Evolutionary Γ-convergence

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Analysis of Multi-Scale Systems Driven by Functionals

CENTRAL Workshop page: $\rightarrow$ Materials

Materials for lecture of Alexander Mielke

Survey Article

A. Mielke (2016):
On evolutionary Γ-convergence for gradient systems.

Chapter 3 (pages 187–249) in the Proceedings of Summer School 2012.

Muntean, Rademacher, Zagaris: *Macroscopic and Large Scale Phenomena: Coarse Graining, Mean Field Limits and Ergodicity.*
Lecture Notes in Applied Math. & Mechanics Vol. 3, Springer 2016.

Overview

Aim of these lectures:

- Evolutionary systems (time-dependent O/PDEs) with multiple scales
  $0 < \varepsilon = 1/n \ll 1$ small parameter
- Describe mathematical methods for limit passage $\varepsilon \to 0$
  ($\varepsilon = h$ contains the case of numerical convergence!)

Restriction:

- only generalized gradient systems
- only very simple applications
- proofs only for the simplest results

General evolutionary equations

Multiscale limit corresponds to interchanging to limits, namely

"$\lim_{\varepsilon\to 0}$" and "$u^\varepsilon(t) = u_0^\varepsilon + \int_0^t \ldots \mathrm{d}s$"

	microsc. system		macrosc. system
	$\dot u_\varepsilon = \boldsymbol{G}_\varepsilon(u_\varepsilon)$		$\dot y = \boldsymbol{G}_0(y)$
initial state	u_ε^0	$\xrightarrow[\boldsymbol{M}_\varepsilon]{\text{upscaling}}$	y^0
time evolution	$\downarrow$		$\downarrow$
	$u_\varepsilon(t) = \boldsymbol{S}_\varepsilon(t, u_\varepsilon^0)$	$\xrightarrow[\boldsymbol{M}_\varepsilon]{\text{upscaling}}$	$y(t) = \boldsymbol{S}_0(t, y^0)$

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Mathematical task: Prove $\lim_{\varepsilon\to 0} \boldsymbol{M}_\varepsilon \circ \boldsymbol{S}_\varepsilon(t,\cdot) = \boldsymbol{S}_0(t, \lim_{\varepsilon\to 0} \boldsymbol{M}_\varepsilon(\cdot))$

We say that **the PDEs $\dot u = G_\varepsilon(u)$ *evolutionary converge to* $\dot u = G_0(u)$.**

Γ-convergence is a purely static concept

At first sight there is no relation to evolution.

- $\mathcal{J}_0,\ \mathcal{J}_\varepsilon : \boldsymbol{X} \to \mathbb{R}$ are a functionals, $\boldsymbol{X}$ sep./refl. Banach space
- If $\mathcal{J}_\varepsilon \xrightharpoonup{\Gamma} \mathcal{J}_0$, then *solutions of minimization problems converge*:

  $\forall\, \ell \in \boldsymbol{X}^*$ we have

$$\left.\begin{array}{c} u_\varepsilon \in \underset{w\in \boldsymbol{X}}{\mathrm{ArgMin}}\Big(\mathcal{J}_\varepsilon(w) - \langle \ell, w\rangle\Big) \\ \text{and } u_\varepsilon \rightharpoonup u \end{array}\right\} \Longrightarrow\ u \in \underset{w\in \boldsymbol{X}}{\mathrm{ArgMin}}\Big(\mathcal{J}_0(w) - \langle \ell, w\rangle\Big)$$

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Evolutionary Γ-convergence means "evolutionary convergence" for time-dependent O/PDEs that are given in terms of functionals.

$\rightsquigarrow$ evolution/dynamics driven by functionals

An example for **evolution driven by functionals**

- the damped wave equation

$$\rho(x)\ddot{u}(t,x) + \delta(x)\dot{u}(t,x) = \mathrm{div}(A(x)\nabla u(t,x)) + f(t,x) \text{ in } \Omega + \text{Dir.B.C.}$$

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What are the relevant functionals?

- kinetic energy $\mathcal{K}(\dot{u}) = \int_\Omega \frac{\rho}{2}\dot{u}^2 \,\mathrm{d}x$
- potential energy $\mathcal{E}(t,u) = \int_\Omega \big(\frac{1}{2}\nabla u \cdot A\nabla u - uf(t)\big)\,\mathrm{d}x$

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- dissipation potential $\mathcal{R}(\dot{u}) = \int_\Omega \frac{\delta}{2}\dot{u}^2 \,\mathrm{d}x$

The PDE is given in terms of the three functionals $\mathcal{K}$, $\mathcal{E}$, and $\mathcal{R}$ via the force balance (cf. lectures by T. Roubíček or E. Davoli)

$$0 \;=\; \underbrace{\frac{\partial}{\partial t}\Big(\mathrm{D}_{\dot{u}}\mathcal{K}(\dot{u})\Big)}_{\text{inertial terms}} + \underbrace{\mathrm{D}_{\dot{u}}\mathcal{R}(\dot{u})}_{\text{dissipation}} + \underbrace{\mathrm{D}_u\mathcal{E}(t,u)}_{\text{potential force}}$$

Slightly more general O/PDE driven by functionals

$$(\mathrm{DE})_\varepsilon \quad 0 = \frac{\partial}{\partial t}\Big(\mathrm{D}_{\dot u}\mathcal{K}_\varepsilon(u,\dot u)\Big) + \mathrm{D}_{\dot u}\mathcal{R}_\varepsilon(u,\dot u) + \mathrm{D}_u\mathcal{E}_\varepsilon(t,u)$$

Naïve hope of evolutionary Γ convergence

$$\left.\begin{array}{c} \mathcal{K}_\varepsilon \xrightarrow{\Gamma} \mathcal{K}_0 \\ \mathcal{R}_\varepsilon \xrightarrow{\Gamma} \mathcal{R}_0 \\ \mathcal{E}_\varepsilon \xrightarrow{\Gamma} \mathcal{E}_0 \end{array}\right\} \Longrightarrow \quad (\mathrm{DE})_\varepsilon \xrightarrow{\text{evol}} (\mathrm{DE})_0$$

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- In general, this is wrong since the convergences need to be “compatible”.
  (In numerics: discretizations of different parts need to be compatible.)
- True goal: Find sufficient compatibility conditions for the convergences.

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Gradient flows = evolution driven by **gradient systems** $(\boldsymbol{X}, \mathcal{E}, \mathbb{G})$

- $u \in \boldsymbol{X}$ = state space (closed convex subset of a reflexive Banach space)
- $\mathcal{E} : [0,T] \times \boldsymbol{X} \to \mathbb{R}_\infty := \mathbb{R} \cup \{\infty\}$ energy functional
- $\mathbb{G}(u) : \mathrm{T}_u\boldsymbol{X} = \boldsymbol{X} \to \mathrm{T}^*_u\boldsymbol{X} = \boldsymbol{X}^*$ metric structure
  (Riemannian tensor with $\mathbb{G}(u) = \mathbb{G}(u)^* \geq 0$)

A gradient system induces a DE via (the force balance)

$$0 = \underbrace{\mathbb{G}(u)\dot{u}}_{\text{visc.force}} + \underbrace{\mathrm{D}_u\mathcal{E}(t,u)}_{\text{rest.force}} \in \mathrm{T}^*_u\boldsymbol{X} = \boldsymbol{X}^*$$

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This gives the equivalent formulation (gradient flow)

$$\dot{u} = -\nabla_{\mathbb{G}}\mathcal{E}(t,u) = -\mathbb{G}(u)^{-1}\mathrm{D}_u\mathcal{E}(t,u) \;\in\; \mathrm{T}_u\boldsymbol{X} = \boldsymbol{X}$$

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2. Gradient systems

$$0 = \mathbb{G}(u)\dot{u} + \mathrm{D}_u\mathcal{E}(t,u) \ \in \boldsymbol{X}^* \qquad \Longleftrightarrow \qquad \dot{u} = -\mathbb{K}(u)\mathrm{D}_u\mathcal{E}(t,u) \ \in \boldsymbol{X}$$

The metric tensor $\mathbb{G}$ is uniquely characterized by a quadratic form, namely the **(primal) dissipation potential**

$$\mathcal{R}(u,\dot{u}) := \tfrac{1}{2}\langle \underbrace{\mathbb{G}(u)\dot{u}}_{\in \boldsymbol{X}^*}, \underbrace{\dot{u}}_{\in \boldsymbol{X}} \rangle \qquad \Longrightarrow \qquad \mathrm{D}_{\dot{u}}\mathcal{R}(u,\dot{u}) = \mathbb{G}(u)\dot{u} \in \boldsymbol{X}^*$$

We introduce the short-hand $\mathbb{K}(u) := \mathbb{G}(u)^{-1}$ and define the **dual dissipation potential**

$$\mathcal{R}^*(u,\xi) := \tfrac{1}{2}\langle \underbrace{\xi}_{\in \boldsymbol{X}^*}, \underbrace{\mathbb{K}(u)\xi}_{\in \boldsymbol{X}} \rangle \qquad \Longrightarrow \qquad \mathrm{D}_{\xi}\mathcal{R}^*(u,\xi) = \mathbb{K}(u)\xi \in \boldsymbol{X}$$

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- Gradient systems can also be denoted by

$$(\boldsymbol{X},\mathcal{E},\mathbb{G}) \ = \ (\boldsymbol{X},\mathcal{E},\mathcal{R}) \ = \ (\boldsymbol{X},\mathcal{E},\mathcal{R}^*) \ = \ (\boldsymbol{X},\mathcal{E},\mathbb{K})$$

- The induced equation can be written as

$$0 = \mathrm{D}_{\dot u}\mathcal{R}(u,\dot u) + \mathrm{D}_u\mathcal{E}(t,u) \in \boldsymbol{X}^* \qquad \Longleftrightarrow \qquad \dot u = \mathrm{D}_\xi\mathcal{R}^*\big(u, -\mathrm{D}_u\mathcal{E}(t,u)\big) \in \boldsymbol{X}$$

Generalized gradient systems $(\boldsymbol{X}, \mathcal{E}, \mathcal{R})$

$\mathcal{R}(u, \dot{u})$ general dissipation potential, which means that

$$\mathcal{R}(u,\cdot){:}\boldsymbol{X} \to [0,\infty] \text{ is convex, lower semi-continuous, and } \mathcal{R}(u,0)=0.$$

The possible dissipative forces are given by the (set-valued) convex subdifferential $\partial_{\dot{u}}\mathcal{R}(u,\dot{u}) = \{\, \xi{\in}\boldsymbol{X}^* \,|\, \forall w{\in}\boldsymbol{X}{:}\ \mathcal{R}(u,w){\geq}\mathcal{R}(u,\dot{u}){+}\langle \xi, w{-}u\rangle \,\}$.

$$\text{(DE)} \quad 0 \in \partial_{\dot{u}}\mathcal{R}(u,\dot{u}) + \mathrm{D}_u\mathcal{E}(t,u)$$

Classical gradient system $\mathcal{R}(u,v) = \frac{1}{2}\langle \mathbb{G}(u)v, v\rangle$ (quadratic)

More general $\mathcal{R}(u,v) = \|\mathbb{A}(u)v\|_B + \frac{1}{2}\|\mathbb{V}(u)v\|_H^2 + \frac{1}{p}\|\mathbb{M}(u)\|_Z^p$

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In **multiscale modeling** on is interested in

Γ-convergence for families of gradient systems $(\boldsymbol{X}, \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon)$

- homogenization
- dimension reductions (plates, interfaces, ...)
- singular perturbations
- numerical approximation $\varepsilon = h \to 0$

Overview

Our working definition for this course:

Definition (Γ-convergence of genenneralized gradient systems = evolutionary Γ-convergence)

We write $(\boldsymbol{X}, \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon) \xrightarrow{\text{evol}} (\boldsymbol{X}, \mathcal{E}_0, \mathcal{R}_0)$ if and only if

$$\left.\begin{array}{c} u^\varepsilon : [0,T] \to \boldsymbol{X} \\ \text{solves } (\boldsymbol{X}, \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon) \\ u^\varepsilon(0) \rightharpoonup u_0, \\ \mathcal{E}_\varepsilon(u^\varepsilon(0)) \to \mathcal{E}_0(u_0) \end{array}\right\} \Longrightarrow \left\{\begin{array}{c} \exists\, u \text{ sln. of } (\boldsymbol{X}, \mathcal{E}_0, \mathcal{R}_0) \text{ with } u(0) = u_0 \\ \text{and a subsequence } \varepsilon_k \to 0 : \\ \forall\, t \in [0,T]: \;\; u^{\varepsilon_k}(t) \rightharpoonup u(t) \\ \mathcal{E}_\varepsilon(u^{\varepsilon_k}(t)) \to \mathcal{E}_0(u(t)) \end{array}\right.$$

Aim: Find conditions of $(\mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon) \rightsquigarrow (\mathcal{E}_0, \mathcal{R}_0)$ to guarantee evolutionary Γ-convergence.

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Why do we want to use gradient structures?

- They displays the physics behind:

  one DE may have several gradient structures

- The gradient structure defines function space

  energy space $u \in \boldsymbol{Z} \;\Leftrightarrow\; \mathcal{E}(t,u) < \infty$

  dynamic space $\dot{u} \in \boldsymbol{X} \;\Leftrightarrow\; \mathcal{R}(\dot{u}) < \infty$

- Using the gradient structure may simplify the proof of showing evolutionary convergence

  However: Evol. Γ-conv. for $(\boldsymbol{X}, \mathcal{E}_\varepsilon, \mathbb{G}_\varepsilon) \underset{\not\Leftarrow}{\Rightarrow}$ Evol. conv. for $\dot{u} = -\nabla_{\mathbb{G}_\varepsilon} \mathcal{E}_\varepsilon(u)$

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- **Most importantly:** We will see an example, where one equation has different gradient structures having evolutionary Γ-limits that do not coincide

$$\dot u = \boldsymbol{G}_\varepsilon(u) \begin{array}{l} \nearrow \dot u = -\mathbb{K}_\varepsilon(u)\mathrm{D}\mathcal{E}_\varepsilon(u) \xrightarrow{\text{evol.}\Gamma\text{-conv.}} \dot u = -\mathbb{K}_0(u)\mathrm{D}\mathcal{E}_0 = \boldsymbol{G}_0(u) \\ \searrow \dot u = -\widetilde{\mathbb{K}}_\varepsilon(u)\mathrm{D}\widetilde{\mathcal{E}}_\varepsilon(u) \xrightarrow{\text{evol.}\Gamma\text{-conv.}} \dot u = -\widetilde{\mathbb{K}}_0(u)\mathrm{D}\widetilde{\mathcal{E}}_0 = \widetilde{\boldsymbol{G}}_0(u) \end{array} \quad \text{different!!}$$

Heat equation $\dot{\theta} = \kappa\Delta\theta$ $\neq$ Diffusion equation $\dot{u} = m\Delta u$

(This will important for coupling reaction-diffusion and heat transfer in one thermodynamic framework. $\rightsquigarrow$ Tutorial)

Heat equation for temperature $\neq$ diffusion equation

Diffusion equation $\dot v = m\Delta v = -\mathbb{K}_{\mathsf{diff}}(v)\mathrm{D}\mathcal{E}_{\mathsf{diff}}(v)$ with

$\mathcal{E}_{\mathsf{diff}}(v) = \int_\Omega v\log v - v\,\mathrm{d}x$ and $\mathbb{K}_{\mathsf{diff}}(v)\xi = -m\,\mathrm{div}(v\nabla\xi)$ JKO/Wasserstein

Pure heat equation for temperature

total entropy $\mathcal{S}(\theta) := \int_\Omega S(\theta(x))\,\mathrm{d}x$

total energy $\mathcal{E}(\theta) := \int_\Omega E(\theta(x))\,\mathrm{d}x$ with Gibbs relation $0 < c(\theta) = E'(\theta) = \frac{1}{\theta}S'(\theta)$

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Physical heat equation: $c(\theta)\dot\theta = \mathrm{div}\,\big(\kappa(\theta)\nabla\theta\big)$

Physical gradient structure $\dot\theta = +\mathbb{K}_{\mathsf{heat}}(\theta)\mathrm{D}\mathcal{S}(\theta)$

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Physical gradient structure $\dot\theta = +\mathbb{K}_{\text{heat}}(\theta)\mathrm{D}\mathcal{S}(\theta)$

Only choice: $\mathbb{K}_{\text{heat}}(\theta)\xi = -\frac{1}{E'(\theta)}\,\mathrm{div}\,\Big(\mu(\theta)\nabla\big(\frac{\xi}{E'(\theta)}\big)\Big)$ (note $\mathbb{K}_{\text{heat}}(\theta)\mathrm{D}\mathcal{E}(\theta) \equiv 0$!)

Using $\frac{S'(\theta)}{E'(\theta)} = \frac{1}{\theta}$ we have to choose $\mu(\theta) = \theta^2\kappa(\theta)$

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Physical gradient structure $\dot\theta = +\mathbb{K}_{\mathsf{heat}}(\theta)\mathrm{D}\mathcal{S}(\theta)$

For coupling it is better to use the **internal energy** $u = E(\theta)$ as variable

$\widehat{\mathcal{E}}(u) = \int_\Omega u(x)\,\mathrm{d}x$ and $\widehat{\mathcal{S}}(u) = \int_\Omega \widehat{S}(u(x))\,\mathrm{d}x$

Equivalent gradient structure $\dot u = \widehat{\mathbb{K}}(u)\mathrm{D}\widehat{\mathcal{S}}(u)$ with $\widehat{\mathbb{K}}(u)\eta = -\,\mathrm{div}\,\big(\widehat{\mu}(u)\nabla\eta\big)$

Inhomogenous diffusion equations

$$\gamma_\varepsilon(x)\dot u(t,x) = \mathrm{div}\,\big(A_\varepsilon(x)\nabla u\big) - f_\varepsilon(x,u(t,x)), \qquad t>0,\ x\in\Omega$$

(& suitable BC)

L^2-type gradient system $(\boldsymbol{X},\mathcal{E}_\varepsilon,\mathbb{G}_\varepsilon)$ with $\boldsymbol{X}=\mathrm{L}^2(\Omega)$

$(\mathbb{G}_\varepsilon v)(x) = \gamma_\varepsilon(x)v(x) \quad \Rightarrow \quad \mathcal{R}_\varepsilon(v) = \int_\Omega \frac12 \gamma_\varepsilon(x)v(x)^2\,\mathrm{d}x$

$\mathcal{E}_\varepsilon(u) = \int_\Omega \frac12 \nabla u\cdot A_\varepsilon(x)\nabla u + F_\varepsilon(x,u(x))\,\mathrm{d}x \qquad F_\varepsilon(x,u) = \int_0^u f_\varepsilon(x,w)\,\mathrm{d}w$

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$\mathcal{E}_\varepsilon(u) = \int_\Omega \frac12 \nabla u\cdot A_\varepsilon(x)\nabla u + F_\varepsilon(x,u(x))\,\mathrm{d}x \qquad F_\varepsilon(x,u)=\int_0^u f_\varepsilon(x,w)\,\mathrm{d}w$

- Homogenization $\gamma_\varepsilon(x) = g(x,\frac{x}{\varepsilon})$ and $A_\varepsilon(x,\frac{x}{\varepsilon})$

  Aim: $\mathcal{R}_{\text{eff}}(v) = \int_\Omega \frac{\gamma_{\text{eff}}}{2} v^2\,\mathrm{d}x$ and $\mathcal{E}_{\text{eff}}(u) = \int_\Omega \frac12 \nabla u\cdot A_{\text{eff}}\nabla u + F_{\text{eff}}(u)\,\mathrm{d}x$

Inhomogenous diffusion equations

$$\gamma_\varepsilon(x)\dot u(t,x) = \mathrm{div}\big(A_\varepsilon(x)\nabla u\big) - f_\varepsilon(x,u(t,x)), \qquad t>0,\ x\in\Omega$$

(& suitable BC)

L^2-type gradient system $(\boldsymbol{X}, \mathcal{E}_\varepsilon, \mathbb{G}_\varepsilon)$ with $\boldsymbol{X} = \mathrm{L}^2(\Omega)$

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- **Homogenization** $\gamma_\varepsilon(x) = g(x,\frac{x}{\varepsilon})$ and $A_\varepsilon(x,\frac{x}{\varepsilon})$

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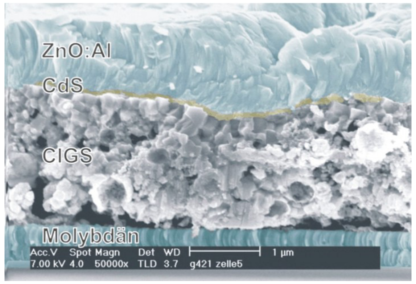
- **Dimension reduction** (modeling of active interfaces)

$$x\in\,]{-1},1[\,\subset\mathbb{R}^1 \qquad A_\varepsilon(x) = \begin{cases}\alpha & \text{for } |x|>\varepsilon/2,\\ \beta\varepsilon & \text{for } |x|<\varepsilon/2\end{cases}$$

$$\mathcal{E}_{\mathsf{eff}}(u) = \int_{-1}^0 \tfrac{\alpha}{2}u_x^2\,\mathrm{d}x + \underbrace{\frac{\beta}{2}\big(u(0^-)-u(0^+)\big)^2}_{\text{gives interface conditions}} + \int_0^1 \tfrac{\alpha}{2}u_x^2\,\mathrm{d}x$$



Fundamental work on evolutionary Γ-convergence:

Sandier-Serfaty 2004 (Comm. Pure Appl. Math.):
Gamma-convergence of gradient flows with applications to Ginzburg-Landau

more recent, nicely readable survey:

Serfaty 2011: Gamma-convergence of gradient flows on Hilbert spaces and metric spaces and applications.

Ginzburg-Landau vortices: $\psi(t,\cdot):\Omega\to\mathbb{C}=\mathbb{R}^2$

$$\text{(GLe)}\quad c_\varepsilon\dot\psi=\Delta\psi+\frac{1}{\varepsilon^2}(1-|\psi|^2)\psi \text{ \& Neum.BC}$$

(GLe) is induced by the gradient system $(\boldsymbol{X},\mathcal{E}_\varepsilon,\mathcal{R}_\varepsilon)$ with $\boldsymbol{X}=\mathrm{H}^1(\Omega;\mathbb{C})$,
$\mathcal{E}_\varepsilon(\psi)=\int_\Omega\frac12|\nabla\psi|^2+\frac{1}{4\varepsilon^2}(1-|\psi|^2)^2\,\mathrm{d}x$, and $\mathcal{R}_\varepsilon(\dot\psi)=\frac{1}{2\log(1/\varepsilon)}\int_\Omega|\dot\psi|^2\,\mathrm{d}x$

Evol.Γ-limit for $\varepsilon\to0$ $\rightsquigarrow$ ODE for vortex positions

Chemical reaction systems with detailed balance

E.g. $2B+C \overset{k_1}{\rightleftharpoons} 2A \qquad A+2C \overset{1/\varepsilon}{\rightleftharpoons} B \qquad A+B+C \overset{k_3}{\rightleftharpoons} 3C$

Fast reaction versus slow reactions $k_1,\ k_3 = O(1)$

$$\begin{pmatrix} \dot c_A \\ \dot c_B \\ \dot c_C \end{pmatrix} = k_1(c_B^2 c_C - c_A^2) \begin{pmatrix} 2 \\ -2 \\ -1 \end{pmatrix} + \frac{1}{\varepsilon}(c_A c_C^2 - c_B) \begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix} + k_3(c_A c_B c_C - c_C^3) \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}$$

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Energy = relative entropy $\mathcal{E}(\boldsymbol{c}) = \sum_{i=A,B,C} \lambda_{\mathrm{Bz}}(c_i) \qquad \lambda_{\mathrm{Bz}}(z) = z\log z - z + 1$

$$\dot{\boldsymbol{c}} = -\mathbb{K}(\boldsymbol{c})\mathrm{D}\mathcal{E}(\boldsymbol{c}) \text{ with } \mathbb{K}_\varepsilon(\boldsymbol{c}) = \mathbb{K}_{1,3}(\boldsymbol{c}) + \frac{1}{\varepsilon}\,\frac{c_A c_C^2 - c_B}{\log(c_A c_C^2/c_B)}\begin{pmatrix} 1 & -1 & 2 \\ -1 & 1 & -2 \\ 2 & -2 & 4 \end{pmatrix}$$

Gradient system $([0,\infty[^3, \mathcal{E}, \mathbb{K})$: $\mathcal{E}$ indep. of ε but $\mathbb{K}_\varepsilon(\boldsymbol{c})$

Evol.Γ-convergence proved in “Disser, Liero, Zinsl 2016 WIAS preprint 2227”.

Overview

$\boldsymbol{X}$ sep./refl. Banach space and functionals $\mathcal{J}_\varepsilon : \boldsymbol{X} \to \mathbb{R}_\infty = \mathbb{R} \cup \{\infty\}$

Definition (Weak/strong Γ-convergence and Mosco convergence)

Weak Γ-convergence: $\mathcal{J}_\varepsilon \xrightharpoonup{\Gamma} \mathcal{J}$ if (G1w) and (G2w) hold:

(G1w) $u_\varepsilon \rightharpoonup u \implies \mathcal{J}(u) \leq \liminf_{\varepsilon \to 0} \mathcal{J}_\varepsilon(u_\varepsilon)$ (liminf estimate)

(G2w) $\forall \widehat{u}\ \exists (\widehat{u}_\varepsilon)_\varepsilon$: $\widehat{u}_\varepsilon \rightharpoonup \widehat{u}$ and $\mathcal{J}(\widehat{u}) = \lim_{\varepsilon \to 0} \mathcal{J}_\varepsilon(\widehat{u}_\varepsilon)$ (ex. recovery seq.)

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Mosco convergence $\mathcal{J}_\varepsilon \xrightarrow{\mathsf{M}} \mathcal{J}$ if $\mathcal{J}_\varepsilon \xrightarrow{\Gamma} \mathcal{J}$ and $\mathcal{J}_\varepsilon \xrightharpoonup{\Gamma} \mathcal{J}$ hold

(or simply (G1w) and (G2s))

The (primal) dissipation potentials $\mathcal{R}(u,\dot u)$ is always **convex in $\dot u$**.

The dual dissipation potential $\mathcal{R}^*$ is always **convex in ξ**.

$$\mathcal{R}^*(u,\xi) := \sup\{\,\langle \xi, v\rangle - \mathcal{R}(u,v) \mid v \in \boldsymbol{X}\,\}$$

Theorem (Attouch 1984)

Let $\boldsymbol{X}$ be a reflexive Banach space and assume that all $\mathcal{F}_n : \boldsymbol{X} \to \mathbb{R}_\infty$ are proper, convex, equicoercive and that $(\mathcal{F}_n^*)^*$. Then,

$$\mathcal{F}_n \xrightarrow{\Gamma} \mathcal{F} \quad \Longleftrightarrow \quad \mathcal{F}_n^* \xrightarrow{\Gamma} \mathcal{F}^*\,.$$

(HU specialist are F. Bethke and N. Farchmin)

In particular, we have $\mathcal{F}_n \xrightarrow{\mathsf{M}} \mathcal{F} \quad \Leftrightarrow \quad \mathcal{F}_n^* \xrightarrow{\mathsf{M}} \mathcal{F}^*$.

Easy to remember via the well-known convergence result of linear functional analysis:

$v_n \rightharpoonup v$ and $\xi_n \to \xi$ implies $\langle \xi_n, v_n\rangle \to \langle \xi, v\rangle$

Overview

$\boldsymbol{X} = \mathbb{R}^2$

$\mathcal{E}_\varepsilon(u) = \frac{1}{2}u_1^2 + \frac{1}{2\varepsilon^2}(u_2 - \varepsilon u_1)^2 = \frac{1}{2}u \cdot A_\varepsilon u$ with $A_\varepsilon = \begin{pmatrix} 2 & -1/\varepsilon \\ -1/\varepsilon & 1/\varepsilon^2 \end{pmatrix}$

$\mathcal{R}_\varepsilon(\dot u) = \frac{1}{2}\dot u_1^2 + \frac{1}{2\varepsilon^\beta}\dot u_2^2 = \frac{1}{2}\dot u \cdot \mathbb{G}_\varepsilon \dot u$ with $\mathbb{G}_\varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & 1/\varepsilon^\beta \end{pmatrix}$

ODE reads $\mathbb{G}_\varepsilon \dot u_\varepsilon = -A_\varepsilon u_\varepsilon$ with $u_\varepsilon(0) = \binom{1}{0}$

Explicit solutions can be calculated for all $\varepsilon > 0$. We find, for all $t \geq 0$,

$\beta \in [0, 2[\,:$ $\quad u_\varepsilon(t) \to \binom{\mathrm{e}^{-t}}{0}$ as $\varepsilon \to 0$

$\beta = 2:$ $\quad u_\varepsilon(t) \to \binom{w(t)}{0}$ as $\varepsilon \to 0$

$\beta > 2:$ $\quad u_\varepsilon(t) \to \binom{\mathrm{e}^{-2t}}{0}$ as $\varepsilon \to 0$

where $w(t) = \frac{1}{2\sqrt{5}}\big((\sqrt{5}+1)\mathrm{e}^{-\mu_1 t} + (\sqrt{5}-1)\mathrm{e}^{-\mu_2 t}\big)$ with $\mu_{1,2} = (3 \pm \sqrt{5})/2$

$$\mathcal{E}_\varepsilon(u) = \frac{1}{2}u_1^2 + \frac{1}{2\varepsilon^2}(u_2 - \varepsilon u_1)^2 = \frac{1}{2}u\cdot A_\varepsilon u \qquad \text{with } A_\varepsilon = \begin{pmatrix} 2 & -1/\varepsilon \\ -1/\varepsilon & 1/\varepsilon^2 \end{pmatrix}$$

$$\mathcal{R}_\varepsilon(\dot u) = \frac{1}{2}\dot u_1^2 + \frac{1}{2\varepsilon^\beta}\dot u_2^2 = \frac{1}{2}\dot u\cdot \mathbb{G}_\varepsilon \dot u \text{ with } \mathbb{G}_\varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & 1/\varepsilon^\beta \end{pmatrix}$$

What are the limits of the functionals?

$$\mathcal{E}_\varepsilon(u) = \tfrac12 u_1^2 + \tfrac{1}{2\varepsilon^2}(u_2{-}\varepsilon u_1)^2 = \tfrac12 u\cdot A_\varepsilon u \qquad \text{with } A_\varepsilon = \begin{pmatrix} 2 & -1/\varepsilon \\ -1/\varepsilon & 1/\varepsilon^2 \end{pmatrix}$$

$$\mathcal{R}_\varepsilon(\dot u) = \tfrac12 \dot u_1^2 + \tfrac{1}{2\varepsilon^\beta}\dot u_2^2 = \tfrac12 \dot u\cdot \mathbb{G}_\varepsilon \dot u \text{ with } \mathbb{G}_\varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & 1/\varepsilon^\beta \end{pmatrix}$$

What are the limits of the functionals?

$$\mathcal{E}_\varepsilon \xrightarrow{\text{pointwise}} \mathcal{E}_{\text{pw}} : u \mapsto \begin{cases} (\tfrac12{+}\tfrac12)u_1^2 & \text{for } u_2 = 0, \\ \infty & \text{otherwise} \end{cases}$$

$$\mathcal{E}_\varepsilon \xrightarrow{\text{M}} \mathcal{E}_0 : u \mapsto \begin{cases} \tfrac12 u_1^2 & \text{for } u_2 = 0, \\ \infty & \text{otherwise} \end{cases} \qquad \mathcal{R}_\varepsilon \xrightarrow{\text{M}} \mathcal{R}_0 : v \mapsto \begin{cases} \tfrac12 v_1^2 & \text{for } v_2 = 0, \\ \infty & \text{otherwise} \end{cases}$$

3. Motivating examples

$$\mathcal{E}_\varepsilon(u) = \tfrac{1}{2}u_1^2 + \tfrac{1}{2\varepsilon^2}(u_2 - \varepsilon u_1)^2 = \tfrac{1}{2}u\cdot A_\varepsilon u \qquad \text{with } A_\varepsilon = \begin{pmatrix} 2 & -1/\varepsilon \\ -1/\varepsilon & 1/\varepsilon^2 \end{pmatrix}$$

$$\mathcal{R}_\varepsilon(\dot u) = \tfrac{1}{2}\dot u_1^2 + \tfrac{1}{2\varepsilon^\beta}\dot u_2^2 = \tfrac{1}{2}\dot u\cdot \mathbb{G}_\varepsilon \dot u \text{ with } \mathbb{G}_\varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & 1/\varepsilon^\beta \end{pmatrix}$$

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$(\mathbb{R}^2, \mathcal{E}, \mathcal{R}_0)$ gives $u(t) = \binom{e^{-t}}{0}$ and $(\mathbb{R}^2, \mathcal{E}_{\text{pw}}, \mathcal{R}_0)$ gives $u(t) = \binom{e^{-2t}}{0}$.

$\boxed{\beta < 2}$ $(\mathbb{R}^2, \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon) \xrightarrow{\text{evol}} (\mathbb{R}^2, \mathcal{E}, \mathcal{R}_0)$

$\boxed{\beta = 2}$ **no** evolutionary Γ convergence

$\boxed{\beta > 2}$ $(\mathbb{R}^2, \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon) \xrightarrow{\text{evol}} (\mathbb{R}^2, \mathcal{E}_{\text{pw}}, \mathcal{R}_0)$

3. Motivating examples

$$\mathcal{E}_\varepsilon(u) = \tfrac{1}{2} u\cdot A_\varepsilon u,\ \ \mathcal{R}_\varepsilon(\dot u) = \tfrac{1}{2}\dot u\cdot \mathbb{G}_\varepsilon \dot u \quad A_\varepsilon = \begin{pmatrix} 2 & -1/\varepsilon \\ -1/\varepsilon & 1/\varepsilon^2 \end{pmatrix},\ \mathbb{G}_\varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & 1/\varepsilon^\beta \end{pmatrix}$$

Reason for non-convergence is seen via the energy-dissipation relation

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_\varepsilon(u_\varepsilon(t)) = \langle \mathrm{D}\mathcal{E}_\varepsilon(u_\varepsilon), \dot u_\varepsilon\rangle = -\langle \mathbb{G}_\varepsilon \dot u_\varepsilon, \dot u_\varepsilon\rangle = -\big((\dot u_{1,\varepsilon})^2 + (\dot u_{2,\varepsilon})^2/\varepsilon^\beta\big)$$

$$\rightsquigarrow \quad \mathcal{E}_\varepsilon(u_\varepsilon(T)) + \int_0^T 2\mathcal{R}_\varepsilon(\dot u_\varepsilon(t))\,\mathrm{d}t = \mathcal{E}_\varepsilon(u_\varepsilon(0)) \ = \ 1 \text{ (finite, indep. of } \varepsilon)$$

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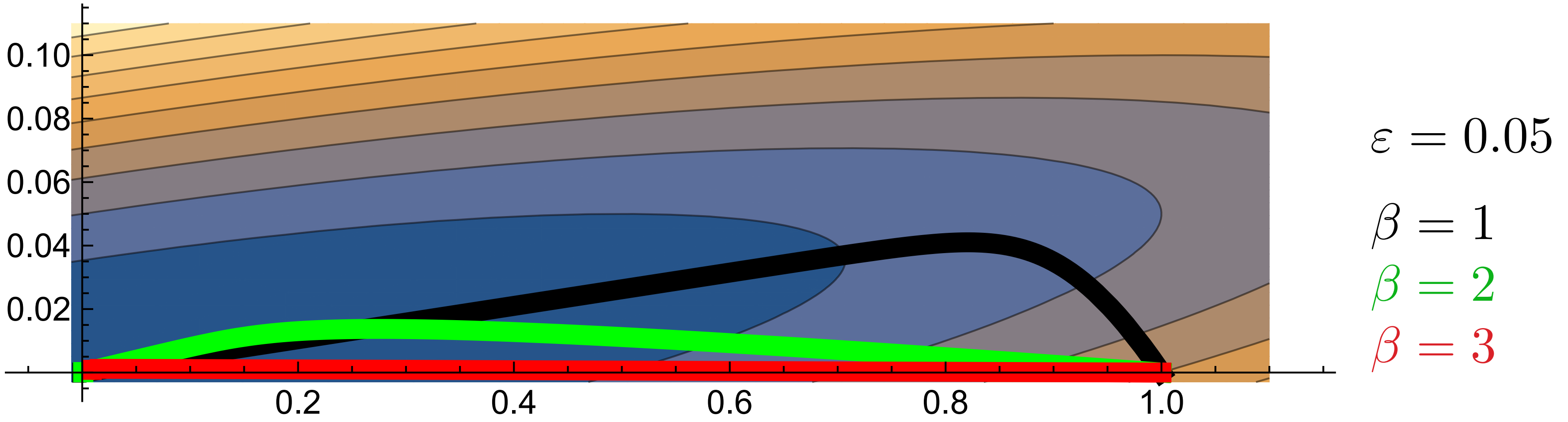
- Energy landspace $\mathcal{E}_\varepsilon(u)$ wants to have $u_2 \approx \varepsilon u_1 \approx \varepsilon \mathrm{e}^{-\lambda t}$
- Dissipation $1 \geq \int_0^T 2\mathcal{R}_\varepsilon(\dot u_\varepsilon(t))\,\mathrm{d}t \geq \int_0^T (\dot u_{2,\varepsilon}(t))^2/\varepsilon^\beta\,\mathrm{d}t$.

$\beta > 2$: "dissipation" doesn't allow solutions to move away from $u_2 \equiv 0$.



Overview

We consider **one-dimensional homogenization** of a parabolic equation on $x \in \Omega =]0,\ell[$ for $t > 0$:

$$c(\tfrac{x}{\varepsilon})\dot{u}(t,x) = \big(a(\tfrac{x}{\varepsilon})u_x(t,x)\big)_x - b(\tfrac{x}{\varepsilon})u(t,x) \qquad u_x(t,0) = 0 = u_x(t,\ell)$$

where $a, b, c \in \mathrm{L}^\infty(\mathbb{R})$ are 1-periodic and are $\geq c_0 > 0$.

Family of gradient system $(\mathrm{L}^2(\Omega), \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon)$ with

$$\mathcal{E}_\varepsilon(u) = \frac{1}{2}\int_\Omega a(\tfrac{x}{\varepsilon})u_x(x)^2 + b(\tfrac{x}{\varepsilon})u(x)^2 \,\mathrm{d}x, \qquad \mathcal{R}_\varepsilon(v) = \frac{1}{2}\int_\Omega c(\tfrac{x}{\varepsilon})v(x)^2 \,\mathrm{d}x$$

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Family of gradient system $(\mathrm{L}^2(\Omega), \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon)$ with

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Aim: Find $\mathcal{E}_{\text{eff}}$ and $\mathcal{R}_{\text{eff}}$ in the form

$$\mathcal{E}_{\text{eff}}(u) = \frac12 \int_\Omega a_{\text{eff}}\, u_x(x)^2 + b_{\text{eff}}\, u^2\,\mathrm{d}x, \qquad \mathcal{R}_{\text{eff}}(v) = \frac12 \int_\Omega c_{\text{eff}}\, v^2\,\mathrm{d}x$$

$a_{\text{eff}} = ?$

$b_{\text{eff}} = ?$

$c_{\text{eff}} = ?$

Quadratic functionals:

$$\Psi_\varepsilon(v) = \frac{1}{2}\int_\Omega v(x)\cdot G_\varepsilon(x)v(x)\,\mathrm{d}x \iff \Psi^*_\varepsilon(x) = \frac{1}{2}\int_\Omega \xi(x)\cdot G_\varepsilon(x)^{-1}\xi(x)\,\mathrm{d}x$$

Lemma (One-dimensional homogenization)
Let $G_\varepsilon(x) = \mathbb{G}(x/\varepsilon)$ with $0 < c_0 \leq \mathbb{G}(y) \leq C_1$ and $\mathbb{G}$ 1-periodic.
In $\mathrm{L}^2(]x_1, x_2[)$ we have

weak-Γ: $\Psi_\varepsilon \xrightharpoonup{\Gamma} \Psi_{\mathsf{harm}} : v \mapsto \frac{1}{2}\int_{x_1}^{x_2} v\cdot G_{\mathsf{harm}}v\,\mathrm{d}x$

strong-Γ: $\Psi_\varepsilon \xrightarrow{\Gamma} \Psi_{\mathsf{arith}} : v \mapsto \frac{1}{2}\int_{x_1}^{x_2} v\cdot G_{\mathsf{arithm}}v\,\mathrm{d}x$

with $G_{\mathsf{harm}} = \big(\int_0^1 \mathbb{G}(y)^{-1}\,\mathrm{d}y\big)^{-1} \leq G_{\mathsf{arithm}} = \int_0^1 \mathbb{G}(y)\,\mathrm{d}y$.

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Proof of weak-Γ: Assume $v_\varepsilon \rightharpoonup v$ in $\mathrm{L}^2(]a, b[)$.

$$\Psi_\varepsilon(v_\varepsilon) = \frac{1}{2}\int_{x_1}^{x_2} v_\varepsilon\cdot G_\varepsilon v_\varepsilon\,\mathrm{d}x =$$

$$\frac{1}{2}\int_{x_1}^{x_2} \underbrace{(\mathbb{G}_\varepsilon v_\varepsilon - G_{\text{ha}}v)\cdot\mathbb{G}_\varepsilon^{-1}(G_\varepsilon v_\varepsilon - G_{\text{ha}}v)}_{\geq 0} + 2\underbrace{\mathbb{G}_\varepsilon v_\varepsilon\cdot\mathbb{G}_\varepsilon^{-1}}_{=v_\varepsilon\rightharpoonup v} G_{\text{ha}}v - G_{\text{ha}}v\cdot \underbrace{G_\varepsilon^{-1}}_{\overset{*}{\rightharpoonup} G_{\text{ha}}^{-1}} G_{\text{ha}}v\,\mathrm{d}x$$

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with $G_{\mathsf{harm}} = \Big(\int_0^1 \mathbb{G}(y)^{-1}\,\mathrm{d}y\Big)^{-1} \le G_{\mathsf{arithm}} = \int_0^1 \mathbb{G}(y)\,\mathrm{d}y$.

Proof of weak-Γ: Assume $v_\varepsilon \rightharpoonup v$ in $\mathrm{L}^2(]a,b[)$.
$\Psi_\varepsilon(v_\varepsilon) = \frac{1}{2}\int_{x_1}^{x_2} v_\varepsilon\cdot G_\varepsilon v_\varepsilon\,\mathrm{d}x =$

$$\frac{1}{2}\int_{x_1}^{x_2} \underbrace{(\mathbb{G}_\varepsilon v_\varepsilon - G_{\mathsf{ha}}v)\cdot\mathbb{G}_\varepsilon^{-1}(G_\varepsilon v_\varepsilon - G_{\mathsf{ha}}v)}_{\ge 0} + 2\underbrace{\mathbb{G}_\varepsilon v_\varepsilon\cdot\mathbb{G}_\varepsilon^{-1}}_{=v_\varepsilon \rightharpoonup v} G_{\mathsf{ha}}v - G_{\mathsf{ha}}v\cdot\underbrace{G_\varepsilon^{-1}}_{\stackrel{*}{\rightharpoonup} G_{\mathsf{ha}}^{-1}} G_{\mathsf{ha}}v\,\mathrm{d}x$$

Hence, $\liminf_{\varepsilon\to 0}\Psi_\varepsilon(v_\varepsilon) \ge \frac{1}{2}\int_{x_1}^{x_2} \quad 0 \quad + 2v\cdot G_{\mathsf{ha}}v \quad - \quad v\cdot G_{\mathsf{ha}}v \quad \mathrm{d}x = \Psi_{\mathsf{harm}}(v)$

Quadratic functionals:

$$\Psi_\varepsilon(v) = \frac{1}{2}\int_\Omega v(x)\cdot G_\varepsilon(x)v(x)\,\mathrm{d}x \iff \Psi^*_\varepsilon(x) = \frac{1}{2}\int_\Omega \xi(x)\cdot G_\varepsilon(x)^{-1}\xi(x)\,\mathrm{d}x$$

Lemma (One-dimensional homogenization)
Let $G_\varepsilon(x) = \mathbb{G}(x/\varepsilon)$ with $0 < c_0 \leq \mathbb{G}(y) \leq C_1$ and $\mathbb{G}$ 1-periodic.
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Given $\widehat{v}$ choose the recovery sequence $\widehat{v}_\varepsilon = G_\varepsilon^{-1} G_{\text{ha}}\widehat{v} \rightharpoonup \widehat{v}$ and first term $= 0$.
Hence, $\Psi_\varepsilon(\widehat{v}_\varepsilon) = \int_{x_1}^{x_2} \quad 0 \quad + G_{\text{ha}}\widehat{v}\cdot G_\varepsilon^{-1} G_{\text{ha}}\widehat{v}\,\mathrm{d}x \;\to\; \Psi_{\text{harm}}(\widehat{v})$

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Proof of strong-Γ is much simpler:
If $v_\varepsilon \to v$ in $\mathrm{L}^2(]a,b[)$, then

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Result is compatible with Attouch's theorem: $\boxed{\Psi_\varepsilon \stackrel{\Gamma}{\rightharpoonup} \Psi_{\text{harm}} \iff \Psi^*_\varepsilon \stackrel{\Gamma}{\rightarrow} \Psi^*_{\text{harm}}}$

For this, simply use $\text{arith}(\mathbb{G}^{-1}) = \text{harm}(\mathbb{G})^{-1}$.

One-dimensional homogenization for parabolic equation on $x \in \Omega =]0, \ell[$:

$$c(\tfrac{x}{\varepsilon})\dot u(t,x) = \big(a(\tfrac{x}{\varepsilon})u_x(t,x)\big)_x - b(\tfrac{x}{\varepsilon})u(t,x) \qquad u_x(t,0) = 0 = u_x(t,\ell)$$

where $a, b, c \in \mathrm{L}^\infty(\mathbb{R})$ are 1-periodic and are $\geq c_0 > 0$.

Gradient system $(\mathrm{L}^2(\Omega), \mathcal{E}_\varepsilon, \Psi_\varepsilon)$ with

$$\mathcal{E}_\varepsilon(u) = \frac12 \int_\Omega a(\tfrac{x}{\varepsilon})u_x(x)^2 + b(\tfrac{x}{\varepsilon})u(x)^2 \,\mathrm{d}x, \qquad \Psi_\varepsilon(v) = \frac12 \int_\Omega c(\tfrac{x}{\varepsilon})v(x)^2 \,\mathrm{d}x$$

- $\Psi_\varepsilon \xrightharpoonup{\Gamma} \Psi_{\text{harm}}$ or $\Psi_\varepsilon \xrightarrow{\Gamma} \Psi_{\text{arith}}$ in the **dynamic space** $\mathrm{L}^2(\Omega)$
- Analogously the energy satisfies in the **energy space** $\mathrm{H}^1(\Omega) \Subset \mathrm{L}^2(\Omega)$

$$\mathcal{E}_\varepsilon \xrightharpoonup{\Gamma} \mathcal{E}_{\text{ha}} : u \mapsto \frac12 \int_\Omega a_{\text{harm}} u_x^2 + b_{\text{arith}} u^2 \,\mathrm{d}x \qquad \text{(weakly in } \mathrm{H}^1(\Omega))$$

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We will use later: $\mathcal{E}_\varepsilon \xrightarrow{\mathrm{M}} \mathcal{E}_{\text{ha}}$ (Mosco in $\mathrm{L}^2(\Omega)$)

⇝ expected limit eqn $c_{\text{eff}}\, u_t = a_{\text{harm}} u_{xx} - b_{\text{arith}} u$ with $c_{\text{eff}} \in \{c_{\text{harm}}, c_{\text{arith}}\}$

Overview

Overview

Legendre-Fenchel theory for a reflexive Banach space

$\Psi : \boldsymbol{X} \to \mathbb{R}_\infty$ proper, convex, lower semicontinuous

Legendre transform $\Psi^* = \mathcal{L}\Psi : \boldsymbol{X}^* \to \mathbb{R}_\infty$ with

$\Psi^*(\xi) := \sup\{ \langle \xi, v\rangle - \Psi(v) \,|\, v \in \boldsymbol{X} \}$

Basic properties:

- $\mathcal{L}(\mathcal{L}\Psi) = \Psi$ or $\Psi^{**} = \Psi$
- Young-Fenchel estimate: $\forall v \in \boldsymbol{X}\ \forall \xi \in \boldsymbol{X}^* :\ \Psi(v) + \Psi^*(\xi) \geq \langle \xi, v\rangle$
- $\Psi(v) = \frac{1}{2}\langle Gv, v\rangle \implies \Psi^*(\xi) = \frac{1}{2}\langle \xi, G^{-1}\xi\rangle$
- $\Psi(v) = \frac{1}{p}\|v\|_{\boldsymbol{X}}^p \implies \Psi^*(\xi) = \frac{1}{p^*}\|\xi\|_{\boldsymbol{X}^*}^{p^*}$ for $1 < p < \infty$, $p^* = p/(p{-}1)$

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$\Psi^{**} = \Psi$ and $\Psi(v) + \Psi^*(\xi) \geq \langle \xi, v\rangle$

Subdifferential of convex Ψ

$\partial\Psi(v) = \{ \eta \in \boldsymbol{X}^* \mid \forall\, w \in \boldsymbol{X} : \ \Psi(w) \geq \Psi(v) + \langle \eta, w{-}v\rangle \} \subset \boldsymbol{X}^*$

If $\Psi \in \mathrm{C}^1(\boldsymbol{X};\mathbb{R})$ and convex, then $\partial\Psi(v) = \{\mathrm{D}\Psi(v)\}$.

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Theorem (Fenchel equivalence)

(i) $\xi \in \partial\Psi(v) \iff$ *(ii)* $v \in \partial\Psi^*(\xi) \iff$ *(iii)* $\Psi(v) + \Psi^*(\xi) \leq \langle \xi, v \rangle$

4. Energy-dissipation formulations

(i) $\xi \in \partial\Psi(v) \iff$ (ii) $v \in \partial\Psi^*(\xi) \iff$ (iii) $\Psi(v) + \Psi^*(\xi) \leq \langle \xi, v \rangle$

Generalized gradient system $(\boldsymbol{X}, \mathcal{E}, \mathcal{R})$

Energy funct. $\mathcal{E} : [0,T]\times\boldsymbol{X} \to \mathbb{R}_\infty$, dissipation pot. $\mathcal{R}(u,\cdot) : \boldsymbol{X} \to [0,\infty]$

(i) $0 \in \partial_{\dot u}\mathcal{R}(u(t),\dot u(t)) + \mathrm{D}\mathcal{E}(t,u(t)) \ \in \boldsymbol{X}^*$ for a.a. $t \in [0,T]$

force balance in $\boldsymbol{X}^*$ Biot's equation 1954

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force balance in $\boldsymbol{X}^*$ — Biot's equation 1954

Dual dissipation potential $\mathcal{R}^*(u,\xi) = \mathcal{L}(\mathcal{R}(u,\cdot))(\xi)$

(ii) $\dot u(t) \in \partial_\xi\mathcal{R}^*(u(t), -\mathrm{D}\mathcal{E}(t, u(t))) \ \in \boldsymbol{X}$ for a.a. $t \in [0,T]$

rate equation in $\boldsymbol{X}$ — Onsager's equation 1931

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(iii) $\mathcal{R}(u(t),\dot u(t))+\mathcal{R}^*(u(t),-\mathrm{D}\mathcal{E}(t,u(t))) \leq \langle -\mathrm{D}\mathcal{E}(t,u(t)),\dot u(t)\rangle$

power balance in $\mathbb{R}$ (equivalent to equality by Young-Fenchel)

De Giorgi's (Ψ,Ψ^*) formulation 1980

4. Energy-dissipation formulations

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Theorem (Energy-Dissipation Principle (EDP), De Giorgi'80)

Assume that $\mathcal{E}$ satisfies the ***chain rule on $\boldsymbol{X}$***, *then $u \in \mathrm{W}^{1,1}([0,T];\boldsymbol{X})$ solves (i) or (ii) if and only if* ***(EDE)*** *holds:*

(EDE)
$$\mathcal{E}(T,u(T)) + \int_0^T \mathcal{R}(u(t),\dot u(t)) + \mathcal{R}^*(u(t), -\mathrm{D}\mathcal{E}(t,u(t)))\,\mathrm{d}t \le \mathcal{E}(0,u(0)) + \int_0^T \partial_s\mathcal{E}(s,u(s))\,\mathrm{d}s$$

Final energy + dissipated energy = initial energy + external work

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Proof: $\int_0^T -\langle \mathrm{D}\mathcal{E}(t,u),\dot u\rangle\,\mathrm{d}t \overset{\text{YF}}{\leq} \int_0^T \mathcal{R}(u,\dot u) + \mathcal{R}^*(u,-\mathrm{D}\mathcal{E})\,\mathrm{d}t$

$$\overset{\text{(EDE)}}{\leq} \mathcal{E}(0,u(0)) + \int_0^T \partial_t \mathcal{E}\,\mathrm{d}t - \mathcal{E}(T,u(T)) \overset{\text{Ch.Rule}}{=} \int_0^T -\langle \mathrm{D}\mathcal{E}(t,u),\dot u\rangle\,\mathrm{d}t$$

$\Rightarrow$ all estimates are equalities $\Rightarrow$ Young-Fenchel estimate is equality a.e. QED

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Fundamental and more general tool **Chain-Rule Estimate (CR)**

$\mathcal{E}:\boldsymbol{X}\to\mathbb{R}_\infty$ satisfies **CRE**, if

$$\left.\begin{aligned} u\in \mathrm{W}^{1,p}([0,T];\boldsymbol{X}),\ \xi \in \mathrm{L}^{p'}([0,T];\boldsymbol{X}^*) \\ \xi(t)\in\partial\mathcal{E}(u(t)) \end{aligned}\right\} \Longrightarrow \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(u(t)) \geq \langle \xi(t),\dot u(t)\rangle$$

(e.g. always true for lsc and convex $\mathcal{E}(\cdot)$)

Overview

$0 \in \partial_{\dot u}\mathcal{R}_\varepsilon(u,\dot u) + \mathrm{D}\mathcal{E}_\varepsilon(u) \overset{\text{Fenchel}}{\Longleftrightarrow}$ (EDE) = Energy-Dissipation Estimate

(EDE) $\mathcal{E}_\varepsilon(u^\varepsilon(t)) + \int_0^T \mathcal{R}_\varepsilon(u^\varepsilon,\dot u^\varepsilon) + \mathcal{R}^*_\varepsilon(u^\varepsilon, -\mathrm{D}\mathcal{E}_\varepsilon(u^\varepsilon))\,\mathrm{d}t \leq \mathcal{E}_\varepsilon(u^\varepsilon(0))$

Evolutionary Γ convergence based on (EDP)

- Sandier-Serfaty'04 (general approach)
- here: improved version of M-Rossi-Savare'12 (CVPDE) $\mathcal{R}(u,v) = \Psi(v)$

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Theorem (Mosco convergence implies evolutionary Γ-convergence)

$\boldsymbol{X}$ reflexive, $\exists\, c, C, \lambda_c > 0,\ p > 1$ such that $\mathcal{E}_\varepsilon(\cdot) + \lambda_c\|\cdot\|^2_{\boldsymbol{X}}$ is convex, $\Psi_\varepsilon(v) \ge c\|v\|^p_{\boldsymbol{X}} - C,\ \Psi^*_\varepsilon(\xi) \ge c\|\xi\|^p_{\boldsymbol{X}^*} - C,\ \mathcal{E}_\varepsilon(u) \ge c\|u\|_{\boldsymbol{Z}} - C$ with $\boldsymbol{Z} \Subset \boldsymbol{X}$

$$\Big(\mathcal{E}_\varepsilon \xrightarrow{M} \mathcal{E}_0\ \&\ \Psi_\varepsilon \xrightarrow{\Gamma} \Psi_0 \text{ in } \boldsymbol{X}\Big) \implies (\boldsymbol{X}, \mathcal{E}_\varepsilon, \Psi_\varepsilon) \xrightarrow{evol} (\boldsymbol{X}, \mathcal{E}_0, \Psi_0)$$

Compatibility: $\mathcal{E}_\varepsilon \xrightarrow{\mathrm{M}} \mathcal{E}_0$ and $\Psi_\varepsilon \xrightarrow{\Gamma} \Psi_0$ in SAME topology $\boldsymbol{X}$

Theorem (Mosco convergence implies evolutionary Γ-convergence)

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Application to our two simple problems (coercivity of Ψ defines $\boldsymbol{X}$)

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Application to our two simple problems (coercivity of Ψ defines $\boldsymbol{X}$)

- M-Rossi-Savare'12 (CVPDE) uses the much stronger statement $\Psi_\varepsilon \xrightarrow{M} \Psi_0$ (but allows state dependence via $\mathcal{R}_\varepsilon(u, \dot u)$)
- Sina Reichelt 2014 (not contained in my survey article, see Reichelt-Liero SIMA'16 and excercise in tutorial)

$\Psi_\varepsilon \xrightarrow{\Gamma} \Psi_0$ is sufficient

Theorem (Mosco convergence implies evolutionary Γ-convergence)

$\boldsymbol{X}$ *reflexive,* $\exists\, c, C, \lambda_{\rm c} > 0,\ p > 1$ *such that* $\mathcal{E}_\varepsilon(\cdot) + \lambda_{\rm c}\|\cdot\|^2_{\boldsymbol{X}}$ *is convex,*
$\Psi_\varepsilon(v) \geq c\|v\|^p_{\boldsymbol{X}} - C,\ \ \Psi^*_\varepsilon(\xi) \geq c\|\xi\|^p_{\boldsymbol{X}^*} - C,\ \ \mathcal{E}_\varepsilon(u) \geq c\|u\|_{\boldsymbol{Z}} - C$ *with* $\boldsymbol{Z} \Subset \boldsymbol{X}$

$$\Big(\mathcal{E}_\varepsilon \xrightarrow{\mathsf{M}} \mathcal{E}_0 \ \&\ \Psi_\varepsilon \xrightarrow{\Gamma} \Psi_0 \text{ in } \boldsymbol{X}\Big) \quad \Longrightarrow \quad (\boldsymbol{X}, \mathcal{E}_\varepsilon, \Psi_\varepsilon) \xrightarrow{\textit{evol}} (\boldsymbol{X}, \mathcal{E}_0, \Psi_0)$$

Application to our two simple problems (coercivity of Ψ defines $\boldsymbol{X}$)

ODE model on $\boldsymbol{X} = \mathbb{R}^2$

We always have $\mathcal{E}_\varepsilon \xrightarrow{\mathsf{M}} \mathcal{E}$ and $\mathcal{R}_\varepsilon \xrightarrow{\mathsf{M}} \mathcal{R}$.

$\mathcal{R}(v) = \frac{1}{2}(v_1^2 + v_2^2/\varepsilon^\beta)$ and $\mathcal{R}^*(\xi) = \frac{1}{2}(\xi_1^2 + \varepsilon^\beta \xi_2^2)$

Theorem is applicable for $\beta = 0$ only,
because of needed equicoercivity of for Ψ^*_ε.

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Application to our two simple problems (coercivity of Ψ defines $\boldsymbol{X}$)

Homogenization: $c\|v\|^2_{\mathrm{L}^2} \leq \Psi_\varepsilon(v) \leq C\|v\|^2_{\mathrm{L}^2} \quad\Longrightarrow\quad \boldsymbol{X} = \mathrm{L}^2(0,\ell)$.

$\mathcal{E}_\varepsilon(u) = \frac{1}{2}\int_0^\ell a_\varepsilon u_x^2 + b_\varepsilon u^2 \,\mathrm{d}x :$ $\qquad \mathcal{E}_\varepsilon \xrightarrow{\mathrm{M}} \mathcal{E}_0$ in $\boldsymbol{X} = \mathrm{L}^2(0,\ell)$ $\quad\oplus$

$\Psi_\varepsilon(v) = \frac{1}{2}\int_0^\ell c(x/\varepsilon) v(x)^2 \,\mathrm{d}x$: $\qquad$ not $\xrightarrow{\mathrm{M}}$, but $\Psi_\varepsilon \xrightarrow{\Gamma} \Psi_{\text{strong}}$ $\quad\oplus$

Theorem is applicable and gives $c_{\text{eff}} = c_{\text{arith}} \geq c_{\text{harm}}$.

$$c_\varepsilon u_t = (a_\varepsilon u_x)_x - b_\varepsilon u \quad\xrightarrow{\text{evol}}\quad c_{\text{arith}} u_t = (a_{\text{harm}} u_x)_x - b_{\text{arith}} u$$

Sketch of proof of theorem: u_ε are solutions of (i) = (EDB)$_\varepsilon$:

$$\mathcal{E}_\varepsilon(u_\varepsilon(T)) + \int_0^T \big(\Psi_\varepsilon(\dot u_\varepsilon)+\Psi^*_\varepsilon(\xi_\varepsilon)\big)\,\mathrm{d}t = \mathcal{E}_\varepsilon(u_\varepsilon(0)) \text{ where } -\xi_\varepsilon(t) \in \mathrm{D}\mathcal{E}_\varepsilon(u_\varepsilon(t))$$

- Uniform coercivity of $\mathcal{E}_\varepsilon$, Ψ_ε. and Ψ^*_ε yield uniform a priori bounds

$$\|u_\varepsilon\|_{\mathrm{L}^\infty([0,T];\boldsymbol{Z})} + \|u_\varepsilon\|_{\mathrm{W}^{1,p}([0,T];\boldsymbol{X})} + \|\xi_\varepsilon\|_{\mathrm{L}^p([0,T];\boldsymbol{X}^*)} \leq C$$

- We find convergent subsequences (still $\varepsilon = \varepsilon_k$)

$$u_\varepsilon(t) \to u(t) \text{ in } \boldsymbol{X},\ u_\varepsilon \rightharpoonup u \text{ in } \mathrm{W}^{1,p}([0,T];\boldsymbol{X}),\ \xi_\varepsilon \rightharpoonup \xi \text{ in } \mathrm{L}^p([0,T];\boldsymbol{X}^*)$$

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- Lower semicontinuity of the dual dissipation (use $\Psi_\varepsilon \xrightarrow{\Gamma} \Psi_0 \iff \Psi^*_\varepsilon \xrightharpoonup{\Gamma} \Psi^*_0$)

Ioffe's lsc result: $\int_0^T \Psi^*_0(\xi)\,\mathrm{d}t \leq \liminf_{\varepsilon\to 0} \int_0^T \Psi^*_\varepsilon(\xi_\varepsilon)\,\mathrm{d}t$

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Ioffe's lsc result: $\int_0^T \Psi^*_0(\xi)\,\mathrm{d}t \le \liminf_{\varepsilon\to 0} \int_0^T \Psi^*_\varepsilon(\xi_\varepsilon)\,\mathrm{d}t$

(Sebastian Hensel is specialist for versions of Ioffe's theorem!)

The integrand $(\varepsilon,\xi) \mapsto \Psi^*_\varepsilon(\xi)$ for the functional $\mathcal{J}(\varepsilon,\xi) := \int_0^T \Psi^*_\varepsilon(\xi)\,\mathrm{d}t$

- is seq. weakly lower semicontinuous
- and convex in $\xi \in \boldsymbol{X}^*$

The convergence $\varepsilon_\varepsilon := \varepsilon \to 0$ is strong, while the convergence $\xi_\varepsilon \rightharpoonup \xi$ is weak.

$\mathcal{E}_\varepsilon(u_\varepsilon(T)) + \int_0^T \big(\Psi_\varepsilon(\dot u_\varepsilon)+\Psi^*_\varepsilon(\xi_\varepsilon)\big)\,\mathrm{d}t = \mathcal{E}_\varepsilon(u_\varepsilon(0))$ where $-\xi_\varepsilon(t) \in \mathrm{D}\mathcal{E}_\varepsilon(u_\varepsilon(t))$

Sketch of proof of theorem (continued):

- $\int_0^T \Psi_0(\dot u(t))\,\mathrm{d}t \leq \liminf_{\varepsilon\to 0} \int_0^T \Psi_\varepsilon(\dot u_\varepsilon(t))\,\mathrm{d}t.$

Ioffe's theorem doesn't apply as $\dot u_\varepsilon \rightharpoonup \dot u$ in $\mathrm{W}^{1,p}(0,T;\boldsymbol{X})$ and $\Psi_\varepsilon \xrightarrow{\Gamma} \Psi_0$.

Reichelt's lemma (see tutorial): It still holds because of $u_\varepsilon(t) \xrightarrow{\boldsymbol{X}} u(t)$.

- Taking $\varepsilon \to 0$ in (EDE)$_\varepsilon$ and using well-prepared initial cond. (i.c.) gives

$$\mathcal{E}_0(u(T)) + \int_0^T \Big(\Psi_0(\dot u)+\Psi^*_0(\xi)\Big)\,\mathrm{d}t \leq \lim_{\varepsilon\to 0} \mathcal{E}_\varepsilon(u_\varepsilon(0)) \overset{\text{(i.c.)}}{=} \mathcal{E}_0(u(0))$$

- We would be finished if we knew $-\xi(t) = \mathrm{D}\mathcal{E}_0(u(t))$ for a.a. $t \in [0,T]$.

$\mathcal{E}_\varepsilon(u_\varepsilon(T)) + \int_0^T \big(\Psi_\varepsilon(\dot u_\varepsilon){+}\Psi^*_\varepsilon(\xi_\varepsilon)\big)\,\mathrm{d}t = \mathcal{E}_\varepsilon(u_\varepsilon(0))$ where $-\xi_\varepsilon(t) \in \mathrm{D}\mathcal{E}_\varepsilon(u_\varepsilon(t))$

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Strong-weak closedness of $\mathrm{D}\mathcal{E}_\varepsilon$ if $\mathcal{E}_\varepsilon \xrightarrow{\mathrm{M}} \mathcal{E}_0$ & $\mathcal{E}_\varepsilon$ λ_c-convex (cf. Attouch'84)

$$u_\varepsilon \to u \text{ in } \boldsymbol{X} \text{ and } -\mathrm{D}\mathcal{E}_\varepsilon(u_\varepsilon) \ni \xi_\varepsilon \rightharpoonup \xi \text{ in } \boldsymbol{X}^* \;\Rightarrow\; -\xi \in \mathrm{D}\mathcal{E}_0(\mathrm{u})$$

- Now the Energy-Dissipation Principle shows
  - that u is a solution and
  - that $\mathcal{E}_\varepsilon(t) \to \mathcal{E}_0(u(t))$ for all $t \in [0,T]$. QED

4. Energy-dissipation formulations

Main tools is Strong-Weak Closedness of the graph of $(\mathrm{D}\mathcal{E}_\varepsilon)_{\varepsilon\in]0,1[}$

(SWC) $\quad u_\varepsilon \to u$ in $\boldsymbol{X}$ and $\mathrm{D}\mathcal{E}_\varepsilon(u_\varepsilon) \ni \xi_\varepsilon \rightharpoonup \xi$ in $\boldsymbol{X}^*$ $\;\Rightarrow\;$ $\xi \in \mathrm{D}\mathcal{E}(\mathrm{u})$

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This is a consequence of Γ-convergence and convexity!

Theorem (Convexity and $\xrightarrow{\Gamma}$ imply (SWC), cf. Attouch 1984)

If all $\mathcal{E}_\varepsilon$ are lsc and convex, then $\mathcal{E}_\varepsilon \xrightarrow{M} \mathcal{E}_0$ implies (SWC).

Proof: Assume $u_\varepsilon \to u$, $\xi_\varepsilon \rightharpoonup \xi$, and $\mathcal{E}_\varepsilon(u_\varepsilon) \to e_*$

Then convexity gives $\qquad (\mathrm{Cvx})_\varepsilon \quad \mathcal{E}_\varepsilon(w) \geq \mathcal{E}_\varepsilon(u_\varepsilon) + \langle \xi_\varepsilon, w{-}u_\varepsilon\rangle$

For given $\widehat{u}$ the Γ_{s}-convergence gives a rec. seq. $\widehat{u}_\varepsilon$ with $\widehat{u}_\varepsilon \to \widehat{u}$, $\mathcal{E}_\varepsilon(\widehat{u}_\varepsilon) \to \mathcal{E}_0(\widehat{u})$

Hence, setting $w = \widehat{u}_\varepsilon$ in $(\mathrm{Cvx})_\varepsilon$ gives $\underbrace{\mathcal{E}_\varepsilon(\widehat{u}_\varepsilon)}_{\to \mathcal{E}_0(\widehat{u})} \geq \underbrace{\mathcal{E}_\varepsilon(u_\varepsilon)}_{\to e_*} + \langle \underbrace{\xi_\varepsilon}_{\rightharpoonup \xi}, \underbrace{\widehat{u}_\varepsilon{-}u_\varepsilon}_{\to \widehat{u}-u}\rangle$

Taking the limit $\varepsilon \to 0$ we obtain the relation $\mathcal{E}_0(\widehat{u}) \geq e_* + \langle \xi, \widehat{u}{-}u\rangle$

Choose $\widehat{u} = u$ we see that $\mathcal{E}_0(u) \geq e_*$ but Γ_{s}-liminf gives $e_* \geq \mathcal{E}_0(u)$. Thus, $e_* = \mathcal{E}_0(u)$ and we conclude $\xi \in \partial\mathcal{E}_0(u)$ as desired. $\square$

(EDE) $\mathcal{E}_\varepsilon(u_\varepsilon(t)) + \int_0^T \big(\Psi_\varepsilon(\dot u_\varepsilon) + \Psi^*_\varepsilon(-\mathrm{D}\mathcal{E}_\varepsilon(u_\varepsilon))\big)\,\mathrm{d}t \leq \mathcal{E}_\varepsilon(u_\varepsilon(0))$

The **Sandier-Serfaty [2004]** approach is more general.

They do assume

neither Strong-Weak Closedness of $(\partial\mathcal{E}_\varepsilon)_{\varepsilon\in[0,1]}$

nor the Mosco convergence of $\Psi_\varepsilon \xrightarrow{\mathsf{M}} \Psi$

Instead they assume

(i) $v_\varepsilon \rightharpoonup v \implies \liminf_{\varepsilon\to} \Psi_\varepsilon(v_\varepsilon) \geq \Psi_0(v)$ (w-Γ-liminf)

(ii) $u_\varepsilon \to u \implies \liminf_{\varepsilon\to} \Psi^*_\varepsilon(\mathrm{D}\mathcal{E}_\varepsilon(u_\varepsilon)) \geq \Psi^*_0(\mathrm{D}\mathcal{E}_0(u))$ (dual w-Γ-liminf)

Clearly, (SWC) & $\Psi_\varepsilon \xrightarrow{\mathsf{M}} \Psi$ imply (i) and (ii) but not vice-versa.

Overview

One equation $\dot u = \mathcal{V}(u)$ may have different gradient structures:

- Gradient structure $\dot u = -\mathbb{K}(u)\mathcal{E}(u)$ is additional physical information.
- Different physical problems may have the same PDE but different GS.

  heat equation $\dot\theta = \Delta\theta \qquad \neq \qquad \dot u = \Delta u$ diffusion equation
- In a multiscale problem only certain GS may have a pE-limit

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  heat equation $\dot\theta = \Delta\theta$ $\quad\neq\quad$ $\dot u = \Delta u$ diffusion equation
- In a multiscale problem only certain GS may have a pE-limit
- Even more dramatic: **Different gradient structures may lead to different effective equations!**

Tartar 1990: Nonlocal homogenization of hyperbolic equations:

$\Omega =]0,\ell[$, $u^\varepsilon(t,x) \in \mathbb{R}$

$\dot u^\varepsilon(t,x) = -a(x/\varepsilon)u^\varepsilon(t,x)$ soln. $u^\varepsilon(t,x) = u^\varepsilon(0,x)\exp\big(-ta(x/\varepsilon)\big)$

Problem $u^\varepsilon(0,\cdot) \rightharpoonup u_0^0 \quad \not\Rightarrow u^\varepsilon(t,\cdot) = u_0^0 \exp(-t\, a_{\text{eff}})$

Philosophy: GS of $\boxed{\dot u^\varepsilon(t,x) = -a(x/\varepsilon)u^\varepsilon(t,x)}$ **is important!**

$(\boldsymbol{X}, \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon)$ with $\boldsymbol{X} = \mathrm{L}^2(\Omega)$

(A) $\mathcal{E}_\varepsilon(u) = \int_\Omega \frac{a(x/\varepsilon)}{2} u(x)^2 \,\mathrm{d}x$ and $\mathcal{R}_\varepsilon(\dot u) = \mathcal{R}(\dot u) = \int_\Omega \frac{1}{2}\dot u(x)^2 \,\mathrm{d}x$

$\mathcal{E}_\varepsilon \xrightarrow{\Gamma} \mathcal{E}_\mathsf{harm} : u \mapsto \int_\Omega \frac{a_\mathsf{harm}}{2} u^2 \,\mathrm{d}x$ $\qquad \mathcal{R}_\varepsilon = \mathcal{R}$

Guess (A) for limit $\boxed{\dot u = -a_\mathsf{harm} u}$ (cf. Braides 2013)

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(B) $\overline{\mathcal{E}}_\varepsilon(u) = \overline{\mathcal{E}}(u) = \int_\Omega \frac{1}{2} u(x)^2 \,\mathrm{d}x$ and $\overline{\mathcal{R}}_\varepsilon(\dot u) = \int_\Omega \frac{1}{2a(x/\varepsilon)} \dot u(x)^2 \,\mathrm{d}x$

$\overline{\mathcal{E}}_\varepsilon = \overline{\mathcal{E}}$ $\qquad \overline{\mathcal{R}}_\varepsilon(\dot u) \xrightarrow{\Gamma} \overline{\mathcal{R}}_0(\dot u) = \int_\Omega \frac{1}{2a_{\mathsf{arith}}} \dot u^2 \,\mathrm{d}x$

Guess (B) for limit $\dot u = -a_{\mathsf{arith}} u$

Is (A) or (B) correct? Or both? or None?

Philosophy: GS of $\dot u^\varepsilon(t,x) = -a(x/\varepsilon)u^\varepsilon(t,x)$ **is important!**

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Guess (B) for limit $\dot u = -a_{\text{arith}} u$

Is (A) or (B) correct? Or both? or None?

Neither $(\mathrm{L}^2(\Omega), \mathcal{E}_\varepsilon, \mathcal{R})$ **nor** $(\mathrm{L}^2(\Omega), \overline{\mathcal{E}}, \overline{\mathcal{R}}_\varepsilon)$ **do** pE**-converge!**

Two other gradient structures inspired by different physics
(namely by transport theory and growth or death of species)

$\boldsymbol{X}_{\mathrm{M}} := \mathrm{M}_{\geq 0}(\overline{\Omega})$ non-negative Radon measures

(C) $\widetilde{\mathcal{E}}_\varepsilon(u) = \int_\Omega a(\frac{x}{\varepsilon})u(x)\,\mathrm{d}x$ and $\widetilde{\mathcal{R}}_\varepsilon(u,\dot u) = \int_\Omega \frac{\dot u(x)^2}{2u(x)}\,\mathrm{d}x$

$$\mathrm{D}_{\dot u}\widetilde{\mathcal{R}}_\varepsilon(u,\dot u) = \frac{\dot u}{u} = -a(\tfrac{x}{\varepsilon}) = -\mathrm{D}\widetilde{\mathcal{E}}_\varepsilon(u)$$ PDE is OK

(D) $\widehat{\mathcal{E}}_\varepsilon(u) = \int_\Omega \frac{1}{a(x/\varepsilon)}u(x)\,\mathrm{d}x$ and $\widehat{\mathcal{R}}_\varepsilon(u,\dot u) = \int_\Omega \frac{\dot u(x)^2}{2a(x/\varepsilon)^2 u(x)}\,\mathrm{d}x$

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4. Energy-dissipation formulations

Two other gradient structures inspired by different physics

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Theorem [Survey'16] (C) $(\boldsymbol{X}_{\rm M},\widetilde{\mathcal{E}}_\varepsilon,\widetilde{\mathcal{R}}_\varepsilon) \xrightarrow{\text{evol}} (w^*)\ (\boldsymbol{X}_{\rm M},\widetilde{\mathcal{E}}_{\sf min},\widetilde{\mathcal{R}}_{\rm H})$ and
(D) $(\boldsymbol{X}_{\rm M},\widehat{\mathcal{E}}_\varepsilon,\widehat{\mathcal{R}}_\varepsilon) \xrightarrow{\text{evol}} (w^*)\ (\boldsymbol{X}_{\rm M},\widehat{\mathcal{E}}_{\sf max},\widehat{\mathcal{R}}_{\sf max})$

(C) $\widetilde{\mathcal{E}}_{\sf min}(u) = \int_\Omega a_{\sf min}u\,{\rm d}x \ \rightsquigarrow\ \dot u = -a_{\sf min}u$
(D) $\widehat{\mathcal{E}}_{\sf max}(u) = \int_\Omega \frac{1}{a_{\sf max}}u\,{\rm d}x \ \rightsquigarrow\ \dot u = -a_{\sf max}u$

Different effective equations depending on choice of GS!

Sketch of proof for case (C) [(D) is analogous, cf. Survey'16]:

- $\widetilde{\mathcal{E}}_\varepsilon(u) = \int_0^\ell a(x/\varepsilon)\,\mathrm{d}u(x)$ is a linear energy functional in $\boldsymbol{X}_\mathrm{M}$
- $\widetilde{\mathcal{R}}_\varepsilon(u,\dot u) = \mathcal{R}_\mathrm{H}(u,\dot u) = \int_\Omega \dot u^2/(2u)\,\mathrm{d}x$ is a state-dependent dissipation potential that induces Hellinger distance $d_\mathrm{H}(u_0,u_1) = 2\|\sqrt{u_1}-\sqrt{u_0}\|_{\mathrm{L}^2}$

(EDB) $\widetilde{\mathcal{E}}_\varepsilon(u_\varepsilon(T)) + \int_0^T \big(\widetilde{\mathcal{R}}_\mathrm{H}(u_\varepsilon,\dot u_\varepsilon) + \mathcal{R}^*_\mathrm{H}(u_\varepsilon, -\mathrm{D}\mathcal{E}_\varepsilon(u_\varepsilon))\big)\,\mathrm{d}t \;=\; \widetilde{\mathcal{E}}_\varepsilon(u_\varepsilon(0))$

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(2) Using $a(x/\varepsilon) \geq a_{\mathsf{min}}$ gives $\widetilde{\mathcal{E}}_\varepsilon(u) \geq \widetilde{\mathcal{E}}_{\mathsf{min}}(u)$
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Sketch of proof for case (C) [(D) is analogous, cf. Survey'16]:

- $\widetilde{\mathcal{E}}_\varepsilon(u) = \int_0^\ell a(x/\varepsilon)\,\mathrm{d}u(x)$ is a linear energy functional in $\boldsymbol{X}_\mathrm{M}$
- $\widetilde{\mathcal{R}}_\varepsilon(u,\dot u) = \mathcal{R}_\mathrm{H}(u,\dot u) = \int_\Omega \dot u^2/(2u)\,\mathrm{d}x$ is a state-dependent dissipation potential that induces Hellinger distance $d_\mathrm{H}(u_0,u_1) = 2\|\sqrt{u_1}-\sqrt{u_0}\|_{\mathrm{L}^2}$

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$$\int_0^T \big(\widetilde{\mathcal{R}}_\varepsilon(u_\varepsilon,\dot u_\varepsilon) + \widetilde{\mathcal{R}}^*_\varepsilon(u_\varepsilon,-\mathrm{D}\mathcal{E}_\varepsilon(u_\varepsilon))\big)\,\mathrm{d}t = \int_0^T\int_0^\ell \big(\tfrac{\dot u_\varepsilon^2}{2u_\varepsilon} + \tfrac{u_\varepsilon}{2}a_\varepsilon^2\big)\,\mathrm{d}x\,\mathrm{d}t$$

Estimate $a_\varepsilon^2 \geq a_{\mathsf{min}}^2$, use $u_\varepsilon \overset{*}{\rightharpoonup} u$ and convexity of $(u,v) \mapsto \frac{v^2}{2u}$ to obtain

$$\liminf_{\varepsilon\to0}\int_0^T\int_0^\ell\big(\tfrac{\dot u_\varepsilon^2}{2u_\varepsilon}+\tfrac{u_\varepsilon}{2}a_\varepsilon^2\big)\mathrm{d}x\,\mathrm{d}t \geq \int_0^T\int_0^\ell\big(\tfrac{\dot u^2}{2u}+\tfrac{u}{2}a_{\mathsf{min}}^2\big)\mathrm{d}x\,\mathrm{d}t = \int_0^T\big(\mathcal{R}_\mathrm{H}(u,\dot u)+\mathcal{R}^*_\mathrm{H}(u,\text{-}\mathrm{D}\widetilde{\mathcal{E}}_{\mathsf{min}}(u))\big)$$

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$$\liminf_{\varepsilon\to0}\int_0^T\int_0^\ell\Big(\frac{\dot u_\varepsilon^2}{2u_\varepsilon}+\frac{u_\varepsilon}{2}a_\varepsilon^2\Big)\mathrm{d}x\,\mathrm{d}t \geq \int_0^T\int_0^\ell\Big(\frac{\dot u^2}{2u}+\frac{u}{2}a_{\mathsf{min}}^2\Big)\mathrm{d}x\,\mathrm{d}t = \int_0^T\big(\mathcal{R}_\mathrm{H}(u,\dot u)+\mathcal{R}^*_\mathrm{H}(u,\text{-}\mathrm{D}\widetilde{\mathcal{E}}_{\mathsf{min}}(u))\big)$$

(1)–(3) show that u is a solution of (EDE) for $(\boldsymbol{X}_\mathrm{M},\mathcal{E}_{\mathsf{min}},\mathcal{R}_\mathrm{H})$. □

Overview

$$\text{(EDE)}\quad \mathcal{E}_\varepsilon(u^\varepsilon(t)) + \int_0^T \mathcal{R}_\varepsilon(u^\varepsilon, \dot u^\varepsilon) + \mathcal{R}^*_\varepsilon(u^\varepsilon, -\mathrm{D}\mathcal{E}_\varepsilon(u^\varepsilon))\,\mathrm{d}t \le \mathcal{E}_\varepsilon(u^\varepsilon(0))$$

EDE is quite flexible
- general $\mathcal{R}_\varepsilon(u,\cdot)$
- λ_c-conv. of $\mathcal{E}_\varepsilon$ not needed
- convergence of individual terms not needed

It suffices to find $(\boldsymbol{X}, \mathcal{E}_0, \mathcal{R}_0)$ and $\mathcal{M}$ such that

- $\mathcal{E}_\varepsilon \xrightarrow{\Gamma} \mathcal{E}_0$
- Chain rule holds for $(\boldsymbol{X}, \mathcal{E}_0, \mathcal{R}_0)$

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- $\mathcal{E}_\varepsilon \xrightarrow{\Gamma} \mathcal{E}_0$
- Chain rule holds for $(\boldsymbol{X}, \mathcal{E}_0, \mathcal{R}_0)$
- $\int_0^T \mathcal{M}(u,\dot u)\,\mathrm{d}t \leq \liminf_\varepsilon \int_0^T \big(\mathcal{R}_\varepsilon(u^\varepsilon,\dot u^\varepsilon){+}\mathcal{R}^*_\varepsilon(u^\varepsilon, -\mathrm{D}\mathcal{E}_\varepsilon(u^\varepsilon))\big)\,\mathrm{d}t$
  (a) $\mathcal{M}(u,v) \geq -\langle \mathrm{D}\mathcal{E}_0(u), v\rangle$ and
  (b) $\mathcal{M}(u,v) = -\langle \mathrm{D}\mathcal{E}_0(u), v\rangle \implies$
  $$\mathcal{R}_0(u,v){+}\mathcal{R}^*_0(u,-\mathrm{D}\mathcal{E}_0(u)) = -\langle \mathrm{D}\mathcal{E}_0(u), v\rangle$$

Remark:
$\mathcal{M}(u,v) \geq \mathcal{R}_0(u,v){+}\mathcal{R}^*_0(u,-\mathrm{D}\mathcal{E}_0(u))$ is suffic. for (a,b) but not necessary!

Even, passage from quadratic $\mathcal{R}_\varepsilon(v) = r_\varepsilon \|v\|_H^{\mathbf{2}}$

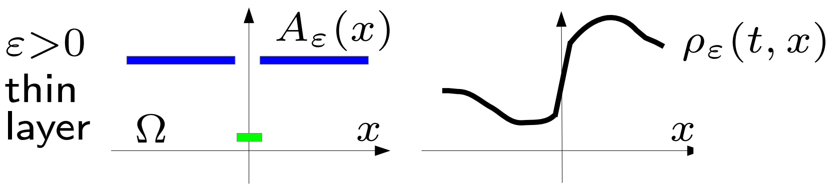
to 1-homogeneous $\mathcal{R}_0(v) = r_0\|v\|_{\boldsymbol{X}}^{\mathbf{1}}$ is possible!

From diffusion to transmission (a case of dimension reduction)

(Liero'12 PhD thesis, Liero-M-Peletier-Renger'2015 WIAS preprint 2148)

Consider diffusion in $]-l,l[$ with much lower mobility in thin layer $]-\varepsilon,\varepsilon[$:

$$\dot u = \mathrm{div}(A_\varepsilon(x)\nabla u) + \text{Neum.BC} \quad \text{with } A_\varepsilon(x) = \begin{cases} a & \text{for } \varepsilon < |x| < l, \\ \varepsilon b & \text{for } |x| \leq \varepsilon \end{cases}$$



$$\mathcal{E}_\varepsilon(u) = \int_\Omega \lambda_\mathrm{B}(u(x))\,\mathrm{d}x \quad \text{with } \lambda_\mathrm{B}(z) = z\log z - z + 1 \geq 0$$

$$\mathcal{R}^*_\varepsilon(u,\xi) = \frac{1}{2}\int_\Omega A_\varepsilon(x)u(x)\xi'(x)^2\,\mathrm{d}x$$

quadratic Wasserstein diffusion

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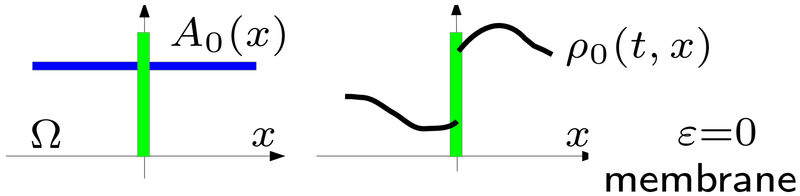

$\mathcal{E}_\varepsilon(u) = \int_\Omega \lambda_{\mathrm{B}}(u(x))\,\mathrm{d}x$

$\mathcal{R}^*_\varepsilon(u,\xi) = \frac{1}{2}\int_\Omega A_\varepsilon(x)u(x)\xi'(x)^2\,\mathrm{d}x$ quadratic Wasserstein diffusion

Theorem (LMPR'15) $(\mathrm{L}^1_\geq(\Omega),\mathcal{E},\mathcal{R}^*_\varepsilon) \overset{\text{evol}}{\rightharpoonup} (\mathrm{L}^1_\geq(\Omega),\mathcal{E},\mathcal{R}^*_0)$

with $\mathcal{R}^*_0(u,\xi) = \frac{a}{2}\int_{]-l,0[} u|\xi'|^2\,\mathrm{d}x + \frac{a}{2}\int_{]0,l[} u|\xi'|^2\,\mathrm{d}x$ Wasserstein diffusion

$+\, b\sqrt{u(0^-)u(0^+)}\Big(\cosh\big(\tfrac{1}{2}(\xi(0^+)-\xi(0^-))\big)-1\Big)$ non-quadratic

Limit gradient system $(\mathrm{L}^1_{\geq}(\Omega), \mathcal{E}, \mathcal{R}^*_0)$ with $\mathcal{E}(u) = \int_{-l}^{l} \lambda_{\mathrm{B}}(u(x))\,\mathrm{d}x$ and

$$\mathcal{R}^*_0(u,\xi) = \frac{a}{2}\int_{]-l,0[} u|\xi'|^2\,\mathrm{d}x + \frac{a}{2}\int_{]0,l[} u|\xi'|^2\,\mathrm{d}x + b\sqrt{u(0^-)u(0^+)}\Big(\cosh\big(\tfrac{1}{2}(\xi(0^+){-}\xi(0^-))\big){-}1\Big)$$

Chemical potential $\xi(x) = \mathrm{D}\mathcal{E}(u)(x) = \log u(x)$

Transmission cond. arises from $\dot{u} = \mathrm{D}_\xi\mathcal{R}^*_0(u, -\mathrm{D}\mathcal{E}(u))$ via integr.by parts:

$$x = 0^+ : \quad au(0^+)\xi'(0^+) = -b\sqrt{u(0^-)u(0^+)}\,\frac{1}{2}\sinh\big(\tfrac{1}{2}(\xi(0^+){-}\xi(0^-))\big)$$

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$$au'(0^+) \quad = \quad -b\big(u(0^+){-}u(0^-)\big)$$

$$x = 0^- : \quad au'(0^-) \quad = \quad +b\big(u(0^+){-}u(0^-)\big)$$

⊖ Linear transmission conditions arise in nontrivial nonlinear way.

⊕ Obtain Marcelin-de Donder kinetics (as used in physics) for membrane.

Since $\mathcal{E}_\varepsilon = \mathcal{E}$ the evol. Γ-convergence follows easily using the next result.

Proposition. Define the time-space functional

$$\mathcal{J}_\varepsilon(u) = \int_0^T \big(\mathcal{R}_\varepsilon(u,\dot u) + \mathcal{R}^*_\varepsilon(u,\text{-}\log u)\big)\,\mathrm{d}x = \int_0^T\!\!\int_{\text{-}l}^{l} \Big(\frac{(\int_x^1 \dot u \,\mathrm{d}y)^2}{2A_\varepsilon(x)u} + \frac{A_\varepsilon(x)(u')^2}{2u} \Big)\,\mathrm{d}x\,\mathrm{d}t,$$

then $\mathcal{J}_\varepsilon \xrightarrow{\Gamma} \mathcal{J}_0$ in $\mathrm{L}^1([0,T]{\times}\Omega)$ with $\mathcal{J}_0(u) = \int_0^T \big(\mathcal{R}_0(u,\dot u) + \mathcal{R}^*_0(u,\text{-}\log u)\big)\mathrm{d}x$.

- The Sandier-Serfaty approach does not work:
  For general u (not solutions $u_\varepsilon \to u$) we have separate Γ-limits
  - $u \mapsto \int_0^T \mathcal{R}_\varepsilon(u,\dot u)\,\mathrm{d}t \xrightarrow{\Gamma} \mathcal{J}_{\mathsf{veloc}} \lneqq \int_0^T \mathcal{R}_0\,\mathrm{d}t$
  - $u \mapsto \int_0^T \mathcal{R}^*_\varepsilon(u,\text{-}\log u)\,\mathrm{d}t \xrightarrow{\Gamma} \mathcal{J}_{\mathsf{slope}} \lneqq \int_0^T \mathcal{R}^*_0(\cdot,\text{-}\log\cdot)\,\mathrm{d}t$
- There is a non-trivial interplay between the two terms,
  recovery sequences for $\mathcal{J}_{\mathsf{veloc}}$ and $\mathcal{J}_{\mathsf{slope}}$ are different: $\mathcal{J}_0 \gneqq \mathcal{J}_{\mathsf{veloc}} + \mathcal{J}_{\mathsf{slope}}$

Idea of the proof of proposition: $\mathfrak{J}_\varepsilon(u) = \int_{-l}^{l} \left(\frac{\left(\int_{-1}^{x} \dot u \,\mathrm{d}y\right)^2}{2A_\varepsilon(x)u} + \frac{A_\varepsilon(x)(u')^2}{2u} \right) \mathrm{d}x$

Blow up of membrane to size 1: $x = X_\varepsilon(\hat x) = \begin{cases} \hat x & \text{for } \hat x \in [-l, -\varepsilon], \\ \frac{\varepsilon(2\hat x - 1)}{1+2\hat\varepsilon} & \text{for } \hat x \in [-\varepsilon, 1+\varepsilon], \\ \hat x - 1 & \text{for } \hat x \in [1+\varepsilon, l+1]. \end{cases}$

Setting $\hat u(\hat x) = u(X_\varepsilon(\hat x))$ and $\widehat{a}_\varepsilon(\hat x) := \frac{A_\varepsilon(X_\varepsilon(\hat x))}{X'_\varepsilon(\hat x)} \in \{a, b\}$ yields transformed fnctl

$$\widehat{\mathfrak{J}}_\varepsilon(\hat u) = \int_{-l}^{l+1} \left(\frac{\left(\int_{-1}^{\hat x} \dot{\hat u}\, X'_\varepsilon(\hat y) \,\mathrm{d}\hat y\right)^2}{2\widehat{a}_\varepsilon(\hat x)\hat u} + \frac{\widehat{a}_\varepsilon(\hat x)(\hat u')^2}{2\hat u} \right) \mathrm{d}\hat x \xrightarrow{\Gamma} \widehat{\mathfrak{J}}_0 := \widehat{\mathfrak{J}}_{[-1,0]} + \widehat{\mathfrak{J}}_{\mathsf{memb}} + \widehat{\mathfrak{J}}_{[1,l+1]}$$

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where $\widehat{\mathfrak{J}}_{\text{memb}}(\hat u) = \int_0^1 \left(\frac{\alpha^2}{2b\widehat u} + \frac{b(\hat u')^2}{2\hat u} \right) \mathrm{d}\hat x$ with $\alpha = \int_{-l}^{0} \dot{\hat u}(\hat y) \, \mathrm{d}\hat y = \text{const.}$

Now we use $\min \left\{ \int_0^1 \frac{\beta^2 + (\hat u')^2}{2\widehat u} \mathrm{d}\hat x \;\middle|\; \begin{array}{l} \widehat u(0) = u(0^-) \\ \widehat u(1) = u(0^+) \end{array} \right\} = \ldots$

$= \sqrt{u(0^-)u(0^+)} \Big(\mathfrak{S}\big(\frac{\beta}{\sqrt{u(0^-)u(0^+)}} \big) + \mathfrak{S}^*\big(\log \frac{u(0^+)}{u(0^-)} \big) \Big)$ with $\mathfrak{S}^*(\xi) = 4\cosh(\frac{1}{2}\xi) - 4$

Overview

4. Energy-dissipation formulations

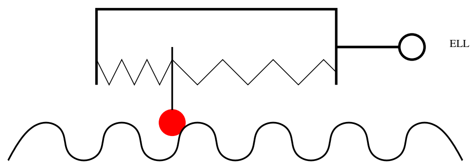
Aim: Derive dry friction as evol. Γ-limit of viscous friction

$(\mathbb{R}, \mathcal{E}_\varepsilon, \Psi_\varepsilon) \xrightarrow{\text{evol}} (\mathbb{R}, \mathcal{E}_0, \Psi_0)$

where $\Psi_\varepsilon(v) = \frac{\varepsilon^\alpha}{2} v^2$ (quadratic)
and $\Psi_0(v) = \rho|v|$ (one-homogeneous)

Here $\mathcal{E}_\varepsilon(t, \cdot)$ is a **wiggly energy landscape**

James '96, Puglisi&Truskinovsky '02,'05



Prandtl Gedankenmodell 1928

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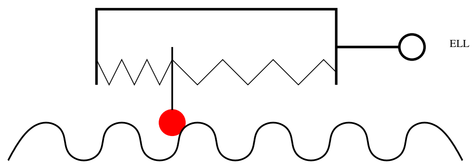
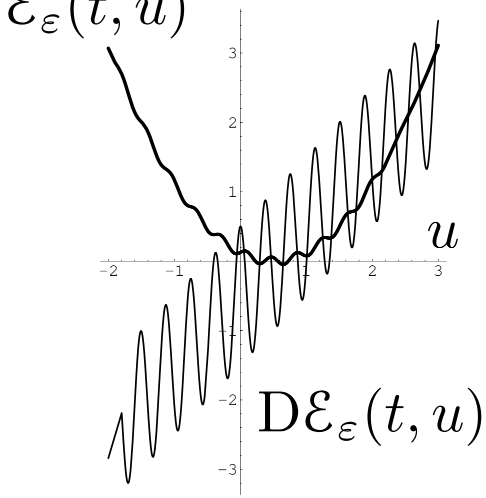


Prandtl Gedankenmodell 1928

Driven gradient system $(\mathbb{R}, \mathcal{E}_\varepsilon, \Psi_\varepsilon)$

$$\mathcal{E}_\varepsilon(t,u) = \underbrace{\tfrac{1}{2}u^2 - \ell(t)u}_{\text{macroscopic part}} + \underbrace{\varepsilon\rho\cos(u/\varepsilon)}_{\text{wiggly part}}$$



$$\varepsilon^\alpha \dot{u} = -\mathrm{D}_u \mathcal{E}_\varepsilon(t,u) = -\big(u - \ell(t)\big) + \rho \sin(u/\varepsilon)$$

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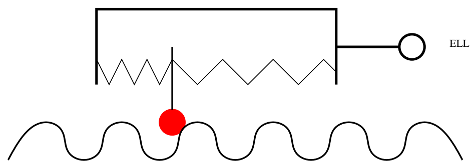

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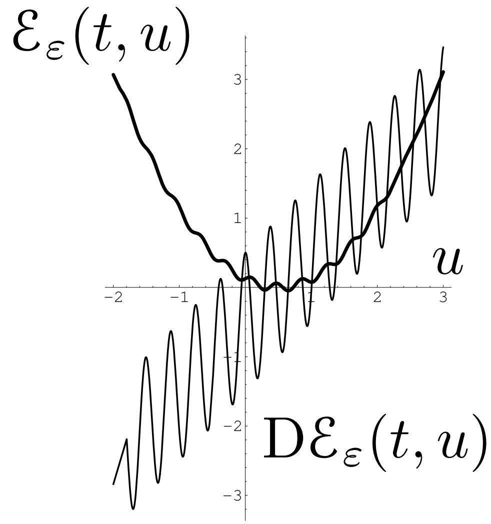


$$\boxed{\varepsilon^\alpha \dot{u} = -\mathrm{D}_u \mathcal{E}_\varepsilon(t,u) = -\big(u - \ell(t)\big) + \rho\sin(u/\varepsilon)}$$

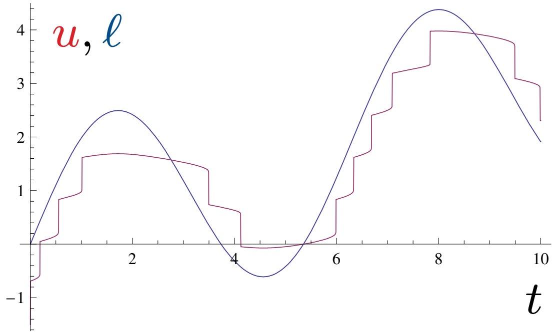
$\mathcal{E}_\varepsilon(t,u) \xrightarrow{\text{pw}} \mathcal{E}_0(t,u) = \frac{1}{2}u^2 - \ell(t)u + 0$ and $\Psi_\varepsilon \to \Psi_0 \equiv 0$

However, $u = \lim u^\varepsilon$ **does not solve** $0 = -\mathrm{D}_u \mathcal{E}_0(t, u(t))$ **!!**

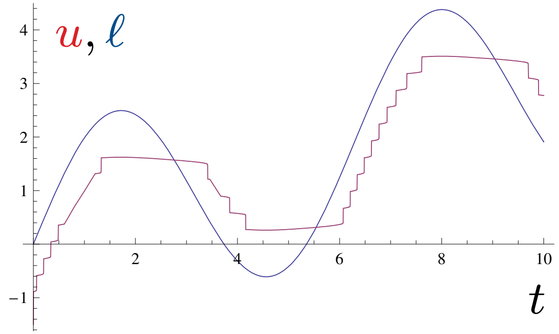
4. Energy-dissipation formulations

Simulation: $\mathcal{E}_\varepsilon(t,u) = \frac{1}{2}u^2 - \ell(t)u - \varepsilon\cos(u/\varepsilon)$,

$\ell(t) = 2\sin t + 0.3\,t$, $q(0) = -1.0$, $\varepsilon^\alpha = 10^{-3}$

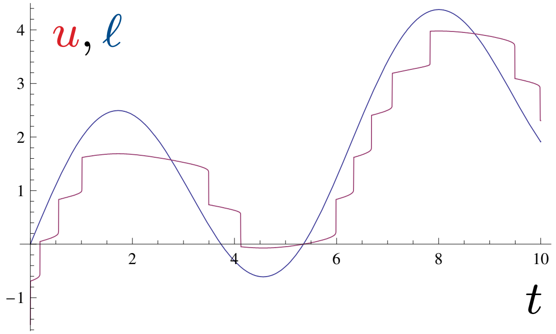


$\varepsilon = 0.125$

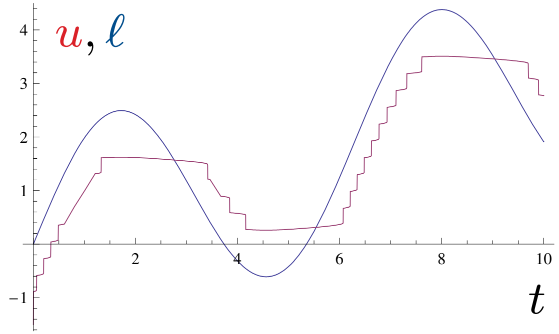


$\varepsilon = 0.05$

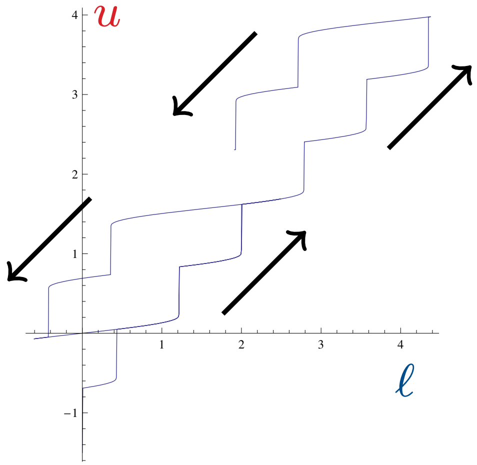
Simulation: $\mathcal{E}_\varepsilon(t,u) = \frac{1}{2}u^2 - \ell(t)u - \varepsilon\cos(u/\varepsilon)$,
$\ell(t) = 2\sin t + 0.3\,t, \quad q(0) = -1.0, \quad \varepsilon^\alpha = 10^{-3}$



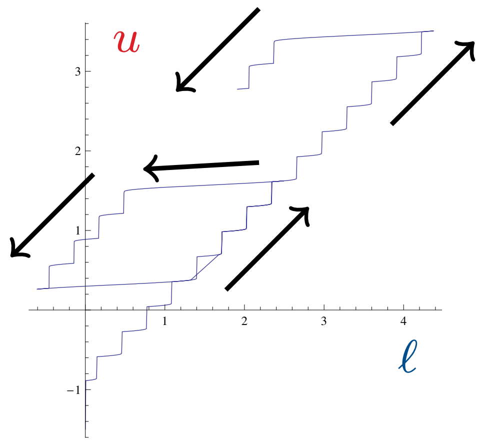
$\varepsilon = 0.125$



$\varepsilon = 0.05$



$\varepsilon = 0.125$



$\varepsilon = 0.05$

For $\varepsilon \to 0$ (vanishing oscillations and vanishing viscosity):
Convergence to a rate-independent hysteresis operator

$$\mathcal{E}_\varepsilon(t,u) = \frac{1}{2}u^2 - \ell(t)u + \varepsilon\rho\cos(u/\varepsilon), \qquad \Psi_\varepsilon(v) = \frac{\varepsilon^\alpha}{2}v^2, \qquad \Psi^*_\varepsilon(\xi) = \frac{1}{2\varepsilon^\alpha}\xi^2$$

Theorem (M'11 Cont. Mech. Thermodyn. / Puglisi-Truskinovsky'05)

$$(\mathbb{R}, \mathcal{E}_\varepsilon, \Psi_\varepsilon) \xrightarrow{\text{evol}} (\mathbb{R}, \mathcal{E}_0, \Psi_0)$$

where $\mathcal{E}_0(u) = \frac{1}{2}u^2 - \ell(t)u$
and $\Psi_0(v) = \rho|v|$

Use (EDE) $\mathcal{E}_\varepsilon(T, u_\varepsilon(T)) + \mathcal{J}_\varepsilon(u_\varepsilon) = \mathcal{E}_\varepsilon(u_\varepsilon(0))$ with

$$\mathcal{J}_\varepsilon(u) = \int_0^T \Psi_\varepsilon(\dot u) + \Psi^*_\varepsilon(-\mathrm{D}\mathcal{E}_\varepsilon(t,u))\,\mathrm{d}t \geq \int_0^T (1-\varepsilon^{\frac{\alpha}{2}})|\dot u||\mathrm{D}\mathcal{E}_\varepsilon(t,u)| + \frac{1/2}{\varepsilon^{\alpha/2}}\mathrm{D}\mathcal{E}_\varepsilon(t,u)^2\,\mathrm{d}t$$

4. Energy-dissipation formulations

$$\mathcal{E}_\varepsilon(t,u) = \frac{1}{2}u^2 - \ell(t)u + \varepsilon\rho\cos(u/\varepsilon), \qquad \Psi_\varepsilon(v) = \frac{\varepsilon^\alpha}{2}v^2, \qquad \Psi^*_\varepsilon(\xi) = \frac{1}{2\varepsilon^\alpha}\xi^2$$

Theorem (M'11 Cont. Mech. Thermodyn. / Puglisi-Truskinovsky'05)

$$(\mathbb{R},\mathcal{E}_\varepsilon,\Psi_\varepsilon) \xrightarrow{\text{evol}} (\mathbb{R},\mathcal{E}_0,\Psi_0)$$

where $\mathcal{E}_0(u) = \frac{1}{2}u^2 - \ell(t)u$
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Proposition: $u^\varepsilon \rightsquigarrow u^0 \implies \liminf_{\varepsilon\to 0} \mathbb{J}_\varepsilon(u^\varepsilon) \geq \int_0^T \mathcal{M}(u^0,\dot u^0,t)\,\mathrm{d}t$ with

$\mathcal{M}(u,v,t) = |v|K(\ell(t)-u) + \chi_{[-\rho,\rho]}(\ell(t)-u)$ and $K(\xi) = \frac{1}{2\pi}\int_0^{2\pi}|\xi+\rho\cos y|\,\mathrm{d}y$

$K(\xi) = |\xi|$ for $|\xi| \geq \rho$ and $K(\xi) \gneqq |\xi|$ for $|\xi| < \rho \implies$

$$\mathcal{M}(u,v,t) \geq |v|\,|\ell(t)-u| \geq -v\,\mathrm{D}\mathcal{E}_0(t,u) \quad\implies\quad \ldots\ldots \quad\implies\quad \Psi_0(v) = \rho|v|$$

Overview

Overview

Ambrosio-Gigli-Savaré'05, Daneri-Savaré'08'10

Gradient system $(\boldsymbol{X}, \mathcal{E}, \mathcal{R})$ with **quadratic** $\mathcal{R}(u,v) = \frac{1}{2}\langle \mathbb{G}(u)v, v\rangle$

- **Geodesic distance** $d_\mathcal{R} : \boldsymbol{X} \times \boldsymbol{X} \to [0,\infty]$ defined via
$$d_\mathcal{R}(u_0,u_1)^2 = \inf\{ \int_0^1 2\mathcal{R}(\widetilde{u},\dot{\widetilde{u}})\,\mathrm{d}s \mid u_0 \overset{\widetilde{u}}{\rightsquigarrow} u_1 \}$$
- $\widetilde{u} : [s_0,s_1] \to \boldsymbol{X}$ is called a **geodesic curve** in $(\boldsymbol{X}, d_\mathcal{R})$
if $d_\mathcal{R}(\widetilde{u}(r),\widetilde{u}(t)) = |t-r| d_\mathcal{R}(\widetilde{u}(s_0),\widetilde{u}(s_1))$ for all $r,t \in [s_0,s_1]$

Ambrosio-Gigli-Savaré'05, Daneri-Savaré'08'10

Gradient system $(\boldsymbol{X},\mathcal{E},\mathcal{R})$ with **quadratic** $\mathcal{R}(u,v)=\frac12\langle\mathbb{G}(u)v,v\rangle$

- **Geodesic distance** $d_\mathcal{R}:\boldsymbol{X}\times\boldsymbol{X}\to[0,\infty]$ defined via
$$d_\mathcal{R}(u_0,u_1)^2=\inf\{\,\int_0^1 2\mathcal{R}(\widetilde u,\dot{\widetilde u})\,\mathrm{d}s \mid u_0 \overset{\widetilde u}{\rightsquigarrow} u_1\,\}$$
- $\widetilde u:[s_0,s_1]\to\boldsymbol{X}$ is called a **geodesic curve** in $(\boldsymbol{X},d_\mathcal{R})$
if $d_\mathcal{R}(\widetilde u(r),\widetilde u(t))=|t{-}r|d_\mathcal{R}(\widetilde u(s_0),\widetilde u(s_1))$ for all $r,t\in[s_0,s_1]$
- $\mathcal{E}:\boldsymbol{X}\to\mathbb{R}_\infty$ is called **geodesically λ-convex** on $(\boldsymbol{X},d_\mathcal{R})$ if
$s\mapsto\mathcal{E}(\widetilde u(s))-\lambda\frac{d_\mathcal{R}(\widetilde u(s_0),\widetilde u(s))^2}{2}$ is convex on $[s_0,s_1]$ for all geod. $\widetilde u$

Trivial but useful and important case: Hilbert spaces!!

$\mathbb{G}(u)=\mathbb{G}_\varepsilon$ =const. $\Longrightarrow$ $d_{\mathcal{R}_\varepsilon}(u_0,u_1)=\|u_1{-}u_0\|_{\mathbb{G}_\varepsilon}$ with $\|w\|^2_{\mathbb{G}_\varepsilon}=\langle\mathbb{G}_\varepsilon w,w\rangle$

Then, $\mathcal{E}$ geod. λ-convex on $(\boldsymbol{X},d_{\mathbb{G}_\varepsilon})$ $\iff$ $\mathrm{D}^2\mathcal{E}\geq\lambda\mathbb{G}_\varepsilon$

Formulations used so far:

(i) $0 \in \mathbb{G}(u)\dot{u} + \mathrm{D}\mathcal{E}(u)$ (ii) $\dot{u} = -\nabla_{\mathbb{G}}\mathcal{E}(u) = -\mathbb{K}(u)\mathrm{D}\mathcal{E}(u)$ (iii) ....

(EDE) $\mathcal{E}(u(T)) + \int_0^T \mathcal{R}(u,\dot{u}) + \mathcal{R}^*(u, -\mathrm{D}\mathcal{E}(u))\,\mathrm{d}t \leq \mathcal{E}(u(0))$

Truely derivative-free reformulation for λ-convex gradient system

Theorem [AGS'05] (Benilan'72: Hilbert-space case $d = d_{\mathbb{G}_{\text{const}}}$)

If $(\boldsymbol{X}, \mathcal{E}, \mathbb{G})$ is geodesically λ-convex, then

(i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (EDE) $\Leftrightarrow$ **(EVI)**$_\lambda$ $\Leftrightarrow$ **(EVI')**$_\lambda$

where

(EVI)$_\lambda$ $\frac{1}{2}\frac{\mathrm{d}^+}{\mathrm{d}t} d_{\mathbb{G}}(u(t),w)^2 + \frac{\lambda}{2} d_{\mathbb{G}}(u(t),w)^2 + \mathcal{E}(u(t)) \leq \mathcal{E}(w)$

for $t > 0$, $w \in \boldsymbol{X}$

(EVI')$_\lambda$ $\frac{\mathrm{e}^{\lambda\tau}}{2} d_{\mathbb{G}}(u(t{+}\tau),w)^2 - \frac{1}{2} d_{\mathbb{G}}(u(t),w)^2$

$\leq \frac{\mathrm{e}^{\lambda\tau}-1}{\lambda}\big(\mathcal{E}(w) - \mathcal{E}(u(t{+}\tau))\big)$ for $t, \tau > 0$, $w \in \boldsymbol{X}$

Formulations used so far:

(i) $0 \in \mathbb{G}(u)\dot{u} + \mathrm{D}\mathcal{E}(u)$ (ii) $\dot{u} = -\nabla_{\mathbb{G}}\mathcal{E}(u) = -\mathbb{K}(u)\mathrm{D}\mathcal{E}(u)$ (iii) ....

(EDE) $\mathcal{E}(u(T)) + \int_0^T \mathcal{R}(u,\dot{u}) + \mathcal{R}^*(u, -\mathrm{D}\mathcal{E}(u))\,\mathrm{d}t \leq \mathcal{E}(u(0))$

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$\leq \frac{\mathrm{e}^{\lambda\tau}-1}{\lambda}\big(\mathcal{E}(w) - \mathcal{E}(u(t{+}\tau))\big)$ for $t, \tau > 0$, $w \in \boldsymbol{X}$

Exercise:

(a) Prove (EDE) $\Leftrightarrow$ (EVI)$_\lambda$ (b) Prove (EVI)$_\lambda$ $\Leftrightarrow$ (EVI'$_\lambda$

5. Evolutionary variational inequality (EVI)

Formulations used so far:

(i) $0 \in \mathbb{G}(u)\dot{u} + \mathrm{D}\mathcal{E}(u)$ (ii) $\dot{u} = -\nabla_{\mathbb{G}}\mathcal{E}(u) = -\mathbb{K}(u)\mathrm{D}\mathcal{E}(u)$ (iii) ....

(EDE) $\mathcal{E}(u(T)) + \int_0^T \mathcal{R}(u,\dot{u}) + \mathcal{R}^*(u,-\mathrm{D}\mathcal{E}(u))\,\mathrm{d}t \leq \mathcal{E}(u(0))$

Truely derivative-free reformulation for λ-convex gradient system

Theorem [AGS'05] (Benilan'72: Hilbert-space case $d = d_{\mathbb{G}_{\text{const}}}$)

If $(\boldsymbol{X}, \mathcal{E}, \mathbb{G})$ is geodesically λ-convex, then

(i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (EDE) $\Leftrightarrow$ **(EVI)**$_\lambda$ $\Leftrightarrow$ **(EVI')**$_\lambda$

where

(EVI)$_\lambda$ $\frac{1}{2}\frac{\mathrm{d}^+}{\mathrm{d}t} d_{\mathbb{G}}(u(t),w)^2 + \frac{\lambda}{2} d_{\mathbb{G}}(u(t),w)^2 + \mathcal{E}(u(t)) \leq \mathcal{E}(w)$

for $t > 0$, $w \in \boldsymbol{X}$

(EVI')$_\lambda$ $\frac{\mathrm{e}^{\lambda\tau}}{2} d_{\mathbb{G}}(u(t{+}\tau),w)^2 - \frac{1}{2} d_{\mathbb{G}}(u(t),w)^2$

$\leq \frac{\mathrm{e}^{\lambda\tau}-1}{\lambda}\big(\mathcal{E}(w) - \mathcal{E}(u(t{+}\tau))\big)$ for $t,\tau > 0$, $w \in \boldsymbol{X}$

⊕ no derivatives of $\mathcal{E}_\varepsilon$ and $\mathcal{R}_\varepsilon$ appear ⇝ ideal for Γ-convergence

⊕ no time derivative $\dot{u}$ is involved

(EVI')$_\lambda$ $\frac{e^{\lambda\tau}}{2} d_\varepsilon(u(t{+}\tau),w)^2 - \frac{1}{2} d_\varepsilon(u(t),w)^2 \ \leq\ \frac{e^{\lambda\tau}-1}{\lambda}\big(\mathcal{E}_\varepsilon(w) - \mathcal{E}_\varepsilon(u(t{+}\tau))\big)$

Theorem (Savaré'11 (personal communication))

If $(\boldsymbol{X}, \mathcal{E}_\varepsilon, d_\varepsilon)$ is geodesically λ-convex, $\mathcal{E}_\varepsilon$ $\boldsymbol{X}$-coercive (both unif. in ε), $\mathcal{E}_\varepsilon \xrightarrow{\Gamma} \mathcal{E}$, and $d_\varepsilon \xrightarrow{\text{cont}} d$ in $\boldsymbol{X}$, then $(\boldsymbol{X}, \mathcal{E}_\varepsilon, d_\varepsilon) \xrightarrow{\text{evol}} (\boldsymbol{X}, \mathcal{E}, d)$.

(Convergence of the whole sequence u^ε to u, since solutions are unique.)

(EVI')$_\lambda$ $\frac{e^{\lambda\tau}}{2} d_\varepsilon(u(t+\tau), w)^2 - \frac{1}{2} d_\varepsilon(u(t), w)^2 \leq \frac{e^{\lambda\tau}-1}{\lambda}\big(\mathcal{E}_\varepsilon(w) - \mathcal{E}_\varepsilon(u(t+\tau))\big)$

Theorem (Savaré'11 (personal communication))

If $(\boldsymbol{X}, \mathcal{E}_\varepsilon, d_\varepsilon)$ is geodesically λ-convex, $\mathcal{E}_\varepsilon$ $\boldsymbol{X}$-coercive (both unif. in ε), $\mathcal{E}_\varepsilon \xrightharpoonup{\Gamma} \mathcal{E}$, and $d_\varepsilon \xrightharpoonup{\text{cont}} d$ in $\boldsymbol{X}$, then $(\boldsymbol{X}, \mathcal{E}_\varepsilon, d_\varepsilon) \xrightarrow{\text{evol}} (\boldsymbol{X}, \mathcal{E}, d)$.

(Convergence of the whole sequence u^ε to u, since solutions are unique.)

The relatively strong assumption $d_\varepsilon \xrightharpoonup{\text{cont}} d$ in $\boldsymbol{X}$ means
$u_\varepsilon \rightharpoonup u$ & $w_\varepsilon \rightharpoonup w$ in $\boldsymbol{X}$ $\quad\Longrightarrow\quad$ $d_\varepsilon(u_\varepsilon, w_\varepsilon) \to d(u, w)$

This can be weakened to
Gromov-Hausdorff convergence $(\boldsymbol{X}, d_\varepsilon) \xrightarrow{\text{GH}} (\boldsymbol{X}, d)$.

(EVI')$_\lambda$ $\frac{e^{\lambda\tau}}{2} d_\varepsilon(u(t{+}\tau), w)^2 - \frac{1}{2} d_\varepsilon(u(t), w)^2 \ \leq \ \frac{e^{\lambda\tau}-1}{\lambda}\big(\mathcal{E}_\varepsilon(w) - \mathcal{E}_\varepsilon(u(t{+}\tau))\big)$

Theorem (Savaré'11 (personal communication))

If $(\boldsymbol{X}, \mathcal{E}_\varepsilon, d_\varepsilon)$ is geodesically λ-convex, $\mathcal{E}_\varepsilon$ $\boldsymbol{X}$-coercive (both unif. in ε), $\mathcal{E}_\varepsilon \xrightharpoonup{\Gamma} \mathcal{E}$, and $d_\varepsilon \xrightharpoonup{cont} d$ in $\boldsymbol{X}$, then $(\boldsymbol{X}, \mathcal{E}_\varepsilon, d_\varepsilon) \xrightarrow{evol} (\boldsymbol{X}, \mathcal{E}, d)$.

(Convergence of the whole sequence u^ε to u, since solutions are unique.)

Sketch of proof: u_ε solves (EVI')$_\lambda$ for $(\boldsymbol{X}, \mathcal{E}_\varepsilon, d_\varepsilon)$

- ε-uniform bounds from (EVI')$_\lambda$ $\Longrightarrow$ $u_{\varepsilon_k}(t) \rightharpoonup u(t)$ for all $t \in [0, T]$
- Pass to the limit in (EVI')$_\lambda$ using

  recovery sequence $w_\varepsilon \rightharpoonup w$ with $\mathcal{E}_\varepsilon(w_\varepsilon) \to \mathcal{E}(w)$

  $\Rightarrow$ $d_\varepsilon(u_\varepsilon(t{+}\tau), w_\varepsilon) \to d(u(t{+}\tau), w)$ and $d_\varepsilon(u_\varepsilon(t), w_\varepsilon) \to d(u(t), w)$

  $\Rightarrow$ $\mathcal{E}(u(t{+}\tau)) \leq \liminf_{\varepsilon\to 0} \mathcal{E}_\varepsilon(u_\varepsilon(t{+}\tau))$ by Γ-liminf estimate
- Hence, $u : [0, T] \to \boldsymbol{X}$ satisfies (EVI')$_\lambda$ for $(\boldsymbol{X}, \mathcal{E}, d)$ QED

Overview