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A priori and a posteriori error analysis of the lowest-order NCVEM for second-order linear indefinite elliptic problems

Carsten Carstensen[1] · Rekha Khot[2] · Amiya K. Pani[2]

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Abstract
The nonconforming virtual element method (NCVEM) for the approximation of the weak solution to a general linear second-order non-selfadjoint indefinite elliptic PDE in a polygonal domain Ω is analyzed under reduced elliptic regularity. The main tool in the a priori error analysis is the connection between the nonconforming virtual element space and the Sobolev space $H^1_0(\Omega)$ by a right-inverse J of the interpolation operator I_h. The stability of the discrete solution allows for the proof of existence of a unique discrete solution, of a discrete inf-sup estimate and, consequently, for optimal error estimates in the H^1 and L^2 norms. The explicit residual-based a posteriori error estimate for the NCVEM is reliable and efficient up to the oscillation terms. Numerical experiments on different types of polygonal meshes illustrate the robustness of an error estimator and support the improved convergence rate of an adaptive mesh-refinement in comparison to the uniform mesh-refinement.

Mathematics Subject Classification 65N12 · 65N15 · 65N30 · 65N50

1 Introduction

The nonconforming virtual element method approximates the weak solution $u \in H^1_0(\Omega)$ to the second-order linear elliptic boundary value problem

✉ Carsten Carstensen
cc@math.hu-berlin.de

Rekha Khot
rekhamp@math.iitb.ac.in

Amiya K. Pani
akp@math.iitb.ac.in

1 Department of Mathematics, Humboldt-Universität zu Berlin, 10099 Berlin, Germany

2 Department of Mathematics, Indian Institute of Technology Bombay, Powai, Mumbai 400076, India

$$\mathcal{L}u := -\mathrm{div}(\mathbf{A}\nabla u + \mathbf{b}u) + \gamma u = f \quad \text{in} \quad \Omega \tag{1.1}$$

for a given $f \in L^2(\Omega)$ in a bounded polygonal Lipschitz domain $\Omega \subset \mathbb{R}^2$ subject to homogeneous Dirichlet boundary conditions.

1.1 General introduction

The virtual element method (VEM) introduced in [4] is one of the well-received polygonal methods for approximating the solutions to partial differential equations (PDEs) in the continuation of the mimetic finite difference method [7]. This method is becoming increasingly popular [1, 3, 5, 6, 16, 17] for its ability to deal with fairly general polygonal/polyhedral meshes. On the account of its versatility in shape of polygonal domains, the local finite-dimensional space (the space of shape functions) comprises non-polynomial functions. The novelty of this approach lies in the fact that it does not demand for the explicit construction of non-polynomial functions and the knowledge of degrees of freedom along with suitable projections onto polynomials is sufficient to implement the method.

Recently, Beirão da Veiga et al. discuss a conforming VEM for the indefinite problem (1.1) in [6]. Cangiani et al. [17] develop a nonconforming VEM under the additional condition

$$0 \le \gamma - \frac{1}{2}\mathrm{div}(\mathbf{b}), \tag{1.2}$$

which makes the bilinear form coercive and significantly simplifies the analysis. The two papers [6, 17] prove a priori error estimates for a solution $u \in H^2(\Omega) \cap H_0^1(\Omega)$ in a convex domain Ω. The a priori error analysis for the nonconforming VEM in [17] can be extended to the case when the exact solution $u \in H^{1+\sigma}(\Omega) \cap H_0^1(\Omega)$ with $\sigma > 1/2$ as it is based on traces. This paper shows it for all $\sigma > 0$ and circumvents any trace inequality. Huang et al. [31] discuss a priori error analysis of the nonconforming VEM applied to Poisson and Biharmonic problems for $\sigma > 0$. An a posteriori error estimate in [16] explores the conforming VEM for (1.1) under the assumption (1.2). There are a few contributions [9, 16, 34] on residual-based a posteriori error control for the conforming VEM. This paper presents a priori and a posteriori error estimates for the nonconforming VEM without (1.2), but under the assumption that the Fredholm operator $\mathcal{L}$ is injective.

1.2 Assumptions on (1.1)

This paper solely imposes the following assumptions (A1)–(A3) on the coefficients $\mathbf{A}, \mathbf{b}, \gamma$ and the operator $\mathcal{L}$ in (1.1) with $f \in L^2(\Omega)$.

(A1) The coefficients $\mathbf{A}_{jk}, \mathbf{b}_j, \gamma$ for $j, k = 1, 2$ are piecewise Lipschitz continuous functions. For any decomposition $\mathcal{T}$ (admissible in the sense of Sect. 2.1) and any polygonal domain $P \in \mathcal{T}$, the coefficients $\mathbf{A}, \mathbf{b}, \gamma$ are bounded pointwise a.e. by $\|\mathbf{A}\|_\infty$, $\|\mathbf{b}\|_\infty$, $\|\gamma\|_\infty$ and their piecewise first derivatives by $|\mathbf{A}|_{1,\infty}$, $|\mathbf{b}|_{1,\infty}$, $|\gamma|_{1,\infty}$.

(A2) There exist positive constants a_0 and a_1 such that, for a.e. $x \in \Omega$, $\mathbf{A}(x)$ is SPD and

$$a_0|\xi|^2 \leq \sum_{j,k=1}^{2} \mathbf{A}_{jk}(x)\xi_j\xi_k \leq a_1|\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^2. \tag{1.3}$$

(A3) The linear operator $\mathcal{L} : H_0^1(\Omega) \to H^{-1}(\Omega)$ is injective, i.e., zero is not an eigenvalue of $\mathcal{L}$.

Since the bounded linear operator $\mathcal{L}$ is a Fredholm operator [30,p. 321], (A3) implies that $\mathcal{L}$ is bijective with bounded inverse $\mathcal{L}^{-1} : H^{-1}(\Omega) \to H_0^1(\Omega)$. The Fredholm theory also entails the existence of a unique solution to the adjoint problem, that is, for every $g \in L^2(\Omega)$, there exists a unique solution $\Phi \in H_0^1(\Omega)$ to

$$\mathcal{L}^*\Phi := -\operatorname{div}(\mathbf{A}\nabla\Phi) + \mathbf{b}\cdot\nabla\Phi + \gamma\Phi = g. \tag{1.4}$$

The bounded polygonal Lipschitz domain Ω, the homogeneous Dirichlet boundary conditions, and (A1)–(A2) lead to some $0 < \sigma \leq 1$ and positive constants C_{reg} and C^*_{reg} (depending only on σ, Ω and coefficients of $\mathcal{L}$) such that, for any $f, g \in L^2(\Omega)$, the unique solution u to (1.1) and the unique solution Φ to (1.4) belong to $H^{1+\sigma}(\Omega) \cap H_0^1(\Omega)$ and satisfy

$$\|u\|_{1+\sigma,\Omega} \leq C_{\text{reg}}\|f\|_{L^2(\Omega)} \quad \text{and} \quad \|\Phi\|_{1+\sigma,\Omega} \leq C^*_{\text{reg}}\|g\|_{L^2(\Omega)}. \tag{1.5}$$

(The restriction $\sigma \leq 1$ is for convenience owing to the limitation to first-order convergence of the scheme.)

1.3 Weak formulation

Given the coefficients $\mathbf{A}$, $\mathbf{b}$, γ with (A1)–(A2), define, for all $u, v \in V := H_0^1(\Omega)$,

$$a(u,v) := (\mathbf{A}\nabla u, \nabla v)_{L^2(\Omega)}, \quad b(u,v) := (u, \mathbf{b}\cdot\nabla v)_{L^2(\Omega)}, \quad c(u,v) := (\gamma u, v)_{L^2(\Omega)} \tag{1.6}$$

and

$$B(u,v) := a(u,v) + b(u,v) + c(u,v) \tag{1.7}$$

(with piecewise versions a_{pw}, b_{pw}, c_{pw} and B_{pw} for ∇ replaced by the piecewise gradient ∇_{pw} and local contributions a^P, b^P, c^P defined in Sect. 3.1 throughout this paper). The weak formulation of the problem (1.1) seeks $u \in V$ such that

$$B(u,v) = (f,v)_{L^2(\Omega)} \quad \text{for all } v \in V. \tag{1.8}$$

Assumptions (A1)–(A3) imply that the bilinear form $B(\cdot,\cdot)$ is continuous and satisfies an inf-sup condition [11]

$$0 < \beta_0 := \inf_{0\neq v\in V} \sup_{0\neq w\in V} \frac{B(v,w)}{\|v\|_{1,\Omega}\|w\|_{1,\Omega}}. \tag{1.9}$$

1.4 Main results and outline

Section 2 introduces the VEM and guides the reader to the first-order nonconforming VEM on polygonal meshes. It explains the continuity of the interpolation operator and related error estimates in detail. Section 3 starts with the discrete bilinear forms and their properties, followed by some preliminary estimates for the consistency error and the nonconformity error. The nonconformity error uses a new conforming companion operator resulting in the well-posedness of the discrete problem for sufficiently fine meshes. Section 4 proves the discrete inf-sup estimate and optimal a priori error estimates. Section 5 discusses both reliability and efficiency of an explicit residual-based a posteriori error estimator. Numerical experiments in Sect. 6 for three computational benchmarks illustrate the performance of an error estimator and show the improved convergence rate in adaptive mesh-refinement.

1.5 Notation

Throughout this paper, standard notation applies to Lebesgue and Sobolev spaces H^m with norm $\|\cdot\|_{m,\mathcal{D}}$ (resp. seminorm $|\cdot|_{m,\mathcal{D}}$) for $m>0$, while $(\cdot,\cdot)_{L^2(\mathcal{D})}$ and $\|\cdot\|_{L^2(\mathcal{D})}$ denote the L^2 scalar product and L^2 norm on a domain $\mathcal{D}$. The space $C^0(\mathcal{D})$ consists of all continuous functions vanishing on the boundary of a domain $\mathcal{D}$. The dual space of $H^1_0(\Omega)$ is denoted by $H^{-1}(\Omega)$ with dual norm $\|\cdot\|_{-1}$. An inequality $A \lesssim B$ abbreviates $A \le CB$ for a generic constant C, that may depend on the coefficients of $\mathcal{L}$, the universal constants σ, ρ (from (M2) below), but that is independent of the mesh-size. Let $\mathcal{P}_k(\mathcal{D})$ denote the set of polynomials of degree at most $k\in\mathbb{N}_0$ defined on a domain $\mathcal{D}$ and let Π_k denote the piecewise L^2 projection on $\mathcal{P}_k(\mathcal{T})$ for any admissible partition $\mathcal{T}\in\mathbb{T}$ (hidden in the notation Π_k). The notation $H^s(P) := H^s(\mathrm{int}P)$ for a compact polygonal domain P means the Sobolev space H^s [30] defined in the interior $\mathrm{int}(P)$ of P throughout this paper. The outward normal derivative is denoted by $\frac{\partial\bullet}{\partial\mathbf{n}_P} = \mathbf{n}_P\cdot\nabla\bullet$ for the exterior unit normal vector $\mathbf{n}_P$ along the boundary ∂P of the domain P.

2 First-order virtual element method on a polygonal mesh

This section describes class of admissible partitions of Ω into polygonal domains and the lowest-order nonconforming virtual element method for the problem (1.1) [3, 17].

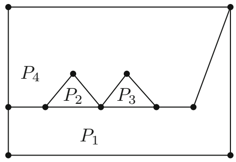

Fig. 1 Polygonal domains P_1 and P_2 share one edge, while P_1 and P_4 share three edges

2.1 Polygonal meshes

A polygonal domain P in this paper is a non-void compact simply-connected set P with polygonal boundary ∂P so that $\mathrm{int}(P)$ is a Lipschitz domain. The polygonal boundary ∂P is a simple closed polygon described by a finite sequence of distinct points. The set $\mathcal{N}(\partial P) = \{z_1, z_2, \ldots, z_J\}$ of nodes of a polygon P is enumerated with $z_{J+1} := z_1$ such that $E(j) := \mathrm{conv}\{z_j, z_{j+1}\}$ defines an edge and all J edges cover the boundary $\partial P = E(1) \cup \cdots \cup E(J)$ with an intersection $E(j) \cap E(j+1) = \{z_{j+1}\}$ for $j = 1, \ldots, J-1$ and $E(J) \cap E(1) = z_1$ with $\mathrm{dist}(E(j), E(k)) > 0$ for all distinct indices $j \neq k$.

Let $\mathbb{T}$ be a family of partitions of $\overline{\Omega}$ into polygonal domains, which satisfies the conditions (M1)–(M2) with a universal positive constant ρ.

(M1) Admissibility. Any two distinct polygonal domains P and P' in $\mathcal{T} \in \mathbb{T}$ are disjoint or share a finite number of edges or vertices (Fig. 1).

(M2) Mesh regularity. Every polygonal domain P of diameter h_P is star-shaped with respect to every point of a ball of radius greater than equal to ρh_P and every edge E of P has a length $|E|$ greater than equal to ρh_P.

Here and throughout this paper, $h_{\mathcal{T}}|_P := h_P$ denotes the piecewise constant mesh-size and $\mathbb{T}(\delta) := \{\mathcal{T} \in \mathbb{T} : h_{\max} \leq \delta \leq 1\}$ with the maximum diameter $h_{\max}$ of the polygonal domains in $\mathcal{T}$ denotes the subclass of partitions of $\overline{\Omega}$ into polygonal domains of maximal mesh-size $\leq \delta$. Let $|P|$ denote the area of polygonal domain P and $|E|$ denote the length of an edge E. With a fixed orientation to a polygonal domain P, assign the outer unit normal $\mathbf{n}_P$ along the boundary ∂P and $\mathbf{n}_E := \mathbf{n}_P|_E$ for an edge E of P. Let $\mathcal{E}$ (resp. $\widehat{\mathcal{E}}$) denote the set of edges E of $\mathcal{T}$ (resp. of $\widehat{\mathcal{T}}$) and $\mathcal{E}(P)$ denote the set of edges of polygonal domain $P \in \mathcal{T}$. For a polygonal domain P, define

$$\mathrm{mid}(P) := \frac{1}{|P|}\int_P x\,dx \quad \text{and} \quad \mathrm{mid}(\partial P) := \frac{1}{|\partial P|}\int_{\partial P} x\,ds.$$

Let $\mathcal{P}_k(\mathcal{T}) := \{v \in L^2(\Omega) : \forall P \in \mathcal{T} \ \ v|_P \in \mathcal{P}_k(P)\}$ for $k \in \mathbb{N}_0$ and Π_k denote the piecewise L^2 projection onto $\mathcal{P}_k(\mathcal{T})$. The notation Π_k hides its dependence on $\mathcal{T}$ and also assume Π_k applies componentwise to vectors. Given a decomposition $\mathcal{T} \in \mathbb{T}$ of Ω and a function $f \in L^2(\Omega)$, its oscillation reads

$$\mathrm{osc}_k(f, P) := \|h_P(1 - \Pi_k) f\|_{L^2(P)} \quad \text{and}$$
$$\mathrm{osc}_k(f, \mathcal{T}) := \left(\sum_{P \in \mathcal{T}} \|h_P(1 - \Pi_k) f\|^2_{L^2(P)}\right)^{1/2}$$

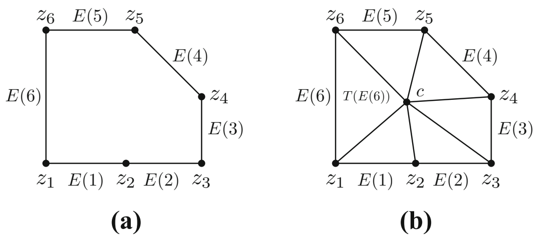

Fig. 2 **a** Polygon P and **b** its sub-triangulation $\widehat{\mathcal{T}}(P)$

with $\operatorname{osc}(f, \bullet) := \operatorname{osc}_0(f, \bullet)$.

Remark 1 (consequence of mesh regularity assumption) There exists an interior node c in the sub-triangulation $\widehat{\mathcal{T}}(P) := \{T(E) = \operatorname{conv}(c, E) : E \in \mathcal{E}(P)\}$ of a polygonal domain P with $h_{T(E)} \le h_P \le C_{\mathrm{sr}} h_{T(E)}$ as illustrated in Fig. 2. Each polygonal domain P can be divided into triangles so that the resulting sub-triangulation $\widehat{\mathcal{T}}|_P := \widehat{\mathcal{T}}(P)$ of $\mathcal{T}$ is shape-regular. The minimum angle in the sub-triangulation solely depends on ρ [13,Sec. 2.1].

Lemma 2.1 (Poincaré–Friedrichs inequality) *There exists a positive constant C_{PF}, that depends solely on ρ, such that*

$$\|f\|_{L^2(P)} \le C_{\mathrm{PF}} h_P |f|_{1,P} \tag{2.1}$$

holds for any $f \in H^1(P)$ with $\sum_{j\in J} \int_{E(j)} f\, ds = 0$ for a nonempty subset $J \subseteq \{1, \dots, m\}$ of indices in the notation $\partial P = E(1) \cup \dots \cup E(m)$ of Fig. 2. The constant C_{PF} depends exclusively on the number $m := |\mathcal{E}(P)|$ of the edges in the polygonal domain P and the quotient of the maximal area divided by the minimal area of a triangle in the triangulation $\widehat{\mathcal{T}}(P)$.

Some comments on C_{PF} for anisotropic meshes are in order before the proof gives an explicit expression for C_{PF}.

Example 2.1 Consider a rectangle P with a large aspect ratio divided into four congruent sub-triangles all with vertex $c = \operatorname{mid}(P)$. Then, $m = 4$ and the quotient of the maximal area divided by the minimal area of a triangle in the criss-cross triangulation $\widehat{\mathcal{T}}(P)$ is one. Hence $C_{\mathrm{PF}} \le 1.4231$ (from the proof below) is independent of the aspect ratio of P.

Proof of Lemma 2.1 The case $J = \{1, \dots, m\}$ with $f \in H^1(P)$ and $\int_{\partial P} f\, ds = 0$ is well-known cf. e.g. [13, Sec. 2.1.5], and follows from the Bramble-Hilbert lemma [14, Lemma 4.3.8] and the trace inequality [13, Sec. 2.1.1]. The remaining part of the proof shows the inequality (2.1) for the case $J \subseteq \{1, \dots, m\}$. The polygonal domain P and its triangulation $\widehat{\mathcal{T}}(P)$ from Fig. 2 has the center c and the nodes $z_1, \dots, z_m$ for the $m := |\mathcal{E}(P)| = |\widehat{\mathcal{T}}(P)|$ edges $E(1), \dots, E(m)$ and the triangles $T(1), \dots, T(m)$ with $T(j) = T(E(j)) = \operatorname{conv}\{c, E(j)\} = \operatorname{conv}\{c, z_j, z_{j+1}\}$ for $j = 1, \dots, m$. Here and throughout this proof, all indices are understood modulo m, e.g., $z_0 = z_m$. The proof uses the trace identity

$$\fint_{E(j)} f\,ds = \fint_{T(j)} f\,dx + \frac{1}{2}\fint_{T(j)} (x-c)\cdot\nabla f(x)\,dx \tag{2.2}$$

for $f \in H^1(P)$ as in the lemma. This follows from an integration by parts and the observation that $(x-c)\cdot\mathbf{n}_F = 0$ on $F \in \mathcal{E}(T(j))\setminus E(j)$ and the height $(x-c)\cdot\mathbf{n}_{E(j)} = \frac{2|T(j)|}{|E(j)|}$ of the edge $E(j)$ in the triangle $T(j)$, for $x \in E(j)$; cf. [24, Lemma 2.1] or [25, Lemma 2.6] for the remaining details. Another version of the trace identity (2.2) concerns $\operatorname{conv}\{z_j, c\} =: F(j) = \partial T(j-1)\cap\partial T(j)$ and reads

$$\begin{aligned}\fint_{F(j)} f\,ds &= \fint_{T(j-1)} f\,dx + \frac{1}{2}\fint_{T(j-1)} (x-z_{j-1})\cdot\nabla f(x)\,dx\\ &= \fint_{T(j)} f\,dx + \frac{1}{2}\fint_{T(j)} (x-z_{j+1})\cdot\nabla f(x)\,dx\end{aligned} \tag{2.3}$$

in $T(j-1)$ and $T(j)$. The three trace identities in (2.2)–(2.3) are rewritten with the following abbreviations, for $j = 1,\dots m$,

$$\begin{aligned}&x_j := \fint_{E(j)} f\,ds, \quad f_j := \fint_{T(j)} f\,dx, \quad a_j := \frac{1}{2}\fint_{T(j)} (x-c)\cdot\nabla f(x)\,dx,\\ &b_j := \frac{1}{2}\fint_{T(j)} (x-z_j)\cdot\nabla f(x)\,dx, \quad c_j := \frac{1}{2}\fint_{T(j)} (x-z_{j+1})\cdot\nabla f(x)\,dx.\end{aligned}$$

Let $t_{\min} = \min_{T\in\widehat{\mathcal{T}}(P)}|T|$ and $t_{\max} = \max_{T\in\widehat{\mathcal{T}}(P)}|T|$ abbreviate the minimal and maximal area of a triangle in $\widehat{\mathcal{T}}(P)$ and let $\widehat{\Pi}_0 f \in \mathcal{P}_0(\widehat{\mathcal{T}}(P))$ denote the piecewise integral means of f with respect to the triangulation $\widehat{\mathcal{T}}(P)$. The Poincaré inequality in a triangle with the constant $C_{\mathrm{P}} := 1/j_{1,1}$ and the first positive root $j_{1,1}\approx 3.8317$ of the Bessel function J_1 from [24, Thm. 2.1] allows for

$$\|f-\widehat{\Pi}_0 f\|_{L^2(T(j))} \le C_{\mathrm{P}} h_{T(j)} |f|_{1,T(j)} \quad \text{for } j = 1,\dots,m.$$

Hence $\|f-\widehat{\Pi}_0 f\|_{L^2(P)} \le C_{\mathrm{P}} h_P |f|_{1,P}$. This and the Pythagoras theorem (with $f-\widehat{\Pi}_0 f \perp \mathcal{P}_0(\widehat{\mathcal{T}}(P))$ in $L^2(P)$) show

$$\|f\|^2_{L^2(P)} = \|\widehat{\Pi}_0 f\|^2_{L^2(P))} + \|f-\widehat{\Pi}_0 f\|^2_{L^2(P))} \le \|\widehat{\Pi}_0 f\|^2_{L^2(P))} + C_{\mathrm{P}}^2 h_P^2 |f|^2_{1,P}. \tag{2.4}$$

It remains to bound the term $\|\widehat{\Pi}_0 f\|^2_{L^2(P))}$. The assumption on f reads $\sum_{j\in J}\int_{E(j)} f\,ds = \sum_{j\in J}|E(j)|x_j = 0$ for a subset $J\subset\{1,\dots,m\}$ so that $0\in\operatorname{conv}\{|E(1)|x_1,\dots,|E(m)|x_m\}$. It follows $0\in\operatorname{conv}\{x_1,\dots,x_m\}$ and it is known that this implies

$$\sum_{k=1}^{m} x_k^2 \leq \mathcal{M} \sum_{k=1}^{m} (x_k - x_{k-1})^2 \tag{2.5}$$

for a constant $\mathcal{M} = \frac{1}{2(1-\cos(\pi/m))}$ that depends exclusively on m [25, Lemma 4.2]. Recall (2.2) in the form $x_j = f_j + a_j$ to deduce from a triangle inequality and (2.5) that

$$\frac{1}{2}\sum_{j=1}^{m} f_j^2 \leq \sum_{k=1}^{m} x_k^2 + \sum_{\ell=1}^{m} a_\ell^2 \leq \mathcal{M} \sum_{k=1}^{m} (x_k - x_{k-1})^2 + \sum_{\ell=1}^{m} a_\ell^2.$$

This shows that

$$t_{\max}^{-1} \|\widehat{\Pi}_0 f\|_{L^2(P)}^2 = t_{\max}^{-1} \sum_{j=1}^{m} |T(j)| f_j^2 \leq \sum_{j=1}^{m} f_j^2 \leq 2\mathcal{M} \sum_{k=1}^{m} (x_k - x_{k-1})^2 + 2\sum_{\ell=1}^{m} a_\ell^2.$$

Recall (2.2)–(2.3) in the form $f_j - f_{j-1} = b_{j-1} - c_j$ and $x_j - x_{j-1} = f_j - f_{j-1} + a_j - a_{j-1} = b_{j-1} - a_{j-1} + a_j - c_j$ for all $j = 1, \ldots, m$. This and the Cauchy–Schwarz inequality imply the first two estimates in

$$\begin{aligned} 2|x_j - x_{j-1}| &= \left| \fint_{T(j-1)} (c - z_{j-1}) \cdot \nabla f(x)\, dx + \fint_{T(j)} (z_{j+1} - c) \cdot \nabla f(x)\, dx \right| \\ &\leq \max\{|c - z_{j-1}|, |c - z_{j+1}|\} \\ &\quad \left(|T(j-1)|^{-1/2} |f|_{1,T(j-1)} + |T(j)|^{-1/2} |f|_{1,T(j)} \right) \\ &\leq h_P t_{\min}^{-1/2} |f|_{1,T(j-1)\cup T(j)} \end{aligned}$$

with the definition of h_P and $t_{\min}$ in the end. The inequality $\int_{T(j)} |x - c|^2 \, dx \leq \frac{1}{2} h_{T(j)}^2 |T(j)|$ [25, Lemma 2.7] and the Cauchy–Schwarz inequality show, for $j = 1, \ldots, m$, that

$$|a_j| \leq 2^{-3/2} h_{T(j)} |T(j)|^{-1/2} |f|_{1,|T(j)|} \leq 2^{-3/2} h_P t_{\min}^{-1/2} |f|_{1,|T(j)|}.$$

The combination of the previous three displayed estimates result in

$$\begin{aligned} 4h_P^{-2}(t_{\min}/t_{\max}) \|\widehat{\Pi}_0 f\|_{L^2(P)}^2 &\leq 2\mathcal{M} \sum_{k=1}^{m} |f|_{T(k-1)\cup T(k)}^2 + \sum_{\ell=1}^{m} |f|_{1,T(\ell)}^2 \\ &= (4\mathcal{M} + 1)|f|_{1,P}^2. \end{aligned}$$

This and (2.4) conclude the proof with the constant $C_{\mathrm{PF}}^2 = (\mathcal{M} + 1/4)(t_{\max}/t_{\min}) + C_{\mathrm{P}}^2$. $\square$

In the nonconforming VEM, the finite-dimensional space V_h is a subset of the piecewise Sobolev space

$$H^1(\mathcal{T}) := \{v \in L^2(\Omega) : \forall P \in \mathcal{T} \;\; v|_P \in H^1(P)\} \equiv \prod_{P\in\mathcal{T}} H^1(P).$$

The piecewise H^1 seminorm (piecewise with respect to $\mathcal{T}$ hidden in the notation for brevity) reads

$$|v_h|_{1,\mathrm{pw}} := \Big(\sum_{P\in\mathcal{T}} |v_h|_{1,P}^2\Big)^{1/2} \quad \text{for any } v_h \in H^1(\mathcal{T}).$$

2.2 Local virtual element space

The first nonconforming virtual element space [3] is a subspace of harmonic functions with edgewise constant Neumann boundary values on each polygon. The extended nonconforming virtual element space [1, 17] reads

$$\widehat{V}_h(P) := \left\{ v_h \in H^1(P) : \;\; \Delta v_h \in \mathcal{P}_1(P) \;\; \text{and} \;\; \forall E \in \mathcal{E}(P) \;\; \frac{\partial v_h}{\partial \mathbf{n}_P}\Big|_E \in \mathcal{P}_0(E) \right\}. \tag{2.6}$$

Definition 2.2 (Ritz projection) Let Π_1^∇ be the Ritz projection from $H^1(P)$ onto the affine functions $\mathcal{P}_1(P)$ in the H^1 seminorm defined, for $v_h \in H^1(P)$, by

$$(\nabla \Pi_1^\nabla v_h - \nabla v_h, \nabla \chi)_{L^2(P)} = 0 \quad \text{for all } \chi \in \mathcal{P}_1(P)$$
$$\text{and} \quad \int_{\partial P} \Pi_1^\nabla v_h \, ds = \int_{\partial P} v_h \, ds. \tag{2.7}$$

Remark 2 (integral mean) For $P \in \mathcal{T}$ and $f \in H^1(P)$, $\nabla \Pi_1^\nabla f = \Pi_0 \nabla f$. (This follows from (2.7.a) and the definition of the L^2 projection operator Π_0 (acting componentwise) onto the piecewise constants $\mathcal{P}_0(P; \mathbb{R}^2)$.)

Remark 3 (representation of Π_1^∇) For $P \in \mathcal{T}$ and $f \in H^1(P)$, the Ritz projection $\Pi_1^\nabla f$ reads

$$(\Pi_1^\nabla f)(x) = \frac{1}{|P|}\Big(\int_{\partial P} f \mathbf{n}_P \, ds\Big) \cdot \Big(x - \mathrm{mid}(\partial P)\Big) + \fint_{\partial P} f \, ds \quad \text{for } x \in P. \tag{2.8}$$

(The proof of (2.8) consists in the verification of (2.7): The equation (2.7.a) follows from Remark 2 with an integration by parts. The equation (2.7.b) follows from the definition of $\mathrm{mid}(\partial P)$ as the barycenter of ∂P.)

The enhanced virtual element spaces [1, 17] are designed with a computable L^2 projection Π_1 onto $\mathcal{P}_1(\mathcal{T})$. The resulting local discrete space under consideration throughout this paper reads

$$V_h(P) := \left\{ v_h \in \widehat{V}_h(P) : v_h - \Pi_1^\nabla v_h \perp \mathcal{P}_1(P) \quad \text{in } L^2(P) \right\} \quad . \tag{2.9}$$

The point in the selection of $V_h(P)$ is that the Ritz projection $\Pi_1^\nabla v_h$ coincides with the L^2 projection $\Pi_1 v_h$ for all $v_h \in V_h(P)$. The degrees of freedom on P are given by

$$\mathrm{dof}_E(v) = \frac{1}{|E|} \int_E v\, ds \quad \text{for all } E \in \mathcal{E}(P) \text{ and } v \in V_h(P). \tag{2.10}$$

Proposition 2.3 *(a) The vector space $\widehat{V}_h(P)$ from (2.6) is of dimension $3 + |\mathcal{E}(P)|$. (b) $V_h(P)$ from (2.9) is of dimension $|\mathcal{E}(P)|$ and the triplet $(P, V_h(P), dof_E : E \in \mathcal{E}(P))$ is a finite element in the sense of Ciarlet* [28].

Proof Let $E(1), \ldots, E(m)$ be an enumeration of the edges $\mathcal{E}(P)$ of the polygonal domain P in a consecutive way as depicted in Fig. 2a and define $W(P) := \mathcal{P}_1(P) \times \mathcal{P}_0(E(1)) \times \cdots \times \mathcal{P}_0(E(m))$. Recall $\widehat{V}_h(P)$ from (2.6) and identify the quotient space $\widehat{V}_h(P)/\mathbb{R} \equiv \left\{ f \in \widehat{V}_h(P) : \int_{\partial P} f\, ds = 0 \right\}$ with all functions in $\widehat{V}_h(P)$ having zero integral over the boundary ∂P of P. Since the space $\widehat{V}_h(P)$ consists of functions with an affine Laplacian and edgewise constant Neumann data, the map

$$S : \widehat{V}_h(P)/\mathbb{R} \to W(P), \qquad f \mapsto \left(-\Delta f, \frac{\partial f}{\partial \mathbf{n}_P}\Big|_{E(1)}, \ldots, \frac{\partial f}{\partial \mathbf{n}_P}\Big|_{E(m)} \right)$$

is well-defined and linear. The compatibility conditions for the existence of a solution of a Laplacian problem with Neumann data show that the image of S is equal to

$$\mathcal{R}(S) = \left\{ (f_1, g_1, \ldots, g_m) \in W(P) : \int_P f_1 dx + \sum_{j=1}^m g_j |E(j)| = 0 \right\}.$$

(The proof of this identity assumes the compatible data $(f_1, g_1, \ldots, g_m)$ from the set on the right-hand side and solves the Neumann problem with a unique solution $\widehat{u}$ in $\widehat{V}_h(P)/\mathbb{R}$ and $S\widehat{u} = (f_1, g_1, \ldots, g_m)$.) It is known that the Neumann problem has a unique solution up to an additive constant and so S is a bijection and the dimension $m + 2$ of $\widehat{V}_h(P)/\mathbb{R}$ is that of $\mathcal{R}(S)$. In particular, dimension of $\widehat{V}_h(P)$ is $m + 3$. This proves (a).

Let $\Lambda_0, \Lambda_1, \Lambda_2 : H^1(P) \to \mathbb{R}$ be linear functionals

$$\Lambda_0 f := \Pi_0 f, \quad \Lambda_j f := \mathcal{M}_j((\Pi_1^\nabla - \Pi_1) f)$$

with $\mathcal{M}_j f := \Pi_0((x_j - c_j) f)$ for $j = 1, 2$ and $f \in H^1(P)$ that determines an affine function $p_1 \in \mathcal{P}_1(P)$ such that $(P, \mathcal{P}_1(P), (\Lambda_0, \Lambda_1, \Lambda_2))$ is a finite element in the sense of Ciarlet. For any edge $E(j) \in \mathcal{E}(P)$, define $\Lambda_{j+2} f = \fint_{E(j)} f\, ds$ as integral mean of the traces of f in $H^1(P)$ on $E(j)$. It is elementary to see that $\Lambda_0, \ldots, \Lambda_{m+2}$ are linearly independent: If f in $\widehat{V}_h(P)$ belongs to the kernel of all the linear functionals, then $\Pi_1^\nabla f = 0$ from (2.8) with $\Lambda_j f = 0$ for each $j = 3, \dots, 2+m$. Since the functionals $\Lambda_j f = 0$ for $j = 1, 2$, $(x_j - c_j)(\Pi_1^\nabla - \Pi_1) f = 0$ and $\Pi_1^\nabla f = 0$ imply $\Pi_1 f = 0$. An integration by parts leads to

$$\|\nabla f\|^2_{L^2(P)} = (-\Delta f, f)_{L^2(P)} + \left(f, \frac{\partial f}{\partial \mathbf{n}_P}\right)_{L^2(\partial P)} = 0.$$

This and $\fint_{\partial P} f\, ds = 0$ show $f \equiv 0$. Consequently, the intersection $\cap_{j=0}^{m+2} \mathrm{Ker}(\Lambda_j)$ of all kernels $\mathrm{Ker}(\Lambda_0), \dots, \mathrm{Ker}(\Lambda)_{m+2}$ is trivial and so that the functionals $\Lambda_0, \dots, \Lambda_{m+2}$ are linearly independent. Since the number of the linear functionals is equal to the dimension of $\widehat{V}_h(P)$, $(P, \widehat{V}_h(P), \{\Lambda_0, \dots, \Lambda_{m+2}\})$ is a finite element in the sense of Ciarlet and there exists a nodal basis $\psi_0, \dots, \psi_{m+2}$ of $\widehat{V}_h(P)$ with

$$\Lambda_j(\psi_k) = \delta_{jk} \quad \text{for all } j, k = 0, \dots, m+2.$$

The linearly independent functions $\psi_3, \dots, \psi_{m+2}$ belong to $V_h(P)$ and so $\dim(V_h(P)) \geq m$. Since $V_h(P) \subset \widehat{V}_h(P)$ and three linearly independent conditions $(1 - \Pi_1^\nabla) v_h \perp \mathcal{P}_1(P)$ in $L^2(P)$ are imposed on $\widehat{V}_h(P)$ to define $V_h(P)$, $\dim(V_h(P)) \leq m$. This shows that $\dim(V_h(P)) = m$ and hence, the linear functionals $\mathrm{dof}_E = \fint_E \bullet\, ds$ for $E \in \mathcal{E}(P)$ form a dual basis of $V_h(P)$. This concludes the proof of (b). $\square$

Remark 4 (stability of L^2 projection) The L^2 projection Π_k for $k = 0, 1$ is H^1 and L^2 stable in $V_h(P)$, in the sense that any v_h in $V_h(P)$ satisfies

$$\|\Pi_k v_h\|_{L^2(P)} \leq \|v_h\|_{L^2(P)} \text{ and } \|\nabla(\Pi_k v_h)\|_{L^2(P)} \leq \|\nabla v_h\|_{L^2(P)}. \tag{2.11}$$

(The first inequality follows from the definition of Π_k. The orthogonality in (2.9) and the definition of Π_1 imply that the Ritz projection Π_1^∇ and the L^2 projection Π_1 coincide on the space $V_h(P)$ for $P \in \mathcal{T}$. This with the definition of the Ritz projection Π_1^∇ verifies the second inequality.)

Definition 2.4 (Fractional order Sobolev space [14]) Let $\alpha := (\alpha_1, \alpha_2)$ denote a multi-index with $\alpha_j \in \mathbb{N}_0$ for $j = 1, 2$ and $|\alpha| := \alpha_1 + \alpha_2$. For a real number m with $0 < m < 1$, define

$$H^{1+m}(\omega) := \left\{ v \in H^1(\omega) : \frac{|v^\alpha(x) - v^\alpha(y)|}{|x - y|^{(1+m)}} \in L^2(\omega \times \omega) \quad \text{for all } |\alpha| = 1 \right\}$$

with v^α as the partial derivative of v of order α. Define the seminorm $|\cdot|_{1+m}$ and Sobolev-Slobodeckij norm $\|\cdot\|_{1+m}$ by

$$|v|^2_{1+m,\omega} = \sum_{|\alpha|=1} \int_\omega \int_\omega \frac{|v^\alpha(x) - v^\alpha(y)|^2}{|x - y|^{2(1+m)}}\, dx\, dy \quad \text{and} \quad \|v\|^2_{1+m,\omega} = \|v\|^2_{1,\omega} + |v|^2_{1+m,\omega}.$$

Proposition 2.5 (approximation by polynomials [29, Thm. 6.1]) *Under the assumption (M2), there exists a positive constant* C_{apx} *(depending on* ρ *and on the polynomial* *degree k) such that, for every* $v \in H^m(P)$, *the* L^2 *projection* Π_k *on* $\mathcal{P}_k(P)$ *for* $k \in \mathbb{N}_0$ *satisfies*

$$\|v - \Pi_k v\|_{L^2(P)} + h_P|v - \Pi_k v|_{1,P} \le C_{\mathrm{apx}} h_P^m |v|_{m,P} \quad \textit{for } 1 \le m \le k+1. \tag{2.12}$$

2.3 Global virtual element space

Define the global nonconforming virtual element space, for any $\mathcal{T} \in \mathbb{T}$, by

$$V_h := \left\{ v_h \in H^1(\mathcal{T}) : \forall P \in \mathcal{T} \;\; v_h|_P \in V_h(P) \quad \text{and} \quad \forall E \in \mathcal{E} \quad \int_E [v_h]_E \, ds = 0 \right\}. \tag{2.13}$$

Let $[\cdot]_E$ denote the jump across an edge $E \in \mathcal{E}$: For two neighboring polygonal domains P^+ and P^- sharing a common edge $E \in \mathcal{E}(P^+) \cap \mathcal{E}(P^-)$, $[v_h]_E := v_{h|P^+} - v_{h|P^-}$, where P^+ denote the adjoint polygonal domain with $\mathbf{n}_{P^+|E} = \mathbf{n}_E$ and P^- denote the polygonal domain with $\mathbf{n}_{P^-|E} = -\mathbf{n}_E$. If $E \subset \partial\Omega$ is a boundary edge, then $[v_h]_E := v_h|_E$.

Example 2.2 If each polygonal domain P is a triangle, then the finite-dimensional space V_h coincides with CR-FEM space. (Since the dimension of the vector space $V_h(P)$ is three and $\mathcal{P}_1(P) \subset V_h(P)$, $V_h(P) = \mathcal{P}_1(P)$ for $P \in \mathcal{T}$.)

Lemma 2.6 *There exists a universal constant* C_{F} *(that depends only on* ρ *from (M2)) such that, for all* $\mathcal{T} \in \mathbb{T}$, *any* $v_h \in V_h$ *from* (2.13) *satisfies*

$$\|v_h\|_{L^2(\Omega)} \le C_{\mathrm{F}} |v_h|_{1,pw}. \tag{2.14}$$

Proof Recall from Remark 1 that $\widehat{\mathcal{T}}$ is a shape regular sub-triangulation of $\mathcal{T}$ into triangles. Since $V_h \subset H^1(\widehat{\mathcal{T}})$ and the Friedrichs' inequality holds for all functions in $H^1(\widehat{\mathcal{T}})$ [14,Thm. 10.6.16], there exists a positive constant C_{F} such that the (first) inequality holds in

$$\|v_h\|_{L^2(\Omega)} \le C_{\mathrm{F}} \left(\sum_{T \in \widehat{\mathcal{T}}} \|\nabla v_h\|_{L^2(T)}^2 \right)^{1/2} = C_{\mathrm{F}} |v_h|_{1,\mathrm{pw}}.$$

The (second) equality follows for $v_h \in H^1(P)$ with $P \in \mathcal{T}$. □

Lemma 2.6 implies that the seminorm $|\cdot|_{1,\mathrm{pw}}$ is equivalent to the norm $\|\cdot\|_{1,\mathrm{pw}} := \|\cdot\|_{L^2(\Omega)}^2 + |\cdot|_{1,\mathrm{pw}}^2$ in V_h with mesh-size independent equivalence constants.

2.4 Interpolation

Definition 2.7 (interpolation operator) Let $(\psi_E : E \in \mathcal{E})$ be the nodal basis of V_h defined by $\mathrm{dof}_E(\psi_E) = 1$ and $\mathrm{dof}_F(\psi_E) = 0$ for all other edges $F \in \mathcal{E} \setminus \{E\}$. The global interpolation operator $I_h : H^1_0(\Omega) \to V_h$ reads

$$I_h v := \sum_{E\in\mathcal{E}} \Big(\fint_E v\, ds\Big)\psi_E \quad \text{for } v \in V.$$

Since a Sobolev function $v \in V$ has traces and the jumps $[v]_E$ vanish across any edge $E \in \mathcal{E}$, the interpolation operator I_h is well-defined. Recall ρ from (M2), C_{PF} from Lemma 2.1, and C_{apx} from Proposition 2.5.

Theorem 2.8 (interpolation error)

(a) *There exists a positive constant C_{Itn} (depending on ρ) such that any $v \in H^1(P)$ and its interpolation $I_h v \in V_h(P)$ satisfy*

$$\|\nabla I_h v\|_{L^2(P)} \le C_{\mathrm{Itn}} \|\nabla v\|_{L^2(P)}.$$

(b) *Any $P \in \mathcal{T} \in \mathbb{T}$ and $v \in H^1(P)$ satisfy $|v - I_h v|_{1,P} \le (1 + C_{\mathrm{Itn}})\|(1 - \Pi_0)\nabla v\|_{L^2(P)}$*
and

$$h_P^{-1}\|(1-\Pi_1 I_h)v\|_{L^2(P)} + |(1-\Pi_1 I_h)v|_{1,P} \le (1 + C_{\mathrm{PF}})\|(1-\Pi_0)\nabla v\|_{L^2(P)}.$$

(c) *The positive constant $C_{\mathrm{I}} := C_{\mathrm{apx}}(1 + C_{\mathrm{Itn}})(1 + C_{\mathrm{PF}})$, any $0 < \sigma \le 1$, and any $v \in H^{1+\sigma}(P)$ with the local interpolation $I_h v|_P \in V_h(P)$ satisfy*

$$\|v - I_h v\|_{L^2(P)} + h_P|v - I_h v|_{1,P} \le C_{\mathrm{I}} h_P^{1+\sigma}|v|_{1+\sigma,P}. \tag{2.15}$$

Proof of (a) The boundedness of the interpolation operator in $V_h(P)$ is mentioned in [17] with a soft proof in its appendix. The subsequent analysis aims at a clarification that C_{I} depends exclusively on the parameter ρ in (M2). The elementary arguments apply to more general situations in particular to 3D. Given $I_h v \in V_h(P)$, $q_1 := -\Delta I_h v \in \mathcal{P}_1(P)$ is affine and $\int_E (v - I_h v)\, ds = 0$. Since $\frac{\partial I_h v}{\partial \mathbf{n}_P}$ is edgewise constant, this shows $\int_E \frac{\partial I_h v}{\partial \mathbf{n}_P}|_E (v - I_h v)\, ds = 0$ for all $E \in \mathcal{E}(P)$ and so $\big\langle \frac{\partial I_h v}{\partial \mathbf{n}_P}, v - I_h v\big\rangle_{\partial P} = 0$. An integration by parts leads to

$$(\nabla I_h v, \nabla(I_h v - v))_{L^2(P)} = (q_1, I_h v - v)_{L^2(P)} = (q_1, \Pi_1^\nabla I_h v - v)_{L^2(P)}$$

with $q_1 \in \mathcal{P}_1(P)$ and $\Pi_1 v_h = \Pi_1^\nabla v_h$ for $v_h \in V_h(P)$ in the last step. Consequently,

$$\begin{aligned}\|\nabla I_h v\|^2_{L^2(P)} &= (\nabla I_h v, \nabla(I_h v - v))_{L^2(P)} + (\nabla I_h v, \nabla v)_{L^2(P)}\\ &= (q_1, \Pi_1^\nabla I_h v - v)_{L^2(P)} + (\nabla I_h v, \nabla v)_{L^2(P)}\\ &\le \|q_1\|_{L^2(P)}\|v - \Pi_1^\nabla I_h v\|_{L^2(P)} + \|\nabla I_h v\|_{L^2(P)}\|\nabla v\|_{L^2(P)}\end{aligned} \tag{2.16}$$

with the Cauchy inequality in the last step. Remarks 2 and 3 on the Ritz projection, and the definition of I_h show

$$\Pi_0 \nabla v = \nabla \Pi_1^\nabla v = |P|^{-1} \int_{\partial P} v\, \mathbf{n}_P\, ds$$
$$= |P|^{-1} \int_{\partial P} I_h v \mathbf{n}_P\, ds = \Pi_0 \nabla I_h v = \nabla \Pi_1^\nabla I_h v. \tag{2.17}$$

The function $f := v - \Pi_1^\nabla I_h v \in H^1(P)$ satisfies $\int_{\partial P} f\, ds = \int_{\partial P} (v - I_h v)\, ds = 0$ and the Poincaré–Friedrichs inequality from Lemma 2.1.a shows

$$\|v - \Pi_1^\nabla I_h v\|_{L^2(P)} \le C_{\mathrm{PF}} h_P \|\nabla(v - \Pi_1^\nabla I_h v)\|_{L^2(P)} = C_{\mathrm{PF}} h_P \|(1 - \Pi_0)\nabla v\|_{L^2(P)} \tag{2.18}$$

with (2.17) in the last step. Let $\phi_c \in S_0^1(\widehat{\mathcal{T}}(P)) := \{w \in C^0(P) : w|_{T(E)} \in \mathcal{P}_1(T(E))\ \text{ for all } E \in \mathcal{E}(P)\}$ denote the piecewise linear nodal basis function of the interior node c with respect to the triangulation $\widehat{\mathcal{T}}(P) = \{T(E) : E \in \mathcal{E}(P)\}$ (cf. Fig. 2b for an illustration of $\widehat{\mathcal{T}}(P)$). An inverse estimate

$$\|f_1\|_{L^2(T(E))} \le C_1 \|\phi_c^{1/2} f_1\|_{L^2(T(E))} \quad \text{for all } f_1 \in \mathcal{P}_1(\widehat{\mathcal{T}}(P))$$

on the triangle $T(E) := \mathrm{conv}(E \cup \{c\})$ holds with the universal constant C_1. A constructive proof computes the mass matrices for T with and without the weight ϕ_c to verify that the universal constant C_1 does not depend on the shape of the triangle $T(E)$. This implies

$$C_1^{-1} \|q_1\|_{L^2(P)}^2 \le (\phi_c q_1, q_1)_{L^2(P)} = (-\Delta I_h v, \phi_c q_1) = (\nabla I_h v, \nabla(\phi_c q_1))_{L^2(P)} \tag{2.19}$$

with an integration by parts for $\phi_c q_1 \in H_0^1(P)$ and $I_h v$ in the last step. The mesh-size independent constant C_2 in the standard inverse estimate

$$h_{T(E)} \|\nabla q_2\|_{L^2(T(E))} \le C_2 \|q_2\|_{L^2(T(E))} \quad \text{for all } q_2 \in \mathcal{P}_2(T(E))$$

depends merely on the angles in the triangle $T(E)$, $E \in \mathcal{E}(P)$, and so exclusively on ρ. With $C_{\mathrm{sr}}^{-1} h_P \le h_{T(E)}$ from Remark 1, this shows

$$C_2^{-1} C_{\mathrm{sr}}^{-1} h_P \|\nabla \phi_c q_1\|_{L^2(P)} \le \|\phi_c q_1\|_{L^2(P)} \le \|q_1\|_{L^2(P)}.$$

This and (2.19) lead to

$$\|q_1\|_{L^2(P)} \le C_1 C_2 C_{\mathrm{sr}} h_P^{-1} \|\nabla I_h v\|_{L^2(P)}. \tag{2.20}$$

The combination with (2.16)–(2.18) proves

$$\|\nabla I_h v\|_{L^2(P)}^2 \le (C_1 C_2 C_{\mathrm{sr}} C_{\mathrm{PF}} \|(1 - \Pi_0)\nabla v\|_{L^2(P)} + \|\nabla v\|_{L^2(P)}) \|\nabla I_h v\|_{L^2(P)}$$

$$\leq (1 + C_1 C_2 C_{\mathrm{sr}} C_{\mathrm{PF}}) \|\nabla v\|_{L^2(P)} \|\nabla I_h v\|_{L^2(P)}.$$

□

Proof of (b) The identity (2.17) reads $\Pi_0 \nabla (1 - I_h) v = 0$ and the triangle inequality results in

$$\begin{aligned} |v - I_h v|_{1,P} &= \|(1 - \Pi_0) \nabla (1 - I_h) v\|_{L^2(p)} \\ &\leq \|(1 - \Pi_0) \nabla v\|_{L^2(P)} + \|(1 - \Pi_0) \nabla I_h v\|_{L^2(P)}. \end{aligned} \tag{2.21}$$

Since I_h is the identity in $\mathcal{P}_1(P)$, it follows $(1 - \Pi_0)\nabla I_h v = (1 - \Pi_0)\nabla I_h (v - \Pi_1^{\nabla} v)$. This and the boundedness of the interpolation operator I_h lead to

$$\begin{aligned} \|(1 - \Pi_0) \nabla I_h v\|_{L^2(P)} &\leq \|\nabla I_h (1 - \Pi_1^{\nabla}) v\|_{L^2(P)} \\ &\leq C_{\mathrm{Itn}} \|\nabla (1 - \Pi_1^{\nabla}) v\|_{L^2(P)} = C_{\mathrm{Itn}} \|(1 - \Pi_0) \nabla v\|_{L^2(P)} \end{aligned} \tag{2.22}$$

with Remark 2 in the last step. The combination of (2.21) and (2.22) proves the first part of (b).

The identity $|(1 - \Pi_1 I_h) v|_{1,P} = \|(1 - \Pi_0) \nabla v\|_{L^2(P)}$ follows from (2.17). Since $\Pi_1 = \Pi_1^{\nabla}$ in V_h and $\fint_{\partial P} v\, ds = \fint_{\partial P} I_h v\, ds = \fint_{\partial P} \Pi_1^{\nabla} I_h v\, ds$, the Poincaré–Friedrichs inequality

$$\|(1 - \Pi_1 I_h) v\|_{L^2(P)} \leq C_{\mathrm{PF}} h_P |(1 - \Pi_1 I_h) v|_{1,P}$$

follows from Lemma 2.1.a. This concludes the proof of (b). □

Proof of (c) This is an immediate consequence of the part (b) with (2.12) and the Poincaré–Friedrichs inequality for $v - I_h v$ (from above) in Lemma 2.1.a. □

3 Preliminary estimates

This subsection formulates the discrete problem along with the properties of the discrete bilinear form such as boundedness and a Gårding-type inequality.

3.1 The discrete problem

Denote the restriction of the bilinear forms $a(\cdot, \cdot)$, $b(\cdot, \cdot)$ and $c(\cdot, \cdot)$ on a polygonal domain $P \in \mathcal{T}$ by $a^P(\cdot, \cdot)$, $b^P(\cdot, \cdot)$ and $c^P(\cdot, \cdot)$. The corresponding local discrete bilinear forms are defined for $u_h, v_h \in V_h(P)$ by

$$a_h^P(u_h, v_h) := (\mathbf{A} \nabla \Pi_1 u_h, \nabla \Pi_1 v_h)_{L^2(P)} + S^P((1 - \Pi_1) u_h, (1 - \Pi_1) v_h), \tag{3.1}$$

$$b_h^P(u_h, v_h) := (\Pi_1 u_h, \mathbf{b} \cdot \nabla \Pi_1 v_h)_{L^2(P)}, \tag{3.2}$$

$$c_h^P(u_h, v_h) := (\gamma \Pi_1 u_h, \Pi_1 v_h)_{L^2(P)}, \tag{3.3}$$

$$B_h^P(u_h, v_h) := a_h^P(u_h, v_h) + b_h^P(u_h, v_h) + c_h^P(u_h, v_h). \tag{3.4}$$

Choose the stability term $S^P(u_h, v_h)$ as a symmetric positive definite bilinear form on $V_h(P) \times V_h(P)$ for a positive constant C_s independent of P and h_P satisfying

$$C_s^{-1} a^P(v_h, v_h) \leq S^P(v_h, v_h) \leq C_s a^P(v_h, v_h) \quad \text{for all } v_h \in V_h(P) \text{ with } \Pi_1 v_h = 0. \tag{3.5}$$

For some positive constant approximation $\overline{\mathbf{A}}_P$ of $\mathbf{A}$ over P and the number $N_P := |\mathcal{E}(P)|$ of the degrees of freedom (2.10) of $V_h(P)$, a standard example of a stabilization term from [4],[36,Sec. 4.3] with a scaling coefficient $\overline{\mathbf{A}}_P$ reads

$$S^P(v_h, w_h) := \overline{\mathbf{A}}_P \sum_{r=1}^{N_P} \mathrm{dof}_r(v_h)\mathrm{dof}_r(w_h) \quad \text{for all } v_h, w_h \in V_h. \tag{3.6}$$

Note that an approximation $\overline{\mathbf{A}}_P$ is a positive real number (not a matrix) and can be chosen as $\sqrt{a_0 a_1}$ with the positive constants a_0 and a_1 from (**A2**). For $f \in L^2(\Omega)$ and $v_h \in V_h$, define the right-hand side functional f_h on V_h by

$$(f_h, v_h)_{L^2(P)} := (f, \Pi_1 v_h)_{L^2(P)}. \tag{3.7}$$

The sum over all the polygonal domains $P \in \mathcal{T}$ reads

$$\begin{aligned}
a_h(u_h, v_h) &:= \sum_{P \in \mathcal{T}} a_h^P(u_h, v_h), \; b_h(u_h, v_h) := \sum_{P \in \mathcal{T}} b_h^P(u_h, v_h),\\
c_h(u_h, v_h) &:= \sum_{P \in \mathcal{T}} c_h^P(u_h, v_h), \; s_h(u_h, v_h) := \sum_{P \in \mathcal{T}} S^P((1-\Pi_1)u_h, (1-\Pi_1)v_h),\\
B_h(u_h, v_h) &:= \sum_{P \in \mathcal{T}} B_h^P(u_h, v_h), \; (f_h, v_h)_{L^2(\Omega)} := \sum_{P \in \mathcal{T}} (f_h, v_h)_{L^2(P)} \quad \text{for all } u_h, v_h \in V_h.
\end{aligned}$$

The discrete problem seeks $u_h \in V_h$ such that

$$B_h(u_h, v_h) = (f_h, v_h)_{L^2(\Omega)} \quad \text{for all } v_h \in V_h. \tag{3.8}$$

Remark 5 (polygonal mesh with small edges) The conditions (M1)–(M2) are well established and apply throughout the paper. The sub-triangulation $\widehat{\mathcal{T}}$ may not be shape-regular without the edge condition $|E| \geq \rho h_P$ for an edge $E \in \mathcal{T}(P)$ and $P \in \mathcal{T}$, but satisfies the maximal angle condition and the arguments employed in the proof of [8,Lemma 6.3] can be applied to show (2.20) in Theorem 2.8.a. For more general star-shaped polygon domains with short edges, the recent anisotropic analysis [8, 15, 18] indicates that the stabilization term has to be modified as well to avoid a logarithmic factor in the optimal error estimates.

3.2 Properties of the discrete bilinear form

The following proposition provides two main properties of the discrete bilinear form B_h.

Proposition 3.1 *There exist positive universal constants M, α and a universal non-negative constant β depending on the coefficients $\boldsymbol{A}, \boldsymbol{b}, \gamma$ such that*

(a) Boundedness: $|B_h(u_h, v_h)| \leq M|u_h|_{1,pw}|v_h|_{1,pw}$ for all $u_h, v_h \in V_h$.
(b) Gårding-type inequality: $\alpha|v_h|^2_{1,pw} - \beta\|v_h\|^2_{L^2(\Omega)} \leq B_h(v_h, v_h)$ for all $v_h \in V_h$.

Proof of (a) The upper bound of the coefficients from the assumption (A1), the Cauchy–Schwarz inequality, the stability (2.11) of Π_1, and the definition (3.5) of the stabilization term imply the boundedness of B_h with $M := (1 + C_s)\|\mathbf{A}\|_\infty + C_{\mathrm{F}}\|\mathbf{b}\|_\infty + C_{\mathrm{F}}^2\|\gamma\|_\infty$. The details of the proof follow as in [6, Lemma 5.2] with the constant C_{F} from Lemma 2.6. □

Proof of (b) The first step shows that $a_h(\cdot, \cdot)$ is coercive. For $v_h \in V_h(P)$, $\Pi_1 v_h = \Pi_1^\nabla v_h$ and $\nabla\Pi_1 v_h \perp \nabla(v_h - \Pi_1^\nabla v_h)$ in $L^2(P; \mathbb{R}^2)$. This orthogonality, the assumption (A2), and the definition of the stability term (3.5) with the constant $C_s^{-1} \leq 1$ imply for $\alpha_0 = a_0 C_s^{-1}$ that

$$\begin{aligned}
\alpha_0|v_h|^2_{1,\mathrm{pw}} &\leq a_0\|\nabla_{\mathrm{pw}}\Pi_1 v_h\|^2_{L^2(\Omega)} + a_0 C_s^{-1}\|\nabla_{\mathrm{pw}}(1 - \Pi_1)v_h\|^2_{L^2(\Omega)} \\
&\leq \Big(\mathbf{A}\nabla_{\mathrm{pw}}\Pi_1 v_h, \nabla_{\mathrm{pw}}\Pi_1 v_h)_{L^2(\Omega)} + C_s^{-1}(\mathbf{A}\nabla_{\mathrm{pw}}(1 - \Pi_1)v_h, \nabla_{\mathrm{pw}}(1-\Pi_1)v_h\Big)_{L^2(\Omega)} \\
&\leq (\mathbf{A}\nabla_{\mathrm{pw}}\Pi_1 v_h, \nabla_{\mathrm{pw}}\Pi_1 v_h)_{L^2(\Omega)} + s_h((1 - \Pi_1)v_h, (1 - \Pi_1)v_h) = a_h(v_h, v_h).
\end{aligned} \tag{3.9}$$

The Cauchy–Schwarz inequality, (2.11), and the Young inequality lead to

$$\begin{aligned}
&|b_h(v_h, v_h) + c_h(v_h, v_h)| \\
&\quad\leq \|\mathbf{b}\|_\infty\|\Pi_1 v_h\|_{L^2(\Omega)}\|\nabla_{\mathrm{pw}}\Pi_1 v_h\|_{L^2(\Omega)} + \|\gamma\|_\infty\|\Pi_1 v_h\|^2_{L^2(\Omega)} \\
&\quad leq\|\mathbf{b}\|_\infty\|v_h\|_{L^2(\Omega)}|v_h|_{1,\mathrm{pw}} + \|\gamma\|_\infty\|v_h\|^2_{L^2(\Omega)} \\
&\quad\leq \frac{\|\mathbf{b}\|^2_\infty}{2\alpha_0}\|v_h\|^2_{L^2(\Omega)} + \frac{\alpha_0}{2}|v_h|^2_{1,\mathrm{pw}} + \|\gamma\|_\infty\|v_h\|^2_{L^2(\Omega)}.
\end{aligned} \tag{3.10}$$

The combination of (3.9)–(3.10) proves

$$\frac{\alpha_0}{2}|v_h|^2_{1,\mathrm{pw}} - \left(\frac{\|\mathbf{b}\|^2_\infty}{2\alpha_0} + \|\gamma\|_\infty\right)\|v_h\|^2_{L^2(\Omega)} \leq B_h(v_h, v_h).$$

This concludes the proof of (b) with $\alpha = \frac{\alpha_0}{2}$ and $\beta = \frac{\|\mathbf{b}\|^2_\infty}{2\alpha_0} + \|\gamma\|_\infty$. □

Remark 6 ($\|\cdot\|_h \approx |\cdot|_{1,\mathrm{pw}}$) The discrete space V_h of the nonconforming VEM is endowed with the natural norm $\|\cdot\|_h := a_h(\cdot, \cdot)^{1/2}$ induced by the scalar product a_h.

The boundedness of a_h is proven in (a), while (3.9) shows the converse estimate in the equivalence $\|\cdot\|_h \approx |\cdot|_{1,\mathrm{pw}}$ in V_h, namely

$$\alpha_0|v_h|^2_{1,\mathrm{pw}} \le a_h(v_h, v_h) \le \|\mathbf{A}\|_\infty(1 + C_s)|v_h|^2_{1,\mathrm{pw}} \quad \text{for all } v_h \in V_h.$$

3.3 Consistency error

This subsection discusses the consistency error between the continuous bilinear form B and the corresponding discrete bilinear form B_h. Recall the definition $B^P(\cdot,\cdot) \equiv a^P(\cdot,\cdot)+b^P(\cdot,\cdot)+c^P(\cdot,\cdot)$ and $B^P_h(\cdot,\cdot) \equiv a^P_h(\cdot,\cdot)+b^P_h(\cdot,\cdot)+c^P_h(\cdot,\cdot)$ for a polygonal domain $P \in \mathcal{T}$ from Sect. 2.1.

Lemma 3.2 (consistency)

(a) There exists a positive constant C_{cst} (depending only on ρ) such that any $v \in H^1(\Omega)$ and $w_h \in V_h$ satisfy

$$B^P(\Pi_1 v, w_h) - B^P_h(\Pi_1 v, w_h) \le C_{\mathrm{cst}}\, h_P\|v\|_{1,P}|w_h|_{1,P} \quad \textit{for all } P \in \mathcal{T}. \tag{3.11}$$

(b) Any $f \in L^2(\Omega)$ and $f_h := \Pi_1 f$ satisfy

$$\|f - f_h\|_{V_h^*} := \sup_{0\ne v_h\in V_h} \frac{(f - f_h, v_h)_{L^2(\Omega)}}{\|v_h\|_{1,pw}} \le C_{\mathrm{PF}}\,\mathrm{osc}_1(f, \mathcal{T}). \tag{3.12}$$

Proof Observe that $S^P((1-\Pi_1)\Pi_1 v, (1-\Pi_1)w_h) = 0$ follows from $(1-\Pi_1)\Pi_1 v = 0$. The definition of B^P and B^P_h show

$$B^P(\Pi_1 v, w_h) - B^P_h(\Pi_1 v, w_h) =: T_1 + T_2 + T_3. \tag{3.13}$$

The term T_1 in (3.13) is defined as the difference of the contributions from a^P and a^P_h. Their definitions prove the equality (at the end of the first line below) and the definition of Π_1 prove the next equality in

$$\begin{aligned} T_1 &:= a^P(\Pi_1 v, w_h) - a^p_h(\Pi_1 v, w_h) = (\mathbf{A}\nabla\Pi_1 v, \nabla(1 - \Pi_1)w_h)_{L^2(P)} \\ &= ((\mathbf{A} - \Pi_0\mathbf{A})(\nabla\Pi_1 v), \nabla(1 - \Pi_1)w_h)_{L^2(P)} \le h_P|\mathbf{A}|_{1,\infty}|v|_{1,P}|w_h|_{1,P}. \end{aligned}$$

The last inequality follows from the Cauchy–Schwarz inequality, the Lipschitz continuity of $\mathbf{A}$, and the stabilities $\|\nabla\Pi_1 v_h\|_{L^2(P)} \le \|\nabla v_h\|_{L^2(P)}$ and $\|\nabla(1 - \Pi_1)w_h\|_{L^2(P)} \le \|\nabla w_h\|_{L^2(P)}$ from Remark 4. Similar arguments apply to T_2 from the differences of b^P and b^P_h, and T_3 from those of c^P and c^P_h in (3.13). This leads to

$$\begin{aligned} T_2 &:= b^P(\Pi_1 v, w_h) - b^P_h(\Pi_1 v, w_h) \\ &= ((\mathbf{b} - \Pi_0\mathbf{b})\Pi_1 v, \nabla(1 - \Pi_1)w_h)_{L^2(P)} \\ &\quad + ((\Pi_0\mathbf{b})(1 - \Pi_0)(\Pi_1 v), \nabla(1 - \Pi_1)w_h)_{L^2(P)} \end{aligned}$$

$$\leq (|\mathbf{b}|_{1,\infty} + C_{\mathrm{apx}}\|\mathbf{b}\|_\infty) h_P \|v\|_{1,P} |w_h|_{1,P},$$

$$T_3 := c^P(\Pi_1 v, w_h) - c_h^P(\Pi_1 v, w_h) = (\gamma \Pi_1 v, (1-\Pi_1) w_h)_{L^2(P)}$$

$$\leq C_{\mathrm{PF}} \|\gamma\|_\infty h_P \|v\|_{L^2(P)} |w_h|_{1,P}.$$

The inequality for the last step in T_2 follows from the Cauchy–Schwarz inequality, the Lipschitz continuity of $\mathbf{b}$, the estimate $\|(1-\Pi_0)\Pi_1 v\|_{L^2(P)} \leq \|(1-\Pi_0)v\|_{L^2(P)} \leq C_{\mathrm{apx}} h_P |v|_{1,P}$ from (2.12), and the above stabilities $\|\nabla \Pi_1 v_h\|_{L^2(P)} \leq \|\nabla v_h\|_{L^2(P)}$ and $\|\nabla(1-\Pi_1) w_h\|_{L^2(P)} \leq \|\nabla w_h\|_{L^2(P)}$. The inequality for the last step in T_3 follows from the Cauchy–Schwarz inequality, $\|\Pi_1 v\|_{L^2(P)} \leq \|v\|_{L^2(P)}$ from (2.11) and the Poincaré–Friedrichs inequality in Lemma 2.1.a for $w_h - \Pi_1 w_h$ with $\int_{\partial P}(w_h - \Pi_1 w_h)\, ds = 0$ from $\Pi_1 = \Pi_1^\nabla$ in V_h. The combination of the above estimates shows (3.11). The proof of (3.12) adapts the arguments in the above analysis of T_3 and the definition of $\mathrm{osc}_1(f, \mathcal{T})$ in Sect. 2.1 for the proof of

$$(f - f_h, w_h)_{L^2(P)} = (f - \Pi_1 f, w_h - \Pi_1 w_h)_{L^2(P)} \leq C_{\mathrm{PF}} |w_h|_{1,P}\, \mathrm{osc}_1(f, P).$$

This concludes the proof. □

3.4 Nonconformity error

Enrichment operators play a vital role in the analysis of nonconforming finite element methods [12]. For any $v_h \in V_h$, the objective is to find a corresponding function $J v_h \in H_0^1(\Omega)$. The idea is to map the VEM nonconforming space into the Crouzeix-Raviart finite element space

$$\mathrm{CR}_0^1(\widehat{\mathcal{T}}) := \{v \in \mathcal{P}_1(\widehat{\mathcal{T}}) : \forall\, E \in \widehat{\mathcal{E}} \quad v \text{ is continuous at } \mathrm{mid}(E) \quad \text{and} \\ \forall\, E \in \mathcal{E}(\partial\Omega) \quad v(\mathrm{mid}(E)) = 0\}$$

with respect to the shape-regular triangulation $\widehat{\mathcal{T}}$ from Remark 1. Let ψ_E be the edge-oriented basis functions of $\mathrm{CR}_0^1(\widehat{\mathcal{T}})$ with $\psi_E(\mathrm{mid} E) = 1$ and $\psi_E(\mathrm{mid} F) = 0$ for all other edges $F \in \widehat{\mathcal{E}} \setminus \{E\}$. Define the interpolation operator $I_{\mathrm{CR}} : V_h \to \mathrm{CR}_0^1(\widehat{\mathcal{T}})$, for $v_h \in V_h$, by

$$I_{\mathrm{CR}} v_h = \sum_{F \in \widehat{\mathcal{E}}} \left(⨍_F v_h\, ds \right) \psi_F. \tag{3.14}$$

The definition of V_h implies $\int_F [v_h]\, ds = 0$ for $v_h \in V_h$ and for all $F \in \mathcal{E}$. Since $v_h|_P \in H^1(P)$, it follows $\int_F [v_h]\, ds = 0$ for all $F \in \widehat{\mathcal{E}} \setminus \mathcal{E}$. This shows $\int_F v_{h|T^\pm}\, ds$ is unique for all edges $F = \partial T^+ \cap \partial T^- \in \widehat{\mathcal{E}}$ and, consequently, $I_{\mathrm{CR}} v_h$ is well-defined (independent of the choice of traces selected in the evaluation of $⨍_F v_h\, ds = ⨍_F v_h|_{T^+}\, ds = ⨍_F v_h|_{T^-}\, ds$). The approximation property of I_{CR} on each $T \in \widehat{\mathcal{T}}$ reads

$$h_T^{-1} \|v_h - I_{\mathrm{CR}} v_h\|_{L^2(T)} + |v_h - I_{\mathrm{CR}} v_h|_{1,T} \leq 2 |v_h|_{1,T} \tag{3.15}$$

(cf. [23, Thm 2.1] or [21, Thm 4] for explicit constants). Define an enrichment operator $E_h : \mathrm{CR}_0^1(\widehat{\mathcal{T}}) \to H_0^1(\Omega)$ by averaging the function values at each interior vertex z, that is,

$$E_h v_{\mathrm{CR}}(z) = \frac{1}{|\widehat{\mathcal{T}}(z)|} \sum_{T \in \widehat{\mathcal{T}}(z)} v_{\mathrm{CR}}|_T(z) \tag{3.16}$$

and zero on boundary vertices. In (3.16) the set $\widehat{\mathcal{T}}(z) := \{T \in \widehat{\mathcal{T}} | z \in T\}$ of neighboring triangles has the cardinality $|\widehat{\mathcal{T}}(z)| \geq 3$.

The following lemma describes the construction of a modified companion operator $J : V_h \to H_0^1(\Omega)$, which is a right-inverse of the interpolation operator I_h from Definition 2.7.

Lemma 3.3 (conforming companion operator) *There exists a linear map $J : V_h \to H_0^1(\Omega)$ and a universal constant $C_{\mathrm{J}} \lesssim 1$ such that any $v_h \in V_h$ satisfies $I_h J v_h = v_h$ and*

(a) $\fint_E J v_h \, ds = \fint_E v_h \, ds$ *for any edge* $E \in \widehat{\mathcal{E}}$,
(b) $\nabla_{pw}(v_h - J v_h) \perp \mathcal{P}_0(\mathcal{T}; \mathbb{R}^2)$ *in* $L^2(\Omega; \mathbb{R}^2)$,
(c) $v_h - J v_h \perp \mathcal{P}_1(\mathcal{T})$ *in* $L^2(\Omega)$,
(d) $\|h_{\mathcal{T}}^{-1}(v_h - J v_h)\|_{L^2(\Omega)} + |v_h - J v_h|_{1,pw} \leq C_{\mathrm{J}} |v_h|_{1,pw}$.

Design of J in Lemma 3.3 Given $v_h \in V_h$, let $v_{\mathrm{CR}} := I_{\mathrm{CR}} v_h \in \mathrm{CR}_0^1(\widehat{\mathcal{T}})$. There exists an operator $J' : \mathrm{CR}_0^1(\widehat{\mathcal{T}}) \to H_0^1(\Omega)$ from [22, Prop. 2.3] such that any $v_{\mathrm{CR}} \in \mathrm{CR}_0^1(\widehat{\mathcal{T}})$ satisfies

(a') $\fint_E J' v_{CR} \, ds = \fint_E v_{CR} \, ds$ for any edge $E \in \widehat{\mathcal{E}}$,
(b') $\int_P \nabla_{\mathrm{pw}}(v_{\mathrm{CR}} - J' v_{\mathrm{CR}}) \, dx = 0$ for all $P \in \mathcal{T}$,
(c') $\|h_{\widehat{\mathcal{T}}}^{-1}(v_{\mathrm{CR}} - J' v_{\mathrm{CR}})\|_{L^2(\Omega)} + |v_{\mathrm{CR}} - J' v_{\mathrm{CR}}|_{1,\mathrm{pw}} \leq C_{\mathrm{J}'} \min_{v \in H_0^1(\Omega)} |v_{\mathrm{CR}} - v|_{1,\mathrm{pw}}$

with a universal constant $C_{\mathrm{J}'}$ from [25]. Set $v := J' I_{\mathrm{CR}} v_h \in V := H_0^1(\Omega)$. Recall that $\widehat{\mathcal{T}}(P)$ is a shape-regular triangulation of P into a finite number of triangles. For each $T \in \widehat{\mathcal{T}}(P)$, let $b_T \in W_0^{1,\infty}(T)$ denote the cubic bubble-function $27\lambda_1\lambda_2\lambda_3$ for the barycentric co-ordinates $\lambda_1, \lambda_2, \lambda_3 \in \mathcal{P}_1(T)$ of T with $\fint_T b_T \, dx = 9/20$ and $\|\nabla b_T\|_{L^2(T)} \lesssim h_T^{-1}|T|^{1/2} \approx 1$. Let b_T be extended by zero outside T and, for $P \in \mathcal{T}$, define

$$b_P := \frac{20}{9} \sum_{T \in \widehat{\mathcal{T}}(P)} b_T \in W_0^{1,\infty}(P) \subset W_0^{1,\infty}(\Omega) \tag{3.17}$$

with $\fint_P b_P \, dx = 1$ and $\|\nabla b_P\|_{L^2(P)} \lesssim h_P^{-1}|P|^{1/2} \approx 1$. Let $v_P \in \mathcal{P}_1(\mathcal{T})$ be the Riesz representation of the linear functional $\mathcal{P}_1(\mathcal{T}) \to \mathbb{R}$ defined by $w_1 \mapsto (v_h - v, w_1)_{L^2(\Omega)}$ for $w_1 \in \mathcal{P}_1(\mathcal{T})$ in the Hilbert space $\mathcal{P}_1(\mathcal{T})$ endowed with the weighted L^2

scalar product $(b_P\bullet,\bullet)_{L^2(P)}$. Hence v_P exists uniquely and satisfies $\Pi_1(v_h - v) = \Pi_1(b_P v_P)$. Given the bubble-functions $(b_P : P \in \mathcal{T})$ from (3.17) and the above functions $(v_P : P \in \mathcal{T})$ for $v_h \in V_h$, define

$$J v_h := v + \sum_{P\in\mathcal{T}} v_P b_P \in V. \tag{3.18}$$

□

Proof of (a) Since b_P vanishes at any $x \in E \in \mathcal{E}$, it follows for any $E \in \widehat{\mathcal{E}}$ that

$$\fint_E J v_h\, ds = \fint_E v\, ds = \fint_E J' v_{\mathrm{CR}}\, ds = \fint_E v_{\mathrm{CR}}\, ds = \fint_E v_h\, ds,$$

where the definition of $v = J' v_{\mathrm{CR}}$, (a), and $v_{\mathrm{CR}} = I_{\mathrm{CR}} v_h$ lead to the second, third, and fourth equality. This proves (a). □

Proof of (b) An integration by parts and (b) show, for all $v_h \in V_h$ with $J v_h$ from (3.18), that

$$\begin{aligned}\int_P \nabla J v_h\, dx &= \int_{\partial P} J v_h \mathbf{n}_P\, ds = \sum_{E\in\mathcal{E}(P)} \Big(\int_E J v_h \mathbf{n}_E\, ds\Big)\\ &= \sum_{E\in\mathcal{E}(P)} \Big(\int_E v_h \mathbf{n}_E\, ds\Big) = \int_P \nabla v_h\, dx.\end{aligned}$$

Since this holds for all $P \in \mathcal{T}$, it proves (b). □

Proof of (c) This is $\Pi_1 v_h = \Pi_1 J v_h$ and guaranteed by the design of J in (3.18). □

Proof of (d) This relies on the definition of J in (3.18) and J' with (c′). Since (a) allows for $\int_{\partial P}(v_h - J v_h)\, ds = 0$, the Poincaré–Friedrichs inequality from Lemma 2.1.a implies

$$h_P^{-1}\|v_h - J v_h\|_{L^2(P)} \le C_{\mathrm{PF}}|v_h - J v_h|_{1,P}.$$

Hence it remains to prove $|v_h - J v_h|_{1,\mathrm{pw}} \lesssim |v_h|_{1,\mathrm{pw}}$. Triangle inequalities with $v_h, J v_h, v = J' v_{\mathrm{CR}}$ and $v_{\mathrm{CR}} = I_{\mathrm{CR}} v_h$ show the first and second inequality in

$$\begin{aligned}|v_h - J v_h|_{1,\mathrm{pw}} - |v - J v_h|_{1,\mathrm{pw}} &\le |v - v_h|_{1,\mathrm{pw}}\\ &\le |v_h - I_{\mathrm{CR}} v_h|_{1,\mathrm{pw}} + |v_{\mathrm{CR}} - J' v_{\mathrm{CR}}|_{1,\mathrm{pw}}\\ &\le (1 + C_{\mathrm{J}'})|v_h|_{1,\mathrm{pw}}\end{aligned} \tag{3.19}$$

with (b′) for $|v_{\mathrm{CR}}|_{1,\mathrm{pw}} = \|\Pi_0 \nabla_{\mathrm{pw}} v_h\|_{L^2(\Omega)} \le \|\nabla_{\mathrm{pw}} v_h\|_{L^2(\Omega)} = |v_h|_{1,\mathrm{pw}}$ in the last step. The equivalence of norms in the finite-dimensional space $\mathcal{P}_1(P)$ assures the existence of a positive constant C_b, independent of h_P, such that any $\chi \in \mathcal{P}_1(P)$ satisfies the inverse inequalities

$$C_b^{-1}\|\chi\|_{L^2(P)}^2 \le (b_P, \chi^2)_{L^2(P)} \le C_b\|\chi\|_{L^2(P)}^2, \tag{3.20}$$

$$C_b^{-1}\|\chi\|_{L^2(P)} \le \|b_P\chi\|_{L^2(P)} + h_P\|\nabla(b_P\chi)\|_{L^2(P)} \le C_b\|\chi\|_{L^2(P)}. \tag{3.21}$$

These estimates are completely standard on shape-regular triangles [2,p. 27] or [37]; so they hold on each $T \in \widehat{\mathcal{T}}$ and, by definition of b_P, their sum is (3.20)–(3.21). The analysis of the term $|v - Jv_h|_{1,\mathrm{pw}}$ starts with one $P \in \mathcal{T}$ and (3.18) for

$$|v - Jv_h|_{1,P} = |v_P b_P|_{1,P} \le C_b h_P^{-1}\|v_P\|_{L^2(P)} \tag{3.22}$$

with (3.21) in the last step. The estimate (3.20) leads to the first inequality in

$$\begin{aligned} C_b^{-1}\|v_P\|_{L^2(P)}^2 &\le (b_P v_P, v_P)_{L^2(P)} = (v_h - v, v_P)_{L^2(P)} \\ &\le \|v_h - v\|_{L^2(P)}\|v_P\|_{L^2(P)}. \end{aligned}$$

The equality results from $\Pi_1(v_h - v) = \Pi_1(v_P b_P)$ and $v_P \in \mathcal{P}_1(\mathcal{T})$, while the last step is the Cauchy–Schwarz inequality. Consequently, $\|v_P\|_{L^2(P)} \le C_b\|v_h - v\|_{L^2(P)}$. This and (3.22) show

$$|v - Jv_h|_{1,\mathrm{pw}} \le C_b^2\|h_{\mathcal{T}}^{-1}(v - v_h)\|_{L^2(\Omega)} \le C_b^2 C_{\mathrm{PF}}|v - v_h|_{1,\mathrm{pw}}$$

with $\int_{\partial P}(v - v_h)\,ds = 0$ from (a) and hence the Poincaré–Friedrichs inequality for $v - v_h$ from Lemma 2.1.a in the last step. Recall $|v - v_h|_{1,\mathrm{pw}} \lesssim |v_h|_{1,\mathrm{pw}}$ from (3.19) to conclude $|v - Jv_h|_{1,\mathrm{pw}} \lesssim |v_h|_{1,\mathrm{pw}}$ from the previous displayed inequality. This concludes the proof of (d). □

Proof (Proof of $I_h J = \mathrm{id}$ in V_h) Definition 2.7 and Lemma 3.3.a show, for all $v_h \in V_h$, that

$$I_h J v_h = \sum_{E\in\mathcal{E}} \left(\fint_E J v_h\,ds\right)\psi_E = \sum_{E\in\mathcal{E}} \left(\fint_E v_h\,ds\right)\psi_E = v_h.$$

This concludes the proof of Lemma 3.3. □

Since V_h is not a subset of $H_0^1(\Omega)$ in general, the substitution of discrete function v_h in the weak formulation leads to a nonconformity error.

Lemma 3.4 (nonconformity error) *There exist positive universal constants $C_{\mathrm{NC}}, C_{\mathrm{NC}}^*$ (depending on the coefficients $\boldsymbol{A}, \boldsymbol{b}$ and the universal constants ρ, σ) such that all $f, g \in L^2(\Omega)$ and all $\mathcal{T} \in \mathbb{T}(\delta)$ (with the assumption $h_{max} \le \delta \le 1$) satisfy* (a) *and* (b).

(a) The solution $u \in H^{1+\sigma}(\Omega) \cap H_0^1(\Omega)$ to (1.1) satisfies

$$\sup_{0 \neq v_h \in V_h} \frac{|B_{pw}(u, v_h) - (f, v_h)_{L^2(\Omega)}|}{\|v_h\|_{1,pw}} \leq C_{\mathrm{NC}} h_{max}^{\sigma} \|f\|_{L^2(\Omega)}. \tag{3.23}$$

(b) The solution $\Phi \in H^{1+\sigma}(\Omega) \cap H_0^1(\Omega)$ to the dual problem (1.4) satisfies

$$\sup_{0 \neq v_h \in V_h} \frac{|B_{pw}(v_h, \Phi) - (g, v_h)_{L^2(\Omega)}|}{\|v_h\|_{1,pw}} \leq C_{\mathrm{NC}}^{*} h_{max}^{\sigma} \|g\|_{L^2(\Omega)}. \tag{3.24}$$

Proof of (a) Given $v_h \in V_h$, define $Jv_h \in V$ and the piecewise averages $\overline{\mathbf{A}} := \Pi_0(\mathbf{A})$, $\overline{\mathbf{b}} := \Pi_0(\mathbf{b})$, and $\overline{\gamma} := \Pi_0(\gamma)$ of the coefficients $\mathbf{A}$, $\mathbf{b}$, and γ. The choice of test function $v := Jv_h \in V$ in the weak formulation (1.8) having extra properties provides the terms with oscillations in the further analysis. Abbreviate $\boldsymbol{\sigma} := \mathbf{A}\nabla u + \mathbf{b}u$. The weak formulation (1.8), Lemma 3.3.b–c, and the Cauchy–Schwarz inequality reveal that

$$\begin{aligned} B_{\mathrm{pw}}(u, v_h) - (f, v_h)_{L^2(\Omega)} &= B_{\mathrm{pw}}(u, v_h - Jv_h) - (f, v_h - Jv_h)_{L^2(\Omega)} \\ &\leq \|\boldsymbol{\sigma} - \Pi_0\boldsymbol{\sigma}\|_{L^2(\Omega)} \|\nabla_{\mathrm{pw}}(1 - J)v_h\|_{L^2(\Omega)} \\ &+ \|h_{\mathcal{T}}(1 - \Pi_1)(f - \gamma u)\|_{L^2(\Omega)} \|h_{\mathcal{T}}^{-1}(1 - J)v_h\|_{L^2(\Omega)}. \end{aligned} \tag{3.25}$$

The first term on the right-hand side of (3.25) involves the factor

$$\begin{aligned} \|\boldsymbol{\sigma} - \Pi_0\boldsymbol{\sigma}\|_{L^2(\Omega)} &\leq \|\mathbf{A}\nabla u - \Pi_0(\mathbf{A}\nabla u)\|_{L^2(\Omega)} + \|\mathbf{b}u - \Pi_0(\mathbf{b}u)\|_{L^2(\Omega)} \\ &\leq \|(\mathbf{A} - \overline{\mathbf{A}})\nabla u + \overline{\mathbf{A}}(1 - \Pi_0)\nabla u\|_{L^2(\Omega)} \\ &\quad + \|(\mathbf{b} - \overline{\mathbf{b}})u + \overline{\mathbf{b}}(1 - \Pi_0)u\|_{L^2(\Omega)} \\ &\leq \Big(h_{\max}(|\mathbf{A}|_{1,\infty} + |\mathbf{b}|_{1,\infty}) + C_{\mathrm{apx}}(h_{\max}^{\sigma}\|\mathbf{A}\|_{\infty} + h_{\max}\|\mathbf{b}\|_{\infty})\Big) \\ &\quad \|u\|_{1+\sigma,\Omega}. \end{aligned}$$

The last inequality follows from the Lipschitz continuity of the coefficients $\mathbf{A}$ and $\mathbf{b}$, and the estimate (2.12). Lemma 3.3.d leads to the estimates $\|\nabla_{\mathrm{pw}}(1 - J)v_h\|_{L^2(\Omega)} \leq C_J |v_h|_{1,\mathrm{pw}}$ and

$$\|h_{\mathcal{T}}(1 - \Pi_1)(f - \gamma u)\|_{L^2(\Omega)} \|h_{\mathcal{T}}^{-1}(1 - J)v_h\|_{L^2(\Omega)} \leq \mathrm{osc}_1(f - \gamma u, \mathcal{T}) C_J |v_h|_{1,\mathrm{pw}}.$$

The substitution of the previous estimates in (3.25) with $h_{\max} \leq 1$ (from $\delta \leq 1$ by assumption) and the regularity (1.5) show

$$B_{\mathrm{pw}}(u, v_h) - (f, v_h) \leq C_{\mathrm{NC}} h_{\max}^{\sigma} \|f\|_{L^2(\Omega)} \|v_h\|_{1,\mathrm{pw}}$$

with $C_{\mathrm{NC}} := C_J\Big((|\mathbf{A}|_{1,\infty} + |\mathbf{b}|_{1,\infty} + C_{\mathrm{apx}}(\|\mathbf{A}\|_{\infty} + \|\mathbf{b}\|_{\infty}) + \|\gamma\|_{\infty})C_{\mathrm{reg}} + 1\Big)$. This concludes the proof of Lemma 3.4.a. □

Proof of (b) The solution $\Phi \in V$ to (1.4) satisfies $B(v, \Phi) = (g, v)_{L^2(\Omega)}$ for all $v \in V$. This implies

$$B_{\mathrm{pw}}(v_h, \Phi) - (g, v_h)_{L^2(\Omega)} = B_{\mathrm{pw}}(v_h - Jv_h, \Phi) - (g, v_h - Jv_h)_{L^2(\Omega)}.$$

The arguments in the proof of (a) lead to the bound (3.24) with

$$C_{\mathrm{NC}}^* := C_J\Big((|\mathbf{A}|_{1,\infty} + C_{\mathrm{apx}}\|\mathbf{A}\|_\infty + \|\mathbf{b}\|_\infty + \|\gamma\|_\infty)C_{\mathrm{reg}}^* + 1\Big).$$

The remaining analogous details are omitted in the proof of Lemma 3.4.b for brevity. □

4 A priori error analysis

This section focuses on the stability, existence, and uniqueness of the discrete solution u_h. The a priori error analysis uses the discrete inf-sup condition.

4.1 Existence and uniqueness of the discrete solution

Theorem 4.1 (stability) *There exist positive constants $\delta \le 1$ and C_{stab} (depending on $\alpha, \beta, \sigma, \rho$, and C_{F}) such that, for all $\mathcal{T} \in \mathbb{T}(\delta)$ and for all $f \in L^2(\Omega)$, the discrete problem* (3.8) *has a unique solution $u_h \in V_h$ and*

$$|u_h|_{1,pw} \le C_{\mathrm{stab}}\|f_h\|_{V_h^*}.$$

Proof In the first part of the proof, suppose there exists some solution $u_h \in V_h$ to the discrete problem (3.8) for some $f \in L^2(\Omega)$. (This is certainly true for all $f \equiv 0 \equiv u_h$, but will be discussed for all those pairs at the end of the proof and shall lead to the uniqueness of discrete solutions.) Since u_h satisfies a Gårding-type inequality in Proposition 3.1.b,

$$\alpha|u_h|_{1,\mathrm{pw}}^2 \le \beta\|u_h\|_{L^2(\Omega)}^2 + B_h(u_h, u_h) = \beta\|u_h\|_{L^2(\Omega)}^2 + (f_h, u_h)_{L^2(\Omega)}.$$

This, (2.14), and the definition of the dual norm in (3.12) lead to

$$\alpha|u_h|_{1,\mathrm{pw}} \le \beta C_{\mathrm{F}}\|u_h\|_{L^2(\Omega)} + \|f_h\|_{V_h^*}. \tag{4.1}$$

Given $g := u_h \in L^2(\Omega)$, let $\Phi \in V \cap H^{1+\sigma}(\Omega)$ solve the dual problem $\mathcal{L}^*\Phi = g$ and let $I_h\Phi \in V_h$ be the interpolation of Φ from Sect. 2.4. Elementary algebra shows

$$\begin{aligned}\|u_h\|_{L^2(\Omega)}^2 = \Big((g, u_h)_{L^2(\Omega)} - B_{\mathrm{pw}}(u_h, \Phi)\Big) + B_{\mathrm{pw}}(u_h, \Phi - I_h\Phi)\\ + \Big(B_{\mathrm{pw}}(u_h, I_h\Phi) - B_h(u_h, I_h\Phi)\Big) + (f_h, I_h\Phi)_{L^2(\Omega)}.\end{aligned} \tag{4.2}$$

Rewrite a part of the third term corresponding to diffusion on the right-hand side of (4.2) as

$$
\begin{aligned}
a^P(u_h, I_h\Phi) - a_h^P(u_h, I_h\Phi) &= (\mathbf{A}\nabla u_h, \nabla(1-\Pi_1)I_h\Phi)_{L^2(P)} \\
&+ (\nabla(1-\Pi_1)u_h, (\mathbf{A}-\Pi_0\mathbf{A})(\nabla\Pi_1 I_h\Phi))_{L^2(P)} \\
&- S^P\big((1-\Pi_1)u_h, (1-\Pi_1)I_h\Phi\big).
\end{aligned}
$$

The Cauchy–Schwarz inequality in the semi-scalar product $S^P(\bullet,\bullet)$, and (3.5) with the upper bound $\|\mathbf{A}\|_\infty$ for the coefficient $\mathbf{A}$ in $a^P(\bullet,\bullet)$ lead to the estimate

$$
\begin{aligned}
C_s^{-1}S^P\big((1-\Pi_1)u_h, (1-\Pi_1)I_h\Phi\big) &\le |(1-\Pi_1)u_h|_{1,P}|(1-\Pi_1)I_h\Phi|_{1,P} \\
&\le \|\mathbf{A}\|_\infty|u_h|_{1,P}\Big(\|\nabla(I_h\Phi-\Phi)\|_{L^2(P)} + \|\nabla(1-\Pi_1 I_h)\Phi\|_{L^2(P)}\Big) \\
&\le \|\mathbf{A}\|_\infty C_{\mathrm{apx}}\Big(2 + C_{\mathrm{PF}} + C_{\mathrm{Itn}}\Big)h_P^\sigma|u_h|_{1,P}|\Phi|_{1+\sigma,P} \qquad (4.3)
\end{aligned}
$$

with Theorem 2.8.b followed by (2.12) in the final step. This and Theorem 2.8 imply that

$$
\begin{aligned}
|a^P(u_h, I_h\Phi) - a_h^P(u_h, I_h\Phi)| &\le h_P^\sigma|u_h|_{1,P}\|\Phi\|_{1+\sigma,P} \\
&\times\Big(\|\mathbf{A}\|_\infty C_{\mathrm{apx}}(2+C_{\mathrm{PF}}+C_{\mathrm{Itn}})(1+C_s) + |\mathbf{A}|_{1,\infty}C_{\mathrm{Itn}}\Big).
\end{aligned}
$$

The terms $b^P - b_h^P$ and $c^P - c_h^P$ are controlled by

$$
\begin{aligned}
&|b^P(u_h, I_h\Phi) - b_h^P(u_h, I_h\Phi)| + |c^P(u_h, I_h\Phi) - c_h^P(u_h, I_h\Phi)| \\
&\quad\le h_P^\sigma\|\Phi\|_{1+\sigma,P}\big(\|\mathbf{b}\|_\infty(C_{\mathrm{apx}}(2+C_{\mathrm{PF}}+C_{\mathrm{Itn}})\|u_h\|_{L^2(P)} + C_{\mathrm{Itn}}C_{\mathrm{PF}}|u_h|_{1,P}) \\
&\qquad\qquad + \|\gamma\|_\infty C_{\mathrm{PF}}(C_{\mathrm{Itn}}\|u_h\|_{L^2(P)} + |u_h|_{1,P})\big).
\end{aligned}
$$

The combination of the previous four displayed estimates with Lemma 2.6 leads to an estimate for P. The sum over all polygonal domains $P \in \mathcal{T}$ reads

$$
B_{\mathrm{pw}}(u_h, I_h\Phi) - B_h(u_h, I_h\Phi) \le C_d h_{\max}^\sigma|u_h|_{1,\mathrm{pw}}\|\Phi\|_{1+\sigma,\Omega} \qquad (4.4)
$$

with a universal constant C_d. The bound for (4.2) results from Lemma 3.4.b for the first term, the boundedness of B_{pw} (with a universal constant $M_b := \|\mathbf{A}\|_\infty + C_{\mathrm{F}}\|\mathbf{b}\|_\infty + C_{\mathrm{F}}^2\|\gamma\|_\infty$) and (2.15) for the second term, (4.4) for the third term, and Theorem 2.8.a for the last term on the right-hand side of (4.2). This shows

$$
\|u_h\|_{L^2(\Omega)}^2 \le \Big(C_{\mathrm{NC}}^* + C_{\mathrm{I}}M_b + C_d\Big)h_{\max}^\sigma|u_h|_{1,\mathrm{pw}}\|\Phi\|_{1+\sigma,\Omega} + C_{\mathrm{Itn}}\|f_h\|_{V_h^*}\|\Phi\|_{1,\Omega}.
$$

This and the regularity estimate (1.5) lead to $C_3 = C_{\mathrm{NC}}^* + C_{\mathrm{I}}M_b + C_d$ in

$$
\|u_h\|_{L^2(\Omega)} \le C_3\,C_{\mathrm{reg}}^* h_{\max}^\sigma|u_h|_{1,\mathrm{pw}} + C_{\mathrm{Itn}}\|f_h\|_{V_h^*}.
$$

The substitution of this in (4.1) proves

$$\alpha|u_h|_{1,\mathrm{pw}} \le \beta C_{\mathrm{F}} C_3 C^*_{\mathrm{reg}} h^\sigma_{\max} |u_h|_{1,\mathrm{pw}} + (\beta C_{\mathrm{F}} C_{\mathrm{Itn}} + 1)\|f_h\|_{V_h^*}. \tag{4.5}$$

For all $0 < h_{\max} \le \delta := (\frac{\alpha}{2\beta C_{\mathrm{F}} C_3 C^*_{\mathrm{reg}}})^{1/\sigma}$, the constant $\overline{c} = (1 - \frac{\beta}{\alpha} C_{\mathrm{F}} C_3 C^*_{\mathrm{reg}} h^\sigma_{\max})$ is positive and $C_{\mathrm{stab}} := \frac{\beta C_{\mathrm{F}} C_{\mathrm{Itn}}+1}{\alpha - \beta C_{\mathrm{F}} C_3 C^*_{\mathrm{reg}} h_0^\sigma}$ is well-defined. This leads in (4.5) to

$$|u_h|_{1,\mathrm{pw}} \le C_{\mathrm{stab}} \|f_h\|_{V_h^*}. \tag{4.6}$$

In the last part of the proof, suppose $f_h \equiv 0$ and let u_h be any solution to the resulting homogeneous linear discrete system. The stability result (4.6) proves $u_h \equiv 0$. Hence, the linear system of equations (3.8) has a unique solution and the coefficient matrix is regular. This proves that there exists a unique solution u_h to (3.8) for any right-hand side $f_h \in V_h^*$. The combination of this with (4.6) concludes the proof. □

An immediate consequence of Theorem 4.1 is the following discrete inf-sup estimate.

Theorem 4.2 (discrete inf-sup) *There exist* $0 < \delta \le 1$ *and* $\overline{\beta}_0 > 0$ *such that, for all* $\mathcal{T} \in \mathbb{T}(\delta)$,

$$\overline{\beta}_0 \le \inf_{0 \ne u_h \in V_h} \sup_{0 \ne v_h \in V_h} \frac{B_h(u_h, v_h)}{|u_h|_{1,pw} |v_h|_{1,pw}}. \tag{4.7}$$

Proof Define the operator $\mathcal{L}_h : V_h \to V_h^*, v_h \mapsto B_h(v_h, \bullet)$. The stability Theorem 4.1 can be interpreted as follows: For any $f_h \in V_h^*$ there exists $u_h \in V_h$ such that $\mathcal{L}_h u_h = f_h$ and

$$\overline{\beta}_0 |u_h|_{1,\mathrm{pw}} \le \|f_h\|_{V_h^*} = \sup_{0 \ne v_h \in V_h} \frac{(f_h, v_h)}{|v_h|_{1,\mathrm{pw}}} = \sup_{0 \ne v_h \in V_h} \frac{B_h(u_h, v_h)}{|v_h|_{1,\mathrm{pw}}}.$$

The discrete problem $B_h(u_h, \bullet) = (f_h, \bullet)_{L^2(\Omega)}$ has a unique solution in V_h. Therefore, f_h and u_h are in one to one correspondence and the last displayed estimate holds for any $u_h \in V_h$. The infimum over $u_h \in V_h$ therein proves (4.7) with $\overline{\beta}_0 = C_{\mathrm{stab}}^{-1}$. □

4.2 A priori error estimates

This subsection establishes the error estimate in the energy norm $|\cdot|_{1,\mathrm{pw}}$ and in the L^2 norm. The discrete inf-sup condition allows for an error estimate in the H^1 norm and an Aubin–Nitsche duality argument leads to an error estimate in the L^2 norm.

Recall $u \in H_0^1(\Omega)$ is a unique solution of (1.8) and $u_h \in V_h$ is a unique solution of (3.8). Recall the definition of the bilinear form $s_h(\cdot, \cdot)$ from Sect. 3.1 and define the induced seminorm $|v_h|_{\mathrm{s}} := s_h(v_h, v_h)^{1/2}$ for $v_h \in V_h$ as a part of the norm $\|\cdot\|_h$ from Remark 6.

Theorem 4.3 (error estimate) *Set* $\boldsymbol{\sigma} := \boldsymbol{A}\nabla u + \boldsymbol{b}u \in H(div, \Omega)$. *There exist positive constants* C_4, C_5, *and* δ *such that, for all* $\mathcal{T} \in \mathbb{T}(\delta)$, *the discrete problem* (3.8) *has a unique solution* $u_h \in V_h$ *and*

$$\begin{aligned}
&|u - u_h|_{1,pw} + |u - \Pi_1 u_h|_{1,pw} + h_{\max}^{-\sigma}(\|u - u_h\|_{L^2(\Omega)} + \|u - \Pi_1 u_h\|_{L^2(\Omega)}) \\
&\quad + |u_h|_{\mathrm{s}} + |I_h u - u_h|_{\mathrm{s}} \\
&\quad \le C_4\Big(\|(1 - \Pi_0)\boldsymbol{\sigma}\|_{L^2(\Omega)} + \|(1 - \Pi_0)\nabla u\|_{L^2(\Omega)} + \mathrm{osc}_1(f - \gamma u, \mathcal{T})\Big) \\
&\quad \le C_5 h_{\max}^{\sigma}\|f\|_{L^2(\Omega)}.
\end{aligned} \tag{4.8}$$

Proof Step 1 (initialization). Let $I_h u \in V_h$ be the interpolation of u from Definition 2.7. The discrete inf-sup condition (4.7) for $I_h u - u_h \in V_h$ leads to some $v_h \in V_h$ with $|v_h|_{1,\mathrm{pw}} \le 1$ such that

$$\overline{\beta}_0 |I_h u - u_h|_{1,\mathrm{pw}} = B_h(I_h u - u_h, v_h).$$

Step 2 (error estimate for $|u - u_h|_{1,\mathrm{pw}}$). Rewrite the last equation with the continuous and the discrete problem (1.8) and (3.8) as

$$\overline{\beta}_0 |I_h u - u_h|_{1,\mathrm{pw}} = B_h(I_h u, v_h) - B(u, v) + (f, v)_{L^2(\Omega)} - (f_h, v_h)_{L^2(\Omega)}.$$

This equality is rewritten with the definition of $B(u, v)$ in (1.7), the definition of $B_h(I_h u, v_h)$ in Sect. 3.1, and with $f_h = \Pi_1 f$. Recall $v := J v_h \in V$ from Lemma 3.3 and recall $\nabla_{\mathrm{pw}} \Pi_1 I_h u = \Pi_0 \nabla u$ from (2.17). This results in

$$\begin{aligned}
\mathrm{LHS} &:= \overline{\beta}_0 |I_h u - u_h|_{1,\mathrm{pw}} - s_h((1 - \Pi_1) I_h u, (1 - \Pi_1) v_h) \\
&= (\mathbf{A}\Pi_0 \nabla u + \mathbf{b}\Pi_1 I_h u, \nabla_{\mathrm{pw}} \Pi_1 v_h)_{L^2(\Omega)} + (\gamma \Pi_1 I_h u, \Pi_1 v_h)_{L^2(\Omega)} \\
&\quad - (\boldsymbol{\sigma}, \nabla v)_{L^2(\Omega)} + (f - \gamma u, v)_{L^2(\Omega)} - (f, \Pi_1 v_h)_{L^2(\Omega)}.
\end{aligned}$$

Abbreviate $w := v - \Pi_1 v_h$ and observe the orthogonalities $\nabla_{\mathrm{pw}} w \perp \mathcal{P}_0(\mathcal{T}; \mathbb{R}^2)$ in $L^2(\Omega; \mathbb{R}^2)$ and $w \perp \mathcal{P}_1(\mathcal{T})$ in $L^2(\Omega)$ from Lemma 3.3.b-c and the definition of Π_1 with $\Pi_1 = \Pi_1^\nabla$ in V_h. Lemma 3.3.d, the bound $|(1 - \Pi_1^\nabla) v_h|_{1,\mathrm{pw}} \le |v_h|_{1,\mathrm{pw}} \le 1$, and the Poincaré–Friedrichs inequality for $v_h - \Pi_1^\nabla v_h$ from Lemma 2.1.a lead to

$$|w|_{1,\mathrm{pw}} \le |v - v_h|_{1,\mathrm{pw}} + |v_h - \Pi_1 v_h|_{1,\mathrm{pw}} \le C_{\mathrm{J}} + 1, \tag{4.9}$$

$$\|h_{\mathcal{T}}^{-1} w\|_{L^2(\Omega)} \le \|h_{\mathcal{T}}^{-1}(v - v_h)\|_{L^2(\Omega)} + \|h_{\mathcal{T}}^{-1}(v_h - \Pi_1 v_h)\|_{L^2(\Omega)} \le C_{\mathrm{J}} + C_{\mathrm{PF}}. \tag{4.10}$$

Elementary algebra and the above orthogonalities prove that

$$\begin{aligned}
\mathrm{LHS} = {} & ((\mathbf{A} - \Pi_0 \mathbf{A})(\Pi_0 - 1)\nabla u + \mathbf{b}(\Pi_1 I_h u - u), \nabla_{\mathrm{pw}} \Pi_1 v_h)_{L^2(\Omega)} \\
& - ((1 - \Pi_0)\boldsymbol{\sigma}, \nabla_{\mathrm{pw}} w)_{L^2(\Omega)} + (\gamma(\Pi_1 I_h u - u), \Pi_1 v_h)_{L^2(\Omega)} \\
& + (h_{\mathcal{T}}(1 - \Pi_1)(f - \gamma u), h_{\mathcal{T}}^{-1} w)_{L^2(\Omega)}
\end{aligned}$$

$$\leq \Big(|\mathbf{A}|_{1,\infty} + (1 + C_{\mathrm{PF}})(\|\mathbf{b}\|_\infty + C_{\mathrm{F}}\|\gamma\|_\infty)\Big) h_{\max}\|(1-\Pi_0)\nabla u\|_{L^2(\Omega)}$$
$$+(C_{\mathrm{J}}+1)\|(1-\Pi_0)\boldsymbol{\sigma}\|_{L^2(\Omega)} + (C_{\mathrm{J}} + C_{\mathrm{PF}})\mathrm{osc}_1(f-\gamma u, \mathcal{T}) \tag{4.11}$$

with the Lipschitz continuity of $\mathbf{A}$, Lemma 2.8.b, the stabilities of Π_1 from (2.11), and (4.9)–(4.10) in the last step. The definition of stability term (3.5) and Theorem 2.8.b lead to

$$\begin{aligned}
&C_s^{-1} s_h((1-\Pi_1)I_h u, (1-\Pi_1)v_h)\\
&\quad \leq \|\mathbf{A}\|_\infty |(1-\Pi_1)I_h u|_{1,\mathrm{pw}} |(1-\Pi_1)v_h|_{1,\mathrm{pw}}\\
&\quad \leq \|\mathbf{A}\|_\infty (|I_h u - u|_{1,\mathrm{pw}} + |u - \Pi_1 I_h u|_{1,\mathrm{pw}})|v_h|_{1,\mathrm{pw}}\\
&\quad \leq \|\mathbf{A}\|_\infty (2 + C_{\mathrm{Itn}} + C_{\mathrm{PF}})\|(1-\Pi_0)\nabla u\|_{L^2(\Omega)}|v_h|_{1,\mathrm{pw}}.
\end{aligned} \tag{4.12}$$

The triangle inequality, the bound (2.15) for the term $|u - I_h u|_{1,\mathrm{pw}}$, and (4.11)–(4.12) for the term $|I_h u - u_h|_{1,\mathrm{pw}}$ conclude the proof of (4.8) for the term $|u - u_h|_{1,\mathrm{pw}}$.
Step 3 (duality argument). To prove the bound for $u - u_h$ in the L^2 norm with a duality technique, let $g := I_h u - u_h \in L^2(\Omega)$. The solution $\Phi \in H_0^1(\Omega) \cap H^{1+\sigma}(\Omega)$ to the dual problem (1.4) satisfies the elliptic regularity (1.5),

$$\|\Phi\|_{1+\sigma,\Omega} \leq C_{\mathrm{reg}}^* \|I_h u - u_h\|_{L^2(\Omega)}. \tag{4.13}$$

Step 4 (error estimate for $\|u - u_h\|_{L^2(\Omega)}$). Let $I_h\Phi \in V_h$ be the interpolation of Φ from Definition 2.7. Elementary algebra reveals the identity

$$\begin{aligned}
\|g\|_{L^2(\Omega)}^2 &= ((g,g)_{L^2(\Omega)} - B_{\mathrm{pw}}(g,\Phi)) + B_{\mathrm{pw}}(g, \Phi - I_h\Phi)\\
&\quad + (B_{\mathrm{pw}}(g, I_h\Phi) - B_h(g, I_h\Phi)) + B_h(g, I_h\Phi).
\end{aligned} \tag{4.14}$$

The bound (4.4) with g as the first argument shows

$$B_{\mathrm{pw}}(g, I_h\Phi) - B_h(g, I_h\Phi) \leq C_d h_{\max}^\sigma |g|_{1,\mathrm{pw}} \|\Phi\|_{1+\sigma,\Omega}.$$

This controls the third term in (4.14), Lemma 3.4.b controls the first term, the boundedness of B_{pw} and the interpolation error estimate (2.15) control the second term on the right-hand side of (4.14). This results in

$$\|I_h u - u_h\|_{L^2(\Omega)}^2 \leq (C_{\mathrm{NC}}^* + C_{\mathrm{I}} M_b + C_d) h_{\max}^\sigma |g|_{1,\mathrm{pw}} \|\Phi\|_{1+\sigma,\Omega} + B_h(g, I_h\Phi). \tag{4.15}$$

It remains to bound $B_h(g, I_h\Phi)$. The continuous and the discrete problem (1.8) and (3.8) imply

$$B_h(g, I_h\Phi) = B_h(I_h u, I_h\Phi) - B(u,\Phi) + (f,\Phi)_{L^2(\Omega)} - (f_h, I_h\Phi)_{L^2(\Omega)}.$$

The definition of B_h and Π_0 lead to

$$B_h(g, I_h\Phi) - s_h((1-\Pi_1)I_h u, (1-\Pi_1)I_h\Phi)$$

$$
\begin{aligned}
&= ((\mathbf{A}-\Pi_0\mathbf{A})(\Pi_0-1)\nabla u+\mathbf{b}(\Pi_1 I_h u-u),\nabla_{\mathrm{pw}}\Pi_1 I_h\Phi)_{L^2(\Omega)}\\
&\quad+(\gamma(\Pi_1 I_h u-u),\Pi_1 I_h\Phi)_{L^2(\Omega)}-((1-\Pi_0)\boldsymbol{\sigma},\nabla_{\mathrm{pw}}(1-\Pi_1 I_h)\Phi)_{L^2(\Omega)}\\
&\quad+(f-\gamma u,\Phi-\Pi_1 I_h\Phi)_{L^2(\Omega)}.
\end{aligned}\tag{4.16}
$$

The bound for the stability term as in (4.12) is

$$
\begin{aligned}
&s_h((1-\Pi_1)I_h u,(1-\Pi_1)I_h\Phi)\\
&\quad\le C_s\|\mathbf{A}\|_\infty|(1-\Pi_1)I_h u|_{1,\mathrm{pw}}|(1-\Pi_1)I_h\Phi|_{1,\mathrm{pw}}\\
&\quad\le C_s\|\mathbf{A}\|_\infty(2+C_{\mathrm{Itn}}+C_{\mathrm{PF}})^2C_{\mathrm{apx}}h_{\max}^\sigma\|(1-\Pi_0)\nabla u\|_{L^2(\Omega)}|\Phi|_{1+\sigma,\Omega}.
\end{aligned}\tag{4.17}
$$

Step 5 (oscillation). The last term in (4.16) is of optimal order $O(h_{\max}^{1+\sigma})$, but the following arguments allow to write it as an oscillation. Recall the bubble-function $b_{\mathcal{T}}|_P := b_P \in H_0^1(P)$ from (3.17) extended by zero outside P. Given $\Psi := \Phi - \Pi_1 I_h\Phi$, let $\Psi_1 \in \mathcal{P}_1(\mathcal{T})$ be the Riesz representation of the linear functional $\mathcal{P}_1(\mathcal{T}) \to \mathbb{R}$ defined by $w_1 \mapsto (\Psi, w_1)_{L^2(\Omega)}$ in the Hilbert space $\mathcal{P}_1(\mathcal{T})$ endowed with the weighted scalar product $(b_{\mathcal{T}}\bullet,\bullet)_{L^2(\Omega)}$. That means $\Pi_1(b_{\mathcal{T}}\Psi_1)=\Pi_1\Psi$. The identity $(f-\gamma u, b_{\mathcal{T}}\Psi_1)_{L^2(\Omega)} = (\boldsymbol{\sigma},\nabla(b_{\mathcal{T}}\Psi_1))_{L^2(\Omega)}$ follows from (1.8) with the test function $b_{\mathcal{T}}\Psi_1 \in H_0^1(\Omega)$. The L^2 orthogonalities $\Psi - b_{\mathcal{T}}\Psi_1 \perp \mathcal{P}_1(\mathcal{T})$ in $L^2(\Omega)$ and $\nabla(b_{\mathcal{T}}\Psi_1) \perp \mathcal{P}_0(\mathcal{T};\mathbb{R}^2)$ in $L^2(\Omega;\mathbb{R}^2)$ allow the rewriting of the latter identity as

$$
\begin{aligned}
(f-\gamma u,\Psi)_{L^2(\Omega)} &= (h_{\mathcal{T}}(1-\Pi_1)(f-\gamma u),h_{\mathcal{T}}^{-1}(\Psi-b_{\mathcal{T}}\Psi_1))_{L^2(\Omega)}\\
&\quad+((1-\Pi_0)\boldsymbol{\sigma},\nabla(b_{\mathcal{T}}\Psi_1))_{L^2(\Omega)}\\
&\le \mathrm{osc}_1(f-\gamma u,\mathcal{T})\|h_{\mathcal{T}}^{-1}(\Psi-b_{\mathcal{T}}\Psi_1)\|_{L^2(\Omega)}\\
&\quad+\|(1-\Pi_0)\boldsymbol{\sigma}\|_{L^2(\Omega)}|b_{\mathcal{T}}\Psi|_{1,\mathrm{pw}}.
\end{aligned}\tag{4.18}
$$

It remains to control the terms $\|h_{\mathcal{T}}^{-1}(\Psi - b_{\mathcal{T}}\Psi_1)\|_{L^2(\Omega)}$ and $|b_{\mathcal{T}}\Psi|_{1,\mathrm{pw}}$. Since the definition of I_h and the definition of Π_1^∇ with $\Pi_1 = \Pi_1^\nabla$ in V_h imply $\int_{\partial P}\Psi\,ds = \int_{\partial P}(\Phi-\Pi_1 I_h\Phi)\,ds = 0$, this allows the Poincaré–Friedrichs inequality for Ψ from Lemma 2.1.a on each $P\in\mathcal{T}$. This shows

$$
\|h_{\mathcal{T}}^{-1}\Psi\|_{L^2(\Omega)} \le C_{\mathrm{PF}}|\Psi|_{1,\mathrm{pw}} \le C_{\mathrm{PF}}C_{\mathrm{apx}}h_{\max}^\sigma|\Phi|_{1+\sigma,\Omega}\tag{4.19}
$$

with Theorem 2.8.b and (2.12) in the last inequality. Since $b_P\Psi_1 \in H_0^1(P)$ for $P\in\mathcal{T}$, the Poincaré–Friedrichs inequality from Lemma 2.1.a leads to

$$
\|h_P^{-1}(b_P\Psi_1)\|_{L^2(P)} \le C_{\mathrm{PF}}|b_P\Psi_1|_{1,P}.\tag{4.20}
$$

The first estimate in (3.20), the identity $\Pi_1(b_{\mathcal{T}}\Psi_1)=\Pi_1\Psi$, and the Cauchy–Schwarz inequality imply

$$
\begin{aligned}
C_b^{-1}\|h_P^{-1}\Psi_1\|_{L^2(P)}^2 &\le \|h_P^{-1}b_P^{1/2}\Psi_1\|_{L^2(P)}^2 = (h_P^{-1}\Psi_1,h_P^{-1}\Psi)_{L^2(P)}\\
&\le \|h_P^{-1}\Psi_1\|_{L^2(P)}\|h_P^{-1}\Psi\|_{L^2(P)}.
\end{aligned}
$$

This proves $\|h_P^{-1}\Psi_1\|_{L^2(P)} \le C_b\|h_P^{-1}\Psi\|_{L^2(P)}$. The second estimate in (3.21) followed by the first estimate in (3.20) leads to the first inequality and the arguments as above lead to the second inequality in

$$\begin{aligned} C_b^{-3/2}|b_P\Psi_1|_{1,P} &\le \|h_P^{-1}b_P^{1/2}\Psi_1\|_{L^2(P)} \le \|h_P^{-1}\Psi_1\|_{L^2(P)}^{1/2}\|h_P^{-1}\Psi\|_{L^2(P)}^{1/2} \\ &\le C_b^{1/2}\|h_P^{-1}\Psi\|_{L^2(P)} \end{aligned}$$

with $\|h_P^{-1}\Psi_1\|_{L^2(P)}^{1/2} \le C_b^{1/2}\|h_P^{-1}\Psi\|_{L^2(P)}^{1/2}$ from above in the last step. The combination of the previous displayed estimate and (4.18)–(4.20) results with $C_6 := C_{\mathrm{PF}}C_{\mathrm{apx}}(1 + C_b^2(1 + C_{\mathrm{PF}}))$ in

$$(f - \gamma u, \Psi)_{L^2(\Omega)} \le C_6(\mathrm{osc}_1(f - \gamma u, \mathcal{T}) + \|(1 - \Pi_0)\boldsymbol{\sigma}\|_{L^2(\Omega)})h_{\max}^{\sigma}|\Phi|_{1+\sigma,\Omega}. \tag{4.21}$$

Step 6 (continued proof of estimate for $\|u - u_h\|_{L^2(\Omega)}$). The estimate in Step 2 for $|g|_{1,\mathrm{pw}}$, (4.15)–(4.17), and (4.21) with the regularity (4.13) show

$$\begin{aligned} &\|I_h u - u_h\|_{L^2(\Omega)} \\ &\quad \lesssim h_{\max}^{\sigma}\Big(\|(1 - \Pi_0)\nabla u\|_{L^2(\Omega)} + \|(1 - \Pi_0)\boldsymbol{\sigma}\|_{L^2(\Omega)} + \mathrm{osc}_1(f - \gamma u, \mathcal{T})\Big). \end{aligned} \tag{4.22}$$

Rewrite the difference $u - u_h = (u - I_h u) + (I_h u - u_h)$, and apply the triangle inequality with (2.15) for the first term

$$\|u - I_h u\|_{L^2(\Omega)} \le C_{\mathrm{I}} h_{\max}^{1+\sigma}|u|_{1+\sigma,\Omega}.$$

This and (4.22) for the second term $I_h u - u_h$ conclude the proof of the estimate for the term $h_{\max}^{-\sigma}\|u - u_h\|_{L^2(\Omega)}$ in (4.8) .
Step 7 (stabilisation error $|u_h|_{\mathrm{s}}$ and $|I_h u - u_h|_{\mathrm{s}}$). The triangle inequality and the upper bound of the stability term (3.5) lead to

$$|u_h|_{\mathrm{s}} \le |I_h u - u_h|_{\mathrm{s}} + |I_h u|_{\mathrm{s}} \le C_s^{1/2}\|\mathbf{A}\|_\infty^{1/2}(|I_h u - u_h|_{1,\mathrm{pw}} + |(1 - \Pi_1)I_h u|_{1,\mathrm{pw}})$$

with $|(1 - \Pi_1)(I_h u - u_h)|_{1,\mathrm{pw}} \le |I_h u - u_h|_{1,\mathrm{pw}}$ in the last inequality. The arguments as in (4.12) prove that $|(1 - \Pi_1)I_h u|_{1,\mathrm{pw}} \le (2 + C_{\mathrm{Itn}} + C_{\mathrm{PF}})\|(1 - \Pi_0)\nabla u\|_{L^2(\Omega)}$. This and the arguments in Step 2 for the estimate of $|I_h u - u_h|_{1,\mathrm{pw}}$ show the upper bound in (4.8) for the terms $|u_h|_{\mathrm{s}}$ and $|I_h u - u_h|_{\mathrm{s}}$.
Step 8 (error estimate for $u - \Pi_1 u_h$). The VEM solution u_h is defined by the computed degrees of freedom given in (2.10), but the evaluation of the function itself requires expansive additional calculations. The later are avoided if u_h is replaced by the Ritz projection $\Pi_1 u_h$ in the numerical experiments. The triangle inequality leads to

$$|u - \Pi_1 u_h|_{1,\mathrm{pw}} \le |u - u_h|_{1,\mathrm{pw}} + |u_h - \Pi_1 u_h|_{1,\mathrm{pw}}. \tag{4.23}$$

A lower bound of the stability term (3.5) and the assumption (A2) imply

$$|u_h - \Pi_1 u_h|_{1,P} \le a_0^{-1/2} C_s^{1/2} S^P((1-\Pi_1)u_h, (1-\Pi_1)u_h)^{1/2}. \tag{4.24}$$

This shows that the second term in (4.23) is bounded by $|u_h|_{\mathrm{s}}$. Hence Step 2 and Step 7 prove the estimate for $|u - \Pi_1 u_h|_{1,\mathrm{pw}}$. Since $\int_{\partial P}(u_h - \Pi_1 u_h)\,ds = 0$ from the definition of Π_1^∇ and $\Pi_1 = \Pi_1^\nabla$ in V_h, the combination of Poincaré–Friedrichs inequality for $u_h - \Pi_1 u_h$ from Lemma 2.1.a and (4.24) result in

$$C_{\mathrm{PF}}^{-1} a_0^{1/2} C_s^{-1/2} \|u_h - \Pi_1 u_h\|_{L^2(P)} \le h_P S^P((1-\Pi_1)u_h, (1-\Pi_1)u_h)^{1/2}. \tag{4.25}$$

The analogous arguments for $\|u - \Pi_1 u_h\|_{L^2(\Omega)}$, (4.25), and the estimate for $|u_h|_{\mathrm{s}}$ prove the bound (4.8) for the term $h_{\max}^{-\sigma}\|u - \Pi_1 u_h\|_{L^2(\Omega)}$. This concludes the proof of Theorem 4.3. □

5 A posteriori error analysis

This section presents the reliability and efficiency of a residual-type a posteriori error estimator.

5.1 Residual-based explicit a posteriori error control

Recall $u_h \in V_h$ is the solution to the problem (3.8), and the definition of jump $[\cdot]_E$ along an edge $E \in \mathcal{E}$ from Section 2. For any polygonal domain $P \in \mathcal{T}$, set

$$\begin{aligned}
\eta_P^2 &:= h_P^2 \|f - \gamma \Pi_1 u_h\|_{L^2(P)}^2 && \text{(Volume residual)},\\
\zeta_P^2 &:= S^P((1-\Pi_1)u_h, (1-\Pi_1)u_h) && \text{(Stabilization)},\\
\Lambda_P^2 &:= \|(1-\Pi_0)(\mathbf{A}\nabla\Pi_1 u_h + \mathbf{b}\Pi_1 u_h)\|_{L^2(P)}^2 && \text{(Inconsistency)},\\
\Xi_P^2 &:= \sum_{E\in\mathcal{E}(P)} |E|^{-1}\|[\Pi_1 u_h]_E\|_{L^2(E)}^2 && \text{(Nonconformity)}.
\end{aligned}$$

These local quantities $\bullet|_P$ form a family $(\bullet|_P : P \in \mathcal{T})$ over the index set $\mathcal{T}$ and their Euclid vector norm $\bullet|_{\mathcal{T}}$ enters the upper error bound: $\eta_{\mathcal{T}} := (\sum_{P\in\mathcal{T}} \eta_P^2)^{1/2}$, $\zeta_{\mathcal{T}} := (\sum_{P\in\mathcal{T}} \zeta_P^2)^{1/2}$, $\Lambda_{\mathcal{T}} := (\sum_{P\in\mathcal{T}} \Lambda_P^2)^{1/2}$, and $\Xi_{\mathcal{T}} := (\sum_{P\in\mathcal{T}} \Xi_P^2)^{1/2}$. The following theorem provides an upper bound to the error $u - u_h$ in the H^1 and the L^2 norm. Recall the elliptic regularity (1.5) with the index $0 < \sigma \le 1$, and recall the assumption $h_{\max} \le 1$ from Sect. 2.1.

Theorem 5.1 (reliability) *There exist positive constants C_{rel1} and C_{rel2} (both depending on ρ) such that*

$$C_{\mathrm{rel1}}^{-2} |u - u_h|_{1,pw}^2 \le \eta_{\mathcal{T}}^2 + \zeta_{\mathcal{T}}^2 + \Lambda_{\mathcal{T}}^2 + \Xi_{\mathcal{T}}^2 \tag{5.1}$$

and
$$\|u - u_h\|^2_{L^2(\Omega)} \le C^2_{\mathrm{rel2}} \sum_{P\in\mathcal{T}} \Big(h_P^{2\sigma} (\eta_P^2 + \zeta_P^2 + \Lambda_P^2 + \Xi_P^2) \Big). \tag{5.2}$$

The proof of this theorem in Sect. 5.3 relies on a conforming companion operator elaborated in the next subsection. The upper bound in Theorem 5.1 is efficient in the following local sense, where $\omega_E := \mathrm{int}(\cup\mathcal{T}(E))$ denotes the patch of an edge E and consists of the one or the two neighbouring polygons in the set $\mathcal{T}(E) := \{P' \in \mathcal{T} : E \subset \partial P'\}$ that share E. Recall $\boldsymbol{\sigma} = \mathbf{A}\nabla u + \mathbf{b}u$ from Sect. 4.2 and the data-oscillation $\mathrm{osc}_1(f, P) := \|h_P(1 - \Pi_1)f\|_{L^2(P)}$ from Sect. 2.1.

Theorem 5.2 (local efficiency up to oscillation) *The quantities η_P, ζ_P, Λ_P, and Ξ_P from Theorem 5.1 satisfy*

$$\zeta_P^2 \lesssim |u - u_h|^2_{1,P} + |u - \Pi_1 u_h|^2_{1,P}, \tag{5.3}$$
$$\eta_P^2 \lesssim \|u - u_h\|^2_{1,P} + |u - \Pi_1 u_h|^2_{1,P} + \|(1 - \Pi_0)\boldsymbol{\sigma}\|^2_{L^2(P)} + \mathrm{osc}_1^2(f - \gamma u, P), \tag{5.4}$$
$$\Lambda_P^2 \lesssim \|u - u_h\|^2_{1,P} + |u - \Pi_1 u_h|^2_{1,P} + \|(1 - \Pi_0)\boldsymbol{\sigma}\|^2_{L^2(P)}, \tag{5.5}$$
$$\Xi_P^2 \lesssim \sum_{E\in\mathcal{E}(P)} \sum_{P'\in\omega_E} (\|u - u_h\|^2_{1,P'} + |u - \Pi_1 u_h|^2_{1,P'}). \tag{5.6}$$

The proof of Theorem 5.2 follows in Sect. 5.4. The reliability and efficiency estimates in Theorem 5.1 and 5.2 lead to an equivalence up to the approximation term

$$\mathrm{apx} := \|\boldsymbol{\sigma} - \Pi_0\boldsymbol{\sigma}\|_{L^2(\Omega)} + \mathrm{osc}_1(f - \gamma u, \mathcal{T}).$$

Recall the definition of $|u_h|_{\mathrm{s}}$ from Sect. 4.2. In this paper, the norm $|\cdot|_{1,\mathrm{pw}}$ in the nonconforming space V_h has been utilised for simplicity and one alternative is the norm $\|\cdot\|_h$ from Remark 6 induced by a_h. Then it appears natural to have the total error with the stabilisation term as

$$\begin{aligned}\text{total error} := |u - u_h|_{1,\mathrm{pw}} + |u - \Pi_1 u_h|_{1,\mathrm{pw}} + h_{\max}^{-\sigma}\|u - u_h\|_{L^2(\Omega)} \\ + h_{\max}^{-\sigma}\|u - \Pi_1 u_h\|_{L^2(\Omega)} + |u_h|_{\mathrm{s}}.\end{aligned}$$

The point is that Theorem 4.3 assures that total error + apx converges with the expected optimal convergence rate.

Corollary 5.3 *(equivalence) The* estimator $:= \eta_{\mathcal{T}} + \zeta_{\mathcal{T}} + \Lambda_{\mathcal{T}} + \Xi_{\mathcal{T}} \approx$ total error + apx.

Proof Theorem 5.2 motivates apx and shows

$$\begin{aligned}\text{estimator} &\lesssim \|u - u_h\|_{1,\mathrm{pw}} + \|\boldsymbol{\sigma} - \Pi_0\boldsymbol{\sigma}\|_{L^2(\Omega)} + \mathrm{osc}_1(f - \gamma u, \mathcal{T}) + |u_h|_{\mathrm{s}} \\ &\le \text{total error} + \text{apx}.\end{aligned}$$

This proves the first inequality $\lesssim$ in the assertion. Theorem 5.1, the estimates in Sect. 5.3.3.1, and the definition of $|u_h|_s$ show total error $\lesssim$ estimator. The first of the terms in apx is

$$\begin{aligned} &\|\boldsymbol{\sigma} - \Pi_0\boldsymbol{\sigma}\|_{L^2(\Omega)} \\ &\quad \leq \|\boldsymbol{\sigma} - \Pi_0\boldsymbol{\sigma}_h\|_{L^2(\Omega)} \leq \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{L^2(\Omega)} + \|(1 - \Pi_0)\boldsymbol{\sigma}_h\|_{L^2(\Omega)}. \end{aligned}$$

The definition of $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}_h$ plus the triangle and the Cauchy–Schwarz inequality show

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{L^2(\Omega)} \leq \|\mathbf{A}\|_\infty |u - \Pi_1 u_h|_{1,\mathrm{pw}} + \|\mathbf{b}\|_\infty \|u - \Pi_1 u_h\|_{L^2(\Omega)} \lesssim \|u - \Pi_1 u_h\|_{1,\mathrm{pw}}.$$

The upper bound is $\lesssim$ estimator as mentioned above. Since the term $\|(1 - \Pi_0)\boldsymbol{\sigma}_h\|_{L^2(\Omega)} = \Lambda_{\mathcal{T}}$ is a part of the estimator, $\|(1 - \Pi_0)\boldsymbol{\sigma}\|_{L^2(\Omega)} \lesssim$ estimator. The other term in apx is

$$\begin{aligned} \mathrm{osc}_1(f - \gamma u, \mathcal{T}) &\leq \mathrm{osc}_1(f - \gamma \Pi_1 u_h, \mathcal{T}) + \|h_{\mathcal{T}}\gamma(u - \Pi_1 u_h)\|_{L^2(\Omega)} \\ &\leq \eta_{\mathcal{T}} + \|\gamma\|_\infty h_{\max} \|u - \Pi_1 u_h\|_{L^2(\Omega)} \lesssim \text{estimator}. \end{aligned}$$

□

Section 5 establishes the a posteriori error analysis of the nonconforming VEM. Related results are known for the conforming VEM and the nonconforming FEM.

Remark 7 (comparison with nonconforming FEM) Theorem 5.1 generalizes a result for the nonconforming FEM in [19,Thm. 3.4] from triangulations into triangles to those in polygons (recall Example 2.2). The only difference is the extra stabilization term that can be dropped in the nonconforming FEM.

Remark 8 (comparison with conforming VEM) The volume residual, the inconsistency term, and the stabilization also arise in the a posteriori error estimator for the conforming VEM in [16,Thm. 13]. But it also includes an additional term with normal jumps compared to the estimator (5.1). The extra nonconformity term in this paper is caused by the nonconformity $V_h \not\subset V$ in general.

5.2 Enrichment and conforming companion operator

The link from the nonconforming approximation $u_h \in V_h$ to a global Sobolev function in $H_0^1(\Omega)$ can be designed with the help of the underlying refinement $\widehat{\mathcal{T}}$ of the triangulation $\mathcal{T}$ (from Sect. 2). The interpolation $I_{\mathrm{CR}} : V + V_h \to \mathrm{CR}_0^1(\widehat{\mathcal{T}})$ in the Crouzeix-Raviart finite element space $\mathrm{CR}_0^1(\widehat{\mathcal{T}})$ from Sect. 3.4 allows for a right-inverse J'. A companion operator $J' \circ I_{\mathrm{CR}} : V_h \to H_0^1(\Omega)$ acts as displayed

$$V_h \xrightarrow{\;\;I_{\mathrm{CR}}\;\;} \mathrm{CR}_0^1(\widehat{\mathcal{T}}) \xrightarrow{\;\;J'\;\;} H_0^1(\Omega)$$

$$V_h \xrightarrow{\;\;I_{\mathrm{CR}}\;\;} \mathrm{CR}_0^1(\widehat{\mathcal{T}}) \xrightarrow{\;\;J'\;\;} H_0^1(\Omega)$$

Define an enrichment operator $E_{\rm pw}:\mathcal{P}_1(\widehat{\mathcal{T}})\to S_0^1(\widehat{\mathcal{T}})$ by averaging nodal values: For any vertex z in the refined triangulation $\widehat{\mathcal{T}}$, let $\widehat{\mathcal{T}}(z)=\{T\in\widehat{\mathcal{T}}: z\in T\}$ denote the set of $|\widehat{\mathcal{T}}(z)|\geq 1$ many triangles that share the vertex z, and define

$$E_{\rm pw}v_1(z)=\frac{1}{|\widehat{\mathcal{T}}(z)|}\sum_{T\in\widehat{\mathcal{T}}(z)} v_1|_T(z)$$

for an interior vertex z (and zero for a boundary vertex z according to the homogeneous boundary conditions). This defines $E_{\rm pw}v_1$ at any vertex of a triangle T in $\widehat{\mathcal{T}}$, and linear interpolation then defines $E_{\rm pw}v_1$ in $T\in\widehat{\mathcal{T}}$, so that $E_{\rm pw}v_1\in S_0^1(\widehat{\mathcal{T}})$. Huang et al. [31] design an enrichment operator by an extension of [32] to polygonal domains, while we deduce it from a sub-triangulation. The following lemma provides an approximation property of the operator $E_{\rm pw}$.

Lemma 5.4 *There exists a positive constant C_{En} that depends only on the shape regularity of $\widehat{\mathcal{T}}$ such that any $v_1\in\mathcal{P}_1(\mathcal{T})$ satisfies*

$$\|h_{\mathcal{T}}^{-1}(1-E_{pw})v_1\|_{L^2(\Omega)}+|(1-E_{pw})v_1|_{1,pw}\leq C_{\rm En}\left(\sum_{E\in\mathcal{E}}|E|^{-1}\|[v_1]_E\|^2_{L^2(E)}\right)^{1/2}. \tag{5.7}$$

Proof There exists a positive constant C_{En} independent of h and v_1 [32,p. 2378] such that

$$\|h_{\widehat{\mathcal{T}}}^{-1}(1-E_{\rm pw})v_1\|_{L^2(\Omega)}+\left(\sum_{T\in\widehat{\mathcal{T}}}\|\nabla(1-E_{\rm pw})v_1\|^2_{L^2(T)}\right)^{1/2}$$
$$\leq C_{\rm En}\left(\sum_{E\in\widehat{\mathcal{E}}}|E|^{-1}\|[v_1]_E\|^2_{L^2(E)}\right)^{1/2}.$$

Note that any edge $E\in\mathcal{E}$ is unrefined in the sub-triangulation $\widehat{\mathcal{T}}$. Since $v_{1|P}\in H^1(P)$ is continuous in each polygonal domain $P\in\mathcal{T}$ and $h_T\leq h_P$ for all $T\in\widehat{\mathcal{T}}(P)$, the above inequality reduces to (5.7). This concludes the proof. □

Recall the L^2 projection Π_1 onto the piecewise affine functions $\mathcal{P}_1(\mathcal{T})$ from Sect. 2. An enrichment operator $E_{\rm pw}\circ\Pi_1: V_h\to H_0^1(\Omega)$ acts as displayed

$$V_h\xrightarrow{\Pi_1}\mathcal{P}_1(\mathcal{T})\hookrightarrow\mathcal{P}_1(\widehat{\mathcal{T}})\xrightarrow{E_{\rm pw}}H_0^1(\Omega)$$

5.3 Proof of Theorem 5.1

5.3.1 Reliable H^1 error control

Define $E_1 u_h := E_{\mathrm{pw}} \Pi_1 u_h \in H_0^1(\Omega)$ so that $u - E_1 u_h \in H_0^1(\Omega)$. The inf-sup condition (1.9) leads to some $v \in H_0^1(\Omega)$ with $\|v\|_{1,\Omega} \le 1$ and

$$\beta_0 \|u - E_1 u_h\|_{1,\Omega} = B(u - E_1 u_h, v) = ((f, v)_{L^2(\Omega)} - B_{\mathrm{pw}}(\Pi_1 u_h, v)) \\ + B_{\mathrm{pw}}(\Pi_1 u_h - E_1 u_h, v) \tag{5.8}$$

with $B(u, v) = (f, v)_{L^2(\Omega)}$ from (1.8) and the piecewise version B_{pw} of B in the last step. The definition of B_h from Sect. 3.1 and the discrete problem (3.8) with $v_h = I_h v$ imply

$$B_{\mathrm{pw}}(\Pi_1 u_h, \Pi_1 I_h v) + s_h((1 - \Pi_1) u_h, (1 - \Pi_1) I_h v) \\ = B_h(u_h, I_h v) = (f, \Pi_1 I_h v)_{L^2(\Omega)}. \tag{5.9}$$

Abbreviate $w := v - \Pi_1 I_h v$ and $\boldsymbol{\sigma}_h := \mathbf{A} \nabla_{\mathrm{pw}} \Pi_1 u_h + \mathbf{b} \Pi_1 u_h$. This and (5.9) simplify

$$(f, v)_{L^2(\Omega)} - B_{\mathrm{pw}}(\Pi_1 u_h, v) = (f, w)_{L^2(\Omega)} - B_{\mathrm{pw}}(\Pi_1 u_h, w) \\ + s_h((1 - \Pi_1) u_h, (1 - \Pi_1) I_h v) \\ = (f - \gamma \Pi_1 u_h, w)_{L^2(\Omega)} - ((1 - \Pi_0) \boldsymbol{\sigma}_h, \nabla_{\mathrm{pw}} w)_{L^2(\Omega)} \\ + s_h((1 - \Pi_1) u_h, (1 - \Pi_1) I_h v) \tag{5.10}$$

with $\int_P \nabla w \, dx = 0$ for any $P \in \mathcal{T}$ from (2.17) in the last step. Recall the notation η_P, Λ_P, and ζ_P from Sect. 5.1. The Cauchy–Schwarz inequality and Theorem 2.8.b followed by $\|(1 - \Pi_0) \nabla v\|_{L^2(\Omega)} \le |v|_{1,\Omega} \le 1$ in the second step show

$$(f - \gamma \Pi_1 u_h, w)_{L^2(P)} \le \eta_P h_P^{-1} \|w\|_{L^2(P)} \le (1 + C_{\mathrm{PF}}) \eta_P, \tag{5.11}$$

$$((1 - \Pi_0) \boldsymbol{\sigma}_h, \nabla w)_{L^2(P)} \le \Lambda_P |w|_{1,P} \le (1 + C_{\mathrm{PF}}) \Lambda_P. \tag{5.12}$$

The upper bound $\|\mathbf{A}\|_\infty$ of the coefficient $\mathbf{A}$, (3.5), and the Cauchy–Schwarz inequality for the stabilization term lead to the first inequality in

$$C_s^{-1/2} S^P((1 - \Pi_1) u_h, (1 - \Pi_1) I_h v) \\ \le \|\mathbf{A}\|_\infty^{1/2} S^P((1 - \Pi_1) u_h, (1 - \Pi_1) u_h)^{1/2} |(1 - \Pi_1) I_h v|_{1,P} \\ \le \|\mathbf{A}\|_\infty^{1/2} (2 + C_{\mathrm{PF}} + C_{\mathrm{Itn}}) \zeta_P. \tag{5.13}$$

The second inequality in (5.13) follows as in (4.3) and with $\|(1 - \Pi_0) \nabla v\|_{L^2(P)} \le 1$. Recall the boundedness constant M_b of B_{pw} from Sect. 4.1 and deduce from (5.7) and the definition of $\Xi_{\mathcal{T}}$ from Sect. 5.1 that

$$B_{\mathrm{pw}}(\Pi_1 u_h - E_1 u_h, v) \le M_b |\Pi_1 u_h - E_1 u_h|_{1,\mathrm{pw}} \le M_b C_{\mathrm{En}} \Xi_{\mathcal{T}}. \tag{5.14}$$

The substitution of (5.10)–(5.14) in (5.8) reveals that

$$\|u - E_1 u_h\|_{1,\Omega} \leq C_7(\eta_{\mathcal{T}} + \Lambda_{\mathcal{T}} + \zeta_{\mathcal{T}} + \Xi_{\mathcal{T}}) \tag{5.15}$$

with $\beta_0 C_7 = 1 + C_{\mathrm{PF}} + C_s^{1/2}\|\mathbf{A}\|_\infty^{1/2}(2 + C_{\mathrm{PF}} + C_{\mathrm{Itn}}) + M_b C_{\mathrm{En}}$. The combination of (4.24), (5.15) and (5.7) leads in the triangle inequality

$$|u - u_h|_{1,\mathrm{pw}} \leq |u - E_1 u_h|_{1,\Omega} + |E_1 u_h - \Pi_1 u_h|_{1,\mathrm{pw}} + |\Pi_1 u_h - u_h|_{1,\mathrm{pw}}$$

to (5.1) with $C_{\mathrm{rel1}}/2 = C_7 + C_{\mathrm{En}} + a_0^{-1/2} C_s^{1/2}$.

5.3.2 Reliable L^2 error control

Recall I_{CR} from (3.14) and J' from the proof of Lemma 3.3, and define $E_2 u_h := J' I_{\mathrm{CR}} u_h \in H_0^1(\Omega)$ from Sect. 5.2. Let $\Psi \in H_0^1(\Omega) \cap H^{1+\sigma}(\Omega)$ solve the dual problem $B(v, \Psi) = (u - E_2 u_h, v)_{L^2(\Omega)}$ for all $v \in V$ and recall (from (1.5)) the regularity estimate

$$\|\Psi\|_{1+\sigma,\Omega} \leq C^*_{\mathrm{reg}} \|u - E_2 u_h\|_{L^2(\Omega)}. \tag{5.16}$$

The substitution of $v := u - E_2 u_h \in V$ in the dual problem shows

$$\|u - E_2 u_h\|^2_{L^2(\Omega)} = B(u - E_2 u_h, \Psi).$$

The algebra in (5.8)–(5.10) above leads with $v = \Psi$ to the identity

$$\begin{aligned}
&\|u - E_2 u_h\|^2_{L^2(\Omega)} - s_h((1 - \Pi_1)u_h, (1 - \Pi_1) I_h \Psi) \\
&\quad = (f - \gamma \Pi_1 u_h, \Psi - \Pi_1 I_h \Psi)_{L^2(\Omega)} - ((1 - \Pi_0)\boldsymbol{\sigma}_h, \nabla_{\mathrm{pw}}(\Psi - \Pi_1 I_h \Psi))_{L^2(\Omega)} \\
&\qquad + B_{\mathrm{pw}}(\Pi_1 u_h - E_2 u_h, \Psi).
\end{aligned} \tag{5.17}$$

The definition of I_{CR} and J' proves the first and second equality in

$$\int_E u_h \, ds = \int_E I_{\mathrm{CR}} u_h \, ds = \int_E E_2 u_h \, ds \quad \text{for all } E \in \mathcal{E}.$$

This and an integration by parts imply $\int_P \nabla(u_h - E_2 u_h)\, dx = 0$ for all $P \in \mathcal{T}$. Hence Definition 2.2 of Ritz projection $\Pi_1^\nabla = \Pi_1$ in V_h shows $\int_P \nabla(\Pi_1 u_h - E_2 u_h)\, ds = 0$ for all $P \in \mathcal{T}$. This L^2 orthogonality $\nabla_{\mathrm{pw}}(\Pi_1 u_h - E_2 u_h) \perp \mathcal{P}_0(\mathcal{T}; \mathbb{R}^2)$ and the definition of B_{pw} in the last term of (5.17) result with elementary algebra in

$$\begin{aligned}
B_{\mathrm{pw}}(\Pi_1 u_h - E_2 u_h, \Psi) &= ((\mathbf{A} - \Pi_0 \mathbf{A})\nabla_{\mathrm{pw}}(\Pi_1 u_h - E_2 u_h), \nabla \Psi)_{L^2(\Omega)} \\
&\quad + (\nabla_{\mathrm{pw}}(\Pi_1 u_h - E_2 u_h), (\Pi_0 \mathbf{A})(1 - \Pi_0)\nabla \Psi)_{L^2(\Omega)} \\
&\quad + (\Pi_1 u_h - E_2 u_h, \mathbf{b} \cdot \nabla \Psi + \gamma \Psi)_{L^2(\Omega)}.
\end{aligned} \tag{5.18}$$

The triangle inequality and (c) from the proof of Lemma 3.3 imply the first inequality in

$$\begin{aligned}|\Pi_1 u_h - E_2 u_h|_{1,\mathrm{pw}} &\le |\Pi_1 u_h - I_{\mathrm{CR}} u_h|_{1,\mathrm{pw}} + C_{\mathrm{J}'} \min_{v \in V} |I_{\mathrm{CR}} u_h - v|_{1,\mathrm{pw}} \\ &\le |\Pi_1 u_h - I_{\mathrm{CR}} u_h|_{1,\mathrm{pw}} + C_{\mathrm{J}'} |I_{\mathrm{CR}} u_h - E_1 u_h|_{1,\mathrm{pw}} \\ &\le |\Pi_1 u_h - I_{\mathrm{CR}} u_h|_{1,\mathrm{pw}} + C_{\mathrm{J}'} (|I_{\mathrm{CR}} u_h - \Pi_1 u_h|_{1,\mathrm{pw}} \\ &\quad + |\Pi_1 u_h - E_1 u_h|_{1,\mathrm{pw}}) \\ &\le (1 + C_{\mathrm{J}'}) |u_h - \Pi_1 u_h|_{1,\mathrm{pw}} + C_{\mathrm{J}'} |\Pi_1 u_h - E_1 u_h|_{1,\mathrm{pw}}. \end{aligned} \tag{5.19}$$

The second estimate in (5.19) follows from $E_1 u_h \in V$, the third is a triangle inequality, and eventually $|\Pi_1 u_h - I_{\mathrm{CR}} u_h|_{1,\mathrm{pw}} \le |u_h - \Pi_1 u_h|_{1,\mathrm{pw}}$ results from the orthogonality $\nabla_{\mathrm{pw}}(u_h - I_{\mathrm{CR}}) \perp \mathcal{P}_0(\widehat{\mathcal{T}}; \mathbb{R}^2)$ and $\Pi_1 u_h \in \mathcal{P}_1(\mathcal{T})$. The Cauchy–Schwarz inequality, the Lipschitz continuity of $\mathbf{A}$, and the approximation estimate $\|(1-\Pi_0)\nabla\Psi\|_{L^2(P)} \le C_{\mathrm{apx}} h_P^\sigma |\Psi|_{1+\sigma,P}$ in (5.18) lead to the first inequality in

$$\begin{aligned} B_{\mathrm{pw}}(\Pi_1 u_h - E_2 u_h, \Psi) \le \sum_{P \in \mathcal{T}} \Big(&(h_P |\mathbf{A}|_{1,\infty} + \|\mathbf{A}\|_\infty C_{\mathrm{apx}} h_P^\sigma) |\Pi_1 u_h - E_2 u_h|_{1,P} \\ &+ \|\Pi_1 u_h - E_2 u_h\|_{L^2(P)} (\|\mathbf{b}\|_\infty + \|\gamma\|_\infty) \Big) \|\Psi\|_{1+\sigma,P} \\ \le \sum_{P \in \mathcal{T}} &\Big(h_P |\mathbf{A}|_{1,\infty} + \|\mathbf{A}\|_\infty C_{\mathrm{apx}} h_P^\sigma + C_{\mathrm{PF}} (\|\mathbf{b}\|_\infty + \|\gamma\|_\infty) h_P \Big) \\ &|\Pi_1 u_h - E_2 u_h|_{1,P} \|\Psi\|_{1+\sigma,P} \\ \le C_8 \sum_{P \in \mathcal{T}} &h_P^\sigma ((1 + C_{\mathrm{J}'}) |u_h - \Pi_1 u_h|_{1,P} \\ &+ C_{\mathrm{J}'} |\Pi_1 u_h - E_1 u_h|_{1,P}) \|\Psi\|_{1+\sigma,P}. \end{aligned} \tag{5.20}$$

The second inequality in (5.20) follows from the Poincaré–Friedrichs inequality in Lemma 2.1.a for $\Pi_1 u_h - E_2 u_h$ with $\int_{\partial P} (\Pi_1 u_h - E_2 u_h)\, ds = 0$ (from above); the constant $C_8 := |\mathbf{A}|_{1,\infty} + C_{\mathrm{apx}} \|\mathbf{A}\|_\infty + C_{\mathrm{PF}}(\|\mathbf{b}\|_\infty + \|\gamma\|_\infty)$ results from (5.19) and $h_P \le h_P^\sigma$ (recall $h_{\max} \le 1$). Lemma 5.4 with $v_1 = \Pi_1 u_h$ and (4.24) in (5.20) show

$$\begin{aligned} &B_{\mathrm{pw}}(\Pi_1 u_h - E_2 u_h, \Psi) \\ &\quad \le C_8 \sum_{P \in \mathcal{T}} h_P^\sigma ((1 + C_{\mathrm{J}'}) a_0^{-1/2} C_s^{1/2} \zeta_P + C_{\mathrm{J}'} C_{\mathrm{En}} \Xi_P) \|\Psi\|_{1+\sigma,P}. \end{aligned} \tag{5.21}$$

Rewrite (5.11)–(5.13) with $w = \Psi - \Pi_1 I_h \Psi$ and $h_P^{-1} \|w\|_{L^2(P)} + |w|_{1,P} \le (1 + C_{\mathrm{PF}}) \|(1 - \Pi_0)\nabla\Psi\|_{L^2(P)} \le C_{\mathrm{apx}} (1 + C_{\mathrm{PF}}) h_P^\sigma |\Psi|_{1+\sigma,P}$ from (2.12). This and (5.21) lead in (5.17) to

$$\|u - E_2 u_h\|_{L^2(\Omega)}^2 \le C_9 \sum_{P \in \mathcal{T}} h_P^\sigma (\eta_P + \zeta_P + \Lambda_P + \Xi_P) \|\Psi\|_{1+\sigma,P}$$

for $C_9 := C_{\text{apx}}(1 + C_{\text{PF}} + C_s^{1/2}\|\mathbf{A}\|_\infty^{1/2}(2 + C_{\text{PF}} + C_{\text{Itn}})) + C_8((1 + C_{\text{J}'})a_0^{-1/2}C_s^{1/2} + C_{\text{J}'}C_{\text{En}})$. This and the regularity (5.16) result in

$$\|u - E_2u_h\|_{L^2(\Omega)} \le C_9C^*_{\text{reg}}\sum_{P\in\mathcal{T}} h_P^\sigma(\eta_P + \zeta_P + \Lambda_P + \Xi_P). \tag{5.22}$$

The arguments in the proof of (5.20)–(5.21) also lead to

$$\|E_2u_h - \Pi_1u_h\|_{L^2(\Omega)} \le C_{\text{PF}}((1 + C_{\text{J}'})a_0^{-1/2}C_s^{1/2} + C_{\text{J}'}C_{\text{En}})\sum_{P\in\mathcal{T}} h_P(\zeta_P + \Xi_P). \tag{5.23}$$

The combination of (4.25), (5.22)–(5.23) and the triangle inequality

$$\|u - u_h\|_{L^2(\Omega)} \le \|u - E_2u_h\|_{L^2(\Omega)} + \|E_2u_h - \Pi_1u_h\|_{L^2(\Omega)} + \|\Pi_1u_h - u_h\|_{L^2(\Omega)}$$

lead to (5.2) with $C_{rel2}/2 = C_9C^*_{\text{reg}} + C_{\text{PF}}\big((2 + C_{\text{J}'})a_0^{-1/2}C_s^{1/2} + C_{\text{J}'}C_{\text{En}}\big)$. This concludes the proof of the L^2 error estimate in Theorem 5.1.

5.3.3 Comments

5.3.3.1 Estimator for $u - \Pi_1u_h$

The triangle inequality with (5.1) and (4.24) provide an upper bound for H^1 error

$$\frac{1}{2}|u - \Pi_1u_h|^2_{1,\text{pw}} \le |u - u_h|^2_{1,\text{pw}} + |(1 - \Pi_1)u_h|^2_{1,\text{pw}} \le 2C^2_{\text{rel1}}(\eta^2_{\mathcal{T}} + \zeta^2_{\mathcal{T}} + \Lambda^2_{\mathcal{T}} + \Xi^2_{\mathcal{T}}).$$

The same arguments for an upper bound of the L^2 error in Theorem 5.1 show that

$$\begin{aligned}\frac{1}{2}\|u - \Pi_1u_h\|^2_{L^2(\Omega)} &\le \|u - u_h\|^2_{L^2(\Omega)} + \|(1 - \Pi_1)u_h\|^2_{L^2(\Omega)}\\ &\le C^2_{\text{rel2}}\sum_{P\in\mathcal{T}} h_P^{2\sigma}(\eta_P^2 + 2\zeta_P^2 + \Lambda_P^2 + \Xi_P^2).\end{aligned}$$

The numerical experiments do not display C_{rel1} and C_{rel2}, and directly compare the error $H1e := |u - \Pi_1u_h|_{1,\text{pw}}$ in the piecewise H^1 norm and the error $L2e := \|u - \Pi_1u_h\|_{L^2(\Omega)}$ in the L^2 norm with the upper bound $H1\mu$ and $L2\mu$ (see, e.g., Fig. 5).

5.3.3.2 Motivation and discussion of apx

We first argue that those extra terms have to be expected and utilize the abbreviations $\boldsymbol{\sigma} := \mathbf{A}\nabla u + \mathbf{b}u$ and $g := f - \gamma u$ for the exact solution $u \in H^1_0(\Omega)$ to (1.8), which reads

$$(\boldsymbol{\sigma}, \nabla v)_{L^2(\Omega)} = (g, v)_{L^2(\Omega)} \quad \text{for all } v \in H^1_0(\Omega). \tag{5.24}$$

Recall the definition of $s_h(\cdot,\cdot)$ from Sect. 3.1. The discrete problem (3.8) with the discrete solution $u_h \in V_h$ assumes the form

$$(\boldsymbol{\sigma}_h, \nabla\Pi_1 v_h)_{L^2(\Omega)} + s_h((1-\Pi_1)u_h, (1-\Pi_1)v_h) = (g_h, \Pi_1 v_h)_{L^2(\Omega)} \text{ for all } v_h \in V_h \tag{5.25}$$

for $\boldsymbol{\sigma}_h := \mathbf{A}\nabla\Pi_1 u_h + \mathbf{b}\Pi_1 u_h$, and $g_h := f - \gamma\Pi_1 u_h$. Notice that $\boldsymbol{\sigma}_h$ and g_h may be replaced in (5.25) by $\Pi_0\boldsymbol{\sigma}_h$ and $\Pi_1 g_h$ because the test functions $\nabla\Pi_1 v_h$ and $\Pi_1 v_h$ belong to $\mathcal{P}_0(\mathcal{T};\mathbb{R}^2)$ and $\mathcal{P}_1(\mathcal{T})$ respectively. In other words, the discrete problems (3.8) and (5.25) do not see a difference of $\boldsymbol{\sigma}_h$ and g_h compared to $\Pi_0\boldsymbol{\sigma}_h$ and $\Pi_1 g_h$ and so the errors $\boldsymbol{\sigma}_h - \Pi_0\boldsymbol{\sigma}_h$ and $g_h - \Pi_1 g_h$ may arise in a posteriori error control. This motivates the a posteriori error term $\|\boldsymbol{\sigma}_h - \Pi_0\boldsymbol{\sigma}_h\|_{L^2(\Omega)} = \Lambda_{\mathcal{T}}$ as well as the approximation terms $\boldsymbol{\sigma} - \Pi_0\boldsymbol{\sigma}$ and $g - \Pi_1 g$ on the continuous level. The natural norm for the dual variable $\boldsymbol{\sigma}$ is L^2 and that of g is H^{-1} and hence their norms form the approximation term apx as defined in Sect. 5.1.

Example 5.1 ($\mathbf{b}=0$) The term $(1-\Pi_0)\boldsymbol{\sigma}$ may not be visible in case of no advection $\mathbf{b}=0$ at least if $\mathbf{A}$ is piecewise constant. Suppose $\mathbf{A} \in \mathcal{P}_0(\mathcal{T};\mathbb{R}^{2\times 2})$ and estimate

$$\|(1-\Pi_0)(\mathbf{A}\nabla u)\|_{L^2(\Omega)} \le \|\mathbf{A}\|_\infty \|(1-\Pi_0)\nabla u\|_{L^2(\Omega)} \lesssim |u - \Pi_1 u_h|_{1,\mathrm{pw}}.$$

If $\mathbf{A}$ is not constant, there are oscillation terms that can be treated properly in adaptive mesh-refining algorithms, e.g., in [27].

Example 5.2 (γ piecewise constant) While the data approximation term $\mathrm{osc}_1(f,\mathcal{T})$ [10] is widely accepted as a part of the total error in the approximation of nonlinear problems, the term $\mathrm{osc}_1(\gamma u,\mathcal{T}) = \|\gamma h_{\mathcal{T}}(u - \Pi_1 u)\|_{L^2(\Omega)} \lesssim h_{\max}^{1+\sigma}\|f\|_{L^2(\Omega)}$ is of higher order and may even be absorbed in the overall error analysis for a piecewise constant coefficient $\gamma \in \mathcal{P}_0(\mathcal{T})$. In the general case $\gamma \in L^\infty(\Omega)\backslash\mathcal{P}_0(\mathcal{T})$, however, $\mathrm{osc}_1(u,\mathcal{T})$ leads in particular to terms with $\|\gamma - \Pi_0\gamma\|_{L^\infty(\Omega)}$.

5.3.3.3 Higher-order nonconforming VEM

The analysis applied in Theorem 5.1 can be extended to the nonconforming VEM space of higher order $k \in \mathbb{N}$ (see [17, Sec. 4] for the definition of discrete space). Since the projection operators $\nabla\Pi_k^\nabla$ and $\Pi_{k-1}\nabla$ are not the same for general k, and the first operator does not lead to optimal order of convergence for $k \ge 3$, the discrete formulation uses $\Pi_{k-1}\nabla$ (cf. [6, Rem. 4.3] for more details). The definition and approximation properties of the averaging operator E_{pw} extend to the operator $E^k : \mathcal{P}_k(\widehat{\mathcal{T}}) \to H_0^1(\Omega)$ (see [32, p. 2378] for a proof). The identity (5.9) does not hold in general, but algebraic calculations lead to

$$\eta_P^2 := h_P^2\|f - \gamma\Pi_k u_h\|_{L^2(P)}^2, \quad \Lambda_P^2 := \|(1-\Pi_{k-1})(\mathbf{A}\Pi_{k-1}\nabla u_h + \mathbf{b}\Pi_k u_h)\|_{L^2(P)}^2$$
$$\zeta_P^2 := S^P((1-\Pi_k)u_h, (1-\Pi_k)u_h), \qquad \Xi_P^2 := \sum_{E\in\mathcal{E}(P)} |E|^{-1}\|[\Pi_k u_h]_E\|_{L^2(E)}^2.$$

The analysis developed for the upper bound of L^2 norm also extends to the general case. The model problem is chosen in 2D for the simplicity of the presentation. The results of this work can be extended to the three-dimensional case with appropriate modifications. The present analysis holds for any higher regularity index $\sigma > 0$ and avoids any trace inequality for higher derivatives. This is possible by a medius analysis in the form of companion operators [26].

5.3.3.4 Inhomogeneous boundary data

The error estimator for general Dirichlet condition $u|_{\partial\Omega} = g \in H^{1/2}(\partial\Omega)$ can be obtained with some modifications of [33] in Theorem 5.1. The only difference is in the modified jump contributions of the boundary edges in the nonconformity term

$$\Xi_{\mathcal{T}}^2 = \sum_{E\in\mathcal{E}(\Omega)} |E|^{-1}\|[\Pi_1 u_h]\|_{L^2(E)}^2 + \sum_{E\in\mathcal{E}(\partial\Omega)} |E|^{-1}\|g - \Pi_1 u_h\|_{L^2(E)}^2.$$

5.4 Proof of Theorem 5.2

Recall the notation $\boldsymbol{\sigma} = \mathbf{A}\nabla u + \mathbf{b}u$ and $\boldsymbol{\sigma}_h = \mathbf{A}\nabla\Pi_1 u_h + \mathbf{b}\Pi_1 u_h$ from Sect. 5.3.

Proof of 5.3 The upper bound (3.5) for the stabilisation term and the triangle inequality show

$$\zeta_P^2 \le C_s|(1-\Pi_1)u_h|_{1,P}^2 \le 2C_s(|u-u_h|_{1,P}^2 + |u-\Pi_1 u_h|_{1,P}^2).$$

This concludes the proof of (5.3). □

Proof of (5.5) The definition of Λ_P, Π_0, and the triangle inequality lead to

$$\begin{aligned}\Lambda_P = \|\boldsymbol{\sigma}_h - \Pi_0\boldsymbol{\sigma}_h\|_{L^2(P)} &\le \|\boldsymbol{\sigma}_h - \Pi_0\boldsymbol{\sigma}\|_{L^2(P)} \\ &\le \|\mathbf{A}\nabla(\Pi_1 u_h - u) + \mathbf{b}(\Pi_1 u_h - u)\|_{L^2(P)} \\ &\quad + \|(1-\Pi_0)\boldsymbol{\sigma}\|_{L^2(P)}. \end{aligned} \tag{5.26}$$

The upper bound $\|\mathbf{A}\|_\infty$ and $\|\mathbf{b}\|_\infty$ for the coefficients and the triangle inequality lead to

$$\begin{aligned}\Lambda_P - \|(1-\Pi_0)\boldsymbol{\sigma}\|_{L^2(P)} &\le (\|\mathbf{A}\|_\infty + \|\mathbf{b}\|_\infty)\|\Pi_1 u_h - u\|_{1,P} \\ &\le (\|\mathbf{A}\|_\infty + \|\mathbf{b}\|_\infty)(\|u_h - \Pi_1 u_h\|_{1,P} + \|u-u_h\|_{1,P}) \le C_{10}(\zeta_P + \|u-u_h\|_{1,P}) \end{aligned} \tag{5.27}$$

with $\|u_h - \Pi_1 u_h\|_{1,P} \leq (1 + h_P C_{\mathrm{PF}}) a_0^{-1/2} C_s^{1/2} \zeta_P$ from (4.24)–(4.25) and with $C_{10} := (\|\mathbf{A}\|_\infty + \|\mathbf{b}\|_\infty)((1 + h_P C_{\mathrm{PF}}) a_0^{-1/2} C_s^{1/2} + 1)$. This followed by (5.3) concludes the proof of (5.5). ◻

Recall the bubble-function $b_{\mathcal{T}}|_P = b_P$ supported on a polygonal domain $P \in \mathcal{T}$ from (3.17) as the sum of interior bubble-functions supported on each triangle $T \in \widehat{\mathcal{T}}(P)$.

Proof of (5.4) Rewrite the term

$$f - \gamma \Pi_1 u_h = \Pi_1(f - \gamma \Pi_1 u_h) + (1 - \Pi_1)(f - \gamma \Pi_1 u_h) =: R + \theta, \tag{5.28}$$

and denote $R_P := R|_P$ and $\theta_P := \theta|_P$. The definition of $B_{\mathrm{pw}}(u - \Pi_1 u_h, v)$ and the weak formulation $B(u, v) = (f, v)_{L^2(\Omega)}$ from (1.8) for any $v \in V$ imply

$$B_{\mathrm{pw}}(u - \Pi_1 u_h, v) + (\boldsymbol{\sigma}_h, \nabla v)_{L^2(\Omega)} = (f - \gamma \Pi_1 u_h, v)_{L^2(\Omega)} = (R + \theta, v)_{L^2(\Omega)}. \tag{5.29}$$

Since $b_P R_P$ belongs to $H_0^1(\Omega)$ (extended by zero outside P), $v := b_P R_P \in V$ is admissible in (5.29). An integration by parts proves that $(\Pi_0 \boldsymbol{\sigma}_h, \nabla(b_P R_P))_{L^2(P)} = 0$. Therefore, (5.29) shows

$$\begin{aligned}(R_P, b_P R_P)_{L^2(P)} = B^P(u - \Pi_1 u_h, b_P R_P) - (\theta_P, b_P R_P)_{L^2(P)} \\ + ((1 - \Pi_0)\boldsymbol{\sigma}_h, \nabla(b_P R_P))_{L^2(P)}.\end{aligned}$$

The substitution of $\chi = R_P = \Pi_1(f - \gamma \Pi_1 u_h)|_P \in \mathcal{P}_1(P)$ in (3.20) and the previous identity with the boundedness of B and the Cauchy–Schwarz inequality lead to the first two estimates in

$$\begin{aligned}C_b^{-1}\|R_P\|_{L^2(P)}^2 &\leq (R_P, b_P R_P)_{L^2(P)} \\ &\leq \Big(M_b |u - \Pi_1 u_h|_{1,P} + \|(1 - \Pi_0)\boldsymbol{\sigma}_h\|_{L^2(P)}\Big)|b_P R_P|_{1,P} + \|\theta_P\|_{L^2(P)} \|b_P R_P\|_{L^2(P)} \\ &\leq C_b\Big(M_b |u - \Pi_1 u_h|_{1,P} + \Lambda_P + h_P \|\theta_P\|_{L^2(P)}\Big) h_P^{-1} \|R_P\|_{L^2(P)}.\end{aligned}$$

The last inequality follows from the definition of Λ_P, and (3.21) with $\chi = R_P$. This proves that $C_b^{-2} h_P \|R_P\|_{L^2(P)} \leq M_b |u - \Pi_1 u_h|_{1,P} + \Lambda_P + h_P \|\theta_P\|_{L^2(P)}$. Recall η_P from Sect. 5.1 and $\eta_P = h_P \|f - \gamma \Pi_1 u_h\|_{L^2(P)} \leq h_P \|R_P\|_{L^2(P)} + h_P \|\theta_P\|_{L^2(P)}$ from the split in (5.28) and the triangle inequality. This and the previous estimate of $h_P \|R_P\|_{L^2(P)}$ show the first estimate in

$$\begin{aligned}\eta_P &\leq C_b^2 (M_b |u - \Pi_1 u_h|_{1,P} + \Lambda_P) + (C_b^2 + 1) h_P \|\theta_P\|_{L^2(P)} \\ &\leq (C_b^2 + 1)\Big(M_b |u - \Pi_1 u_h|_{1,P} + \Lambda_P + h_P \|(f - \gamma \Pi_1 u_h) - \Pi_1(f - \gamma u)\|_{L^2(P)}\Big) \\ &\leq (C_b^2 + 1)\Big((M_b + h_P \|\gamma\|_\infty)\|u - \Pi_1 u_h\|_{1,P} + \Lambda_P + \mathrm{osc}_1(f - \gamma u, P)\Big).\end{aligned}$$

The second step results from the definition of $\theta_P = (1-\Pi_1)(f-\gamma\Pi_1 u_h)|_P$ in (5.28) followed by the L^2 orthogonality of Π_1, and the last step results from an elementary algebra with the triangle inequality and $\operatorname{osc}_1(f-\gamma u, P) = h_P\|(1-\Pi_1)(f-\gamma u)\|_{L^2(P)}$ from Sect. 5.1. The triangle inequality for the term $u-\Pi_1 u_h$ and the estimate of $\|u_h - \Pi_1 u_h\|_{1,P}$ as in (5.27) lead to

$$C_{11}^{-1}\eta_P \le \|u-u_h\|_{1,P} + \zeta_P + \Lambda_P + \operatorname{osc}_1(f-\gamma u, P)$$

with $C_{11} := (C_b^2+1)(M_b+h_P\|\gamma\|_\infty)((1+h_P C_{\mathrm{PF}})a_0^{-1/2}C_s^{1/2})+1)$. The combination of (5.3) and (5.5) in the last displayed estimate concludes the proof of (5.4). ◻

Proof of (5.6) Recall for $u \in H_0^1(\Omega)$ and $u_h \in V_h$ that $⨍_E u\,ds$ and $⨍_E u_h\,ds$ are well defined for all edges $E \in \mathcal{E}$, and so the constant $\alpha_E := ⨍_E(u-u_h)\,ds$ is uniquely defined as well. Since the jump of $u-\alpha_E$ across any edge $E \in \mathcal{E}$ vanishes, $[\Pi_1 u_h]_E = [\Pi_1 u_h - u + \alpha_E]_E$. Recall $\omega_E = \operatorname{int}(P^+ \cup P^-)$ for $E \in \mathcal{E}(\Omega)$ and $\omega_E = \operatorname{int}(P)$ for $E \in \mathcal{E}(\partial\Omega)$ from Sect. 5.1. The trace inequality $\|v\|_{L^2(E)}^2 \le C_T(|E|^{-1}\|v\|_{L^2(\omega_E)}^2 + |E|\,\|\nabla v\|_{L^2(\omega_E)}^2)$ (cf. [13,p. 554]) leads to

$$\begin{aligned}&|E|^{-1/2}\|[\Pi_1 u_h]_E\|_{L^2(E)}\\ &\quad\le C_T\left(|E|^{-1}\|\Pi_1 u_h - u + \alpha_E\|_{L^2(\omega_E)} + \|\nabla_{\mathrm{pw}}(\Pi_1 u_h - u)\|_{L^2(\omega_E)}\right).\end{aligned}$$

This and the triangle inequality show the first estimate in

$$\begin{aligned}&|E|^{-1/2}\|[\Pi_1 u_h]_E\|_{L^2(E)}\\ &\quad\le C_T\Big(|E|^{-1}(\|u_h-\Pi_1 u_h\|_{L^2(\omega_E)} + \|u_h - u + \alpha_E\|_{L^2(\omega_E)})\\ &\qquad + \|\nabla_{\mathrm{pw}}(u_h - \Pi_1 u_h)\|_{L^2(\omega_E)} + \|\nabla_{\mathrm{pw}}(u-u_h)\|_{L^2(\omega_E)}\Big).\end{aligned}\tag{5.30}$$

The estimates (4.24)–(4.25) control the term $\|u_h - \Pi_1 u_h\|_{1,P}$ as in (5.27), and the Poincaré–Friedrichs inequality from Lemma 2.1.b for $u_h - u + \alpha_E$ with $\int_E (u_h - u + \alpha_E)\,ds = 0$ (by the definition of α_E) implies that $\|u_h - u + \alpha_E\|_{L^2(P)} \le C_{\mathrm{PF}} h_P |u_h - u|_{1,P}$. This with the mesh assumption $h_P \le \rho^{-1}|E|$ and (5.30) result in

$$\begin{aligned}|E|^{-1/2}\|[\Pi_1 u_h]_E\|_{L^2(E)} \le C_T((C_{\mathrm{PF}}\rho^{-1}+1)a_0^{-1/2}C_s^{1/2} + C_{\mathrm{PF}} + 1)\\ \sum_{P' \in \omega_E}(\Lambda_{P'} + |u-u_h|_{1,P'}).\end{aligned}$$

Since this holds for any edge $E \in \mathcal{E}(P)$, the sum over all these edges and the bound (5.3) in the above estimate conclude the proof of (5.6). ◻

Remark 9 (convergence rates of L^2 error control for $0 < \sigma \le 1$) The efficiency estimates (5.4)–(5.6) with a multiplication of $h_P^{2\sigma}$ show that the local quantity $h_P^{2\sigma}(\eta_P^2 + \Lambda_P^2 + \Xi_P^2)$ converges to zero with the expected convergence rate.

Remark 10 (efficiency up to stabilisation and oscillation for L^2 error control when $\sigma = 1$) For convex domains and $\sigma = 1$, there is even a local efficiency result that is briefly described in the sequel: The arguments in the above proof of (5.4)–(5.5) lead to

$$h_P^2\eta_P^2 \lesssim \|u-u_h\|_{L^2(P)}^2 + h_P^2(\zeta_P^2 + \mathrm{osc}_1^2(f-\gamma u, P) + \|(1-\Pi_0)\boldsymbol{\sigma}\|_{L^2(P)}^2),$$
$$h_P^2\Lambda_P^2 \lesssim \|u-u_h\|_{L^2(P)}^2 + h_P^2(\zeta_P^2 + \|\mathbf{A}-\Pi_0\mathbf{A}\|_{L^\infty(P)}^2\|f\|_{L^2(\Omega)}^2 + \|(1-\Pi_0)\mathbf{b}u\|_{L^2(P)}^2).$$

The observation $[\Pi_1 u_h]_E = [\Pi_1 u_h - u]_E$ for the term Ξ_P, the trace inequality, and the triangle inequality show, for any $E \in \mathcal{E}$, that

$$|E|^{1/2}\|[\Pi_1 u_h]_E\|_{L^2(E)} \le C_T\Big(\|u_h - \Pi_1 u_h\|_{L^2(\omega_E)} + \|u-u_h\|_{L^2(\omega_E)}$$
$$+|E|(\|\nabla\Pi_1(u-u_h)\|_{L^2(\omega_E)} + \|\nabla(u-\Pi_1 u)\|_{L^2(\omega_E)})\Big).$$

The bound (4.25) for the first term and the inverse estimate $\|\nabla\chi\|_{L^2(P)} \le C_{\mathrm{inv}}h_P^{-1}\|\chi\|_{L^2(P)}$ for $\chi \in \mathcal{P}_k(P)$ for the third term result in

$$|E|^{1/2}\|[\Pi_1 u_h]_E\|_{L^2(E)} \lesssim \|u-u_h\|_{L^2(\omega_E)} + |E|\sum_{P'\in\omega_E}\Big(\|\nabla(1-\Pi_1)u\|_{L^2(P')} + \Lambda_{P'}\Big).$$

The mesh assumption (M2) implies that $h_P^2\,\Xi_P^2 \le \rho^{-1}\sum_{E\in\mathcal{E}(P)}|E|\;\|[\Pi_1 u_h]_E\|_{L^2(E)}^2$. This and the above displayed inequality prove the efficiency estimate for $h_P^2\,\Xi_P^2$.

6 Numerical experiments

This section manifests the performance of the a posteriori error estimator and an associated adaptive mesh-refining algorithm with Dörfler marking [37]. The numerical results investigate three computational benchmarks for the indefinite problem (1.1).

6.1 Adaptive algorithm

Input: initial partition $\mathcal{T}_0$ of Ω.
For $\ell = 0, 1, 2, \ldots$ do

1. **SOLVE**. Compute the discrete solution u_h to (3.8) with respect to $\mathcal{T}_\ell$ for $\ell = 0, 1, 2\ldots$ (cf. [5] for more details on the implementation).
2. **ESTIMATE**. Compute all the four terms $\eta_\ell := \eta_{\mathcal{T}_\ell}$, $\zeta_\ell := \zeta_{\mathcal{T}_\ell}$, $\Lambda_\ell := \Lambda_{\mathcal{T}_\ell}$ and $\Xi_\ell := \Xi_{\mathcal{T}_\ell}$, which add up to the upper bound (5.1).
3. **MARK**. Mark the polygons P in a subset $\mathcal{M}_\ell \subset \mathcal{T}_\ell$ with minimal cardinality and

$$H1\mu_\ell^2 := H1\mu^2(\mathcal{T}_\ell) := \eta_\ell^2 + \zeta_\ell^2 + \Lambda_\ell^2 + \Xi_\ell^2 \le 0.5\sum_{P\in\mathcal{M}_\ell}(\eta_P^2 + \zeta_P^2 + \Lambda_P^2 + \Xi_P^2).$$

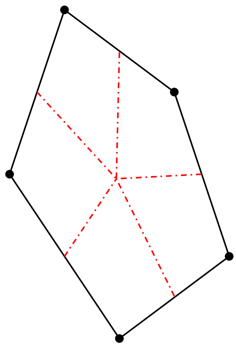

Fig. 3 Refinement of a polygon into quadrilaterals

4. **REFINE** - Refine the marked polygon domains by connecting the mid-point of the edges to the centroid of respective polygon domains and update $\mathcal{T}_\ell$. (cf. Fig. 3 for an illustration of the refinement strategy.)

end do

Output: The sequences $\mathcal{T}_\ell$, and the bounds $\eta_\ell, \zeta_\ell, \Lambda_\ell, \Xi_\ell$, and $H1\mu_\ell$ for $\ell = 0, 1, 2, \ldots$.

The adaptive algorithm is displayed for mesh adaption in the energy error H^1. Replace estimator $H1\mu_\ell$ in the algorithm by $L2\mu_\ell$ (the upper bound in (5.2)) for local mesh-refinement in the L^2 error. Both uniform and adaptive mesh-refinement run to compare the empirical convergence rates and provide numerical evidence for the superiority of adaptive mesh-refinement. Note that uniform refinement means all the polygonal domains are refined. In all examples below, $\overline{\mathbf{A}}_P = 1$ in (3.6). The numerical realizations are based on a MATLAB implementation explained in [35] with a Gauss-like cubature formula over polygons. The cubature formula is exact for all bivariate polynomials of degree at most $2n-1$, so the choice $n \geq (k+1)/2$ leads to integrate a polynomial of degree k exactly. The quadrature errors in the computation of examples presented below appear negligible for the input parameter $n = 5$.

6.2 Square domain (smooth solution)

This subsection discusses the problem (1.1) with the coefficients $\mathbf{A} = I$, $\mathbf{b} = (x, y)$ and $\gamma = x^2 + y^3$ on a square domain $\Omega = (0, 1)^2$, and the exact solution

$$u = 16x(1-x)y(1-y)\arctan(25x - 100y + 50)$$

with $f = \mathcal{L}u$. Since $\gamma - \frac{1}{2}\mathrm{div}(\mathbf{b}) = x^2 + y^3 - 1$ is not always positive on Ω, this is an indefinite problem. Initially, the error and the estimators are large because of an internal layer around the line $25x - 100y + 50 = 0$ with large first derivative of u resolved after few refinements as displayed in Fig. 4-5.

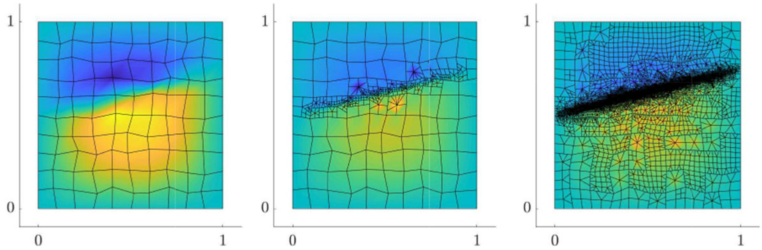

Fig. 4 Output $\mathcal{T}_1$, $\mathcal{T}_8$, $\mathcal{T}_{15}$ of the adaptive algorithm

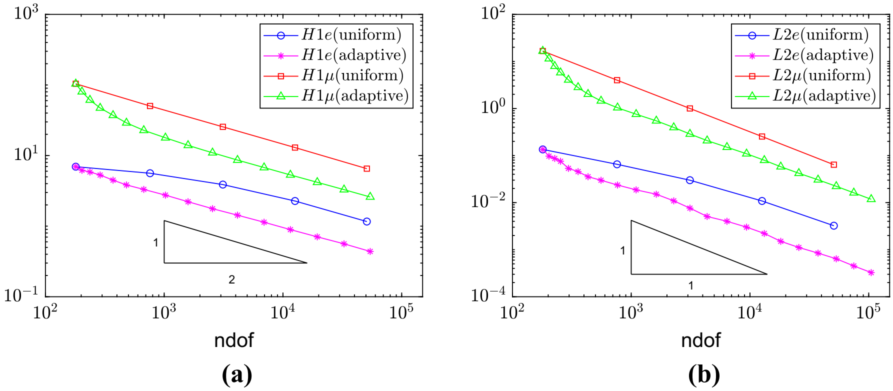

Fig. 5 Convergence history plot of estimator μ and error $e := u - \Pi_1 u_h$ in the **a** piecewise H^1 norm, **b** L^2 norm versus number ndof of degrees of freedom for both uniform and adaptive refinement

6.3 L-shaped domain (non-smooth solution)

This subsection shows an advantage of using adaptive mesh-refinement over uniform meshing for the problem (1.1) with the coefficients as $\mathbf{A} = I$, $\mathbf{b} = (x, y)$ and $\gamma = -4$ on a L-shaped domain $\Omega = (-1, 1)^2 \backslash [0, 1) \times (-1, 0]$ and the exact solution

$$u = r^{2/3} \sin\left(\frac{2\theta}{3}\right)$$

with $f := \mathcal{L}u$. Since the exact solution is not zero along the boundary $\partial\Omega$, the error estimators are modified according to Sect. 5.3.3.4. Since $\gamma - \frac{1}{2}\mathrm{div}(\mathbf{b}) = -5 < 0$, the problem is non-coercive. Observe that with increase in number of iterations, refinement is more at the singularity as highlighted in Fig. 6. Since the exact solution u is in $H^{(5/3)-\epsilon}(\Omega)$ for all $\epsilon > 0$, from a priori error estimates the expected order of convergence in H^1 norm is $1/3$ and in L^2 norm is at least $2/3$ with respect to number of degrees of freedom for uniform refinement. Figure 7 shows that uniform refinement gives the sub-optimal convergence rate, whereas adaptive refinement lead to optimal convergence rates (1/2 for H^1 norm and 5/6 in L^2 norm).

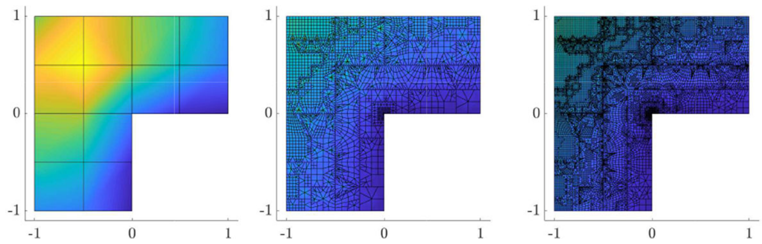

Fig. 6 Output $\mathcal{T}_1$, $\mathcal{T}_{10}$, $\mathcal{T}_{15}$ of the adaptive refinement

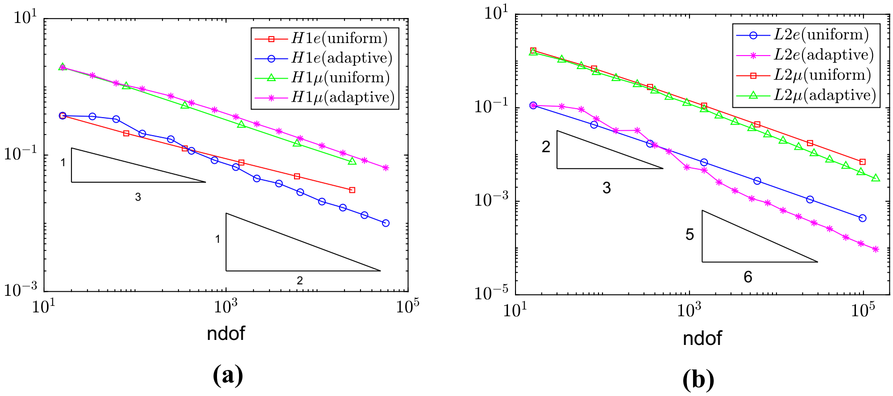

Fig. 7 Convergence history plot of estimator μ and error $e := u - \Pi_1 u_h$ in the **a** piecewise H^1 norm, **b** L^2 norm vs number ndof of degrees of freedom for both uniform and adaptive refinement

6.4 Helmholtz equation

This subsection considers the exact solution $u = 1 + \tanh(-9(x^2 + y^2 - 0.25))$ to the problem

$$-\Delta u - 9u = f \qquad \text{in} \quad \Omega = (-1, 1)^2.$$

There is an internal layer around the circle centered at (0, 0) and of radius 0.25 where the second derivatives of u are large because of steep increase in the solution resulting in the large error at the beginning, and this gets resolved with refinement as displayed in Fig. 8-9.

6.5 Conclusion

The three computational benchmarks provide empirical evidence for the sharpness of the mathematical a priori and a posteriori error analysis in this paper and illustrate the superiority of adaptive over uniform mesh-refining. The empirical convergence rates in all examples for the H^1 and L^2 errors coincide with the predicted convergence rates in Theorem 4.3, in particular, for the non-convex domain and reduced elliptic regularity. The a posteriori error bounds from Theorem 5.1 confirm these convergence

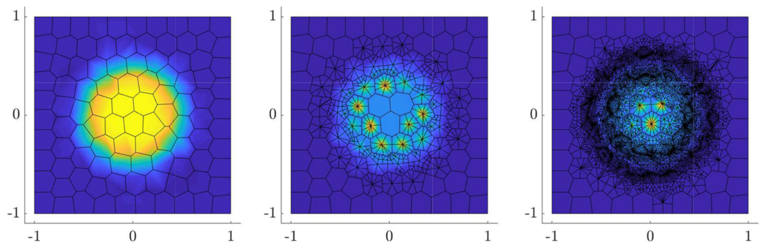

Fig. 8 Output $\mathcal{T}_1, \mathcal{T}_5, \mathcal{T}_{11}$ of the adaptive refinement

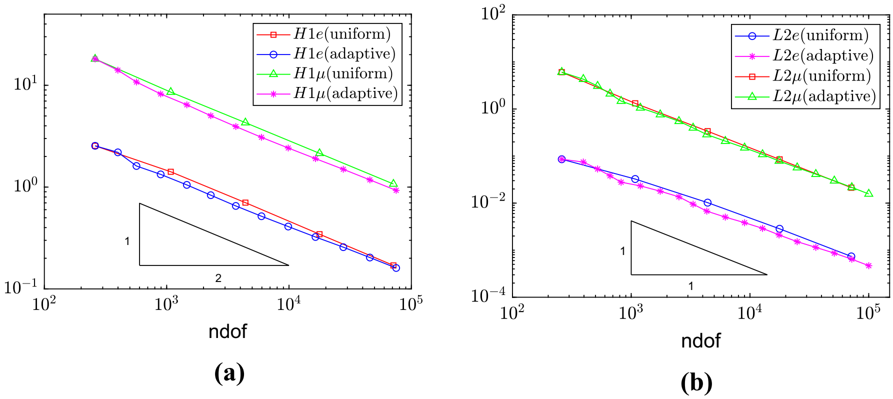

Fig. 9 Convergence history plot of estimator μ and error $e := u - \Pi_1 u_h$ in the **a** piecewise H^1 norm, **b** L^2 norm vs number ndof of degrees of freedom for both uniform and adaptive refinement

rates as well. The ratio of the error estimator μ_ℓ by the H^1 error e_ℓ, sometimes called efficiency index, remains bounded up to a typical value 6; we regard this as a typical overestimation factor for the residual-based a posteriori error estimate. Recall that the constant C_{reg} has not been displayed so the error estimator μ_ℓ does not provide a guaranteed error bound. Figures 10 and 11 display the four different contributions volume residual $(\sum_P \eta_P^2)^{1/2}$, stabilization term $(\sum_P \zeta_P^2)^{1/2}$, inconsistency term $(\sum_P \Lambda_P^2)^{1/2}$ and the nonconformity term $(\sum_P \Xi_P^2)^{1/2}$ that add up to the error estimator μ_ℓ. We clearly see that all four terms converge with the overall rates that proves that none of them is a higher-order term and makes it doubtful that some of those terms can be neglected. The volume residual clearly dominates the a posteriori error estimates, while the stabilisation term remains significantly smaller for the natural stabilisation (with undisplayed parameter one). The proposed adaptive mesh-refining algorithm leads to superior convergence properties and recovers the optimal convergence rates. This holds for the first example with optimal convergence rates in the large pre-asymptotic computational range as well as in the second with suboptimal convergence rates under uniform mesh-refining according to the typical corner singularity and optimal convergence rates for the adaptive mesh-refining. The third example with the Helmholtz equation and a moderate wave number shows certain moderate local mesh-refining in Fig. 8 but no large improvement over the optimal convergence rates for uniform mesh-refining. The adaptive refinement generates hanging nodes because of the way

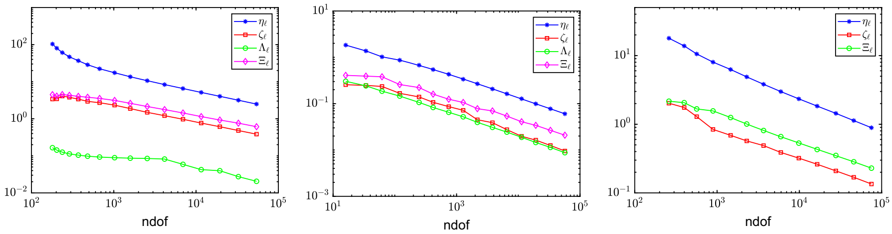

Fig. 10 Estimator components corresponding to the error $H1e = |u - \Pi_1 u_h|_{1,\mathrm{pw}}$ of the adaptive refinement presented in Subsection 6.2–6.4

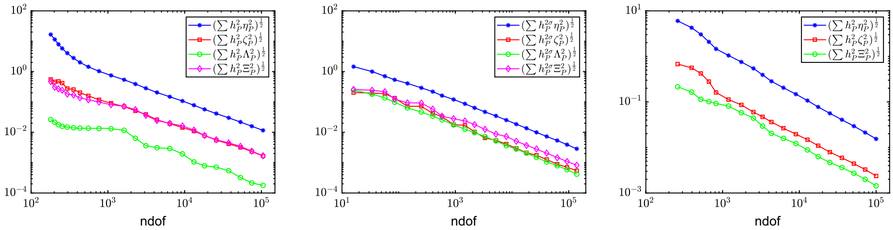

Fig. 11 Estimator components corresponding to the error $L2e = \|u - \Pi_1 u_h\|_{L^2(\Omega)}$ of the adaptive refinement presented in Subsection 6.2–6.4

refinement strategy is defined, but this is not troublesome in VEM setting as hanging node can be treated as a just another vertex in the decompostion of domain. However, an increasing number of hanging nodes with further mesh refinements may violate the mesh assumption (M2), but numerically the method seems robust without putting any restriction on the number of hanging nodes. The future work on the theoretical investigation of the performance of adaptive mesh-refining algorithm is clearly motivated by the successful numerical experiments. The aforementioned empirical observation that the stabilisation terms do not dominate the a posteriori error estimates raises the hope for a possible convergence analysis of the adaptive mesh-refining strategy with the axioms of adaptivity [20] towards a proof of optimal convergence rates: The numerical results in this section support this conjecture at least for the lowest-order VEM in 2D for indefinite non-symmetric second-order elliptic PDEs.

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