The dPG Paradigm —discontinuous Petrov-Galerkin 4 CENTRAL—

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Outline

- dPG Framework
  - minimal residual method
  - mixed problem
  - a posteriori error analysis
- Applications
  - Poisson model problem
  - Stokes equations
  - linear elasticity
  - Maxwell equations
- Adaptive dPG
  - adaptive least-squares FEM

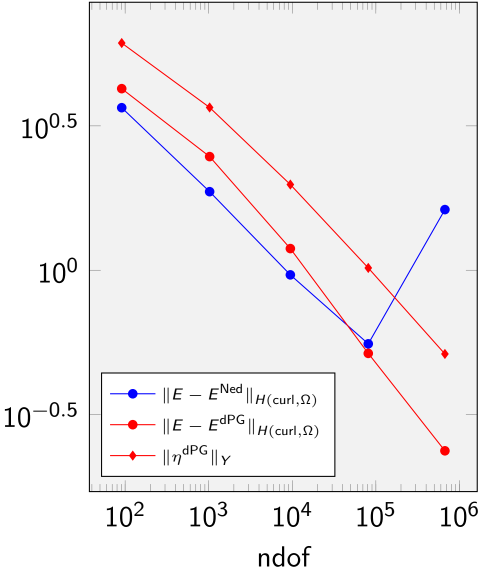
The dPG Methodology

+ instant stability
+ built-in error control
+ general geometries
+ flexible modelling
+ parallel computing

– more degrees of freedom

References. Demkowicz, Gopalakrishnan (since 2010)

[CDG14], [CDG16], and recent publications



dPG Framework

"dPG is a minimal residual method with piecewise discontinuous test functions"

Minimal Residual Method

Suppose $b : X \times Y \to \mathbb{R}$ is a bdd bilinear form on real Hilbert spaces X and Y with inf-sup condition

$$0 < \beta = \inf_{\substack{x \in X \\ \|x\|_X = 1}} \sup_{\substack{y \in Y \\ \|y\|_Y = 1}} b(x, y)$$

Continuous problem (P) with given RHS $F \in Y^*$ seeks

$$u \in X \quad \text{with} \quad b(u, \bullet) = F \text{ in } Y$$

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$$\forall y \in Y\backslash\{0\} \quad b(\bullet, y) \not\equiv 0$$

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$$\forall y \in Y \backslash \{0\} \quad b(\bullet, y) \not\equiv 0$$

so that (P) has a unique solution. The minimal residual method considers

$$u \in \arg\min_{x \in X} \|b(x, \bullet) - F\|_{Y^*}.$$

This is sensitive without any further condition on b bdd bilinear with $\beta > 0$.

Discretization in Minimal Residual Method

Let $X_h \subset X$ and $Y_h \subset Y$ be closed (e.g. finite-dimensional) subspaces with

$$0 < \beta_h := \inf_{\substack{x_h \in X_h \\ \|x_h\|_X = 1}} \sup_{\substack{y_h \in Y_h \\ \|y_h\|_Y = 1}} b(x_h, y_h)$$

Petrov-Galerkin discretization requires a non-degeneracy condition on the discrete level and leads to $\dim X_h = \dim Y_h \in \mathbb{N}_0 \cup \{\infty\}$.

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Petrov-Galerkin discretization requires a non-degeneracy condition on the discrete level and leads to $\dim X_h = \dim Y_h \in \mathbb{N}_0 \cup \{\infty\}$. In what follows, $\dim X_h < \dim Y_h$ and this is *not* a Petrov-Galerkin discretization but suits for a minimal residual method

$$(\mathrm{minRes}_h) \qquad u_h \in \arg\min_{x_h \in X_h} \|b(x_h, \bullet) - F\|_{Y_h^*}$$

Alternative formulation: Seek $(u_h, v_h) \in X_h \times Y_h$ with

$$(\mathrm{M}_h) \qquad \begin{cases} b(x_h, v_h) = 0 & \text{for all } x_h \in X_h \\ (v_h, y_h)_Y + b(u_h, y_h) = F(y_h) & \text{for all } y_h \in Y_h \end{cases}$$

Theorem. $(\mathrm{minRes}_h) \Longleftrightarrow (\mathrm{M}_h)$

Proof. $R_{Y_h} : Y_h \to Y_h^*, y_h \mapsto (y_h, \bullet)_Y$ Riesz map

"$\Longrightarrow$" $u_h \in X_h$ solves (minRes_h) implies for all $t \in \mathbb{R}, x_h \in X_h$

$$\begin{aligned}\|F - b(u_h,\bullet)\|^2_{Y_h^*} &\leqslant \|F - b(u_h + t x_h, \bullet)\|^2_{Y_h^*} \\ &= \| \underbrace{R_{Y_h}^{-1}(F - b(u_h,\bullet))}_{=:\, v_h \Longleftrightarrow (v_h,\bullet)_Y + b(u_h,\bullet) = F \text{ in } Y_h} - t\, R_{Y_h}^{-1} b(x_h,\bullet)\|^2_Y \\ &= \underbrace{\|v_h\|^2_Y}_{= \|F - b(u_h,\bullet)\|^2_{Y_h^*}} - 2t\, b(x_h, v_h) + t^2 \|b(x_h,\bullet)\|^2_{Y_h^*}\end{aligned}$$

Hence $b(\bullet, v_h) = 0$ in X_h □

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$$\begin{aligned}\|F - b(u_h, \bullet)\|^2_{Y_h^*} &\leqslant \|F - b(u_h + tx_h, \bullet)\|^2_{Y_h^*} \\ &= \| \underbrace{R_{Y_h}^{-1}(F - b(u_h, \bullet))}_{=:\, v_h \iff (v_h, \bullet)_Y + b(u_h, \bullet) = F \text{ in } Y_h} - t\, R_{Y_h}^{-1} b(x_h, \bullet)\|^2_Y \\ &= \underbrace{\|v_h\|^2_Y}_{= \|F - b(u_h, \bullet)\|^2_{Y_h^*}} - 2t\, b(x_h, v_h) + t^2 \|b(x_h, \bullet)\|^2_{Y_h^*}\end{aligned}$$

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"$\Longleftarrow$" $(u_h, v_h) \in X_h \times Y_h$ solves (M$_h$) implies

$$\|F - b(u_h + tx_h, \bullet)\|^2_{Y_h^*} = \|v_h\|^2_Y - 2t \underbrace{b(x_h, v_h)}_{=0} + t^2 \|b(x_h, \bullet)\|^2_{Y_h^*}$$ □

dPG as Mixed Problem

Brezzi splitting lemma. (M_h) is well-posed iff

$$0 < \beta_h := \inf_{\substack{x_h \in X_h \\ \|x_h\|_X = 1}} \sup_{\substack{y_h \in Y_h \\ \|y_h\|_Y = 1}} b(x_h, y_h)$$

Fortin criterion: $\beta_h > 0$ is equivalent to the existence of a projection $P : Y \to Y$ (i.e. linear, bdd, idempotent) onto $Y_h = P(Y)$ with annulation property

$$b(\bullet, y - Py) = 0 \quad \text{in } X_h$$

[cf. e.g. FE-book by D.Braess]

Then $$0 < \beta / \|P\| \leqslant \beta_h$$

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General theory of mixed formulations leads to $\gamma_h = \gamma_h(||b||, 1, \beta_h)$:
Solution $u \in X$ to $b(u, \bullet) = F$ and $v = 0$ satisfy best-approximation property in the ansatz space only

$$\|u - u_h\|_X^2 + \|0 - v_h\|_Y^2 \leqslant \|b\| \, \gamma_h^{-1} \min_{x_h \in X_h} \|u - x_h\|_X^2$$

Proof of Fortin Criterion "$\Rightarrow$"

Since $\beta_h > 0$, the discrete mixed problem has a unique solution for all right-hand sides. Given any $y \in \mathrm{Y}$, consider the right-hand side $(F, G) := ((y, \cdot)_{\mathrm{Y}_h}, b(\cdot, y)|_{\mathrm{X}_h}) \in \mathrm{Y}_h^* \times \mathrm{X}_h^*$ and the unique solution $(v_h, u_h) \in \mathrm{Y}_h \times \mathrm{X}_h$ to

(a) $(v_h, \cdot)_{\mathrm{Y}} + b(u_h, \cdot) = (y, \cdot)_{\mathrm{Y}}$ in Y_h.

(b) $b(\cdot, v_h) = b(\cdot, y)$ in X_h.

The map $y \mapsto (v_h, u_h)$ is linear and so $v_h =: Py$ defines $P \in L(\mathrm{Y}, \mathrm{Y})$. If $y \in \mathrm{Y}_h$, then $(y, 0)$ solves (a)-(b). Uniqueness of discrete solutions proves $y = Py$. That is $P = P^2$. The annullation property is (b). $\square$

Proof of Fortin Criterion "$\Leftarrow$"

Given $x_h \in S(\mathrm{X}_h) \subset S(\mathrm{X})$, the inf-sup condition of $\beta > 0$ leads to $y \in S(\mathrm{Y})$ in the Hilbert space Y with

$$\begin{aligned}\beta &\leqslant \|b(x_h,\cdot)\|_{\mathrm{Y}^*} = b(x_h,y) \overset{!}{=} b(x_h,Py)\\ &\leqslant \|b(x_h,\cdot)\|_{\mathrm{Y}_h^*}\underbrace{\|Py\|_{\mathrm{Y}_h}}_{\leqslant\|P\|}.\end{aligned}$$

Hence $\beta/\|P\| \leqslant \|b(x_h,\cdot)\|_{\mathrm{Y}_h^*}$. Since $x_h \in S(\mathrm{X}_h)$ is arbitrary, this proves

$$0 < \beta/\|P\| \leqslant \beta_h := \inf_{x_h\in S(\mathrm{X}_h)} \overbrace{\sup_{y_h\in S(\mathrm{Y}_h)} b(x_h,y_h)}^{\|b(x_h,\cdot)\|_{\mathrm{Y}_h^*}} \qquad \square$$

(R1) P is an oblique projection (not an orthogonal projection in general) and Kato lemma asserts $\|P\| = \|1-P\|$ (provided $P \neq 0,1$).
(R2) The theorem holds in general Banach spaces as pointed out in [Ern, A. and Guermond, J.-L., *A converse to Fortin's Lemma in Banach spaces*, Comptes Rendus de l'Academie des sciences Serie I,2016.]

A Posteriori Error Analysis

Suppose $b(u, \bullet) = F$ is well-posed (i.e. $\beta > 0$ and non-degeneracy condition) on the continuous level. The bilinear form leads to the operator $B_1 : X \to Y^*$, $x \mapsto b(x, \bullet)$ and its dual $B_2 : Y \to X^*$, $y \mapsto b(\bullet, y)$. Well-posedness means that B_1 and B_2 are invertible and and the inverse is bounded by $1/\beta$. Those mapping properties lead to equivalence

$$\|u - u_h\|_X^2 + \|v_h\|_Y^2 \approx \|b(\bullet, v_h)\|_{X*}^2 + \|F - b(u_h, \bullet) + (v_h, \bullet)_Y\|_{Y*}^2$$

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Since B_2 is invertible and bdd,

$$\beta \|v_h\|_Y \leqslant \|b(\bullet, v_h)\|_{X*} \leqslant \|b\| \, \|v_h\|_Y$$

and $\|v_h\|_Y$ is the computable norm of the residual.

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For all $y \in Y$ with norm 1, the annulation operator $P : Y \to Y$ with range $P(Y) = Y_h$ and the discrete equations in (M_h) lead to

$$F(y) - b(u_h, y) + (v_h, y)_Y = F(y - Py) - b(u_h, y - Py) + (v_h, y - Py)_Y$$

Since $b(u_h, y - Py) = 0$ and $|(v_h, y - Py)_Y| \leqslant \|v_h\|_Y\|P\|$, it follows

$$\|u - u_h\|_X^2 + \|v_h\|_Y^2 \approx \underbrace{\|v_h\|_Y^2}_{\text{computable}} + \underbrace{\|F \circ (1 - P)\|_{Y*}^2}_{\text{higher order?}}$$

1. Extreme Example in PMP Shows LS $\subset$ dPG

The Poisson model problem (PMP) seeks $u \in H_0^1(\Omega)$ with $-\Delta\, u = f$ in Ω in the weak sense for a given RHS $f \in L^2(\Omega)$.

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$$p \in H(div, \Omega) \text{ and } u \in H_0^1(\Omega) \text{ with } \quad p = \nabla u \quad \text{and} \quad f + div\, p = 0$$

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For all $(q, v) \in X = H(div, \Omega) \times H^1_0(\Omega)$ define

$$B_1(q, v) := (q - \nabla v, div\, q) \in Y = L^2(\Omega; \mathbb{R}^n) \times L^2(\Omega).$$

The PMP is equivalent to $B_1(p, u) = (0, -f)$. Since $Y \equiv Y^*$, any $(q, v) \in X$ allows for

$$\|b(q, v; \bullet)\|_{Y^*} = \sqrt{||q - \nabla v\|^2 + \|\, div\, q\|^2}$$

The theory of least-squares FEM (LS) shows that this is indeed equivalent to $||(q, v)\|_X$ and, in fact, this dPG method is a LS. This also shows that *any* discretisation $X_h \subset X$ is stable and quasi-optimal.

2. Extreme Example in PMP is Infeasible

In continuation of the PMP, define for any $(q, v) \in X = L^2(\Omega; \mathbb{R}^n) \times H_0^1(\Omega)$

$$B_1(q, v) := (q - \nabla v, div\, q) \in Y = L^2(\Omega; \mathbb{R}^n) \times H^{-1}(\Omega).$$

This leads to a LS with discrete problem (which is always stable)

$$\min_{(q,v)\in X} (\|q - \nabla v\|^2 + \|f + div\, q\|^2_{H^{-1}(\Omega)})$$

The computation of $\|f + div\, q\|_{H^{-1}(\Omega)}$ requires an approximation of the dual norm. The BPX precondition has been suggested to allow a practical variant and is an established LS.

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The computation of $\|f + div\, q\|_{H^{-1}(\Omega)}$ requires an approximation of the dual norm. The BPX precondition has been suggested to allow a practical variant and is an established LS. In general, one requires an approximation of the norm in Y^* by the computable norm in Y_h^* for a large but finite-dimensional space Y_h, the test-search space. This is not a Petrov-Galerkin scheme, so in fact, $\dim X_h < \dim Y_h$ in a minimum residual method. An effective computation, however, requires parallel computing and breaking the test norms in the sense that Y_h is a finite-dimensional space of piecewise discontinuous functions. This allows for a piecewise computation in parallel. The mathematical framework is in product spaces.

Continuous Problem 4 dPG

- $X(T), Y(T)$ real Hilbert spaces for any $T \in \mathcal{P}$ and

$$X \subset \hat{X} := \prod_{T\in\mathcal{P}} X(T) \quad \text{and} \quad Y := \prod_{T\in\mathcal{P}} Y(T)$$

- $b : \hat{X} \times Y \to \mathbb{R}$ is a bounded bilinear form with

$$b((x_T)_{T\in\mathcal{P}}, (y_T)_{T\in\mathcal{P}}) = \sum_{T\in\mathcal{P}} b_T(x_T, y_T)$$

$$0 < \beta = \inf_{\substack{x\in X\\ \|x\|_X=1}} \sup_{\substack{y\in Y\\ \|y\|_Y=1}} b(x,y)$$

- Let $F \in Y^*$ and $u \in X$ satisfy $b(u, \bullet) = F$ in Y

Discretization

Let $X_h(T) \subset X(T)$ and $Y_h(T) \subset Y(T)$ be finite-dimensional subspaces

$$X_h := X \cap \prod_{T \in \mathcal{P}} X_h(T) \quad \text{and} \quad Y_h := \prod_{T \in \mathcal{P}} Y_h(T)$$

"dPG is a minimal residual method

$$(\text{minRes}_h) \qquad u_h \in \arg\min_{x_h \in X_h} \|b(x_h, \bullet) - F\|_{Y_h^*}$$

with piecewise discontinuous test functions"

Alternative formulation. Seek $(u_h, v_h) \in X_h \times Y_h$ with

$$(\text{M}_h) \qquad \begin{cases} b(x_h, v_h) = 0 & \text{for all } x_h \in X_h \\ (v_h, y_h)_Y + b(u_h, y_h) = F(y_h) & \text{for all } y_h \in Y_h \end{cases}$$

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Local annulation. For all $T \in \mathcal{P}$ let $P_T : Y(T) \to Y(T)$ be a bounded linear projection onto $Y_h(T)$ s.t. any $y_T \in Y(T)$ satisfies

$$b_T(\bullet, y_T - P_T y_T) = 0 \quad \text{in } X_h(T)$$

Then
$$0 < \min_{T \in \mathcal{P}} \beta \, \|P_T\|^{-1} \leqslant \beta_h$$

General theory of mixed formulations. For $u \in X$ with $b(u, \bullet) = F$,

$$\|u - u_h\|_X^2 + \|0 - v_h\|_Y^2 \leqslant \|b\| \, \gamma_h^{-1} \min_{x_h \in X_h} \|u - x_h\|_X^2$$

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Suppose $b(u,\bullet) = F$ is well-posed, then

$$\|u-u_h\|_X^2 + \|v_h\|_Y^2 \approx \|b(\bullet, v_h)\|_{X*}^2 + \|F - b(u_h,\bullet) + (v_h,\bullet)_Y\|_{Y*}^2$$

Global annulation. $P := \prod_{T\in\mathcal{P}} P_T$ fulfils $b(\bullet, y - Py) = 0$ in X_h for any $y \in Y$, $\beta_h > 0$, and

$$\|u-u_h\|_X^2 + \|v_h\|_Y^2 \approx \underbrace{\|v_h\|_Y^2}_{\text{computable}} + \underbrace{\|F\circ(1-P)\|_{Y*}^2}_{\text{higher order?}}$$

for all $(x_h, y_h) \in X_h \times Y_h$ replacing (u_h, v_h)

[C-Demkowicz-Gopalakrishnan, SINUM (2014)]

Application to Poisson Model Problem

Simplified dPG for Single Domain

Let $f \in L^2(\Omega)$ in open, bounded, polygonal Lipschitz domain $\Omega \subset \mathbb{R}^2$. Seek $u : \Omega \to \mathbb{R}$ with

$$\text{(PMP)} \qquad -\Delta u = f \quad \text{in } \Omega \qquad \text{and} \qquad u = 0 \quad \text{on } \partial\Omega$$

Test functions $w \in H^1(\Omega)$ require (unknown) boundary term $t = \partial u/\partial \nu$ on $\partial\Omega$ as new variable and lead to well-posed problem: Given $f \in L^2(\Omega)$, seek $(u, t) \in X := H^1_0(\Omega) \times H^{-1/2}(\partial\Omega)$ with

$$\int_\Omega \nabla u \cdot \nabla w \,\mathrm{d}x - \langle t, \gamma_0 w\rangle_{\partial\Omega} = \int_\Omega f\, w \,\mathrm{d}x \qquad \text{for all } w \in Y := H^1(\Omega)$$

Traces

Theorem. $U \subset \mathbb{R}^n$ open, bounded Lipschitz domain

- $\exists\, \gamma_0 \in L(H^1(U); L^2(\partial U))$ with $\gamma_0 w = w|_{\partial U}$ for all $w \in H^1(U) \cap C^0(\overline{U})$
- $H^{1/2}(\partial U) := \gamma_0(H^1(U))$ is a Hilbert space
- Let $H^{-1/2}(\partial U) := (H^{1/2}(\partial U))^*$, then $\exists\, \gamma_\nu \in L(H(\mathrm{div}, U); H^{-1/2}(\partial U))$ surjective with $\gamma_\nu q = q|_{\partial U} \cdot \nu$ for all $q \in C^1(\overline{U}; \mathbb{R}^n)$

Integration by Parts. Any $q \in H(\mathrm{div}, U)$ and $w \in H^1(U)$ satisfy

$$\langle \gamma_\nu q, \gamma_0 w \rangle_{\partial U} = \int_U q \cdot \nabla w \,\mathrm{d}x + \int_U \mathrm{div}\, q \cdot w \,\mathrm{d}x$$

Traces on the Skeleton

Duality lemma.

$$\|g\|_{H^{1/2}(\partial U)} := \min_{\substack{h\in H^1(U)\\ \gamma_0 h=g}} \|h\|_{H^1(U)}, \quad \|t\|_{H^{-1/2}(\partial U)} = \min_{\substack{q\in H(\mathrm{div},U)\\ \gamma_\nu q=t}} \|q\|_{H(\mathrm{div},U)}$$

Let $\mathcal{T}$ be a regular triangulation of Ω

Consequence. For $q = (q_T)_{T\in\mathcal{P}} \in H(\mathrm{div},\mathcal{T}) := \prod_{T\in\mathcal{T}} H(\mathrm{div}, T)$,

$$\gamma_\nu^{\mathcal{T}} q := \prod_{T\in\mathcal{T}} \gamma_\nu q_T$$

Define the Hilbert space $H^{-1/2}(\partial\mathcal{T}) := \gamma_\nu^{\mathcal{T}} H(\mathrm{div},\Omega)$ with norm

$$\|t\|_{H^{-1/2}(\partial\mathcal{T})} = \min_{\substack{q\in H(\mathrm{div},\Omega)\\ \gamma_\nu^{\mathcal{T}} q=t}} \|q\|_{H(\mathrm{div},\Omega)}$$

dPG for PMP and Triangulation $\mathcal{T}$

$X_h(T) := P_1(T) \times P_0(\mathcal{E}(T)) \subset X(T) := H^1(T) \times H^{-1/2}(\partial T)$
$Y_h(T) := P_1(T) \subset Y(T) := H^1(T)$ are Hilbert spaces for any T with

local bilinear form $b_T : X(T) \times Y(T) \to \mathbb{R}$,

$$b_T(u_T, t_T; w_T) = \int_T \nabla u_T \cdot \nabla w_T \,\mathrm{d}x - \langle t_T, \gamma_0 w_T \rangle_{\partial T}$$

Improved version of the duality and splitting lemma [CDG16] lead for

$X := H_0^1(\Omega) \times H^{-1/2}(\partial\mathcal{T}) \subset \prod_{T\in\mathcal{T}} X(T)$ and $Y := \prod_{T\in\mathcal{T}} Y(T)$

to the inf-sup condition

$$0 < \sqrt{1 - \frac{1}{\sqrt{1+\lambda_1}}} \leqslant \beta = \inf_{\substack{x\in X \\ \|x\|_X=1}} \sup_{\substack{y\in Y \\ \|y\|_Y=1}} b(x,y)$$

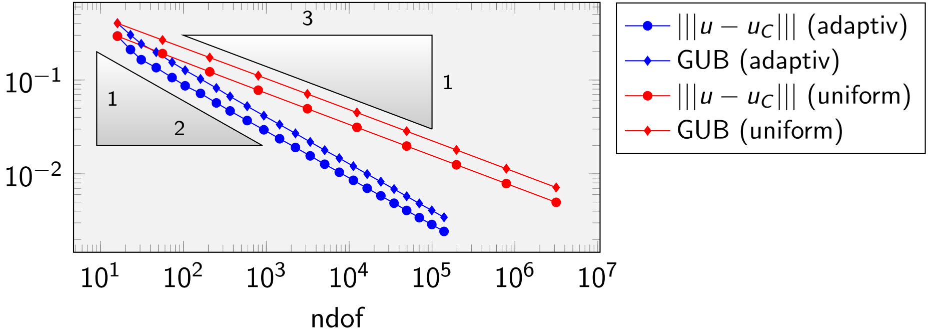
Low-order dPG for PMP

$$X_h \equiv S^1_0(\mathcal{T}) \times P_0(\mathcal{E}) \quad \text{and} \quad Y_h = P_1(\mathcal{T})$$

The nonconforming interpolation $P_T := \mathcal{I}^{\mathrm{nc}}$ has annulation property and

$$\|F \circ (1 - \mathcal{I}^{\mathrm{nc}})\|_{Y^*} \leqslant \sqrt{1/48 + j_{1,1}^{-1}}\, \|h_{\mathcal{T}} f\|_{L^2(\Omega)}.$$

Experiment. $\Omega = (-1,1)^2 \backslash [-1,0]^2$, $f(x,y) = 0$
$u(r,\theta) = r^{2/3} \sin(2(\theta + \pi/2)/3)$ (polar coordinates (r,θ))



Low-order dPG for PMP

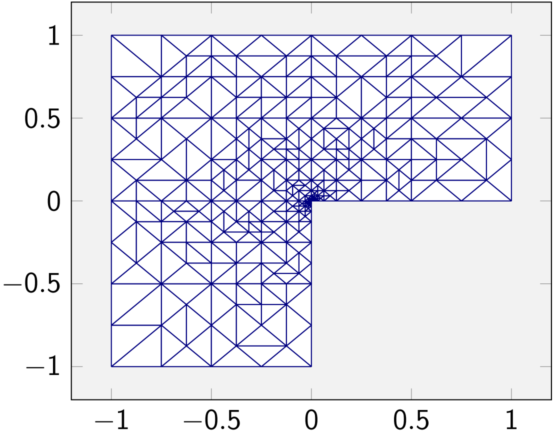


Figure : Triangulation plot with 496 elements (250 degrees of freedom for u_C) for adaptive mesh-refinement with GUB as refinement indicator and $\theta = 0.3$

Application to Stokes Equations

Low-Order dPG for Stokes Equations

The Stokes equations in pseudostress formulation.

Given $f \in L^2(\Omega;\mathbb{R}^n)$ and $g \in H^1(\partial\Omega;\mathbb{R}^n)$ for domain $\Omega \subseteq \mathbb{R}^n$, seek $u \in H^1(\Omega;\mathbb{R}^n)$ and $\boldsymbol{\sigma} \in H(\mathrm{div},\Omega;\mathbb{R}^{n\times n})/\mathbb{R}$ with

$$\begin{aligned} \mathrm{dev}\ \boldsymbol{\sigma} &= \mathrm{D}u && \text{in } \Omega \\ -\mathrm{div}\ \boldsymbol{\sigma} &= f && \text{in } \Omega \\ u &= g && \text{along } \partial\Omega \end{aligned}$$

$$\mathrm{dev}\ A := A - 1/n\ (\mathrm{tr}A)\mathrm{I}_{n\times n}$$

Low-order dPG for Stokes Equations

$$X_h(T) := P_0(T;\mathbb{R}^{n\times n})/\mathbb{R} \times P_0(T;\mathbb{R}^n) \times P_1(\mathcal{E}(T);\mathbb{R}^n) \times P_0(\mathcal{E}(T);\mathbb{R}^n)$$

$$X(T) := L^2(T;\mathbb{R}^{n\times n}) \times L^2(T;\mathbb{R}^n) \times H^{1/2}(\partial T;\mathbb{R}^n) \times H^{-1/2}(\partial T;\mathbb{R}^n)$$

$$X := L^2(\Omega;\mathbb{R}^{n\times n})/\mathbb{R} \times L^2(\Omega;\mathbb{R}^n) \times H_0^{1/2}(\partial\mathcal{T};\mathbb{R}^n) \times H^{-1/2}(\partial\mathcal{T};\mathbb{R}^n)$$

$$\subset \prod_{T\in\mathcal{T}} X(T) \quad \text{while} \quad Y = \prod_{T\in\mathcal{T}} Y(T)$$

$$Y_h(T) := RT_0(T;\mathbb{R}^{n\times n}) \times P_1(T;\mathbb{R}^n)$$

$$Y(T) := H(\mathrm{div}, T;\mathbb{R}^{n\times n}) \times H^1(T;\mathbb{R}^n)$$

For all $T \in \mathcal{T}$ let $b_T : X(T) \times Y(T) \to \mathbb{R}$ with

$$b_T(\boldsymbol{\sigma}, u, s, t; \boldsymbol{\tau}, v) = \int_T \boldsymbol{\sigma} : \mathrm{D}_{\mathrm{nc}} v \,\mathrm{d}x + \int_T \mathrm{dev}\,\boldsymbol{\sigma} : \boldsymbol{\tau} \,\mathrm{d}x + \int_T u \cdot \mathrm{div}_{\mathrm{nc}} \boldsymbol{\tau} \,\mathrm{d}x$$
$$- \langle t, \gamma_0 v\rangle_{\partial T} - \langle \gamma_\nu \boldsymbol{\tau}, s\rangle_{\partial T}$$

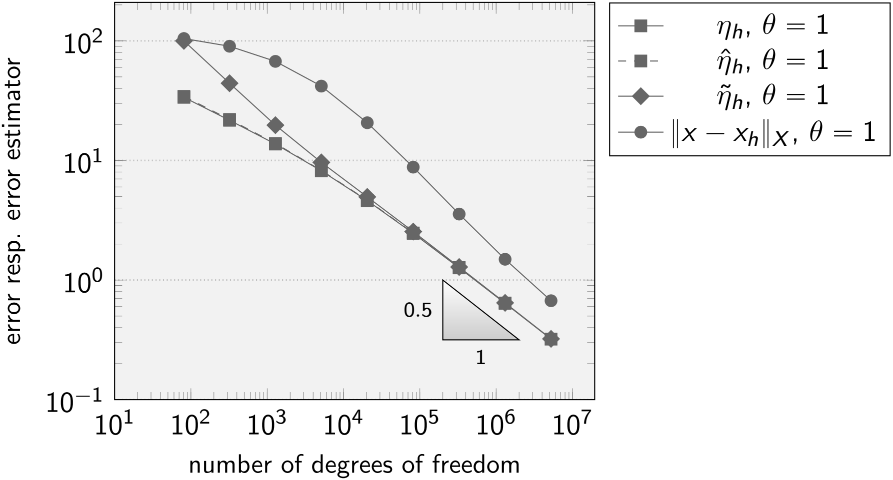
$$F(\boldsymbol{\tau}, v) := \int_\Omega f \cdot v \,\mathrm{d}x + \langle \gamma_\nu^{\mathcal{T}} \boldsymbol{\tau}, \gamma_0^{\mathcal{T}} g\rangle_{\partial\mathcal{T}}$$

Low-Order dPG for Stokes Equations

- β and β_h are explicitly bounded in terms of the Friedrichs, tr-dev-div constant and the inf-sup constant of the mixed FEM $H(\mathrm{div}, \Omega : \mathbb{R}^{n\times n})/\mathbb{R} \times L^2(\Omega : \mathbb{R}^n)$
- the data approximation error $\|F \circ (1-P)\|_{Y_h^*}$ is not necessarily of higher order
- the extension $\hat{Y}_h(T) := RT_0(T; \mathbb{R}^{n\times n})/\mathbb{R} \oplus b_3(T)\mathbb{R}^{n\times n}_{\mathrm{dev}} \times P_1(T; \mathbb{R}^n)$ of $Y_h(T)$ with the cubic bubble $b_3(T)$ guarantees the higher order
- the experiments compare the residual error estimators $\eta_h := \|F - b(x_h, \bullet)\|_{Y_h^*}$, $\hat{\eta}_h := \|F - b(x_h, \bullet)\|_{\hat{Y}_h^*}$, and the up to a generic constant guaranteed bound $\tilde{\eta}_h^2 := \hat{\eta}_h^2 + \mathrm{osc}^2(g', \mathcal{E}(\partial\Omega))$

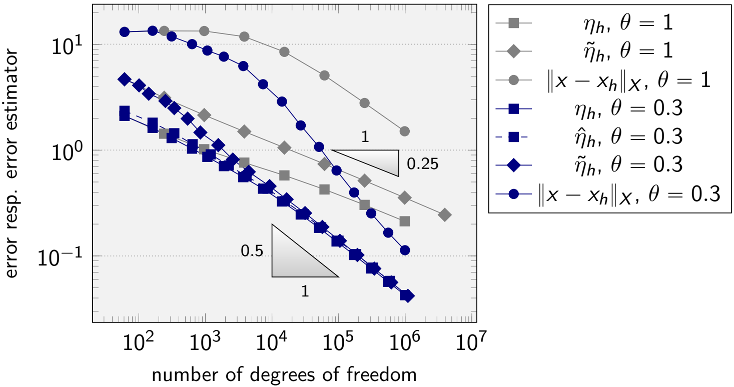
Numerical Example: dPG for Stokes – colliding flow

Experiment. $\Omega = (-1,1)^2$, $f \equiv 0$ with implicit boundary data, for all $(x_1, x_2) \in \Omega$, $u(x_1,x_2) = 4(5x_1x_2^4 - x_1^5,\ 5x_1^4x_2 - x_2^5)$ and $p(x_1,x_2) = 120x_1^2x_2^2 - 20(x_1^4 + x_2^4) - 16/3$.



Numerical Example: dPG for Stokes – L-shaped domain

Experiment. $\Omega = (-1,1)^2 \setminus ([0,1]\times[-1,0])$, $f \equiv 0$,
for all $(r,\varphi) \in [0,\infty) \times [0, 3\pi/2]$, $\omega := 3\pi/2$, $\alpha := 856399/1572864$,
$w(\varphi) := \frac{\sin((1+\alpha)\varphi)\cos(\alpha\omega)}{1+\alpha} - \cos((1+\alpha)\varphi) + \frac{\sin((\alpha-1)\varphi)\cos(\alpha\omega)}{1-\alpha} + \cos((\alpha-1)\varphi)$,
with implicit boundary data, $p(r,\varphi) = \frac{-r^{\alpha-1}((1+\alpha)^2 w'(\varphi) + w'''(\varphi))}{1-\alpha}$,
and $u(r,\varphi) = \begin{pmatrix} r^\alpha((1+\alpha)\sin(\varphi)w(\varphi) + \cos(\varphi)w'(\varphi) \\ -(1+\alpha)\cos(\varphi)w(\varphi) + \sin(\varphi)w'(\varphi) \end{pmatrix}$



Numerical Example: dPG for Stokes – L-shaped domain

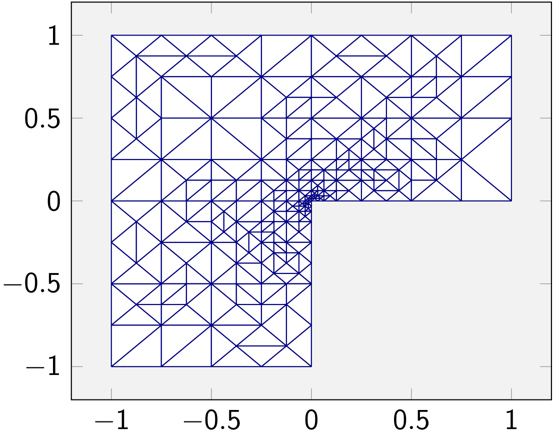
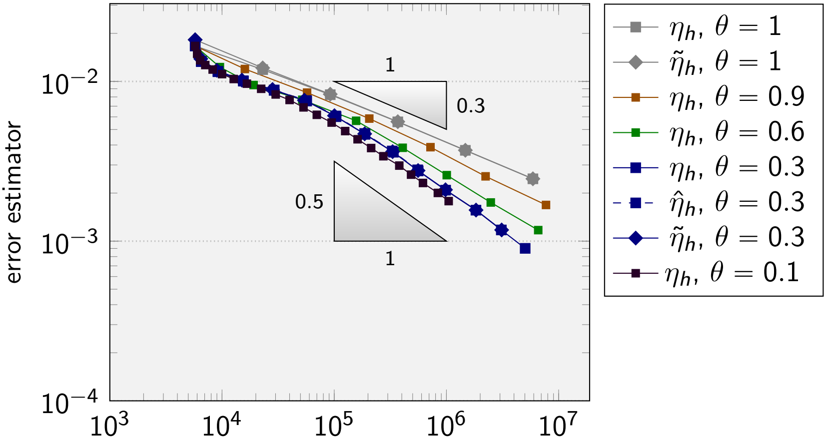


Figure : Triangulation plot with 371 elements (3711 degrees of freedom) for adaptive mesh-refinement with η_ℓ and $\theta = 0.3$

Numerical Example: dPG for Stokes – backward facing step

Experiment. $\Omega = ((-2,8)\times(-1,1))\backslash((-2,0)\times(-1,0))$, $f \equiv 0$,

$$\text{boundary data } g(x_1,x_2) = \begin{cases} 1/10(-x_2(x_2-1),0) & \text{for } x_1 = -2, \\ 1/80(-(x_2-1)(x_2+1),0) & \text{for } x_1 = 8, \\ (0,0) & \text{elsewhere.} \end{cases}$$



Numerical Example: dPG for Stokes – backward facing step

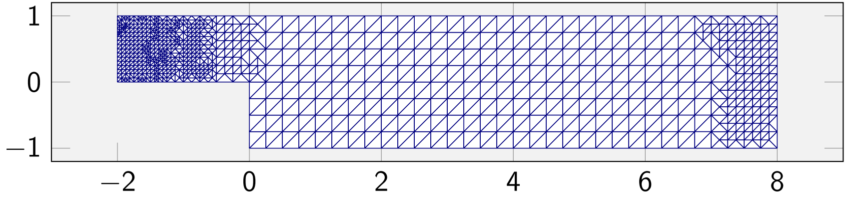


Figure : Triangulation plot with 1551 elements (15511 degrees of freedom) for adaptive mesh-refinement with η_ℓ and $\theta = 0.3$

Example for Linear Elasticity

The Navier-Lamé equations. Seek $u \in H^1(\Omega;\mathbb{R}^n)$ and $\sigma \in H(\operatorname{div},\Omega;\mathbb{S})$ with

$$\begin{aligned} -\operatorname{div}\sigma &= f && \text{in } \Omega,\\ \sigma &= \mathbb{C}\varepsilon(u) && \text{in } \Omega,\\ u &= 0 && \text{on } \Gamma_D,\\ \sigma\nu &= 0 && \text{on } \Gamma_N, \end{aligned}$$

$\varepsilon(u) := \operatorname{sym} \mathrm{D}\, u := (\mathrm{D}\, u + \mathrm{D}\, u^\top)/2$,
$\mathbb{C}(A) := 2\mu A + \lambda \operatorname{tr}(A) I_{n\times n}$.

Low-Order dPG for Linear Elasticity

Hilbert spaces.
$X(T) := L^2(T;\mathbb{S}) \times L^2(T;\mathbb{R}^n) \times H^{1/2}(\partial T;\mathbb{R}^n) \times H^{-1/2}(\partial T;\mathbb{R}^n)$
$Y(T) := H(\mathrm{div}, T;\mathbb{S}) \times H^1(T;\mathbb{R}^n)$
$X := L^2(\Omega;\mathbb{S})/\mathbb{R} \times L^2(\Omega;\mathbb{R}^n) \times H_D^{1/2}(\partial\mathcal{T};\mathbb{R}^n) \times H_N^{-1/2}(\partial\mathcal{T};\mathbb{R}^n)$
$X \subset \prod_{T\in\mathcal{T}} X(T), \quad Y := \prod_{T\in\mathcal{T}} Y(T)$

Bilinear form. For all $T \in \mathcal{T}$ let $b_T : X(T) \times Y(T) \to \mathbb{R}$ with

$$b_T(\sigma, u, s, t; \tau, v) = \int_T \sigma : \varepsilon_{NC}(v)\,\mathrm{d}x + \int_T \mathbb{C}^{-1}\sigma : \tau\,\mathrm{d}x + \int_T u \cdot \mathrm{div}_{\mathrm{nc}}\tau\,\mathrm{d}x$$
$$- \langle t, \gamma_0 v\rangle_{\partial T} - \langle \gamma_\nu \tau, s\rangle_{\partial T}.$$

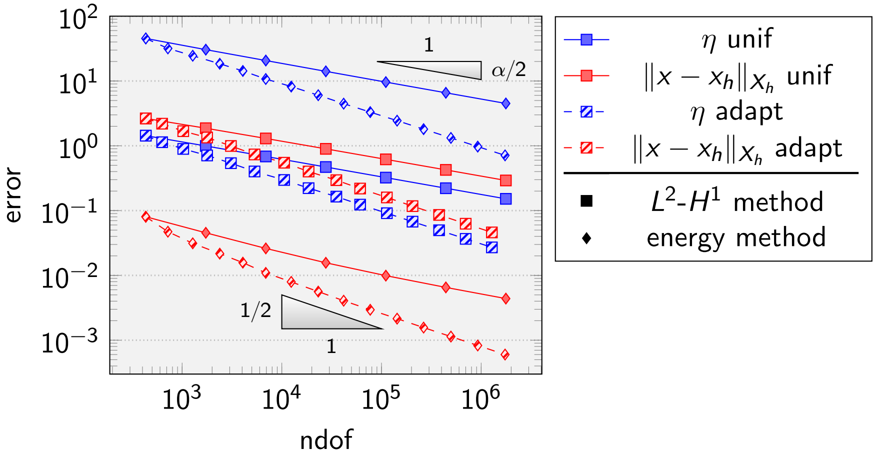
Discretization. $X_h(T) := P_0(T;\mathbb{S})/\mathbb{R} \times P_0(T;\mathbb{R}^n) \times P_1(\mathcal{E}(T);\mathbb{R}^n) \times$

$P_0(\mathcal{E}(T);\mathbb{R}^n)$, $Y_h(T) := \mathrm{sym}\, RT_0(T;\mathbb{R}^{n\times n})/\mathbb{R} \times P_1(T;\mathbb{R}^n)$.

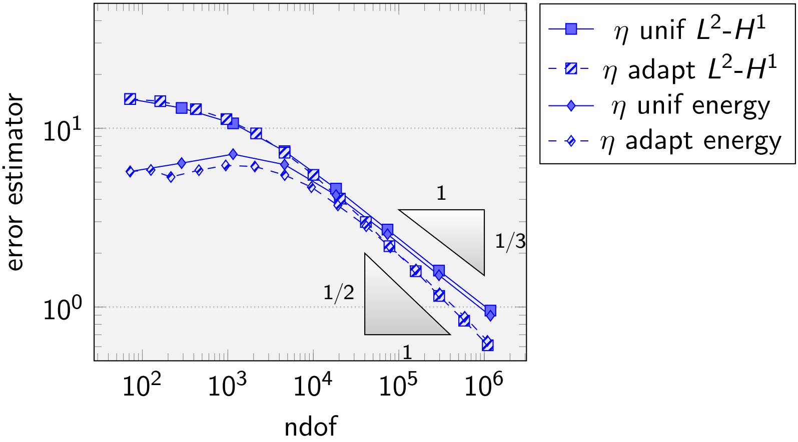
Low-Order dPG for Linear Elasticity

- bounds of β and β_h depend on Friedrichs, Korn, tr-dev-div constant, the inf-sup constant of the mixed FEM and μ, but are *independent* of λ
- canonical choice of norms (e.g., $\|\bullet\|$ on $L^2(\Omega;\mathbb{S})$) lead to locking-free *L^2-H^1 method*
- *energy method* with other, $\mathbb{C}$-dependent norms (e.g., $\|\mathbb{C}^{1/2}\bullet\|$ on $L^2(\Omega;\mathbb{S})$) suffers from locking
- extension like for Stokes equations yield a higher order data approximation error

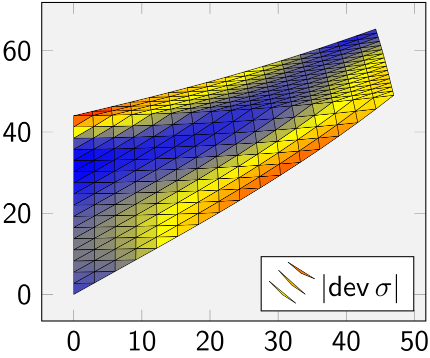
Example: Rotated L-shaped domain with exact solution

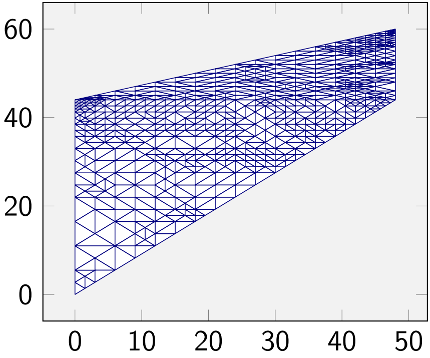


Numerical Example: Cook's membrane

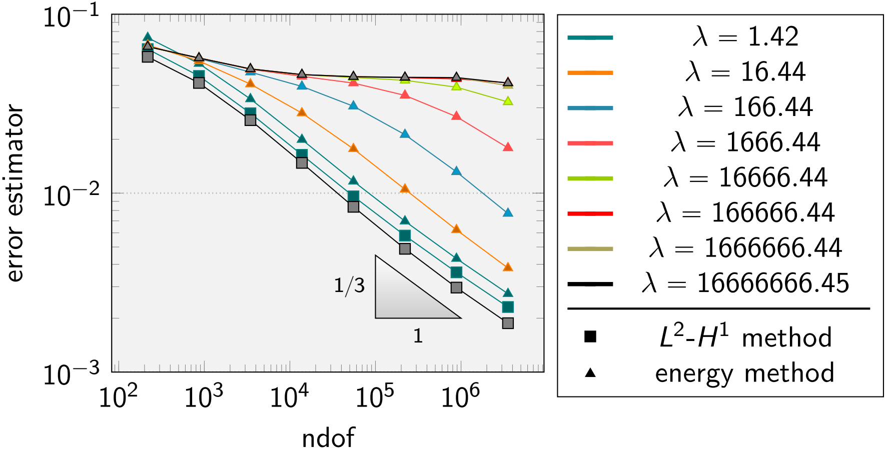


Numerical Example: Cook's membrane





Numerical Example: Locking



Application to Maxwell Equations

Single Domain

$\Omega \subset \mathbb{R}^3$ open, bounded, polyhydral Lipschitz domain. Seek $E : \Omega \to \mathbb{R}^3$ with

(Maxwell) $\quad \operatorname{curl} \operatorname{curl} E - \omega^2 E = J \quad \text{in } \Omega \qquad \text{and} \qquad E \times \nu = 0 \quad \text{on } \partial\Omega$

Test functions $F \in H(\operatorname{curl}, \Omega)$ lead to boundary term $\hat{H} = \operatorname{curl} E \times \nu$ on $\partial\Omega$. Suppose ω^2 is not a Maxwell eigenvalue. The resulting well-posed problem reads:

Given $J \in L^2(\Omega; \mathbb{R}^3)$, seek $(E, \hat{H}) \in X := H_0(\operatorname{curl}, \Omega) \times H^{-1/2}(\operatorname{div}_{\partial\Omega}, \partial\Omega)$

$$\int_\Omega \operatorname{curl} E \cdot \operatorname{curl} F \,\mathrm{d}x - \omega^2 \int_\Omega E \cdot F \,\mathrm{d}x - \langle \pi_\tau F, \hat{H} \rangle_{\partial\Omega} = \int_\Omega J \cdot F \,\mathrm{d}x$$

for all $H(\operatorname{curl}, \Omega)$

Traces

Theorem. $U \subset \mathbb{R}^3$ open, bounded Lipschitz domain

- $\exists\, \gamma_\tau \in L(H(\text{curl}, U); H^{-1/2}(\partial U))$, $\gamma_\tau H = H|_{\partial U} \times \nu$ for $H \in C^\infty(\overline{U}; \mathbb{R}^3)$
- $H^{-1/2}(\text{div}_{\partial U}, \partial U) := \gamma_\tau(H(\text{curl}, U))$ is a Hilbert space
- Let $H^{-1/2}(\text{curl}_{\partial U}, \partial U) := (H^{-1/2}(\text{div}_{\partial U}, \partial U))^*$, then $\exists\, \pi_\tau \in L(H(\text{curl}, U); H^{-1/2}(\text{curl}_{\partial U}, \partial U))$ surjective with $\pi_\tau F = \nu \times (F|_{\partial U} \times \nu)$ for all $F \in C^\infty(\overline{U}; \mathbb{R}^3)$

Integration by parts. Any $F, H \in H(\text{curl}, U)$ satisfy

$$\langle \pi_\tau F, \gamma_\tau H \rangle_{\partial U} = \int_U H \cdot \text{curl}\, F \,\mathrm{d}x - \int_U \text{curl}\, H \cdot F \,\mathrm{d}x$$

Traces on the Skeleton

Duality lemma.

$$\|\hat{H}\|_{H^{-1/2}(\mathrm{div}_{\partial U},\partial U)} := \min_{\substack{H\in H(\mathrm{curl},U)\\ \gamma_\tau H=\hat{H}}} \|H\|_{H(\mathrm{curl},U)}$$

$$\|\hat{F}\|_{H^{-1/2}(\mathrm{curl}_{\partial U},\partial U)} = \min_{\substack{F\in H(\mathrm{curl},U)\\ \gamma_\nu F=\hat{F}}} \|q\|_{H(\mathrm{curl},U)}$$

Let $\mathcal{T}$ be a shape-regular triangulation of Ω into tetrahedra

Consequence. For $H = (H_T)_{T\in\mathcal{T}} \in \prod_{T\in\mathcal{T}} H(\mathrm{curl}, T)$ let

$$\gamma_\tau^{\mathcal{T}} H := \prod_{T\in\mathcal{T}} \gamma_\tau H_T$$

Define the Hilbert space $H^{-1/2}(\mathrm{div}_{\partial\mathcal{T}}, \partial\mathcal{T}) := \gamma_\tau^{\mathcal{T}} H(\mathrm{curl},\Omega)$ with norm

$$\|\hat{H}\|_{H^{-1/2}(\mathrm{div}_{\partial\mathcal{T}},\partial\mathcal{T})} = \min_{\substack{H\in H(\mathrm{curl},\Omega)\\ \gamma_\nu^{\mathcal{T}} H=\hat{H}}} \|H\|_{H(\mathrm{curl},\Omega)}$$

dPG for Maxwell and Triangulation $\mathcal{T}$

$X(T) = H(\text{curl}, T) \times H^{-1/2}(\text{div}_{\partial T}, \partial T)$ and $Y(T) = H(\text{curl}, T)$

$X := H_0(\text{curl}, \Omega) \times H^{-1/2}(\text{div}_{\partial \mathcal{T}}, \partial \mathcal{T}) \subset \prod_{T \in \mathcal{T}} X(T)$ and
$Y := \prod_{T \in \mathcal{T}} Y(T)$ lead in (Maxwell) for all $T \in \mathcal{T}$ to
$b_T : X(T) \times Y(T) \to \mathbb{R}$ with

$$b_T(E_T, \hat{H}_T; F_T) = \int_T \text{curl}\, E_T \cdot \text{curl}\, F_T \,\mathrm{d}x - \omega^2 \int_T E_T \cdot F_T \,\mathrm{d}x - \langle \pi_\tau F_T, \hat{H}_T \rangle_{\partial T}$$

The duality and splitting lemma [CDG16] show for global bilinear form b

$$0 < \beta = \inf_{\substack{x \in X \\ \|x\|_X = 1}} \sup_{\substack{y \in Y \\ \|y\|_Y = 1}} b(x, y)$$

Discretization

Nédélec element. $N_k(T) := P_{k-1}(T;\mathbb{R}^3) \oplus S_k(T;\mathbb{R}^3)$ with

$$S_k(T;\mathbb{R}^3) := \{p \in P_k(T;\mathbb{R}^3) \,|p \text{ homogenous polynomial of degree } k \text{ and } p(x)\cdot x = 0 \text{ in } T\}$$

Discretization with Nédélec elements.

$$X_h(T) := N_k(T) \times \gamma_\tau N_k(T) \quad \text{and} \quad Y_h(T) := N_\ell(T) \quad \text{for } k, \ell \in \mathbb{N}$$

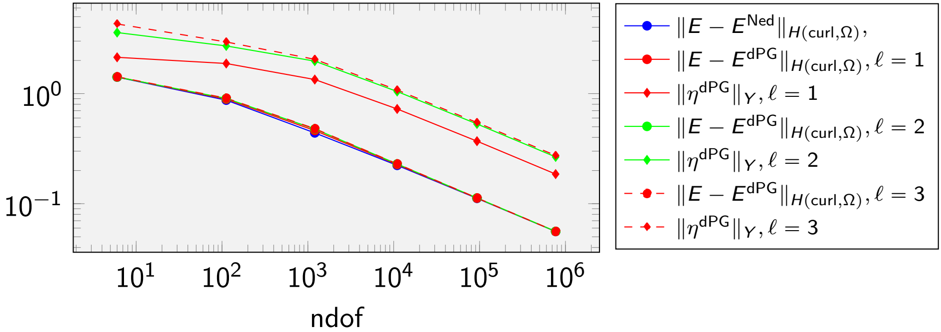
Annulation operator $P_T : Y(T) \to Y_h(T)$ in [CDG16] requires $\ell = k + 3$

Numerical experiments with $k = 1$ seems to work with $\ell = 1$ as well

Numerical Example. Primal dPG for Maxwell

$$X_h(\mathcal{T}) := N_1(\mathcal{T}) \times \gamma_\tau N_1(\mathcal{T}) \quad \text{and} \quad Y_h(\mathcal{T}) := N_\ell(\mathcal{T})$$

Experiment. $\Omega = (0,1)^3$, $\omega^2 = 1$, $E = (\sin \pi x \sin \pi y \sin \pi z, 0, 0)$
$k = 1$, uniform refinement

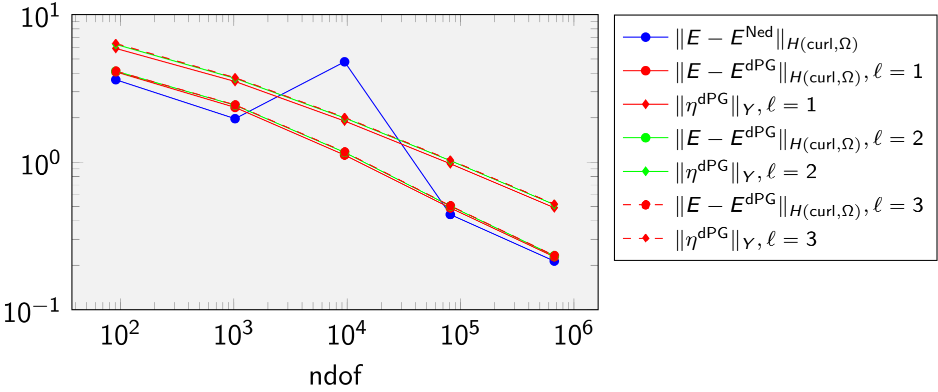


Numerical Example. Primal dPG for Maxwell

$$X_h(T) := N_1(T) \times \gamma_\tau N_1(T) \quad \text{and} \quad Y_h(T) := N_\ell(T)$$

Experiment. $\Omega = (-1,1)^3 \setminus [0,1]^3$ (Fichera's corner domain), $\omega^2 = 3.1$
$E = (e^{i\omega z}, e^{i\omega x}, e^{i\omega y})$

Remark. ω^2 is close to a Maxwell eigenvalue

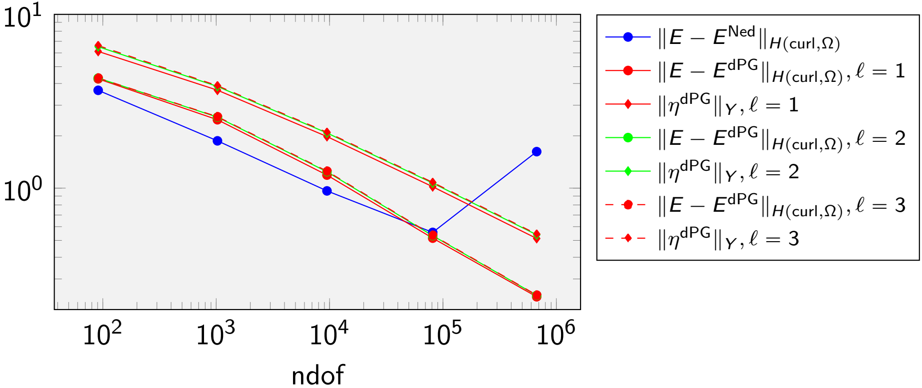


Numerical Example. Primal dPG for Maxwell

$$X_h(T) := N_1(T)\times\gamma_\tau N_1(T) \quad \text{and} \quad Y_h(T) := N_\ell(T)$$

Experiment. $\Omega = (-1,1)^3\backslash[0,1]^3$ (Fichera's corner domain), $\omega^2 = 3.2$
$E = (e^{i\omega z}, e^{i\omega x}, e^{i\omega y})$

Remark. ω^2 is close to a Maxwell eigenvalue

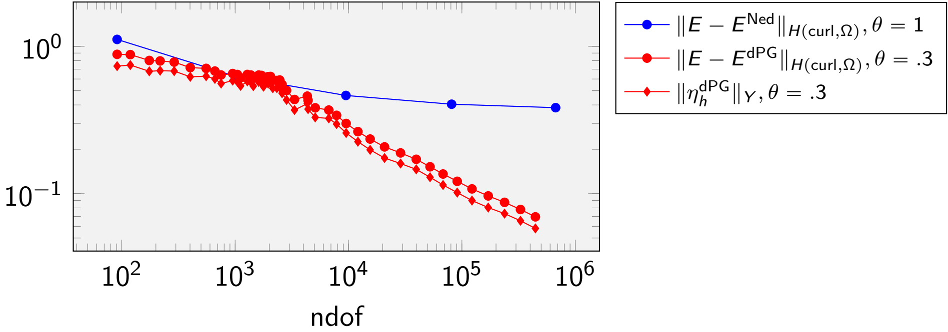


Numerical Example. Singular Solution

$$X_h(\mathcal{T}) := N_1(\mathcal{T}) \times \gamma_\tau N_1(\mathcal{T}) \quad \text{and} \quad Y_h(\mathcal{T}) := N_1(\mathcal{T})$$

Experiment. $\Omega = (-1,1)^3 \backslash [0,1]^3$ (Fichera's corner domain), $\omega^2 = 1$
$E = \nabla p(x,y,z)$ with $p(x,y,z) = (x^2 + y^2 + z^2 + 10^{-6})^{1/4}$

Remark. Singularity in $(x,y,z) = 0$



Numerical Example. [CDG16]

Experiment. $J = 0$ and $\omega^2 = 25$ with boundary condition $\nu \times E = \nu \times E^D$, where $E^D(x, y, z) = (\sin \pi y, 0, 0)$

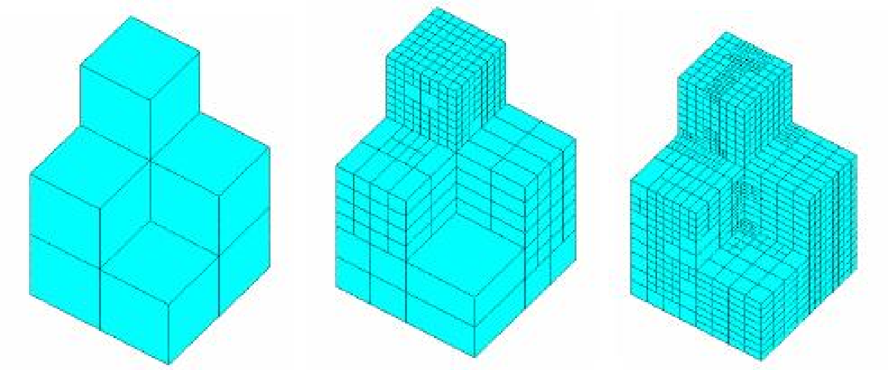
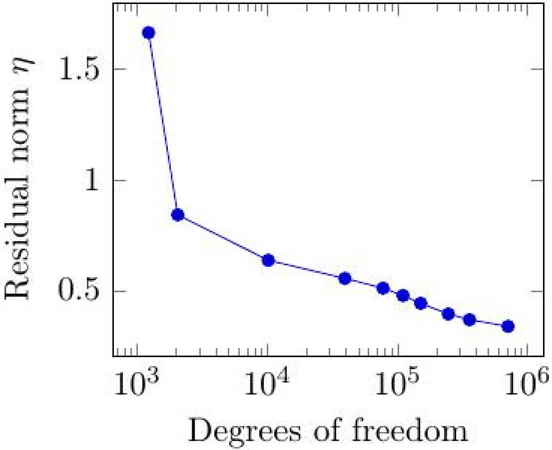


Figure : Iteration 1, 5, and 9

Numerical Example. [CDG16]

Remark. Since ω is large, the initial grid is too coarse for standard discretizations, but adaptive dPG seems to work



Flexible Modelling

There exist equivalent [CDG16] formulations of (Maxwell), e.g. the first-order system

$$i\omega E + \operatorname{curl} H = J \qquad \text{and} \qquad -i\omega H + \operatorname{curl} E = 0$$

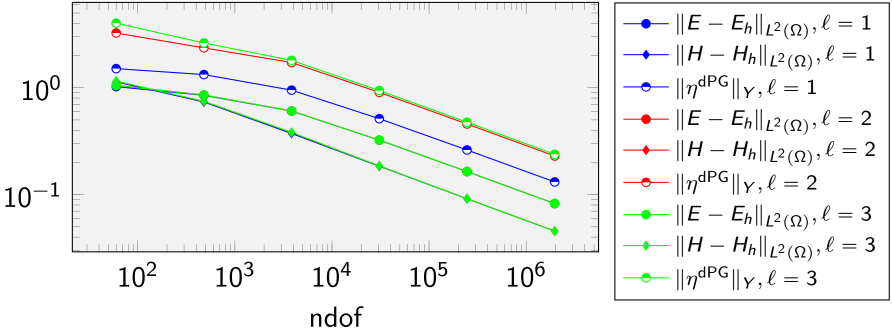
It leads to the dPG method for the ultra-weak formulation

- $X(T) = L^2(T;\mathbb{C}^3) \times L^2(T;\mathbb{C}^3) \times \gamma_\tau H(\operatorname{curl}, T;\mathbb{C}^3) \times \gamma_\tau H(\operatorname{curl}, T;\mathbb{C}^3)$
- $X = L^2(\Omega;\mathbb{C}^3) \times L^2(\Omega;\mathbb{C}^3) \times \gamma_\tau^{\mathcal{T}} H_0(\operatorname{curl}, \Omega;\mathbb{C}^3) \times \gamma_\tau^{\mathcal{T}} H(\operatorname{curl}, \Omega;\mathbb{C}^3)$
- $Y(T) = H(\operatorname{curl}, T;\mathbb{C}^3) \times H(\operatorname{curl}, T;\mathbb{C}^3), \quad Y = \prod_{T\in\mathcal{T}} Y(T)$
- $$\begin{aligned} b_T(E,H,\hat{E},\hat{H};F,G) = {} & i\omega \int_T E\cdot\overline{F}\,\mathrm{d}x + \int_T H\cdot\overline{\operatorname{curl} F}\,\mathrm{d}x - \langle F,\hat{H}\rangle_{\partial T} \\ & - i\omega \int_T H\cdot\overline{G}\,\mathrm{d}x + \int_T E\cdot\overline{\operatorname{curl} G}\,\mathrm{d}x - \langle G,\hat{E}\rangle_{\partial T} \end{aligned}$$

Numerical Example. Ultra-weak dPG for Maxwell

$X_h(T) = P_0(T;\mathbb{C}^3) \times P_0(T;\mathbb{C}^3) \times \gamma_\tau N_1(T;\mathbb{C}) \times \gamma_\tau N_1(T;\mathbb{C})$ and
$Y_h(T) = N_\ell(T;\mathbb{C}^3) \times N_\ell(T;\mathbb{C}^3)$

Experiment. $\Omega = (0,1)^3$, $\omega^2 = 1$, $E = (\sin \pi x \sin \pi y \sin \pi z, 0, 0)$
$k = 1$, uniform refinement



Adaptive Least-Squares FEM

Adaptive LSFEM 4 Stokes – Backward Facing Step

$\Omega = ((-2,8)\times(-1,1))\backslash((-2,0)\times(-1,0))$, $f\equiv 0$,

$$\text{boundary data } g(x_1,x_2) = \begin{cases} 1/10(-x_2(x_2-1),0) & \text{for } x_1=-2,\\ 1/80(-(x_2-1)(x_2+1),0) & \text{for } x_1=8,\\ (0,0) & \text{elsewhere.}\end{cases}$$

$$LS(f;\sigma,u) := \|f+\operatorname{div}\sigma\|_{L^2(\Omega)}\|\operatorname{dev}\sigma - \mathrm{D}\,u\|_{L^2(\Omega)}$$

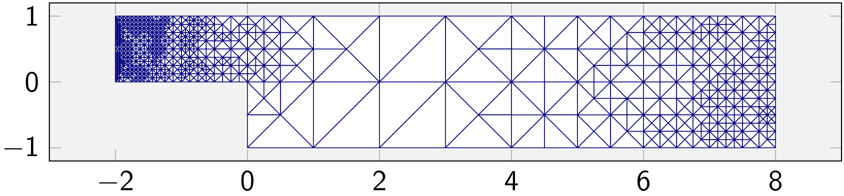
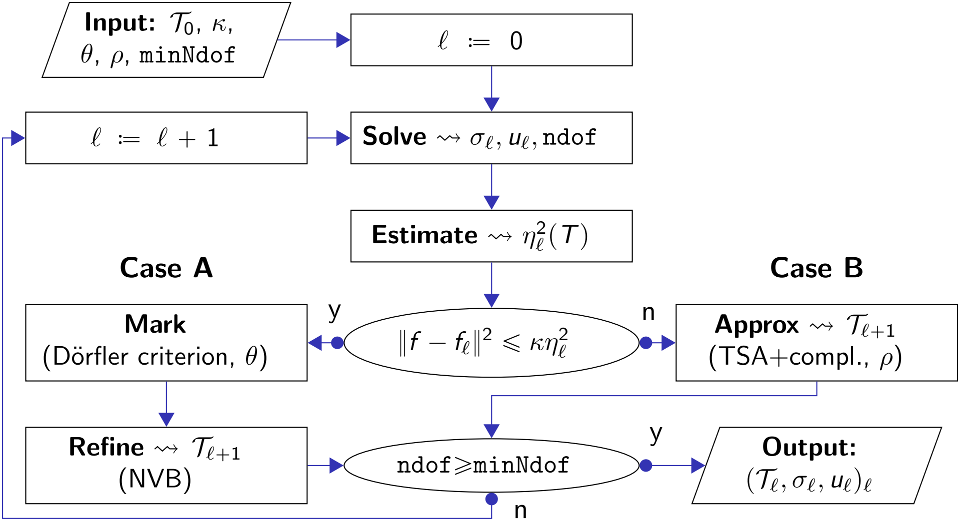


Figure : Triangulation plot with with 1 473 triangles (5 895 degrees of freedom) for adaptive mesh-refinement with η_ℓ and $\theta = 0.5$

Adaptive LSFEM – Adaptive Algorithm with Separate Marking



Adaptive LSFEM – Quasi-Optimal Convergence

Non-linear approximation class $\mathcal{A}_s$ $(u, f) \in \mathcal{A} \times L^2(\Omega; \mathbb{R}^2)$ with

$$|(u,f)|^2_{\mathcal{A}_s} := \sup_{N \in \mathbb{N}} N^{2s} E(u,f,N) < \infty$$

Best possible error

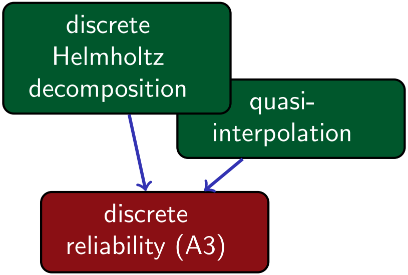
$$E(u,f,N) := \min_{\substack{\mathcal{T} \in \mathbb{T} \\ |\mathcal{T}|-|\mathcal{T}_0| \leqslant N}} \Big(\min_{(\tau_{\mathrm{LS}}, v_{\mathrm{LS}}) \in \Sigma(\mathcal{T}) \times \mathcal{A}(\mathcal{T})} \mathrm{LS}(f; \tau_{\mathrm{LS}}, v_{\mathrm{LS}}) + \mathrm{osc}^2(g', \mathcal{E}(\partial\Omega)) \Big)$$

Optimal convergence rate $\exists 0 < \kappa_0 < \infty$ $\exists 0 < \theta_0 < 1$ $\forall 0 < \kappa \leqslant \kappa_0$ $\forall 0 < \theta \leqslant \theta_0$ $\forall 0 < \rho < 1$ $\forall 0 < s < \infty$,

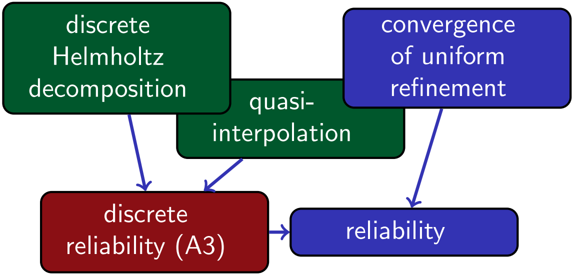
$$\sup_{\ell \in \mathbb{N}} \big(|\mathcal{T}_\ell| - |\mathcal{T}_0|\big)^{2s} \big(\mathrm{LS}(f; \sigma_\ell, u_\ell) + \mathrm{osc}^2(g', \mathcal{E}_\ell(\partial\Omega))\big) \leqslant C_{\mathrm{opt}} |(u,f)|^2_{\mathcal{A}_s}$$

C_{opt} depends solely on $\mathcal{T}_0, s, \kappa, \theta, \rho$

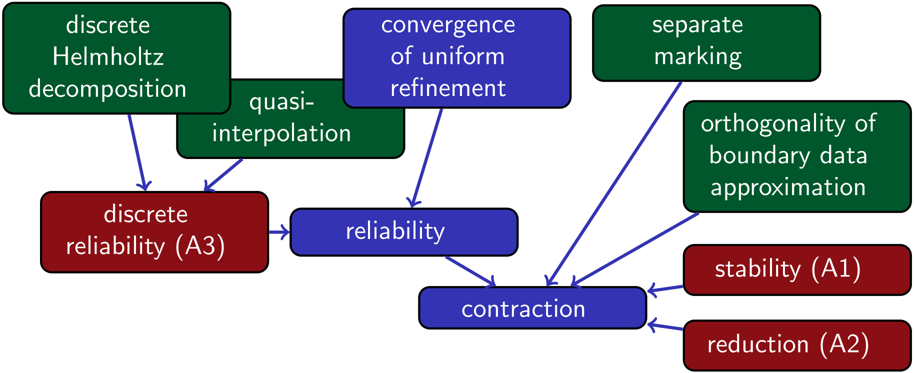
Adaptive LSFEM – Quasi-Optimal Convergence – Proof



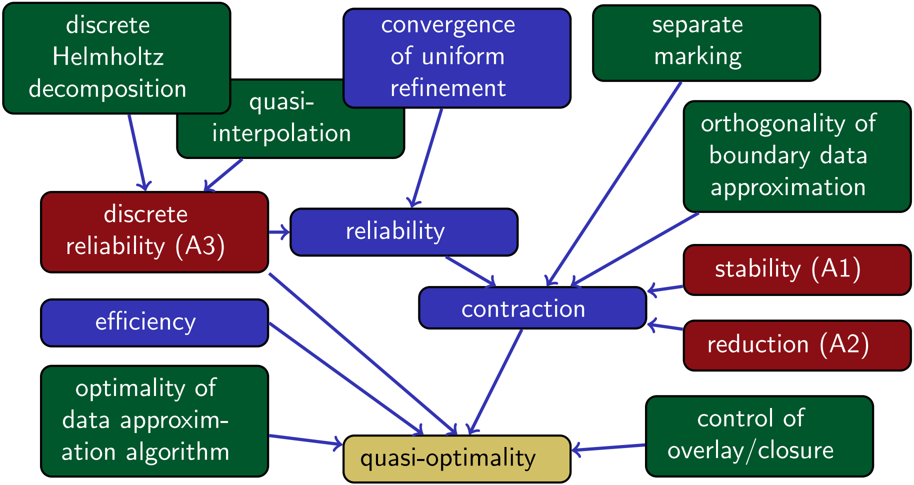
Adaptive LSFEM – Quasi-Optimal Convergence – Proof



Adaptive LSFEM – Quasi-Optimal Convergence – Proof



Adaptive LSFEM – Quasi-Optimal Convergence – Proof



Conclusions

- Comprehensive abstract theory for dPG as a mixed method and/or as a minimum residual method
- More stable and smaller pre-asymptotic range than other/standard methods (e.g. Nédélec-FEM for Maxwell)
- Test search space can be small without loosing stability in the examples presented
- Work in progress on
  - Guaranteed upper error bounds
  - Adaptive mesh design
  - Time-evolving dPG
  - dPG for non-linear problems (e.g. eigenvalue computation)

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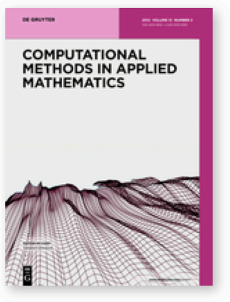

Computational Methods in Applied Mathematics

Editor-in-Chief: Carstensen, Carsten
Managing Editor: Matus, Piotr

4 Issues per year
Mathematical Citation Quotient 2012: 0.34

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