But the situation is different for the stress. First, σ is indeterminate in the subset $\Omega_0 := \{x\in\Omega \mid \nabla u(x)=0\}$, which is simply connected [16]. Furthermore, σ is not unique. Therefore, the best that one can deduce in terms of the regularity of σ would be that, from (1.2), it is continuous in the region $\Omega\backslash\overline{\Omega_0}$. For those situations in which it is possible to construct a continuous extension of σ in Ω_0 that satisfies the momentum equation $\operatorname{div}\sigma + f = 0$, the optimal order of rate of convergence may be recovered. This is demonstrated in the example of Sect. 4.6.

4.6 Example with known solution

Let $\Omega := B(R,0)=\{x\in\mathbb{R}^2 \mid |x|\le R\}$ and $f\equiv C$. Then the exact solution of (1.5) reads [13]

$$u(x)=0 \quad \text{if} \quad g \ge CR/2,$$
$$u(x)=\begin{cases} C(R^2-r^2)/(4\mu)-g(R-r)/\mu & \text{if } 2g/C\le r\le R,\\ C(R-2g/C)^2/(4\mu) & \text{if } 0\le r\le 2g/C\end{cases} \quad \text{if} \quad g<CR/2.$$

One solution $\sigma\in H(\operatorname{div},\Omega)$ of $\sigma\in\partial W(\nabla u)$ with $-\operatorname{div}\sigma = f$ reads $\sigma_0 := -Cx/2$. Let $h := \max h_{\mathcal{T}} := \|h_{\mathcal{T}}\|_{L^\infty(\Omega)}$ denote the maximal mesh-size of an underlying triangulation. Since $\sigma_0 \in \mathrm{RT}_0(\mathcal{T})$, Theorem 4.1 implies

$$\begin{aligned} \operatorname{div}(\sigma_0 - \sigma_h) = 0, \\ \|p - p_h\|^2_{L^2(\Omega)}/2 + |j(p) - j(\Pi_0 p)| \leq \|h_{\mathcal{T}} f\|^2_{L^2(\Omega)}/2 \lesssim h^2, \\ \|u - u_h\|_{L^2(\Omega)} \lesssim \|u - \Pi_0 u\|_{L^2(\Omega)} + \|p - p_h\|_{L^2(\Omega)} \lesssim h. \end{aligned}$$

This proves

$$\|p - p_h\|_{L^2(\Omega)} + \|u - u_h\|_{L^2(\Omega)} + \|\nabla_{\mathrm{NC}}(u - u_{\mathrm{CR}})\|_{L^2(\Omega)} \lesssim h.$$

For comparison, for the conforming P_1 finite element method, [13] proves

$$\|p - p_{\mathrm{C}}\|_{L^2(\Omega)} \lesssim h\sqrt{\log(1/h)}$$

for the gradient $p_{\mathrm{C}} \in P_0(\mathcal{T}; \mathbb{R}^2)$ of the discrete solution of the conforming P_1 finite element method (as described in Sect. 6.1).

5 Generalisation to 3D

This section describes the variational inequality for the 3D Bingham problem with its discretization in Sect. 5.1, the three-field formulation with its discretization in Sect. 5.2 and proves in Sect. 5.3 a priori error bounds.

5.1 Variational inequality

Let $\mathbb{S} = \{A \in \mathbb{R}^{3\times 3} \mid A = A^\top\}$ be the space of symmetric matrices and $\operatorname{sym} A = (A + A^\top)/2$ and define for $q \in L^2(\Omega; \mathbb{R}^{3\times 3})$

$$j(q) = g \int_\Omega |\operatorname{sym}(q)|\, dx.$$

The variational inequality for the Bingham flow problem in 3D seeks $u \in Z := \{w \in H^1_0(\Omega; \mathbb{R}^3) \mid \operatorname{div} w = 0\}$ with

$$\int_\Omega f \cdot (v - u)\, dx \leq \mu \int_\Omega \varepsilon(u) : \varepsilon(v - u)\, dx + j(\nabla v) - j(\nabla u) \quad \text{for all} \quad v \in Z. \tag{5.1}$$

A direct discretization of the variational inequality with P_1 nonconforming finite elements is not possible, since $\int_\Omega \varepsilon_{\mathrm{NC}}(\bullet) : \varepsilon_{\mathrm{NC}}(\bullet)\, dx$ is not positive definite on the P_1 nonconforming finite element space. For homogeneous Dirichlet boundary conditions, a straightforward calculation reveals that

$$2\int_\Omega \varepsilon(u):\varepsilon(v)\,dx = \int_\Omega (\nabla u:\nabla v + \operatorname{div} u \operatorname{div} v)\,dx.$$

Since $\operatorname{div} u = 0$, this leads to the alternative formulation of (5.1): Seek $u \in Z$ with

$$\int_\Omega f\cdot(v-u)\,dx \le (\mu/2)\int_\Omega \nabla u:\nabla(v-u)\,dx \\ + j(\nabla v) - j(\nabla u) \quad \text{for all} \quad v\in Z.$$

Define the energy

$$E_{3D}(v) := \int_\Omega W_{3D}(\nabla v)\,dx - F(v) \quad \text{for all} \quad v \in H^1_0(\Omega;\mathbb{R}^3)$$

with $W_{3D}(A) := (\mu/4)|A|^2 + g|\mathrm{sym} A|$. The unique existence of a solution u follows from the equivalence of (5.1) with the minimization of E_{3D} over Z as in the two-dimensional case. The discretization with P_1 nonconforming finite elements seeks $u_{\mathrm{CR}} \in Z_{\mathrm{CR}}(\mathcal{T}) := \{w_{\mathrm{CR}} \in \mathrm{CR}^1_0(\mathcal{T};\mathbb{R}^3) \mid \operatorname{div}_{\mathrm{NC}} w_{\mathrm{CR}} = 0\}$ with

$$\int_\Omega f\cdot(v_{\mathrm{CR}}-u_{\mathrm{CR}})\,dx \le (\mu/2)\int_\Omega \nabla_{\mathrm{NC}}u_{\mathrm{CR}}:\nabla_{\mathrm{NC}}(v_{\mathrm{CR}}-u_{\mathrm{CR}})\,dx \\ + j(\nabla_{\mathrm{NC}}v_{\mathrm{CR}}) - j(\nabla_{\mathrm{NC}}u_{\mathrm{CR}}) \tag{5.2}$$

for all $v_{\mathrm{CR}} \in Z_{\mathrm{CR}}(\mathcal{T})$. The unique existence of a discrete solution to (5.2) follows with the equivalence to the minimization of

$$E_{\mathrm{NC},3D}(v_{\mathrm{CR}}) := \int_\Omega W_{3D}(\nabla_{\mathrm{NC}}v_{\mathrm{CR}})\,dx - F(v_{\mathrm{CR}}) \quad \text{over} \quad Z_{\mathrm{CR}}(\mathcal{T}).$$

The following theorem is the point of departure for the a priori error analysis from Sect. 5.3.

Theorem 5.1 (Euler–Lagrange equations for 3D Bingham flow) *The solution $u \in H^1_0(\Omega;\mathbb{R}^3)$ to (5.1) satisfies the Euler–Lagrange equations in the sense that there exist $\sigma \in H(\operatorname{div},\Omega;\mathbb{R}^{3\times 3})$ and $\xi \in L^2_0(\Omega)$ with*

$$\sigma - \xi I_{3\times 3} \in \partial W_{3D}(\nabla u) \quad \textit{and} \quad f + \operatorname{div}\sigma = 0 \textit{ a.e. in } \Omega.$$

The discrete solution $u_{\mathrm{CR}} \in \mathrm{CR}^1_0(\mathcal{T};\mathbb{R}^3)$ satisfies the discrete Euler–Lagrange equations in the sense that there exist $\sigma_{\mathrm{CR}} \in P_0(\mathcal{T};\mathbb{R}^{3\times 3})$ and $\xi_0 \in P_0(\mathcal{T}) \cap L^2_0(\Omega)$ with

$$\sigma_{\mathrm{CR}} - \xi_0 I_{3\times 3} \in \partial W_{3D}(\nabla_{\mathrm{NC}}u_{\mathrm{CR}}) \quad \textit{a.e. in } \Omega$$

and

$$(\sigma_{\rm CR}\,, \nabla_{\rm NC} v_{\rm CR})_{L^2(\Omega)} = F(v_{\rm CR}) \quad \textit{for all} \quad v_{\rm CR} \in {\rm CR}^1_0(\mathcal{T};\mathbb{R}^3). \tag{5.3}$$

Proof The proof is analogous to that of [14, Theorem 6.3] and is outlined below. The regularization

$$j_\delta(q) = g \int_\Omega \sqrt{\delta^2 + |{\rm sym}(q)|^2}\, dx \quad \text{for all} \;\; q \in L^2(\Omega;\mathbb{R}^{3\times 3})$$

of j motivates the minimization of

$$E_{\delta,3D}(v) := (\mu/4)\int_\Omega |\nabla v|^2\, dx + j_\delta(\nabla v) - F(v)$$

over Z with unique minimizer u_δ. The arguments of [14, p. 83, l.10–21] lead to

$$u_\delta \to u \;\; \text{strongly in} \;\; H^1_0(\Omega;\mathbb{R}^3) \;\; \text{as} \;\; \delta \to 0.$$

Since j_δ is differentiable on Z, the solution u_δ is characterised by the Euler–Lagrange equations

$$(\mu/2)\int_\Omega \nabla u_\delta \cdot \nabla v\, dx + g\int_\Omega (\varepsilon(u_\delta) : \varepsilon(v)/\sqrt{\delta^2 + |\varepsilon(u_\delta)|^2})\, dx = F(v)$$

for all $v \in Z$. Define $p_\delta := \varepsilon(u_\delta)/\sqrt{\delta^2 + |\varepsilon(u_\delta)|^2}$. Then

$$p_\delta \in \Lambda := \{q \in L^2(\Omega;\mathbb{S}) \mid |q(x)| \le 1 \;\; \text{a.e. and} \;\; {\rm tr}(q) = 0\}$$

and the arguments of [14, p. 84] prove the existence of a weak limit $p \in \Lambda$ with $p_\delta \rightharpoonup p$ weakly in $L^2(\Omega;\mathbb{R}^{3\times 3})$ as $\delta \to 0$,

$$((\mu/2)\nabla u + gp, \nabla v)_{L^2(\Omega)} = F(v) \quad \text{for all} \;\; v \in Z,$$

and $(\mu/2)\nabla u + gp \in \partial W_{3D}(\nabla u)$. In order to involve $\xi \in L^2(\Omega)$ and generalize the equilibrium to all test functions $v \in H^1_0(\Omega;\mathbb{R}^3)$, let $\alpha \in H^1_0(\Omega;\mathbb{R}^3)$ and $\xi \in L^2_0(\Omega)$ be the solutions to the Stokes equations

$$\begin{aligned}(\nabla\alpha, \nabla v)_{L^2(\Omega)} - (\xi,\, {\rm div}\; v)_{L^2(\Omega)} &= F(v) - ((\mu/2)\nabla u + gp, \nabla v)_{L^2(\Omega)},\\ ({\rm div}\;\alpha, \zeta)_{L^2(\Omega)} &= 0\end{aligned}$$

for all $v \in H^1_0(\Omega;\mathbb{R}^3)$ and all $\zeta \in L^2_0(\Omega)$. The choice $v = \alpha \in Z$ proves $\alpha = 0$. Hence, $\sigma := (\mu/2)\nabla u + gp - \xi I_{3\times 3} \in H({\rm div}\,, \Omega;\mathbb{R}^{3\times 3})$ fulfils

$$(\sigma, \nabla v)_{L^2(\Omega)} = F(v) \quad \text{for all} \;\; v \in H^1_0(\Omega;\mathbb{R}^3).$$

The discrete Euler–Lagrange equations (5.3) follow with the same arguments. Indeed, since the discrete spaces are finite dimensional, the convergences are even strong. The existence of $\xi_h \in P_0(\mathcal{T}) \cap L^2_0(\Omega)$ follows from the existence of solutions of the discrete Stokes equations [8]. □

5.2 Three-field formulation

As in two dimensions, define for $\psi = (v, q) \in L^2(\Omega; \mathbb{R}^3) \times L^2(\Omega; \mathbb{R}^{3\times 3})$ and $\tau \in H(\operatorname{div}, \Omega; \mathbb{R}^{3\times 3})$ the bilinear form

$$b(\psi, \tau) = -(v, \operatorname{div} \tau)_{L^2(\Omega)} - (q, \tau)_{L^2(\Omega)}.$$

The three-field formulation seeks $p = \nabla u$ with values in $\mathbb{R}^{3\times 3}_{\text{dev}} := \{A \in \mathbb{R}^{3\times 3} \mid \operatorname{tr}(A) = 0\}$ and in this way incorporates incompressibility. The three-field formulation in 3D seeks $\varphi = (u, p) \in L^2(\Omega; \mathbb{R}^3) \times L^2(\Omega; \mathbb{R}^{3\times 3}_{\text{dev}})$ and $\sigma \in H(\operatorname{div}, \Omega; \mathbb{R}^{3\times 3})$ with

$$\begin{aligned} F(v-u) &\le \mu(p, q-p)_{L^2(\Omega)} + j(q) - j(p) + b(\psi - \varphi, \sigma),\\ b(\varphi, \tau) &= 0 \end{aligned} \tag{5.4}$$

for all $\psi = (v, q) \in L^2(\Omega; \mathbb{R}^3) \times L^2(\Omega; \mathbb{R}^{3\times 3}_{\text{dev}})$ and $\tau \in H(\operatorname{div}, \Omega; \mathbb{R}^{3\times 3})$. Let $\mathrm{RT}_0(\mathcal{T}; \mathbb{R}^{3\times 3})$ denote the space of row-wise Raviart–Thomas functions. The discrete three-field formulation in 3D seeks $\varphi_h = (u_h, p_h) \in P_0(\mathcal{T}; \mathbb{R}^3) \times P_0(\mathcal{T}; \mathbb{R}^{3\times 3}_{\text{dev}})$ and $\sigma_h \in \mathrm{RT}_0(\mathcal{T}; \mathbb{R}^{3\times 3})$ with

$$\begin{aligned} F(v_h - u_h) &\le \mu(p_h, q_h - p)_{L^2(\Omega)} + j(q_h) - j(p_h) + b(\psi_h - \varphi_h, \sigma_h),\\ b(\varphi_h, \tau_h) &= 0 \end{aligned} \tag{5.5}$$

for all $\psi_h = (v_h, q_h) \in P_0(\mathcal{T}; \mathbb{R}^3) \times P_0(\mathcal{T}; \mathbb{R}^{3\times 3}_{\text{dev}})$ and $\tau_h \in \mathrm{RT}_0(\mathcal{T}; \mathbb{R}^{3\times 3})$.

The equivalence of the P_1 nonconforming discretization of the variational inequality and the discretization of the three-field formulation follows with the arguments of Theorem 3.1 and the observation that $\operatorname{tr}(\nabla_{\mathrm{NC}} u_{\mathrm{CR}}(x)) = \operatorname{tr}(p_h(x)) = 0$ implies that $\operatorname{div}_{\mathrm{NC}} u_{\mathrm{CR}} = 0$.

5.3 A priori analysis

This section generalizes Theorems 4.1 and 4.2 of Sect. 4 to 3D.

Theorem 5.2 (Direct analysis of P_1 nonconforming FEM in 3D) *The solution $u \in Z$ of* (5.1) *and the discrete solution $u_{\mathrm{CR}} \in Z_{\mathrm{CR}}(\mathcal{T})$ of* (5.2) *satisfy*

$$\|\nabla_{\mathrm{NC}}(u - u_{\mathrm{CR}})\|_{L^2(\Omega)} \lesssim \|\sigma - \Pi_0 \sigma\|_{L^2(\Omega)} + \operatorname{osc}(f, \mathcal{T}).$$

Proof The crucial points in the proof are analogous to those of Theorem 4.2. The outline given here shows how ξ (the Lagrange multiplier for the incompressibility condition from Theorem 5.1) comes into play.

Let $\sigma \in H(\mathrm{div}, \Omega; \mathbb{R}^{3\times3})$ and $\xi \in L^2_0(\Omega)$ from Theorem 5.1. The sum rule for subderivatives implies $\partial W_{3D}(\nabla u) = \mu\nabla u + \partial j(\nabla u)$. Then $\sigma - \xi I_{3\times3} \in \partial W_{3D}(\nabla u)$ implies for all $q \in L^2(\Omega; \mathbb{R}^{3\times3})$

$$(\sigma - \xi I_{3\times3} - \mu\nabla u, q - \nabla u)_{L^2(\Omega)} \le j(q) - j(\nabla u).$$

For $\sigma_{\mathrm{CR}} \in P_0(\mathcal{T}; \mathbb{R}^{3\times3})$ and $\xi_h \in P_0(\mathcal{T}) \cap L^2_0(\Omega)$ from Theorem 5.1, the same arguments prove for all $q_h \in P_0(\mathcal{T}; \mathbb{R}^{3\times3})$

$$(\sigma_{\mathrm{CR}} - \xi_h I_{3\times3} - \mu\nabla_{\mathrm{NC}} u_{\mathrm{CR}}, q_h - \nabla_{\mathrm{NC}} u_{\mathrm{CR}})_{L^2(\Omega)} \le j(q_h) - j(\nabla_{\mathrm{NC}} u_{\mathrm{CR}}).$$

The choice $q = \nabla_{\mathrm{NC}} u_{\mathrm{CR}}$ and $q_h = \Pi_0\nabla u$ in the two above displayed inequalities and the sum of those prove

$$\begin{aligned}\mu\,\|\nabla_{\mathrm{NC}}(u - u_{\mathrm{CR}})\|^2 &\le j(\Pi_0\nabla u) - j(\nabla u) + \mu(\nabla_{\mathrm{NC}} u_{\mathrm{CR}}, \Pi_0\nabla u - \nabla u)_{L^2(\Omega)}\\ &\quad + (\sigma - \sigma_{\mathrm{CR}}, \nabla_{\mathrm{NC}}(u - u_{\mathrm{CR}}))_{L^2(\Omega)} + (\sigma_{\mathrm{CR}}, \nabla u - \Pi_0\nabla u)_{L^2(\Omega)}\\ &\quad + (\xi - \xi_{\mathrm{CR}}, \mathrm{div}_{\mathrm{NC}}(u - u_{\mathrm{CR}}))_{L^2(\Omega)} + (\xi_{\mathrm{CR}}, \mathrm{div}\, u - \Pi_0\, \mathrm{div}\, u)_{L^2(\Omega)}.\end{aligned}$$

As in the proof of Theorem 4.2, Jensen's inequality [10] yields $j(\Pi_0\nabla u) - j(\nabla u) \le 0$ and, since $\nabla_{\mathrm{NC}} u_{\mathrm{CR}}, \sigma_{\mathrm{CR}}$, and ξ_{CR} are piecewise constant, the third, fifth and seventh term on the right-hand side vanish. The fourth term is estimated by means of a conforming companion operator analogously to the proof of Theorem 5.1. Since $u \in Z$ and $u_{\mathrm{CR}} \in Z_{\mathrm{CR}}(\mathcal{T})$, the remaining term vanishes, namely

$$(\xi - \xi_{\mathrm{CR}}, \mathrm{div}_{\mathrm{NC}}(u - u_{\mathrm{CR}}))_{L^2(\Omega)} = 0.$$

□

Theorem 5.3 *Any solution* $(u, p, \sigma) \in L^2(\Omega; \mathbb{R}^3) \times L^2(\Omega; \mathbb{R}^{3\times3}) \times H(\mathrm{div}, \Omega; \mathbb{R}^{3\times3})$ *of* (5.4) *and any discrete solution* $(u_h, p_h, \sigma_h) \in (P_0(\mathcal{T}; \mathbb{R}^3) \times P_0(\mathcal{T}; \mathbb{R}^{3\times3})) \times \mathrm{RT}_0(\mathcal{T}; \mathbb{R}^{3\times3})$ *of* (5.5) *satisfies*

$$\|\mathrm{div}\,(\sigma - \sigma_h)\|_{L^2(\Omega)} = \|f - \Pi_0 f\|_{L^2(\Omega)}, \tag{5.6}$$

$$\mu\,\|p - \Pi_0 p\|^2_{L^2(\Omega)}/2 + |j(p) - j(\Pi_0 p)| \le \|\sigma - \Pi_0\sigma\|^2_{L^2(\Omega)}/(2\mu), \tag{5.7}$$

$$\begin{aligned}&\mu\,\|p - p_h\|^2_{L^2(\Omega)}/2 + |j(p) - j(\Pi_0 p)|\\ &\le (\min_{\tau_h \in Q(f,\mathcal{T})} \|\sigma - \tau_h\|^2_{L^2(\Omega)} + \|h_{\mathcal{T}} f\|^2_{L^2(\Omega)}/2)/\mu,\end{aligned} \tag{5.8}$$

$$\|u - u_h\|_{L^2(\Omega)} \lesssim \|u - \Pi_0 u\|_{L^2(\Omega)} + \|p - p_h\|_{L^2(\Omega)}. \tag{5.9}$$

Proof The proof of the theorem is analogous to that of Theorem 4.1 and therefore omitted. □

6 Numerical experiments

This section is devoted to numerical experiments for the Bingham flow problem which compare the lowest-order nonconforming and conforming FEM. After a brief introduction into the implementation in Sect. 6.1, the computational benchmarks follow in Sects. 6.2, 6.3 and 6.4. Section 6.5 draws conclusions of the experiments.

6.1 Numerical realisation

The solution of (1.5) is approximated by the regularized problem (3.1) for Crouzeix–Raviart functions and by a conforming approximation $u_{\mathrm{C}} \in P_1(\mathcal{T}) \cap H_0^1$ with

$$E_{\varepsilon,\mathrm{NC}}(u_{\mathrm{C}}) = \min_{v_{\mathrm{C}} \in P_1(\mathcal{T}) \cap H_0^1(\Omega)} E_{\varepsilon,\mathrm{NC}}(v_{\mathrm{C}}). \tag{6.1}$$

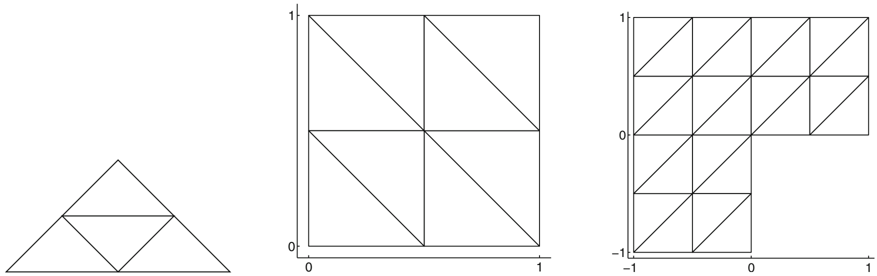
The numerical experiments from Sects. 6.2, 6.3 and 6.4 compare the nonconforming FEM of Sect. 3.2 with the lowest-order conforming FEM and investigate the dependence of the error on the regularization parameter ε.

The representation of a P_1 nonconforming function $v_{\mathrm{CR}} \in \mathrm{CR}_0^1(\mathcal{T})$ with respect to the edge-oriented basis functions $\psi_1, \ldots, \psi_M$ with the number $M := |\mathcal{E}(\Omega)|$ of interior edges in $\mathcal{T}$ and the basis functions defined by $\psi_E(\mathrm{mid}\,(F)) = \delta_{EF})$ reads $v_{\mathrm{CR}} = \sum_{j=1}^{M} x_j \psi_j$ for a coefficient vector $x = (x_1, \ldots, x_M)$. The minimization of $x \mapsto E_{\varepsilon,\mathrm{NC}}(v_{\mathrm{CR}})$ is realised with the Matlab routine `fminunc` (which uses a trust-region algorithm) with input of $E_{\varepsilon,\mathrm{NC}}$, $DE_{\varepsilon,\mathrm{NC}}$, and $D^2 E_{\varepsilon,\mathrm{NC}}$ at x and maximal 2000 iterations and with a termination tolerance of 10^{-15}. A conforming $P_1(\mathcal{T})$ approximation of the minimum serves as the initial guess for the refined triangulation.

For three domains, the discrete problems are solved on a sequence of triangulations $(\mathcal{T}_\ell)_{\ell=1,2,\ldots}$ based on successive uniform red-refinements from Fig. 1a. The right-hand side is $f \equiv 1$, the viscosity $\mu = 1$, and the plasticity yield $g = 0.2$.

Two strategies (a) and (b) are implemented to steer the regularization parameter.

(a) The value of $\varepsilon_j = 10^{-j}$ with $j = 0, 1, \ldots, 5$ is fixed for the sequence of triangulations. The errors for every domain are plotted against the degrees of freedom; the



(a) A red-refinement of a triangle. **(b)** The initial mesh of the square domain from Subsection 6.3. **(c)** The initial mesh of the L-shaped domain from Subsection 6.4.

Fig. 1 A red-refinement of a triangle and initial meshes of the domains from Sects. 6.3 and 6.4

results for $j = 0, \ldots, 5$ correspond to the colours red, green, blue, cyan, magenta, yellow.

(b) The value $\varepsilon \approx \|h_{\mathcal{T}}\|^2_{L^\infty(\Omega)}$ is coupled on the mesh-size of the triangulations. The initial values are chosen as $\varepsilon = 10^{-j}$ for $j = 0, 1, 2, 3$. Given $\mathcal{T}_\ell$ with regularization parameter ε_ℓ, the regularization parameter for $\mathcal{T}_{\ell+1}$ reads $\varepsilon_{\ell+1} = \varepsilon_\ell/4$. Since the refinement is essentially a red-refinement, $4\,\|h_{\mathcal{T}_{\ell+1}}\|^2_{L^\infty(\Omega)} \approx \|h_{\mathcal{T}_\ell}\|^2_{L^\infty(\Omega)}$. The errors for every domain are plotted against the degrees of freedom; the values of $j = 0, \ldots, 3$ correspond to the colours red, green, blue, cyan.

6.2 Circular domain

The first experiment concerns the circular domain $B(1, 0)$ from Sect. 4.6 with $R = 1$. The first three triangulations are depicted in Fig. 2. Given a triangulation $\mathcal{T}_\ell$, the red-refinement $\widetilde{\mathcal{T}}_{\ell+1} := \text{red}(\mathcal{T}_\ell)$ of $\mathcal{T}_\ell$ is computed. Afterwards, the boundary nodes of $\widetilde{\mathcal{T}}_{\ell+1}$ are shifted to $\partial B(1, 0)$. This defines the triangulation $\mathcal{T}_{\ell+1}$.

The errors $\|\nabla_{\text{NC}}(u - u_h)\|$ on the sequence of triangulations are plotted in Fig. 3 against the numbers of degrees of freedom. In the case (b) the errors for different ε are all nearly the same. In the case (a) the differences of the errors for a large number of degrees of freedom are larger than the differences of $\sqrt{g\varepsilon|\Omega|/\mu}$ for different ε. It seems that the discretization error dominates the total error in case (b) and the behaviour of the errors in case (a) leads to the conjecture that the bound of the regularization error from Theorem 4.3 is suboptimal.

6.3 Square domain

The underlying domain in this experiment is the unit square $\Omega = (0, 1)^2$. The initial mesh $\mathcal{T}_1$ is plotted in Fig. 3. Given $\mathcal{T}_\ell$ the triangulation $\mathcal{T}_{\ell+1}$ is defined by $\mathcal{T}_{\ell+1} = \text{red}(\mathcal{T}_\ell)$. Since the exact solution is not known for this domain, the displayed values for the error on a triangulation $\mathcal{T}_\ell$ with regularization parameter ε_ℓ are computed by a conforming reference solution on $\mathcal{T}_{\ell+2} = \text{red}(\text{red}(\mathcal{T}_\ell))$ with finer regularization parameter $\varepsilon_{\text{ref}} = \varepsilon_\ell/64$.

The errors $\|\nabla_{\text{NC}}(u - u_h)\|$ for case (a) and case (b) are plotted in Fig. 4 against the numbers of degrees of freedom. The errors show the same behaviour as for the circular domain.

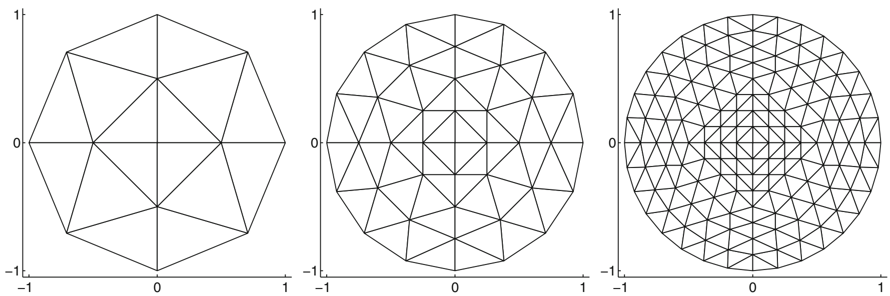
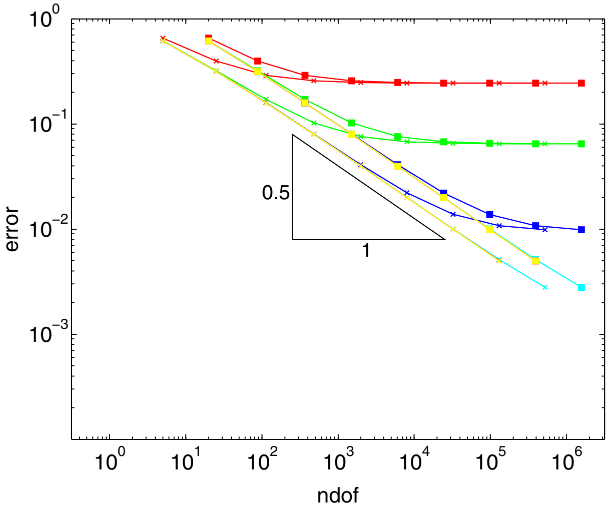
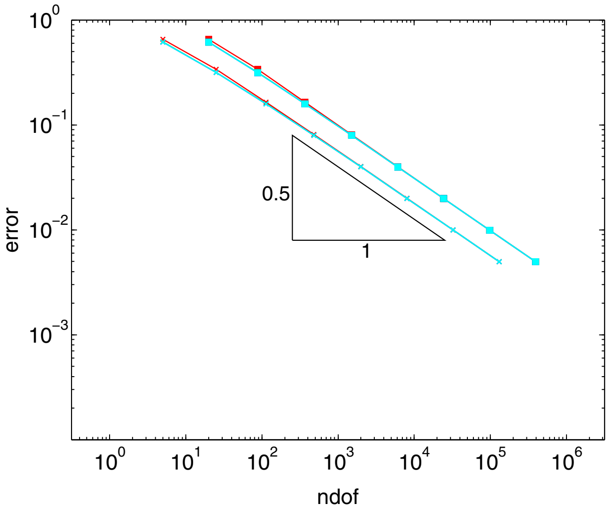


Fig. 2 The first three triangulations of the circular domain from Sect. 6.2



(a) The errors for case (a).

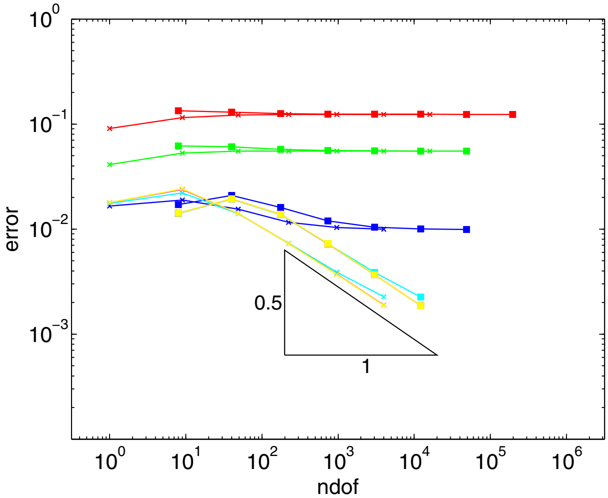


(b) The errors for case (b).

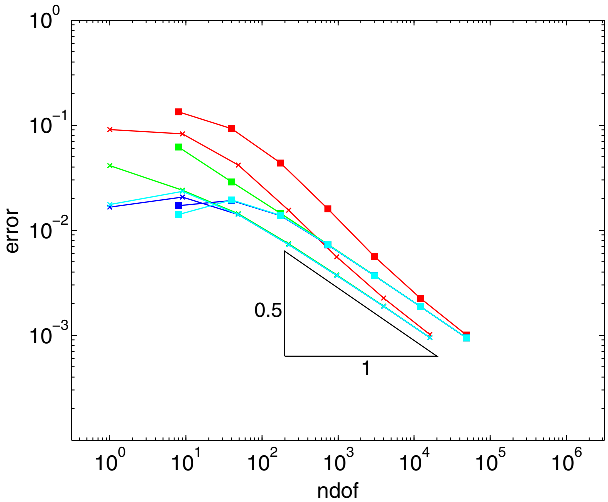
Fig. 3 The errors for the conforming (*times symbol*) and the nonconforming (*black square*) approximation on the circular domain from Sect. 6.2

6.4 L-shaped domain

The underlying domain of this experiment is the L-shaped domain $\Omega = (-1, 1)^2 \setminus ([0, 1] \times [-1, 0])$. The error $\|\nabla_{\mathrm{NC}}(u - u_h)\|$ is computed by a reference solution as in Sect. 6.3. The non-convex domain causes a reduced convergence rate of $1/3$ as can be seen in the convergence history plot of Fig. 5. As in the previous examples, it seems that the regularization error converges to zero faster than anticipated by Theorem 4.3.



(a) The errors for case (a).

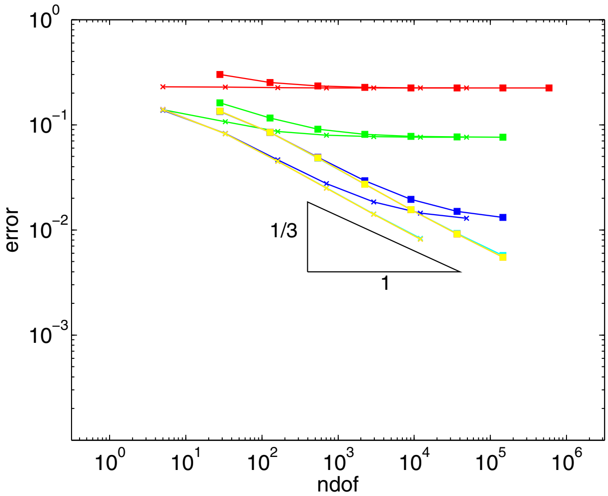


(b) The errors for case (b).

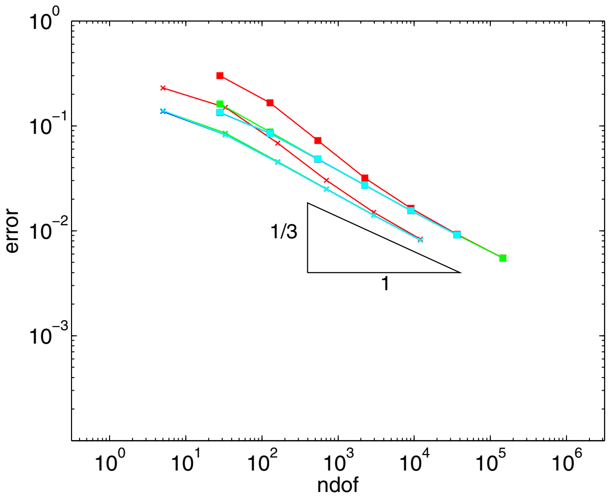
Fig. 4 The errors for the conforming (*times symbol*) and the nonconforming (*black square*) approximation on the unit square from Sect. 6.3

6.5 Conclusions

- The numerical experiments reveal optimal convergence rates for the nonconforming FEM for convex domains. As expected, the re-entrant corner of the L-shaped domain causes a stress singularity which leads to a reduced convergence rate.
- It seems that there is no inferiority of the conforming FEM in the numerical examples, although the mathematical analysis only predicts a suboptimal convergence rate. The convergence rates are comparable in all experiments.



(a) The errors for case (a).



(b) The errors for case (b).

Fig. 5 The errors for the conforming (*times symbol*) and the nonconforming (*black square*) approximation on the L-shaped domain from Sect. 6.4

– Throughout the computations, the regularization error appears to be smaller than predicted by Theorem 4.3.
– The numerical experiments suggest the choice of $\varepsilon \approx \|h_{\mathcal{T}}\|^2_{L^\infty(\Omega)}$. The relatively large initial value of $\varepsilon = 1$ seems to be comparable to small initial values.

Acknowledgments The third named author was supported by the Berlin Mathematical School.

References

1. Arnold, D.N., Falk, R.S.: A uniformly accurate finite element method for the Reissner–Mindlin plate. SIAM J. Numer. Anal. **26**(6), 1276–1290 (1989)
2. Brézis, H.: Monotonicity methods in Hilbert spaces and some applications to nonlinear partial differential equations. In: Zarantonello, E. (ed.) Contributions to Nonlinear Functional Analysis. Academic Press, New York (1971)
3. Carstensen, C., Gallistl, D., Schedensack, M.: Adaptive nonconforming Crouzeix–Raviart FEM for eigenvalue problems. Math. Comput. **84**(293), 1061–1087 (2015)
4. Carstensen, C., Gedicke, J.: Guaranteed lower bounds for eigenvalues. Math. Comput. **83**(290), 2605–2629 (2014)
5. Carstensen, C., Köhler, K., Peterseim, D., Schedensack, M.: Comparison results for the Stokes equations. Appl. Numer. Math. (2014). doi:10.1016/j.apnum.2013.12.005
6. Carstensen, C., Peterseim, D., Schedensack, M.: Comparison results of finite element methods for the Poisson model problem. SIAM J. Numer. Anal. **50**(6), 2803–2823 (2012)
7. Carstensen, C., Schedensack, M.: Medius analysis and comparison results for first-order finite element methods in linear elasticity. IMA J. Numer. Anal. (2014). doi:10.1093/imanum/dru048
8. Crouzeix, M., Raviart, P.A.: Conforming and nonconforming finite element methods for solving the stationary Stokes equations. I. Rev. Française Autom. Inf. Recherche Opér. Sér. Rouge **7**(R-3), 33–75 (1973)
9. Duvaut, G., Lions, J.L., Trémolières, R.: Inequalities in Mechanics and Physics. Springer, Berlin (1976)
10. Evans, L.C.: Graduate Studies in Mathematics. Partial differential equations, vol. 19, 2nd edn. American Mathematical Society, Providence (2010)
11. Falk, R., Mercier, B.: Error estimates for elasto-plastic problems. RAIRO Anal. Numér. **11**(R–2), 135–144 (1977)
12. Fuchs, M., Seregin, G.: Variational methods for problems from plasticity theory and for generalized Newtonian fluids. In: Lecture Notes in Mathematics, vol. 1749. Springer, Berlin (2000)
13. Glowinski, R.: Sur l'approximation d'une inéquation variationnelle elliptique de type Bingham. RAIRO Anal. Numér. **10**(R–3), 13–30 (1976)
14. Glowinski, R.: Scientific Computation. Numerical methods for nonlinear variational problems. Springer, Berlin (2008). (reprint of the 1984 original)
15. Glowinski, R., Lions, J.L., Trémolières, R.: Analyse numérique des inéquations variationnelles. Tome 1. Théorie générale premiéres applications. In: Méthodes Mathématiques de l'Informatique, vol. 5. Dunod, Paris (1976)
16. Mosolov, P.P., Miasnikov, V.P.: Variational methods in the theory of the fluidity of a viscous-plastic medium. J. Appl. Math. Mech. **29**(3), 545–577 (1965)
17. Mosolov, P.P., Miasnikov, V.P.: On stagnant flow regions of a viscous-plastic medium in pipes. J. Appl. Math. Mech. **30**(4), 841–854 (1966)
18. Mosolov, P.P., Miasnikov, V.P.: On qualitative singularities of the flow of a viscoplastic medium in pipes. J. Appl. Math. Mech. **31**(3), 609–613 (1967)
19. Verfürth, R.: A review of a posteriori error estimation and adaptive mesh-refinement techniques. In: Advances in Numerical Mathematics. Wiley, Chichester (1996)