Dimension Reduction Problems for Multiscale Materials in Nonlinear Elasticity.

Part 3: simultaneous homogenization and dimension reduction

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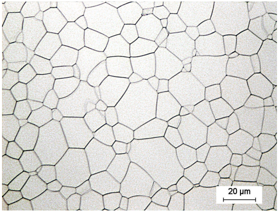
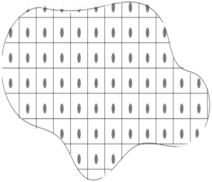
CENTRAL Summerschool
Humboldt-Universität zu Berlin, August 29th-September 2nd 2016.

Overview

- Part 1: dimension reduction problem for homogeneous nonlinearly elastic plates.
- Part 2/tutorial: static Γ-convergence, and the notion of 2-scale convergence.
- Part 3: simultaneous homogenization and dimension reduction.
  - Motivation
  - A little bit of history
  - Homogenization under physical growth conditions

Motivation





In many applications: establish the macroscopic behavior of a material which is "mycroscopically" heterogeneous, in order to study some characteristics of the heterogeneous material (for example its thermal or electrical conductivity).

$$\Downarrow$$

Homogenization problems for thin structures.

Dimension reduction in nonlinear elasticity

Scaling of the applied loads in terms of the thickness parameter

$\Downarrow$

Different scalings of the elastic energy

$\Downarrow$

Different limit models.

Periodic homogenization and dimension reduction

Scaling of the applied loads in terms of the thickness parameter

$\Downarrow$

Different scalings of the elastic energy **&** different ratio thickness/periodicity scale(s)

$\Downarrow$

Different limit models.

A (very) brief history of homogenization and dimension reduction

Seminal papers: membrane regime

$$\left.\begin{array}{l}\text{J-F. Babadjian - M. Baia (2006),}\\ \text{A. Braides - I. Fonseca - G. A. Francfort (2000)}\end{array}\right\}\textbf{p-growth}$$

$$\frac{1}{\beta}|F|^p-\beta\leq W(F)\leq\beta(1+|F|^p).$$

A (very) brief history of homogenization and dimension reduction

Seminal papers: membrane regime

J-F. Babadjian - M. Baia (2006),
A. Braides - I. Fonseca - G. A. Francfort (2000) } **p-growth**

$$\frac{1}{\beta}|F|^p - \beta \leq W(F) \leq \beta(1 + |F|^p).$$

Incompatible with the physical requirement that the energy blows up under very strong compressions.

$$W(F) \to +\infty \quad \text{as } \det F \to 0^+.$$

Homogenization under physical growth conditions for the energy density, at least for models corresponding to very small loads $f^h \approx h^\alpha$, $\alpha > 2$ (Von Kàrmàn plate theories) or $\alpha = 2$ (Kirchhoff plate theories)?

A brief history of homogenization and dimension reduction

Seminal papers: membrane regime

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P. Hornung - S. Neukamm - I. Velčič (2014), S. Neukamm - I. Velčič (2013), I. Velčič (2014), L. Bufford - E.D. - I. Fonseca (2015): homogenization and dimension reduction under **physical growth conditions** for the energy density ($f^h \approx h^\alpha$, $\alpha \geq 2$).

Homogenization with physical growth conditions for a multiscale thin plate

[P. Hornung - S. Neukamm - I. Velčič (2014)], [L. Bufford - E.D. - I. Fonseca (2015)]

Reference configuration:

$$\Omega_h := \omega \times \left(-\frac{h}{2}, \frac{h}{2}\right)$$

- ω=bounded Lipschitz domain in $\mathbb{R}^2$, whose boundary is piecewise C^1,
- $h > 0$=thickness parameter.

Homogenization with physical growth conditions for a multiscale thin plate

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Reference configuration:

$$\Omega_h := \omega \times (-\tfrac{h}{2}, \tfrac{h}{2})$$

- ω=bounded Lipschitz domain in $\mathbb{R}^2$, whose boundary is piecewise C^1,
- $h > 0$=thickness parameter.
- two in plane homogeneity scales - a coarser one and a finer one - $\varepsilon(h)$ and $\varepsilon^2(h)$,
- $\{h\}$ and $\{\varepsilon(h)\}$ are monotone decreasing sequences of positive numbers, $h \to 0$, and $\varepsilon(h) \to 0$ as $h \to 0$.

Homogenization with physical growth conditions

The rescaled nonlinear elastic energy:

$$\mathcal{J}^h(v) := \frac{1}{h}\int_{\Omega_h} W\Big(\frac{x'}{\varepsilon(h)}, \frac{x'}{\varepsilon^2(h)}, \nabla v(x)\Big)\, dx$$

for every deformation $v \in W^{1,2}(\Omega_h;\mathbb{R}^3)$.

Kirchhoff's plate theory: we consider sequences of deformations $\{v^h\} \subset W^{1,2}(\Omega_h;\mathbb{R}^3)$ verifying

$$\limsup_{h\to 0} \frac{\mathcal{J}^h(v^h)}{h^2} < +\infty.$$

Our goal

To identify the effective energy associated to the rescaled elastic energies $\left\{\frac{\mathcal{J}^h(v^h)}{h^2}\right\}$ for different values of

$$\gamma_1 := \lim_{h\to 0} \frac{h}{\varepsilon(h)}$$

and

$$\gamma_2 := \lim_{h\to 0} \frac{h}{\varepsilon^2(h)},$$

i.e. depending on the interaction of the homogeneity scales with the thickness parameter.

Five regimes: $\gamma_1 = +\infty$, $0 < \gamma_1 < +\infty$, $\gamma_1 = 0$ **and** $\gamma_2 = +\infty$, $0 < \gamma_2 < +\infty$, $\gamma_2 = +\infty$.

Assumptions on the stored energy density

$$W : \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{M}^{3\times 3} \to [0, +\infty)$$

(H0) $(\cdot, \cdot, F) \mapsto W(\cdot, \cdot, F)$ is measurable and Q-periodic, $W(y, z, \cdot)$ is continuous,

(H1) $W(y, z, RF) = W(y, z, F)$ for every $F \in \mathbb{M}^{3\times 3}$ and for all $R \in SO(3)$ (**frame indifference**),

(H2) $W(y, z, F) \geq C_1 \,\mathrm{dist}^2(F; SO(3))$ for every $F \in \mathbb{M}^{3\times 3}$ (**nondegeneracy**),

Assumptions on the stored energy density

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(H2) $W(y, z, F) \geq C_1 \operatorname{dist}^2(F; SO(3))$ for every $F \in \mathbb{M}^{3\times 3}$ (**nondegeneracy**),

(H3) there exists $\delta > 0$ such that $W(y, z, F) \leq C_2 \operatorname{dist}^2(F; SO(3))$ for every $F \in \mathbb{M}^{3\times 3}$ with $\operatorname{dist}(F; SO(3)) < \delta$,

(H4) $\lim_{|G|\to 0} \frac{W(y,z,Id+G) - \mathscr{Q}(y,z,G)}{|G|^2} = 0$, where $\mathscr{Q}(y, z, \cdot)$ is a quadratic form on $\mathbb{M}^{3\times 3}$.

Change of variables

We focus on the asymptotic behavior of sequences of deformations $\{u^h\} \subset W^{1,2}(\Omega;\mathbb{R}^3)$ satisfying the uniform energy estimate

$$\mathcal{E}^h(u^h) := \int_\Omega W\Big(\frac{x'}{\varepsilon(h)}, \frac{x'}{\varepsilon^2(h)}, \nabla_h u^h(x)\Big)\,dx \leq Ch^2 \quad \text{for every } h > 0.$$

where $\Omega := \Omega_1 = \omega \times (-\frac{1}{2}, \frac{1}{2})$, and $\nabla_h u(x) := \Big(\nabla' u(x) \Big| \frac{\partial_{x_3} u(x)}{h}\Big)$ for a.e. $x \in \Omega$.

Remark

For W independent of y and z, such scalings of the energy lead to Kirchhoff's nonlinear plate theory [G. Friesecke - R.D James - S. Müller (2006)].

Compactness

Theorem (G. Friesecke - R.D James - S. Müller (2006))

Let $\{u^h\} \subset W^{1,2}(\Omega;\mathbb{R}^3)$ *satisfy the uniform energy estimate. Then, there exists a map* $u \in W^{2,2}(\omega;\mathbb{R}^3)$ *such that, up to subsequences,*

$$u^h - \fint_\Omega u^h(x)\,dx \to u \quad \text{strongly in } L^2(\Omega;\mathbb{R}^3)$$

$$\nabla_h u^h \to (\nabla' u | n_u) \quad \text{strongly in } L^2(\Omega;\mathbb{M}^{3\times 3}),$$

with

$$\partial_{x_\alpha} u(x') \cdot \partial_{x_\beta} u(x') = \delta_{\alpha,\beta} \quad \text{for a.e. } x' \in \omega, \qquad \alpha,\beta \in \{1,2\}$$

and

$$n_u(x') := \partial_{x_1} u(x') \wedge \partial_{x_2} u(x') \quad \text{for a.e. } x' \in \omega.$$

The limit model

Theorem (L. Bufford - E.D. - I. Fonseca (2015))

Let $\gamma_1 \in [0, +\infty]$ *and let* $\gamma_2 = +\infty$. *Let* $\{u^h\} \subset W^{1,2}(\Omega;\mathbb{R}^3)$ *and* $u \in W^{2,2}(\omega;\mathbb{R}^3)$ *be as in Theorem 1. Then*

$$\liminf_{h\to 0} \frac{\mathcal{E}^h(u^h)}{h^2} \geq \mathcal{E}^{\gamma_1}(u).$$

Moreover, for every $u \in W^{2,2}(\omega;\mathbb{R}^3)$ *as in Theorem 1, there exists a sequence* $\{u^h\} \subset W^{1,2}(\Omega;\mathbb{R}^3)$ *such that*

$$\limsup_{h\to 0} \frac{\mathcal{E}^h(u^h)}{h^2} \leq \mathcal{E}^{\gamma_1}(u).$$

The limit model

Theorem (L. Bufford - E.D. - I. Fonseca (2015))

The effective energy is given by

$$\mathcal{E}^{\gamma_1}(u) := \begin{cases} \frac{1}{12}\int_\omega \overline{\mathscr{Q}}^{\gamma_1}_{\mathrm{hom}}(\Pi^u(x'))\,dx' & \textit{if } u \textit{ is as in Theorem 1,} \\ +\infty & \textit{otherwise in } L^2(\Omega;\mathbb{R}^3), \end{cases}$$

where Π^u *is the second fundamental form associated to* u*,*

$$\Pi^u_{\alpha,\beta}(x') := -\partial^2_{\alpha,\beta}u(x')\cdot n_u(x') \quad \textit{for } \alpha,\beta = 1,2,$$

$n_u(x') := \partial_1 u(x') \wedge \partial_2 u(x')$*, and* $\overline{\mathscr{Q}}^{\gamma_1}_{\mathrm{hom}}$ *is a quadratic from dependent on the value of* γ_1*.*

The limit model

Theorem ($0 < \gamma_1 < +\infty$.)

In particular, if $0 < \gamma_1 < +\infty$, *for every* $A \in \mathbb{M}^{2\times 2}_{\rm sym}$

$$\overline{\mathscr{Q}}^{\gamma_1}_{\rm hom}(A) := \inf\Bigg\{ \int_{\left(-\frac12,\frac12\right)\times Q} \mathscr{Q}_{\rm hom}\Bigg(y, \begin{pmatrix} x_3 A + B & 0 \\ 0 & 0 \end{pmatrix} + {\rm sym}\left(\nabla_y \phi_1(x_3,y)\Big|\frac{\partial_{x_3}\phi_1(x_3,y)}{\gamma_1}\right)\Bigg) :$$

$$\phi_1 \in W^{1,2}\left(\left(-\tfrac12,\tfrac12\right); W^{1,2}_{\rm per}(Q;\mathbb{R}^3)\right),\ B \in \mathbb{M}^{2\times 2}_{\rm sym}\Bigg\};$$

where

$$\mathscr{Q}_{\rm hom}(y,C) := \inf\left\{ \int_Q \mathscr{Q}\big(y,z,C + {\rm sym}\left(\nabla\phi_2(z)\big|0\right)\big) : \phi_2 \in W^{1,2}_{\rm per}(Q;\mathbb{R}^3)\right\}$$

for a.e. $y \in Q$, *and for every* $C \in \mathbb{M}^{3\times 3}_{\rm sym}$.

The limit model

Theorem ($\gamma_1 = +\infty$)

If $\gamma_1 = +\infty$, *for every* $A \in \mathbb{M}^{2\times 2}_{\mathrm{sym}}$

$$\overline{\mathscr{Q}}^{\infty}_{\mathrm{hom}}(A) := \inf \Bigg\{ \int_{\left(-\frac{1}{2},\frac{1}{2}\right)\times Q} \mathscr{Q}_{\mathrm{hom}}\Big(y, \begin{pmatrix} x_3 A + B & 0 \\ 0 & 0 \end{pmatrix} + \mathrm{sym}\left(\nabla_y \phi_1(x_3, y) | d(x_3)\right)\Big) : d \in L^2((-\tfrac{1}{2}, \tfrac{1}{2}); \mathbb{R}^3),$$

$$\phi_1 \in L^2((-\tfrac{1}{2}, \tfrac{1}{2}); W^{1,2}_{\mathrm{per}}(Q; \mathbb{R}^3)), \text{ and } B \in \mathbb{M}^{2\times 2}_{\mathrm{sym}} \Bigg\}.$$

The limit model

Theorem ($\gamma_1 = 0$)

If $\gamma_1 = 0$*, for every* $A \in \mathbb{M}^{2\times 2}_{\mathrm{sym}}$

$$\overline{\mathscr{Q}}^0_{\mathrm{hom}}(A) := \inf \Bigg\{ \int_{(-\frac{1}{2},\frac{1}{2})\times Q} \mathscr{Q}_{\mathrm{hom}} \Bigg(y, \begin{pmatrix} x_3 A + B & 0 \\ 0 & 0 \end{pmatrix} + \mathrm{sym} \begin{pmatrix} \mathrm{sym}\,\nabla_y \xi(x_3,y) + x_3 \nabla_y^2 \eta(y) & g_1(x_3,y) \\ & g_2(x_3,y) \\ g_1(x_3,y) \quad g_2(x_3,y) & g_3(x_3,y) \end{pmatrix} \Bigg) :$$

$$\xi \in L^2\big(\big(-\tfrac{1}{2},\tfrac{1}{2}\big); W^{1,2}_{\mathrm{per}}(Q;\mathbb{R}^2)\big),\ \eta \in W^{2,2}_{\mathrm{per}}(Q),$$

$$g_i \in L^2\big(\big(-\tfrac{1}{2},\tfrac{1}{2}\big)\times Q\big), i = 1,2,3, B \in \mathbb{M}^{2\times 2}_{\mathrm{sym}} \Bigg\}.$$

A few questions

- Why are there pointwise minimizations with respect to gradients in the periodicity variables?
- How does the value of γ_1 determine the different minimization problems?
- Where does two-scale convergence come into play?

Proof of the liminf inequality for $\gamma_1 \in (0, +\infty)$(sketch)

1. Convergence of scaled stresses

$$|\sqrt{F^T F} - Id|^2 \leq C\mathrm{dist}^2(F; SO(3)) \leq W(y, z, F)$$

&

Uniform energy estimate

Proof of the liminf inequality for $\gamma_1 \in (0, +\infty)$(sketch)

1. Convergence of scaled stresses

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&

Uniform energy estimate

$$\Downarrow$$

Uniform bound on the L^2-norm of the sequence of linearized stresses

$$E^h(x) := \frac{\sqrt{(\nabla_h u^h(x))^T \nabla_h u^h(x)} - Id}{h}.$$

Proof of the liminf inequality for $\gamma_1 \in (0, +\infty)$(sketch)

1. Convergence of scaled stresses

Linearization of the stored energy density around the identity

$$\Downarrow$$

$$\liminf_{h\to 0} \frac{\mathcal{E}^h(u^h)}{h^2} \approxeq \liminf_{h\to 0} \int_\Omega \mathscr{Q}\Big(\frac{x'}{\varepsilon(h)}, \frac{x'}{\varepsilon^2(h)}, E^h(x)\Big)\, dx.$$

Proof of the liminf inequality for $\gamma_1 \in (0, +\infty)$(sketch)

1. Convergence of scaled stresses

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Key point: to identify the **multiscale** limit of the sequence E^h.
Key ingredient: multiscale convergence adapted to dimension reduction.

Proof of the liminf inequality for $\gamma_1 \in (0, +\infty)$(sketch)

Definition (G. Allaire (1992), D. Lukkassen - G. Nguetseng - P. Wall (2002), G. Nguetseng (1989), G.Allaire - M. Briane (1996))

Let $u \in L^2(\Omega \times Q \times Q)$ and $\{u^h\} \in L^2(\Omega)$. We say that $\{u^h\}$ *converges weakly 3-scale to u* in $L^2(\Omega \times Q \times Q)$, and we write $u^h \xrightarrow{3-s} u$, if

$$\int_\Omega u^h(\xi)\varphi\Big(\xi, \frac{\xi}{\varepsilon(h)}, \frac{\xi}{\varepsilon^2(h)}\Big)\, d\xi \to \int_\Omega \int_Q \int_Q u(\xi,\eta,\lambda)\varphi(\xi,\eta,\lambda)\, d\lambda\, d\eta\, d\xi$$

for every $\varphi \in C_c^\infty(\Omega; C_{\mathrm{per}}(Q \times Q))$.

Proof of the liminf inequality for $\gamma_1 \in (0, +\infty)$(sketch)

Definition (S. Neukamm (2010))

Let $u \in L^2(\Omega \times Q \times Q)$ and $\{u^h\} \in L^2(\Omega)$. We say that $\{u^h\}$ *converges weakly dr-3-scale to u* in $L^2(\Omega \times Q \times Q)$, and we write $u^h \overset{dr-3-s}{\longrightarrow} u$, if

$$\int_\Omega u^h(x)\varphi\Big(x, \frac{x'}{\varepsilon(h)}, \frac{x'}{\varepsilon^2(h)}\Big)\,dx \to \int_\Omega \int_Q \int_Q u(x,y,z)\varphi(x,y,z)\,dz\,dy\,dx$$

for every $\varphi \in C_c^\infty(\Omega; C_{\mathrm{per}}(Q \times Q))$.

Remark

Bounded sequences in L^2 are precompact with respect to multiscale convergence

Proof of the liminf inequality for $\gamma_1 \in (0, +\infty)$(sketch)

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for every $\varphi \in C_c^\infty(\Omega; C_{\mathrm{per}}(Q \times Q))$.

Remark

Bounded sequences in L^2 are precompact with respect to multiscale convergence

Question: how are 3-scale limits, 2-scale limits, and weak L^2-limit related? On the blackboard!

Proof of the liminf inequality for $\gamma_1 \in (0, +\infty)$(sketch)

Theorem (Multiscale limits of scaled gradients)

Let $u, \{u^h\} \subset W^{1,2}(\Omega)$ *be such that*

$$u^h \rightharpoonup u \quad \textit{weakly in } W^{1,2}(\Omega).$$

and

$$\limsup_{h \to 0} \int_\Omega |\nabla_h u^h(x)|^2 \, dx < \infty.$$

Then u *is independent of* x_3.

Proof of the liminf inequality for $\gamma_1 \in (0, +\infty)$(sketch)

Theorem (Multiscale limits of scaled gradients)

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and

$$\limsup_{h \to 0} \int_\Omega |\nabla_h u^h(x)|^2 \, dx < \infty.$$

Then u *is independent of* x_3*.Moreover, there exist* $u_1 \in L^2(\Omega; W^{1,2}_{\mathrm{per}}(Q))$, $u_2 \in L^2(\Omega \times Q; W^{1,2}_{\mathrm{per}}(Q))$, *and* $\bar{u} \in L^2\big(\omega \times Q \times Q; W^{1,2}\big(-\frac{1}{2}, \frac{1}{2}\big)\big)$ *such that, up to the extraction of a (not relabeled) subsequence,*

$$\nabla_h u^h \overset{dr-3-s}{\rightharpoonup} \Big(\nabla' u + \nabla_y u_1 + \nabla_z u_2 \Big| \partial_{x_3} \bar{u}\Big) \quad \textit{weakly dr-3-scale.}$$

Proof of the liminf inequality for $\gamma_1 \in (0, +\infty)$(sketch)

Theorem (Multiscale limits of scaled gradients)

Moreover,

(i) *if* $\gamma_1 = \gamma_2 = +\infty$ *(i.e.* $\varepsilon(h) << h$*), then* $\partial_{y_i} \bar{u} = \partial_{z_i} \bar{u} = 0$*, for* $i = 1, 2$*;*

(ii) *if* $0 < \gamma_1 < +\infty$ *and* $\gamma_2 = +\infty$ *(i.e.* $\varepsilon(h) \sim h$*), then*

$$\bar{u} = \frac{u_1}{\gamma_1};$$

(iii) *if* $\gamma_1 = 0$ *and* $\gamma_2 = +\infty$ *(i.e.* $h << \varepsilon(h) << h^{\frac{1}{2}}$*), then*

$$\partial_{x_3} u_1 = 0 \quad \text{and} \quad \partial_{z_i} \bar{u} = 0, \ i = 1, 2.$$

Proof of the liminf inequality for $\gamma_1 \in (0, +\infty)$(sketch)

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$$\partial_{x_3} u_1 = 0 \quad \text{and} \quad \partial_{z_i} \bar{u} = 0,\ i = 1, 2.$$

Question: why do we have such a structure for multiscale limits of scaled gradients? On the blackboard!

Proof of the liminf inequality for $\gamma_1 \in (0, +\infty)$(sketch)

2. The rigidity estimate

Theorem (G. Friesecke - R.D. James - S. Müller (2002))

Let $\gamma_0 \in (0,1]$ *and let* $h, \delta > 0$ *be such that*

$$\gamma_0 \leq \frac{h}{\delta} \leq \frac{1}{\gamma_0}.$$

There exists a constant C*, depending only on* ω *and* γ_0*, such that for every* $u \in W^{1,2}(\omega; \mathbb{R}^3)$ *there exists a map* $R : \omega \to SO(3)$ *piecewise constant on each cube* $x + \delta Y$*, with* $x \in \delta\mathbb{Z}^2$*, and there exists* $\tilde{R} \in W^{1,2}(\omega; \mathbb{M}^{3\times 3})$ *such that*

$$\|\nabla_h u - R\|^2_{L^2(\Omega;\mathbb{M}^{3\times 3})} + \|R - \tilde{R}\|^2_{L^2(\omega;\mathbb{M}^{3\times 3})} + h^2 \|\nabla' \tilde{R}\|^2_{L^2(\omega;\mathbb{M}^{3\times 3}\times\mathbb{M}^{3\times 3})} \leq C\|\mathrm{dist}(\nabla_h u; SO(3))\|_{L^2(\Omega)}.$$

Proof of the liminf inequality

3. Compactness of linearized strains

$$\gamma_1 := \lim_{h\to 0} \frac{h}{\varepsilon(h)} \in (0,+\infty)$$

$$\Downarrow$$

Apply the theorem with $\delta = \varepsilon(h)$ and construct maps R^h piecewise constant on cubes of size $\varepsilon(h)$ and centers in $\varepsilon(h)\mathbb{Z}^2$ such that

$$\|\nabla_h u^h - R^h\|^2_{L^2(\Omega;\mathbb{M}^{3\times 3})} \leq C\|\mathrm{dist}(\nabla_h u^h; SO(3))\|_{L^2(\Omega)} \leq Ch^2.$$

$$\Downarrow$$

The sequence of linearized strains

$$G^h(x) := \frac{R^h(x')^T \nabla_h u^h(x) - Id}{h}$$

is uniformly bounded in L^2.

Proof of the liminf inequality

4. Stress-strain relation and liminf inequality

$$E^h(x) := \frac{\sqrt{(\nabla_h u^h(x))^T \nabla_h u^h(x)} - Id}{h}$$

$$= \frac{\sqrt{(Id + hR^h(x')G^h(x))^T (Id + hR^h(x')G^h(x))} - Id}{h}$$

$$\approx \mathrm{sym} R^h(x')G^h(x) \approx \mathrm{sym} \frac{\nabla_h u^h(x) - R^h(x')}{h}.$$

The problem becomes:
to identify the multiscale limit of the sequence

$$\mathrm{sym} \frac{\nabla_h u^h - R^h}{h}.$$

Proof of the liminf inequality

5. Identification of the limit strain

Idea: rewrite u^h as

$$u^h(x) =: \bar{u}^h(x') + hx_3\tilde{R}^h(x')e_3 + hr^h(x', x_3)$$

where

$$\bar{u}^h(x') := \int_{-\frac{1}{2}}^{\frac{1}{2}} u^h(x', x_3)\, dx_3.$$

Then

$$\frac{\nabla_h u^h - R^h}{h} = \Big(\frac{\nabla' \bar{u}^h - (R^h)'}{h} + x_3 \nabla' \tilde{R}^h e_3 \Big| \frac{(\tilde{R}^h - R^h)}{h} e_3\Big) + \nabla_h r^h.$$

Proof of the liminf inequality

5. Identification of the limit strain

Bounded sequences in L^2 are precompact with respect to multiscale convergence

$$\Downarrow$$

$$\frac{\nabla' \bar{u}^h - (R^h)'}{h} \xrightarrow{3-s} V \quad \text{weakly 3-scale.}$$

By the results by [P. Hornung - S. Neukamm - I. Velčič (2014)] and the relation between 3-scale limits and 2-scale limits we only need to show

$$V(x', y, z) - \int_Q V(x', y, \xi)\, d\xi = \nabla_z v(x', y, z)$$

for some $v \in L^2(\Omega \times Q; W^{1,2}_{\mathrm{per}}(Q))$...

Proof of the liminf inequality

5. Identification of the limit strain

...that is

$$\int_\Omega \int_Q \int_Q \Big(V(x', y, z) - \int_Q V(x', y, \xi)\, d\xi \Big) : (\nabla')^\perp \varphi(z) \psi(x', y)\, dx\, dy\, dz = 0$$

for every $\varphi \in C^1_{\mathrm{per}}(Q; \mathbb{R}^3)$ and $\psi \in C^\infty_c(\omega; C^\infty_{\mathrm{per}}(Q))$, where

$$(\nabla')^\perp \varphi(z) := \Big(-\partial_{z_2} \varphi(z) | \partial_{z_1} \varphi(z) \Big).$$

Proof of the liminf inequality

5. Identification of the limit strain

...that is

$$\int_\Omega \int_Q \int_Q \Big(V(x', y, z) - \int_Q V(x', y, \xi)\, d\xi \Big) : (\nabla')^\perp \varphi(z) \psi(x', y)\, dx\, dy\, dz = 0$$

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$$\Downarrow$$

Test functions of the form

$$(\nabla')^\perp \varphi\Big(\frac{x'}{\varepsilon^2(h)}\Big) \psi\Big(x', \frac{x'}{\varepsilon(h)}\Big).$$

Proof of the liminf inequality

5. Identification of the limit strain

We need to identify

$$\lim_{h\to 0}\int_{\omega}\frac{\nabla'\bar{u}^h(x')-(R^h)'(x')}{h}:(\nabla')^{\perp}\varphi\Big(\frac{x'}{\varepsilon^2(h)}\Big)\psi\Big(x',\frac{x'}{\varepsilon(h)}\Big)\,dx.$$

Proof of the liminf inequality

5. Identification of the limit strain

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- Step 1:

$$\lim_{h\to 0}\int_\omega \frac{\nabla' \bar{u}^h(x)}{h} : (\nabla')^\perp \varphi\Big(\frac{x'}{\varepsilon^2(h)}\Big)\psi\Big(x', \frac{x'}{\varepsilon(h)}\Big)\,dx = 0.$$

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- Step 2:

$$\int_\Omega\int_Q\int_Q\Big(\int_Q V(x', y, \xi)\, d\xi\Big) : (\nabla')^\perp\varphi(z)\psi(x', y)\, dx\, dy\, dz = 0.$$

Proof of the liminf inequality

5. Identification of the limit strain

We need to identify

$$\lim_{h\to 0}\int_\omega \frac{\nabla'\bar{u}^h(x')-(R^h)'(x')}{h}:(\nabla')^\perp\varphi\Big(\frac{x'}{\varepsilon^2(h)}\Big)\psi\Big(x',\frac{x'}{\varepsilon(h)}\Big)\,dx.$$

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- Step 2:

$$\int_\Omega\int_Q\int_Q\Big(\int_Q V(x',y,\xi)\,d\xi\Big):(\nabla')^\perp\varphi(z)\psi(x',y)\,dx\,dy\,dz=0.$$

- Step 3:

$$\lim_{h\to 0}\int_\omega \frac{(R^h)'(x')}{h}:(\nabla')^\perp\varphi\Big(\frac{x'}{\varepsilon^2(h)}\Big)\psi\Big(x',\frac{x'}{\varepsilon(h)}\Big)\,dx=0$$

Proof of the liminf inequality

5. Identification of the limit strain

Idea: the maps R^h are piecewise constant con cubes of size $\varepsilon(h)$ and centers in $\varepsilon(h)\mathbb{Z}^2$....

Proof of the liminf inequality

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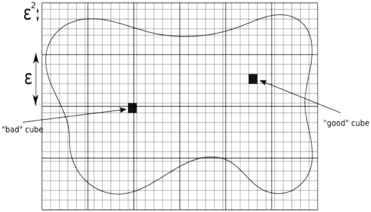
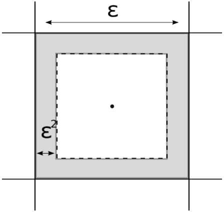
Idea: the maps R^h are piecewise constant con cubes of size $\varepsilon(h)$ and centers in $\varepsilon(h)\mathbb{Z}^2$....
Main difficulty: ...but we have oscillations on cubes of size $\varepsilon^2(h)$ and centers in $\varepsilon^2(h)\mathbb{Z}^2$.

Proof of the liminf inequality

5. Identification of the limit strain

Solution: to distinguish between **"bad cubes"** and **"good cubes"** and show that the measure of the intersection between ω and the set of "bad cubes" converges to zero faster than or comparable to $\varepsilon(h)$, as $h \to 0$.





Final remarks on the case $\gamma_1 = 0$.

- By G. Friesecke, R.D. James and S. Müller's rigidity estimate: work with **sequences of piecewise constant rotations which are constant on cubes of size** $\varepsilon^2(h)$ having centers in the grid $\varepsilon^2(h)\mathbb{Z}^2$.
- To identify the limit multiscale stress we need to deal with **oscillating test functions with vanishing averages on a scale** $\varepsilon(h)$.

Final remarks on the case $\gamma_1 = 0$.

The identification of "good cubes" and "bad cubes" of size $\varepsilon^2(h)$ is not helpful as the contribution of the oscillating test functions on cubes of size $\varepsilon^2(h)$ is not negligible anymore.

We are only able to perform an identification of the multiscale limit in the case $\gamma_2 = +\infty$, extending to the **multiscale setting** the results obtained by **I. Velčič**. The identification of the effective energy in the case in which $\gamma_1 = 0$ and $\gamma_2 \in [0, +\infty)$ remains an open question.

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Thank you for your attention!