PLASTICITY AND DAMAGE — PART II — perfect plasticity with rate dependent damage with a possible healing

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with computational contribution by
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The plot:

Part I: basic scenario: rate-independent plasticity + rate-independent damage

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Part II: perfect plasticity with rate dependent damage with a possible healing

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Part III: rate-independent unidirectional damage with visco-plasticity, thermodynamics, etc.

Part IV: tutorial – further outlooks
(combination with other processes, large strains, etc.)

1 Preliminary ingredients
- Nonsimple materials
- Perfect plasticity (Prandtl-Reuss model)

2 Perfect plasticity in nonsimple materials with damage
- The model
- Weak formulation
- Time discretisation, a-priori estimates, convergence

3 Numerics, simulations, modifications
- Numerics
- Computational simulations
- Some modifications

The concept of 2nd-grade nonsimple materials

(also called complex materials or multipolar solids)

R.A. Toupin 1962, R.D. Mindlin & N.N. Eshel 1968, M. Šilhavý, 1985, P. Podio-Guidugli 2002, P. Podio-Guidugli & M. Vianello 2010, E. Fried & M. E. Gurtin 2006, etc.

The calculus on Γ:

div_S is the surface-divergence operator, which may be introduced as follows: given a vector field $v:\Gamma\to\mathbb{R}^3$, we extend it to a neighborhood of Γ, and we let its surface gradient be defined as $\nabla_\mathrm{S} v=\nabla v\mathbb{P}_\mathrm{S}$, where $\mathbb{P}_\mathrm{S}=\mathbb{I}-n\otimes n$ is the projector on the tangent space of Γ; we then let the surface divergence of v be the scalar field $\mathrm{div}_\mathrm{S} v=\mathbb{P}_\mathrm{S}:\nabla_\mathrm{S} v=\mathrm{tr}(\mathbb{P}_\mathrm{S}\nabla v\mathbb{P}_\mathrm{S})$. Given a tensor field $\mathbb{A}:\Gamma\to\mathbb{R}^{3\times3}$, we let $\mathrm{div}_\mathrm{S}\mathbb{A}:\Gamma\to\mathbb{R}^3$ be the unique vector field such that $\mathrm{div}_\mathrm{S}(\mathbb{A}^T a)=a\cdot\mathrm{div}_\mathrm{S}\mathbb{A}$ for all constant vector fields $a:\Gamma\to\mathbb{R}^3$.

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so that $\mathrm{div}_\mathrm{S}\vec{n}$ is (up to a factor $-\frac{1}{2}$) the mean curvature of the surface Γ.

Consider a quadratic functional:

$$u \mapsto \int_\Omega \frac{1}{2}\mathbb{C}e(u):e(u) + \frac{1}{2}\mathbb{H}\nabla e(u)\vdots\nabla e(u) - g\cdot u\,\mathrm{d}x - \int_{\Gamma_{\rm Neu}} f\cdot u\,\mathrm{d}S$$

to be minimized on $H^2(\Omega;\mathbb{R}^d)$ subject to $u|_{\Gamma_{\rm Dir}} = w_{\rm Dir}$.

How the Euler-Lagrange equation look like?

The weak formulation:

$$\int_\Omega \mathbb{C}e(u):e(v) + \mathbb{H}\nabla e(u)\vdots\nabla e(v)\,\mathrm{d}x = \int_\Omega g\cdot v\,\mathrm{d}x + \int_{\Gamma_{\rm Neu}} f\cdot v\,\mathrm{d}S.$$

$$\forall v\in H^2(\Omega;\mathbb{R}^d),\quad v|_{\Gamma_{\rm Dir}} = 0.$$

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Green's formula:

$$\int_\Omega -\mathrm{div}(\mathbb{C}e(u))\cdot v - \mathrm{div}(\mathbb{H}\nabla e(u)):\nabla v\,\mathrm{d}x = \int_\Omega g\cdot v\,\mathrm{d}x + \int_{\Gamma_{\rm Neu}} f\cdot v\,\mathrm{d}S$$

$$-\int_\Gamma (\mathbb{C}e(u)):(v\otimes\vec{n}) + (\mathbb{H}\nabla e(u)) \vdots (\nabla v\otimes\vec{n})\,\mathrm{d}S.$$

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How the Euler-Lagrange equation look like?

The weak formulation using symmetry of $\mathbb{C}$ and $\mathbb{H}$:

$$\int_\Omega \mathbb{C}e(u) : \nabla v + \mathbb{H}\nabla e(u) \vdots \nabla^2 v \,\mathrm{d}x = \int_\Omega g \cdot v \,\mathrm{d}x + \int_{\Gamma_{\rm Neu}} f \cdot v \,\mathrm{d}S. \qquad \forall v \in H^2(\Omega;\mathbb{R}^d), \quad v|_{\Gamma_{\rm Dir}} = 0.$$

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Green's formula once more:

$$\int_\Omega -\mathrm{div}\big(\mathbb{C}e(u)\big)\cdot v + \mathrm{div}^2\big(\mathbb{H}\nabla e(u)\big)\cdot v\,\mathrm{d}x = \int_\Omega g\cdot v\,\mathrm{d}x + \int_{\Gamma_{\rm Neu}} f\cdot v\,\mathrm{d}S$$

$$-\int_\Gamma \big(\mathbb{C}e(u)\big):(v\otimes\vec{n}) + \big(\mathbb{H}\nabla e(u)\big) \vdots (\nabla v\otimes\vec{n}) - \mathrm{div}\big(\mathbb{H}\nabla e(u)\big):(v\otimes\vec{n})\,\mathrm{d}S.$$

Now we need to re-write the term $\int_\Gamma (\mathbb{H}\nabla e(u)) \,\vdots\, (\nabla v \otimes \vec{n})\,\mathrm{d}S$.

We use a general decomposition $\nabla v = \frac{\partial v}{\partial \vec{n}}\vec{n} + \nabla_{\mathrm{S}} v$ on Γ. Thus:

$$\begin{aligned}\int_\Gamma (\mathbb{H}\nabla e(u)) \,\vdots\, (\vec{n}\otimes\nabla v)\,\mathrm{d}S \\ &= \int_\Gamma \Big((\mathbb{H}\nabla e(u)) : (\vec{n}\otimes\vec{n})\Big)\frac{\partial v}{\partial \vec{n}} + (\mathbb{H}\nabla e(u)) \,\vdots\, (\vec{n}\otimes\nabla_{\mathrm{S}} v)\,\mathrm{d}S \\ &= \int_\Gamma \Big((\mathbb{H}\nabla e(u)) : (\vec{n}\otimes\vec{n})\Big)\frac{\partial v}{\partial \vec{n}} - \mathrm{div}_{\mathrm{S}}\Big((\mathbb{H}\nabla e(u)) \cdot \vec{n}\Big)v \\ &\qquad + (\mathrm{div}_{\mathrm{S}}\vec{n})\Big((\mathbb{H}\nabla e(u)) : (\vec{n}\otimes\vec{n})\Big)v\,\mathrm{d}S.\end{aligned}$$

We used a "surface" Green-type formula:
$\int_\Gamma w{:}((\nabla_{\mathrm{S}} v)\otimes\vec{n})\,\mathrm{d}S = \int_\Gamma (\mathrm{div}_{\mathrm{S}}\vec{n})(w{:}(\vec{n}\otimes\vec{n}))v - \mathrm{div}_{\mathrm{S}}(w{\cdot}\vec{n})v\,\mathrm{d}S$.

Thus:
$$\int_\Omega -\mathrm{div}\big(\mathbb{C}e(u)\big)\cdot v + \mathrm{div}^2\big(\mathbb{H}\nabla e(u)\big)\cdot v\,\mathrm{d}x = \int_\Omega g\cdot v\,\mathrm{d}x + \int_{\Gamma_{\rm Neu}} f\cdot v\,\mathrm{d}S$$
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can be re-written as:
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$$+\,(\mathrm{div}_{\rm s}\vec{n})\Big(\big(\mathbb{H}\nabla e(u)\big):(\vec{n}\otimes\vec{n})\Big)v - \mathrm{div}\big(\mathbb{H}\nabla e(u)\big):(v\otimes\vec{n})\,\mathrm{d}S.$$

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Thus:

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$$-\mathrm{div}\big(\mathbb{C}e(u)\big)+\mathrm{div}^2\big(\mathbb{H}\nabla e(u)\big)=g \qquad \text{on }\Omega,$$
$$\big(\mathbb{C}e(u)\big)\vec n-\mathrm{div}_{\rm s}\Big(\big(\mathbb{H}\nabla e(u)\big)\cdot\vec n\Big)$$
$$+\,(\mathrm{div}_{\rm s}\vec n)\Big(\big(\mathbb{H}\nabla e(u)\big):(\vec n{\otimes}\vec n)\Big)-\mathrm{div}\big(\mathbb{H}\nabla e(u)\big):(v\otimes\vec n)=f \qquad \text{on }\Gamma_{\rm Neu},$$
$$\big(\mathbb{H}\nabla e(u)\big):(\vec n{\otimes}\vec n)=0 \qquad \text{on }\Gamma,$$

and the Dirichlet boundary condition on $\Gamma_{\rm Dir}$.

Thus:
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$$\big(\mathbb{H}\nabla e(u)\big):(\vec n\otimes\vec n) = 0 \qquad \text{on } \Gamma,$$
and the Dirichlet boundary condition on $\Gamma_{\rm Dir}$.

...when choosing v with a compact support in Ω.

Thus: $\displaystyle\int_\Omega -\mathrm{div}\big(\mathbb{C}e(u)\big)\cdot v+\mathrm{div}^2\big(\mathbb{H}\nabla e(u)\big)\cdot v\,\mathrm{d}x=\int_\Omega g\cdot v\,\mathrm{d}x+\int_{\Gamma_{\rm Neu}} f\cdot v\,\mathrm{d}S$
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...we identified the **true traction stress**!

Perfect plasticity (=no hardening, $\mathbb{H}=0$, $b=0$), PRANDTL-REUSS' model:

Space of functions with bounded deformations (P.M. SUQUET, 1978):

$$\mathrm{BD}(\bar{\Omega};\mathbb{R}^d) := \big\{u\in L^1(\Omega;\mathbb{R}^d);\ e(u)\in \mathrm{Meas}(\bar{\Omega};\mathbb{R}^{d\times d}_{\mathrm{sym}})\big\},$$

where $e(u)$ is the distributional symmetric gradient of u.
The state-space is (not a Cartesian product, but):

$$Q_{\mathrm{PR}} = \big\{(u,\pi)\in \mathrm{BD}(\bar{\Omega};\mathbb{R}^d)\times \mathrm{Meas}(\bar{\Omega};\mathbb{R}^{d\times d}_{\mathrm{sym}});\\ e(u)-\pi\in L^2(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}}),\quad u\odot\vec{n}\mathrm{d}S=-\pi \text{ on } \Gamma_{\mathrm{Dir}}\big\},$$

where $a\odot b$ means the symetrised tensorial product $\frac{1}{2}(a\otimes b+b\otimes a)$.
Energetics:

$$\mathcal{E}_{\mathrm{PR}}(t,u,\pi)=\frac{1}{2}\int_\Omega \mathbb{C}\big(e(u)-\pi+2e(w(t))\big):\big(e(u)-\pi\big)\mathrm{d}x,$$

$$\mathcal{R}_{\mathrm{PR}}\Big(\frac{\mathrm{d}\pi}{\mathrm{d}t}\Big)=\int_\Omega R\Big(\frac{\partial\pi}{\partial t}\Big)\,\mathrm{d}x \quad \text{with} \quad R(\dot{\pi})=\delta_P^*(\dot{\pi}),$$

Perfect plasticity (=no hardening, $\mathbb{H}=0$, $b=0$), PRANDTL-REUSS' model:

Space of functions with bounded deformations (P.M. SUQUET, 1978):

$$\mathrm{BD}(\bar\Omega;\mathbb{R}^d):=\big\{u\in L^1(\Omega;\mathbb{R}^d);\ e(u)\in \mathrm{Meas}(\bar\Omega;\mathbb{R}^{d\times d}_{\rm sym})\big\},$$

where $e(u)$ is the distributional symmetric gradient of u.
The state-space is (not a Cartesian product, but):

$$Q_{\rm PR}=\big\{(u,\pi)\in \mathrm{BD}(\bar\Omega;\mathbb{R}^d)\times \mathrm{Meas}(\bar\Omega;\mathbb{R}^{d\times d}_{\rm sym});\\ e(u){-}\pi\in L^2(\Omega;\mathbb{R}^{d\times d}_{\rm sym}),\quad u\odot\vec n\mathrm{d}S=-\pi \text{ on } \Gamma_{\rm Dir}\big\},$$

where $a\odot b$ means the symetrised tensorial product $\frac12(a\otimes b+b\otimes a)$.
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Energetic solutions to Prandtl-Reuss' model:
(G.Dal Maso, A.deSimone, M.G.Mora, 2006)

Assume:
$\Gamma_{\rm Dir}$ has a $(d{-}2$ dimensional$)$ C^2-boundary,
P be convex, bounded, closed neighbourhood of $0 \in \mathbb{R}^{d\times d}_{\rm dev}$,
$\mathbb{C}$ have the special structure so that, with ${\rm dev}\, e := e{-}({\rm tr}\, e)\mathbb{I}/d$,

$$\mathbb{C}e = \mathbb{C}_D{\rm dev}\, e + \kappa({\rm tr}\, e)\mathbb{I} \ \text{ with } \mathbb{C}_D{:}\mathbb{R}^{d\times d}_{\rm dev}{\to}\,\mathbb{R}^{d\times d}_{\rm dev} \text{ positive definite, } \kappa{>}0,$$

$(u_0, \pi_0) \in {\rm BD}(\bar\Omega;\mathbb{R}^d) \times {\rm Meas}(\bar\Omega;\mathbb{R}^{d\times d}_{\rm sym})$ be stable at $t = 0$, and,
the Dirichlet loading $w \in W^{1,1}(I; W^{1/2,2}(\Gamma_{\rm Dir};\mathbb{R}^d))$.

Then:

- there is an energetic solution (u, π) to $(Q_{\rm PR}, \mathcal{E}_{\rm PR}, \mathcal{R}_{\rm PR}, u_0, \pi_0)$.
- The elastic stress $\sigma = \mathbb{C}(e(u){-}\pi)$ is determined uniquely.

No uniqueness in terms of u and π can be expected, however.

Two "inelastic" scenarios on loading: 1) first plasticity, then damage (in Part I)
2) first damage, then plasticity (now).

Yield stress undergoing damage (well doable if ζ rate dependent!),
hardening primarily not important for triggering damage
(perfect plasticity well alowed).

Rate dependent damage 1) allows for modelling also healing phenomena
2) avoids unphysically early jumps
(because, if ζ fixed, plasticity (u, π)
is a "nice" convex-type problem),
3) but needs L^2-regularity of driving stress for damage
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The classical formulation of the Biot inclusion:

no gradient of π, no hardening, but hyperstress $\mathfrak{h}$ and healing force b'.

The governing equation/inclusions read as:

$$\mathrm{div}\big(\mathbb{C}(\zeta)e_{\rm el}-\mathrm{div}\,\mathfrak{h}\big)+g=0 \quad \text{with} \quad \mathfrak{h}=\mathbb{H}\nabla e_{\rm el}, \qquad \text{(momentum equilibrium)}$$

$$\partial\delta^*_{S(\zeta)}\Big(\frac{\partial\pi}{\partial t}\Big)\ni \mathrm{dev}\big(\mathbb{C}(\zeta)e_{\rm el}-\mathrm{div}\,\mathfrak{h}\big) \quad \text{with} \quad e_{\rm el}=e(u)-\pi \qquad \text{(plastic flow rule)}$$

$$\partial a\Big(\frac{\partial\zeta}{\partial t}\Big)+\frac{1}{2}\mathbb{C}'(\zeta)e_{\rm el}:e_{\rm el}$$
$$\qquad -\kappa\,\mathrm{div}\big(|\nabla\zeta|^{r-2}\nabla\zeta\big)+N_{[0,1]}(\zeta)\ni b'(\zeta), \qquad \text{(damage flow rule)}$$

with the boundary conditions:

$$u=w_{\rm Dir} \qquad \text{on } \Gamma_{\rm Dir},$$
$$\big(\mathbb{C}(\zeta)e_{\rm el}-\mathrm{div}\,\mathfrak{h}\big)\cdot\vec{n}-\mathrm{div}_{\rm S}(\mathfrak{h}\vec{n})=f \qquad \text{on } \Gamma_{\rm Neu},$$
$$\nabla\zeta\cdot\vec{n}=0 \quad \text{and} \quad \mathfrak{h}:(\vec{n}\otimes\vec{n})=0 \qquad \text{on } \Gamma$$

Smooth time-dependent Dirichlet boundary conditions $w_{\rm Dir}$ on $\Gamma_{\rm Dir}$ which allows an extension into Q, let us denote it by $u_{\rm Dir}$, such that

$$\big(\mathbb{C}(\zeta)e(u_{\rm Dir}) - {\rm div}\,\mathfrak{h}_{\rm Dir}\big)\cdot\vec{n} - {\rm div}_{\rm S}(\mathfrak{h}_{\rm Dir}\vec{n}) = 0 \qquad \text{on } \Gamma_{\rm Neu}$$

$$\mathfrak{h}_{\rm Dir}{:}(\vec{n}\otimes\vec{n}) = 0 \quad \text{with} \quad \mathfrak{h}_{\rm Dir} = \mathbb{H}\nabla e(u_{\rm Dir}) \qquad \text{on } \Gamma$$

for any admissible ζ, and making a substitution of $u + u_{\rm Dir}$ instead of u.

(does not change the traction force f)

The state space: $\big\{(u,\pi,\zeta)\in{\rm BD}(\bar\Omega;\mathbb{R}^d)\times{\rm Meas}(\bar\Omega;\mathbb{R}^{d\times d}_{\rm dev})\times W^{1,r}(\Omega);$
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The governing functionals:

$$\mathcal{E}(t,u,\pi,\zeta) := \begin{cases} \displaystyle\int_\Omega \frac12\mathbb{C}(\zeta)\big(e(u{+}u_{\rm Dir}(t)){-}\pi\big):\big(e(u{+}u_{\rm Dir}(t)){-}\pi\big) \\ \displaystyle\quad+\frac12\mathbb{H}\nabla\big(e(u{+}u_{\rm Dir}(t)){-}\pi\big)\vdots\nabla\big(e(u{+}u_{\rm Dir}(t)){-}\pi\big) - b(\zeta) \\ \displaystyle\quad+\kappa\,\frac1r|\nabla\zeta|^r - g(t){\cdot}u\,{\rm d}x - \int_{\Gamma_{\rm Neu}} f(t){\cdot}u\,{\rm d}S & \text{if } \zeta\in[0,1] \text{ a.e. on } \Omega, \\ \infty & \text{otherwise,} \end{cases}$$

$$\mathcal{R}\Big(\zeta;\frac{{\rm d}\pi}{{\rm d}t},\frac{{\rm d}\zeta}{{\rm d}t}\Big) := \int_{\bar\Omega}\Big[\delta^*_{S(\zeta)}\Big(\frac{\partial\pi}{\partial t}\Big)\Big]({\rm d}x) + \int_\Omega a\Big(\frac{\partial\zeta}{\partial t}\Big)\,{\rm d}x.$$

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Weak formulation: main fatures:

1) the plastic part (u,π): semistability + energy equality,
2) the damage part: $\nabla\frac{\partial\zeta}{\partial t}$ not controlled, so we need:
$\mathrm{div}(|\nabla\zeta|^{r-2}\nabla\zeta)$ in duality with $\frac{\partial\zeta}{\partial t}$
($\Rightarrow$ the damage flow rule holds even a.e. Q).

The triple (u,π,ζ) with

$$u\in\mathrm{B}([0,T];\mathrm{BD}(\bar\Omega;\mathbb{R}^d)),$$

$$\pi\in\mathrm{B}([0,T];\mathrm{Meas}(\bar\Omega;\mathbb{R}^{d\times d}_{\mathrm{dev}}))\cap\mathrm{BV}([0,T];\mathrm{Meas}(\bar\Omega;\mathbb{R}^{d\times d}_{\mathrm{dev}})),$$

$$\zeta\in\mathrm{B}([0,T];W^{1,r}(\Omega))\cap H^1(0,T;L^2(\Omega))\cap C([0,T]{\times}\bar\Omega)$$

such that also

$$e_{\rm el}=e(u{+}u_{\rm Dir})-\pi\in\mathrm{B}([0,T];H^1(\Omega;\mathbb{R}^{d\times d}))\quad\text{and}$$

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will be called a weak solution if:

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will be called a weak solution if:

1) the semi-stability holds:

$$\mathcal{E}(t,u(t),\pi(t),\zeta(t)) \le \mathcal{E}(t,\widetilde{u},\widetilde{\pi},\zeta(t)) + \mathcal{R}(\zeta(t);\widetilde{\pi}-\pi(t),0)$$

for all $t\in[0,T]$ and for all $(\widetilde{u},\widetilde{\pi})\in \mathrm{BD}(\bar\Omega;\mathbb{R}^d)\times\mathrm{Meas}(\bar\Omega;\mathbb{R}^{d\times d}_{\rm dev})$ with $u|_{\Gamma_{\rm Dir}}\odot\vec{n}\mathrm{d}S+\pi=0$ on $\Gamma_{\rm Dir}$ and with $e(u)-\pi\in H^1(\Omega;\mathbb{R}^{d\times d}_{\rm sym})$,

2) the variational inequality

$$\int_Q a(v) + \Big(\frac12\mathbb{C}'(\zeta)e_{\rm el}:e_{\rm el} - \kappa\,\mathrm{div}\big(|\nabla\zeta|^{r-2}\nabla\zeta\big) - b'(\zeta)+\xi\Big)\Big(v-\frac{\partial\zeta}{\partial t}\Big)\,\mathrm{d}x\mathrm{d}t \ge \int_Q a\Big(\frac{\partial\zeta}{\partial t}\Big)\,\mathrm{d}x\mathrm{d}t,$$

holds for all $v\in L^2(Q)$ and some $\xi\in L^2(Q)$ such that $\xi\in N_{[0,1]}(\zeta)$ on Q,

3) the energy equality holds (with $\widehat{a}(z):=z\partial a(z)$ single-valued, convex):

$$\mathcal{E}(T,u(T),\pi(T),\zeta(T)) + \int_{[0,T]\times\bar\Omega}\Big[\delta^*_{S(\zeta)}\Big(\frac{\partial\pi}{\partial t}\Big)\Big](\mathrm{d}x\mathrm{d}t) + \int_Q \widehat{a}\Big(\frac{\partial\zeta}{\partial t}\Big)\,\mathrm{d}x\mathrm{d}t = \mathcal{E}(0,u_0,\pi_0,\zeta_0) + \int_0^T\frac{\partial\mathcal{E}}{\partial t}(t,u(t),\pi(t),\zeta(t))\,\mathrm{d}t.$$

Main assumptions:

$\Omega\subset\mathbb{R}^d$ bounded C^2-domain, $\Gamma_{\rm Dir}$ has a $(d{-}2)$ dimensional C^2-boundary, $\kappa>0$, $r>d$,

$a:\mathbb{R}\to\mathbb{R}$ convex, smooth on $\mathbb{R}\backslash\{0\}$, $a(0)=0$, and $\exists\,\epsilon>0:\ \ \epsilon|\cdot|^2\le a(\cdot)\le(1{+}|\cdot|^2)/\epsilon$,

$b:[0,1]\to\mathbb{R}$ continuously differentiable, non-decreasing, concave,

$\mathbb{C}:[0,1]\to\mathbb{R}^{d\times d\times d\times d}$ continuously differentiable, positive-semidefinite-valued,

$\forall\,i,j,k,l=1,\dots,d:\quad \mathbb{C}_{ijkl}(\cdot)=\mathbb{C}_{jikl}(\cdot)=\mathbb{C}_{klij}(\cdot)$,

$\forall\,e\in\mathbb{R}^{d\times d}_{\rm sym}:\ \mathbb{C}(\cdot)e{:}e:[0,1]\to\mathbb{R}$ non-decreasing, convex,

$\exists\,\mathbb{C}_{\rm D}(\zeta),\ c_{\rm S}(\zeta):\quad \mathbb{C}(\zeta)e:e=\mathbb{C}_{\rm D}(\zeta){\rm dev}\,e:{\rm dev}\,e+c_{\rm S}(\zeta)({\rm tr}\,e)^2$,

$\mathbb{H}$ positive definite, $\mathbb{H}_{ijkl}=\mathbb{H}_{jikl}=\mathbb{H}_{klij}$,

$\exists\,\mathbb{H}_{\rm D},\ H_{\rm S}:\qquad \mathbb{H}\nabla e\,\vdots\,\nabla e=\mathbb{H}_{\rm D}\nabla{\rm dev}\,e\,\vdots\,\nabla{\rm dev}\,e+H_{\rm S}\nabla{\rm tr}\,e\cdot\nabla{\rm tr}\,e$,

$S(\zeta)=\sigma_{\rm Y}(\zeta)B_1$, $\sigma_{\rm Y}:[0,1]\to(0,\infty)$ continuous nondecreasing, $B_1\subset\mathbb{R}^{d\times d}_{\rm dev}$ a unit ball,

$w_{\rm Dir}\in W^{1,1}(0,T;H^{3/2}(\Gamma_{\rm Dir};\mathbb{R}^d))$ and $\exists\,u_{\rm Dir}\in W^{1,1}(0,T;H^2(\Omega;\mathbb{R}^d))$ and $u_{\rm Dir}|_{\Gamma_{\rm Dir}}=w_{\rm Dir}$,

$g\in W^{1,1}(0,T;L^1(\Omega;\mathbb{R}^d))$, $f\in W^{1,1}(0,T;L^1(\Gamma_{\rm Neu};\mathbb{R}^d))$,

$\exists\,\sigma_{\rm SL}:[0,T]\to L^2(\Omega;\mathbb{R}^{d\times d}_{\rm sym})\ \exists\,\alpha>0:\quad \sigma_{\rm SL}\vec n=g$ on $[0,T]{\times}\Gamma_{\rm Neu}$ and

$\quad{\rm div}\,\sigma_{\rm SL}+f=0$ and $|{\rm dev}\,\sigma_{\rm SL}|\le\sigma_{\rm Y}(0)-\alpha$ on $[0,T]{\times}\Omega$,

$(u_0,\pi_0,\zeta_0)\in{\rm BD}(\bar\Omega;\mathbb{R}^d){\times}{\rm Meas}(\bar\Omega;\mathbb{R}^{d\times d}_{\rm dev}){\times}W^{1,r}(\Omega)$, $0\le\zeta_0\le1$ a.e. on Ω, and

$\forall(\widetilde u,\widetilde\pi)\in{\rm BD}(\bar\Omega;\mathbb{R}^d){\times}{\rm Meas}(\bar\Omega;\mathbb{R}^{d\times d}_{\rm dev})$, $e(\widetilde u){-}\widetilde\pi\in H^1(\Omega;\mathbb{R}^{d\times d}_{\rm sym})$, $\widetilde u\odot\vec n\,{\rm d}S+\widetilde\pi=0$ on $\Gamma_{\rm Dir}$

$\mathcal{E}(0,u_0,\pi_0,\zeta_0)\le\mathcal{E}(0,\widetilde u,\widetilde\pi,\zeta_0)+\mathcal{R}(\zeta_0;0,\widetilde\pi-\pi_0)$, Time discretisation by fractional-step strategy:

$$\mathrm{div}\Big(\mathbb{C}(\zeta_\tau^{k-1})e_{\mathrm{el},\tau}^k - \mathrm{div}\,\mathfrak{h}_\tau^k\Big) + g_\tau^k = 0$$

$$\text{with}\quad e_{\mathrm{el},\tau}^k = e(u_\tau^k{+}u_{\mathrm{Dir}}(k\tau)){-}\pi_\tau^k, \quad \mathfrak{h}_\tau^k = \mathbb{H}\nabla e_{\mathrm{el},\tau}^k, \quad g_\tau^k := g(k\tau),$$

$$N_{S(\zeta_\tau^{k-1})}\Big(\frac{\pi_\tau^k - \pi_\tau^{k-1}}{\tau}\Big) \ni \mathrm{dev}\Big(\mathbb{C}(\zeta_\tau^{k-1})e_{\mathrm{el},\tau}^k - \mathrm{div}\,\mathfrak{h}_\tau^k\Big),$$

$$\partial a\Big(\frac{\zeta_\tau^k - \zeta_\tau^{k-1}}{\tau}\Big) + \frac{1}{2}\mathbb{C}'(\zeta_\tau^k)e_{\mathrm{el},\tau}^k{:}e_{\mathrm{el},\tau}^k - \kappa\,\mathrm{div}\big(|\nabla\zeta_\tau^k|^{r-2}\nabla\zeta_\tau^k\big) + N_{[0,1]}(\zeta_\tau^k) \ni b'(\zeta_\tau^k),$$

together with the corresponding boundary conditions

$$u_\tau^k = 0 \qquad \text{on } \Gamma_{\mathrm{Dir}},$$

$$\big(\mathbb{C}(\zeta_\tau^{k-1})e_{\mathrm{el},\tau}^k - \mathrm{div}\,\mathfrak{h}_\tau^k\big)\cdot\vec{n} - \mathrm{div}_{\mathrm{S}}(\mathfrak{h}_\tau^k\vec{n}) = f_\tau^k, \quad \text{on } \Gamma_{\mathrm{Neu}} \text{ with } f_\tau^k := f(k\tau)$$

$$\nabla\zeta_\tau^k\cdot\vec{n} = 0 \quad \text{and} \quad \mathfrak{h}_\tau^k{:}(\vec{n}\otimes\vec{n}) = 0 \qquad \text{on } \Gamma.$$

to be solved first for (u_τ^k, π_τ^k) and then for ζ_τ^k recursively for $k = 1, ..., T/\tau$.

Given $(\pi_\tau^{k-1}, \zeta_\tau^{k-1})$:
A minimization problem to give (u_τ^k, π_τ^k):

$$\left.\begin{array}{ll}\text{Minimize} & (u,\pi)\mapsto \mathcal{E}(k\tau,u,\pi,\zeta_\tau^{k-1}) + \mathcal{R}(\zeta_\tau^{k-1};\pi-\pi_\tau^{k-1},0)\\ \text{subject to} & u\in \mathrm{BD}(\bar\Omega;\mathbb{R}^d),\quad \pi\in \mathrm{Meas}(\bar\Omega;\mathbb{R}^{d\times d}_{\mathrm{dev}}),\\ & e(u){-}\pi\in H^1(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}}),\quad u\odot\vec{n}\mathrm{d}S+\pi=0 \text{ on } \Gamma_{\mathrm{Dir}},\end{array}\right\}$$

and second minimization problem to give ζ_τ^k:

$$\left.\begin{array}{ll}\text{Minimize} & \zeta\mapsto \mathcal{E}(k\tau,u_\tau^k,\pi_\tau^k,\zeta) + \tau\mathcal{R}\Big(0;0,\dfrac{\zeta-\zeta_\tau^{k-1}}{\tau}\Big)\\ \text{subject to} & \zeta\in W^{1,r}(\Omega),\quad 0\le\zeta\le 1 \ \text{ on } \Omega,\end{array}\right\}$$

Solutions exist by coercivity, convexity, and lower semicontinuity arguments.

If $\mathbb{C}'$ and $-b'$ are nondecreasing (again with respect to the Löwner's ordering) and a is convex, these problems are convex.

We test the discrete inclusions respectively by $u_\tau^k - u_\tau^{k-1}$, $\pi_\tau^k - \pi_\tau^{k-1}$, and $\zeta_\tau^k - \zeta_\tau^{k-1}$.
Relying on the convexity of $\mathcal{E}(k\tau,\cdot,\cdot,\zeta_\tau^{k-1})$ and of $\mathcal{E}(k\tau,u_\tau^k,\pi_\tau^k,\cdot)$, we obtain the estimates

$$\mathcal{E}(k\tau,u_\tau^k,\pi_\tau^k,\zeta_\tau^{k-1}) + \int_{\bar\Omega} \sigma_{\rm Y}(\zeta_\tau^{k-1})|\pi_\tau^k - \pi_\tau^{k-1}|(\mathrm{d}x) \le \mathcal{E}(k\tau,u_\tau^{k-1},\pi_\tau^{k-1},\zeta_\tau^{k-1}),$$

$$\mathcal{E}(k\tau,u_\tau^k,\pi_\tau^k,\zeta_\tau^k) + \tau\int_\Omega \widehat{a}\Big(\frac{\zeta_\tau^k - \zeta_\tau^{k-1}}{\tau}\Big)\,\mathrm{d}x \le \mathcal{E}(k\tau,u_\tau^k,\pi_\tau^k,\zeta_\tau^{k-1})$$

with $\widehat{a}(z) := z\partial a(z)$.

By summing these estimates, we can enjoy the cancellation of the terms $\pm\mathcal{E}(k\tau,u_\tau^k,\pi_\tau^k,\zeta_\tau^{k-1})$, and thus obtain

$$\mathcal{E}(k\tau,u_\tau^k,\pi_\tau^k,\zeta_\tau^k) + \tau\widehat{\mathcal{R}}\Big(\zeta_\tau^{k-1};\frac{\pi_\tau^k - \pi_\tau^{k-1}}{\tau},\frac{\zeta_\tau^k - \zeta_\tau^{k-1}}{\tau}\Big) \le \mathcal{E}(k\tau,u_\tau^{k-1},\pi_\tau^{k-1},\zeta_\tau^{k-1})$$

$$= \mathcal{E}((k-1)\tau,u_\tau^{k-1},\pi_\tau^{k-1},\zeta_\tau^{k-1}) + \int_{(k-1)\tau}^{k\tau} \frac{\partial\mathcal{E}}{\partial t}(t,u_\tau^{k-1},\pi_\tau^{k-1},\zeta_\tau^{k-1})\,\mathrm{d}t$$

with the dissipation rate $\widehat{\mathcal{R}}$ defined as

$$\widehat{\mathcal{R}}\Big(\zeta;\frac{\mathrm{d}\pi}{\mathrm{d}t},\frac{\mathrm{d}\zeta}{\mathrm{d}t}\Big) := \int_{\bar\Omega}\sigma_{\rm Y}(\zeta)\Big|\frac{\partial\pi}{\partial t}\Big|(\mathrm{d}x) + \int_\Omega \widehat{a}\Big(\frac{\partial\zeta}{\partial t}\Big)\,\mathrm{d}x.$$

We test the discrete inclusions respectively by $u_\tau^k - u_\tau^{k-1}$, $\pi_\tau^k - \pi_\tau^{k-1}$, and $\zeta_\tau^k - \zeta_\tau^{k-1}$.
Relying on the convexity of $\mathcal{E}(k\tau, \cdot, \cdot, \zeta_\tau^{k-1})$ and of $\mathcal{E}(k\tau, u_\tau^k, \pi_\tau^k, \cdot)$, we obtain the estimates

$$\mathcal{E}(k\tau, u_\tau^k, \pi_\tau^k, \zeta_\tau^{k-1}) + \int_{\bar\Omega} \sigma_{\rm Y}(\zeta_\tau^{k-1})|\pi_\tau^k - \pi_\tau^{k-1}|({\rm d}x) \le \mathcal{E}(k\tau, u_\tau^{k-1}, \pi_\tau^{k-1}, \zeta_\tau^{k-1}),$$

$$\mathcal{E}(k\tau, u_\tau^k, \pi_\tau^k, \zeta_\tau^k) + \tau \int_\Omega \widehat{a}\Big(\frac{\zeta_\tau^k - \zeta_\tau^{k-1}}{\tau}\Big)\,{\rm d}x \le \mathcal{E}(k\tau, u_\tau^k, \pi_\tau^k, \zeta_\tau^{k-1})$$

with $\widehat{a}(z) := z\partial a(z)$.

By summing these estimates, we can enjoy the cancellation of the terms $\pm\mathcal{E}(k\tau, u_\tau^k, \pi_\tau^k, \zeta_\tau^{k-1})$, and thus obtain

$$\mathcal{E}(k\tau, u_\tau^k, \pi_\tau^k, \zeta_\tau^k) + \tau\widehat{\mathcal{R}}\Big(\zeta_\tau^{k-1}; \frac{\pi_\tau^k - \pi_\tau^{k-1}}{\tau}, \frac{\zeta_\tau^k - \zeta_\tau^{k-1}}{\tau}\Big) \le \mathcal{E}(k\tau, u_\tau^{k-1}, \pi_\tau^{k-1}, \zeta_\tau^{k-1})$$

$$= \mathcal{E}((k-1)\tau, u_\tau^{k-1}, \pi_\tau^{k-1}, \zeta_\tau^{k-1}) + \int_{(k-1)\tau}^{k\tau} \frac{\partial\mathcal{E}}{\partial t}(t, u_\tau^{k-1}, \pi_\tau^{k-1}, \zeta_\tau^{k-1})\,{\rm d}t$$

with the dissipation rate $\widehat{\mathcal{R}}$ defined as

$$\widehat{\mathcal{R}}\Big(\zeta; \frac{{\rm d}\pi}{{\rm d}t}, \frac{{\rm d}\zeta}{{\rm d}t}\Big) := \int_{\bar\Omega} \sigma_{\rm Y}(\zeta)\Big|\frac{\partial\pi}{\partial t}\Big|({\rm d}x) + \int_\Omega \widehat{a}\Big(\frac{\partial\zeta}{\partial t}\Big)\,{\rm d}x.$$

By the discrete Gronwall inequality, we obtain boundedness of $\sup_{t\in[0,T]}\mathcal{E}_\tau(t,\bar u_\tau,\bar\pi_\tau,\bar\zeta_\tau)$ and $\int_0^T\widehat{\mathcal{R}}(\underline{\zeta}_\tau;\frac{\mathrm{d}\pi_\tau}{\mathrm{d}t},\frac{\mathrm{d}\zeta_\tau}{\mathrm{d}t})\,\mathrm{d}t$.

Then, from the coercivity of $\mathcal{E}$ and $\mathcal{R}$, we thus obtain the estimates:

$$
\begin{aligned}
&\big\|\bar u_\tau\big\|_{\mathrm{B}([0,T];\mathrm{BD}(\bar\Omega;\mathbb{R}^d))}\le C,\\
&\big\|\bar\pi_\tau\big\|_{\mathrm{B}([0,T];\mathrm{Meas}(\bar\Omega;\mathbb{R}^{d\times d}_{\mathrm{dev}}))}\le C,\\
&\big\|\bar e_{\mathrm{el},\tau}\big\|_{\mathrm{B}([0,T];H^1(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}}))}\le C,\\
&\big\|\bar\zeta_\tau\big\|_{\mathrm{B}([0,T];W^{1,r}(\Omega))\cap\mathrm{BV}([0,T];L^1(\Omega))\cap L^\infty(Q)}\le C.
\end{aligned}
$$

The same estimate as for $\bar\zeta_\tau$ also holds for $\underline{\zeta}_\tau$.

Having estimated the set $\partial a(\frac{\partial\zeta_\tau}{\partial t})+\frac{1}{2}\mathbb{C}'(\bar\zeta)\bar e_{\mathrm{el},\tau}:\bar e_{\mathrm{el},\tau}-b'(\bar\zeta_\tau)$ in $L^2(Q)$ uniformly in $\tau>0$, we can estimate also $\mathrm{div}(|\nabla\zeta_\tau^k|^{r-2}\nabla\zeta_\tau^k)$ in $L^2(Q)$. For this, we test the damage flow-rule

$$\partial a\Big(\frac{\zeta_\tau^k-\zeta_\tau^{k-1}}{\tau}\Big)+\frac{1}{2}\mathbb{C}'(\zeta_\tau^k)e_{\mathrm{el},\tau}^k:e_{\mathrm{el},\tau}^k-\kappa\,\mathrm{div}\big(|\nabla\zeta_\tau^k|^{r-2}\nabla\zeta_\tau^k\big)+N_{[0,1]}(\zeta_\tau^k)\ni b'(\zeta_\tau^k)$$

by $-\mathrm{div}(|\nabla\zeta_\tau^k|^{r-2}\nabla\zeta_\tau^k)$. An important ingredient, written rather formally:

$$\int_\Omega N_{[0,1]}(\zeta_\tau^k)\big(-\mathrm{div}(|\nabla\zeta_\tau^k|^{r-2}\nabla\zeta_\tau^k)\big)\,\mathrm{d}x=-\int_\Omega\partial\delta_{[0,1]}(\zeta_\tau^k)\big(\mathrm{div}(|\nabla\zeta_\tau^k|^{r-2}\nabla\zeta_\tau^k)\big)\,\mathrm{d}x$$
$$=\int_\Omega\nabla\big(\partial\delta_{[0,1]}(\zeta_\tau^k)\big)\cdot|\nabla\zeta_\tau^k|^{r-2}\nabla\zeta_\tau^k\,\mathrm{d}x$$
$$=\int_\Omega\partial^2\delta_{[0,1]}(\zeta_\tau^k)\cdot\nabla\zeta_\tau^k\cdot|\nabla\zeta_\tau^k|^{r-2}\nabla\zeta_\tau^k\,\mathrm{d}x\geq 0$$

$\Leftarrow$ the positive-semidefineness of the (generalized) Jacobian $\partial^2\delta_{[0,1]}$ (to be proved rigorously by a mollification of $\delta_{[0,1]}$).

Thus we obtain:

$$\big\|\mathrm{div}(|\nabla\bar\zeta_\tau|^{r-2}\nabla\bar\zeta_\tau)\big\|_{L^2(Q)}\leq C.$$

Having estimated the set $\partial a(\frac{\partial \zeta_\tau}{\partial t}) + \frac{1}{2}\mathbb{C}'(\bar\zeta)\bar e_{\mathrm{el},\tau} : \bar e_{\mathrm{el},\tau} - b'(\bar\zeta_\tau)$ in $L^2(Q)$ uniformly in $\tau > 0$, we can estimate also $\mathrm{div}(|\nabla\zeta_\tau^k|^{r-2}\nabla\zeta_\tau^k)$ in $L^2(Q)$.
For this, we test the damage flow-rule

$$\partial a\Big(\frac{\zeta_\tau^k - \zeta_\tau^{k-1}}{\tau}\Big) + \frac{1}{2}\mathbb{C}'(\zeta_\tau^k) e_{\mathrm{el},\tau}^k : e_{\mathrm{el},\tau}^k - \kappa\,\mathrm{div}\big(|\nabla\zeta_\tau^k|^{r-2}\nabla\zeta_\tau^k\big) + N_{[0,1]}(\zeta_\tau^k) \ni b'(\zeta_\tau^k)$$

by $-\mathrm{div}(|\nabla\zeta_\tau^k|^{r-2}\nabla\zeta_\tau^k)$. An important ingredient, written rather formally:

$$\begin{aligned}\int_\Omega N_{[0,1]}(\zeta_\tau^k)\big(-\mathrm{div}(|\nabla\zeta_\tau^k|^{r-2}\nabla\zeta_\tau^k)\big)\,\mathrm{d}x &= -\int_\Omega \partial\delta_{[0,1]}(\zeta_\tau^k)\big(\mathrm{div}(|\nabla\zeta_\tau^k|^{r-2}\nabla\zeta_\tau^k)\big)\,\mathrm{d}x \\ &= \int_\Omega \nabla\big(\partial\delta_{[0,1]}(\zeta_\tau^k)\big)\cdot|\nabla\zeta_\tau^k|^{r-2}\nabla\zeta_\tau^k\,\mathrm{d}x \\ &= \int_\Omega \partial^2\delta_{[0,1]}(\zeta_\tau^k)\cdot\nabla\zeta_\tau^k\cdot|\nabla\zeta_\tau^k|^{r-2}\nabla\zeta_\tau^k\,\mathrm{d}x \geq 0\end{aligned}$$

$\Leftarrow$ the positive-semidefineness of the (generalized) Jacobian $\partial^2\delta_{[0,1]}$ (to be proved rigorously by a mollification of $\delta_{[0,1]}$).

Thus we obtain:

$$\big\|\mathrm{div}(|\nabla\bar\zeta_\tau|^{r-2}\nabla\bar\zeta_\tau)\big\|_{L^2(Q)} \leq C.$$

With the notation $\bar{e}_{\mathrm{el},\tau} = e(\bar{u}_\tau + \bar{u}_{\mathrm{D},\tau}) - \bar{\pi}_\tau$, the discrete solution satisfies:

$$\mathcal{E}(t, \bar{u}_\tau(t), \bar{\pi}_\tau(t), \underline{\zeta}_\tau(t)) \le \mathcal{E}(t, \widetilde{u}, \widetilde{\pi}, \underline{\zeta}_\tau(t)) + \mathcal{R}(\underline{\zeta}_\tau(t); \widetilde{\pi} - \bar{\pi}_\tau(t), 0)$$

for all $t \in [0, T]$ and all admissible $(\widetilde{u}, \widetilde{\pi})$, and

$$\int_Q a(v) + \Big(\frac{1}{2}\mathbb{C}'(\underline{\zeta}_\tau)\bar{e}_{\mathrm{el},\tau} : \bar{e}_{\mathrm{el},\tau} - \kappa\,\mathrm{div}\big(|\nabla\bar{\zeta}_\tau|^{r-2}\nabla\bar{\zeta}_\tau\big) - b'(\bar{\zeta}_\tau) + \bar{\xi}_\tau\Big)\Big(v - \frac{\partial\zeta_\tau}{\partial t}\Big)\,\mathrm{d}x\mathrm{d}t \ge \int_Q a\Big(\frac{\partial\zeta_\tau}{\partial t}\Big)\,\mathrm{d}x\mathrm{d}t$$

holds for all $v \in L^2(Q)$ and for some $\bar{\xi}_\tau \in L^2(Q)$ such that $\bar{\xi}_\tau \in N_{[0,1]}(\bar{\zeta}_\tau)$ a.e. on Q, and eventually the energy (im)balance holds:

$$\mathcal{E}(T, u_\tau(T), \pi_\tau(T), \zeta_\tau(T)) + \int_0^T \widehat{\mathcal{R}}\Big(\underline{\zeta}_\tau; \frac{\mathrm{d}\pi_\tau}{\mathrm{d}t}, \frac{\mathrm{d}\zeta_\tau}{\mathrm{d}t}\Big)\,\mathrm{d}t \le \mathcal{E}(0, u_0, \pi_0, \zeta_0) + \int_0^T \frac{\partial\mathcal{E}}{\partial t}(t, \underline{u}_\tau(t), \underline{\pi}_\tau(t), \underline{\zeta}_\tau(t))\,\mathrm{d}t.$$

Moreover, the a-priori estimate holds:

$\big\|\bar{\xi}_\tau\big\|_{L^2(Q)} \le C.$ $\quad\Big($due to $\bar{\xi}_\tau \in b'(\bar{\zeta}_\tau) - \frac{1}{2}\mathbb{C}'(\underline{\zeta}_\tau)\bar{e}_{\mathrm{el},\tau} : \bar{e}_{\mathrm{el},\tau} + \kappa\,\mathrm{div}(|\nabla\bar{\zeta}_\tau|^{r-2}\nabla\bar{\zeta}_\tau) - \partial a\big(\frac{\partial\zeta_\tau}{\partial t}\big)\Big)$

Convergence:

there is a subsequence and (u,π,ζ,ξ) such that

$$
\begin{aligned}
&\bar u_\tau(t)\to u(t) && \text{weakly* in } \mathrm{BD}(\bar\Omega;\mathbb{R}^d) \text{ for any } t\in[0,T],\\
&\bar\pi_\tau(t)\to \pi(t) && \text{weakly* in } \mathrm{Meas}(\bar\Omega;\mathbb{R}^{d\times d}_{\rm dev}) \text{ for any } t\in[0,T],\\
&\bar e_{\mathrm{el},\tau}(t)=e(\bar u_\tau(t))-\bar\pi_\tau(t) && \\
&\qquad\to e(u(t))-\pi(t)=e_{\rm el}(t) && \text{weakly in } H^1(\Omega;\mathbb{R}^{d\times d}_{\rm sym}) \text{ for any } t\in[0,T],\\
&\underline\zeta_\tau\to\zeta && \text{strongly in } L^\infty(Q) \text{ and}\\
&\bar\zeta_\tau(t)\to\zeta(t) && \text{weakly in } W^{1,r}(\Omega) \text{ for any } t\in[0,T],\\
&\bar\xi_\tau\to\xi && \text{weakly in } L^2(Q).
\end{aligned}
$$

Moreover, any (u,π,ζ) obtained by such a way is a weak solution.

Proof: Step 1: Selection of a converging subsequence.

Banach selection principle:

$$\bar{u}_\tau \to u \qquad \text{weakly* in } L^\infty(0,T;\mathrm{BD}(\bar\Omega;\mathbb{R}^d)),$$

$$\bar{\pi}_\tau \to \pi \qquad \text{weakly* in } L^\infty(0,T;\mathrm{Meas}(\bar\Omega;\mathbb{R}^{d\times d}_{\rm dev})) \cap \mathrm{BV}([0,T];L^1(\Omega;\mathbb{R}^{d\times d}_{\rm dev}))$$

$$\bar{e}_{{\rm el},\tau} = e(\bar{u}_\tau) - \bar{\pi}_\tau \to e_{\rm el} = e(u) - \pi \qquad \text{weakly* in } L^\infty(0,T;H^1(\Omega;\mathbb{R}^{d\times d}_{\rm sym})),$$

$$\zeta_\tau \to \zeta \qquad \text{weakly* in } L^\infty(0,T;W^{1,r}(\Omega)) \cap H^1(0,T;L^2(\Omega)),$$

$$\mathrm{div}(|\nabla\bar{\zeta}_\tau|^{r-2}\nabla\bar{\zeta}_\tau) \to \mathrm{div}(|\nabla\zeta|^{r-2}\nabla\zeta) \quad \text{weakly in } L^2(Q),$$

$$\bar{\xi}_\tau \to \xi \qquad \text{weakly in } L^2(Q).$$

Moreover, for the already selected subsequence, we have also

$$\nabla\zeta_\tau(T) \to \nabla\zeta(T) \quad \text{weakly in } \ L^r(\Omega;\mathbb{R}^d).$$

Moreover, by the BV-estimates and the Helly's selection principle:

$\bar{\pi}_\tau(t) \to \pi(t)$ weakly* in $\mathrm{Meas}(\bar\Omega;\mathbb{R}^{d\times d}_{\rm dev})$

$\bar{\zeta}_\tau(t) \to \zeta(t)$ weakly in $L^2(\Omega)$ (hence weakly in $W^{1,r}(\Omega)$, too).

By the compact embedding $W^{1,r}(\Omega) \Subset C(\bar{\Omega})$ and by the Arzelà-Ascoli modification of the Aubin-Lions theorem, we have the compact embedding

$$C_{\text{weak}}([0,T];W^{1,r}(\Omega)) \cap H^1(0,T;L^2(\Omega)) \Subset C([0,T];C(\bar{\Omega})) = C(\bar{Q}).$$

Thus we obtain $\zeta_\tau \to \zeta$ in $C(\bar{Q})$.

However, we need the uniform convergence not of ζ_τ but of $\underline{\zeta}_\tau$ which occurs in the discrete flow rule $N_{S(\underline{\zeta}_\tau)}(\frac{\partial \pi_\tau}{\partial t}) \ni \mathrm{dev}(\mathbb{C}(\underline{\zeta}_\tau)\bar{e}_{\mathrm{el},\tau} - \mathrm{div}\, \mathfrak{h}_\tau^k$.

However, we cannot directly use the Arzelà-Ascoli type assertion because $\underline{\zeta}_\tau \notin C_{\mathrm{weak}}([0,T];W^{1,r}(\Omega))$. Instead, we need to estimate the difference $\sigma_{\mathrm{Y}}(\underline{\zeta}_\tau)|\frac{\partial \pi_\tau}{\partial t}| - \sigma_{\mathrm{Y}}(\zeta_\tau)|\frac{\partial \pi_\tau}{\partial t}|$. To this goal, relying on uniform continuity of σ_{Y} on $[0,1]$, we need to prove also $\underline{\zeta}_\tau \to \zeta$ in $L^\infty(Q)$.

$\mathrm{BV}([0,T];L^2(\Omega))$-estimate of $\{\underline{\zeta}_\tau\}_{\tau>0} \Rightarrow \underline{\zeta}_\tau(t) \to \zeta_*(t)$ weakly in $L^2(\Omega)$ $\forall\, t\in[0,T]$. (Helly's selection principle)
$\Rightarrow\ \underline{\zeta}_\tau(t) \to \zeta_*(t)$ weakly in $W^{1,r}(\Omega)$, and by $W^{1,r}(\Omega) \Subset C(\bar\Omega)$ also

$$\underline{\zeta}_\tau(t) \to \zeta_*(t) \quad \text{strongly in} \quad C(\bar\Omega) \quad \text{for any} \quad t\in[0,T].$$

The sequence $\{\underline{\zeta}_\tau : [0,T] \to L^2(\Omega)\}_{\tau>0}$ is "equicontinuous" (although particular piecewise constant mappings $\underline{\zeta}_\tau$ are not continuous!) because

$$\big\|\underline{\zeta}_\tau(t_1)-\underline{\zeta}_\tau(t_2)\big\|_{L^2(\Omega)} \le \Big\|\int_{t_1}^{t_2}\frac{\partial\zeta_\tau}{\partial t}\,\mathrm{d}t\Big\|_{L^2(\Omega)} \le \int_{t_1}^{t_2} 1\,\Big\|\frac{\partial\zeta_\tau}{\partial t}\Big\|_{L^2(\Omega)}\mathrm{d}t$$
$$\le \|1\|_{L^2([t_1,t_2])}\Big\|\frac{\partial\zeta_\tau}{\partial t}\Big\|_{L^2(Q)} = |t_1-t_2|^{1/2}\Big\|\frac{\partial\zeta_\tau}{\partial t}\Big\|_{L^2(Q)}$$

for any $0 \le t_1 < t_2 \le T$.

Assume that the selected sequence $\{\underline{\zeta}_\tau\}_{\tau>0} \not\to \zeta_*$ in $L^\infty(0,T;C(\bar\Omega))$.

Thus $\|\underline{\zeta}_\tau - \zeta_*\|_{L^\infty(0,T;C(\bar\Omega))} \geq \epsilon > 0$ for some ϵ and for all $\tau > 0$ and we would get $\|\underline{\zeta}_\tau(t_\tau) - \zeta_*(t_\tau)\|_{C(\bar\Omega)} \geq \epsilon$ for some t_τ.

By compactness of $[0,T]$, we can further select a subsequence and some $t \in [0,T]$ so that $t_\tau \to t$. Then we have $\zeta(t_\tau) \to \zeta_*(t)$ in $C(\bar\Omega)$.

By the above proved equicontinuity, we have also $\underline{\zeta}_\tau(t_\tau) \to \zeta_*(t)$ weakly in $L^2(\Omega)$. By the boundedness of $\{\underline{\zeta}_\tau(t_\tau)\}_{\tau>0}$ in $W^{1,r}(\Omega) \Subset C(\bar\Omega)$, we have also $\underline{\zeta}_\tau(t_\tau) \to \zeta_*(t)$ in $C(\bar\Omega)$.

Then $\|\underline{\zeta}_\tau(t_\tau) - \zeta_*(t_\tau)\|_{C(\bar\Omega)} \to \|\zeta_*(t) - \zeta_*(t)\|_{C(\bar\Omega)} = 0$, a contradiction.

Thus we proved: $\underline{\zeta}_\tau \to \zeta_*$ strongly in $L^\infty(Q)$.

Moreover, $\zeta_* = \zeta$ a.e. on Q (because $\zeta_*(t) = \zeta(t)$ at any continuity point t).

Proof: Step 2: Energy inequality.

we want to pass to the limit in

$$\mathcal{E}(T,u_\tau(T),\pi_\tau(T),\zeta_\tau(T))+\int_0^T\widehat{\mathcal{R}}\Big(\underline{\zeta}_\tau;\frac{\mathrm{d}\pi_\tau}{\mathrm{d}t},\frac{\mathrm{d}\zeta_\tau}{\mathrm{d}t}\Big)\,\mathrm{d}t \\ \le\mathcal{E}(0,u_0,\pi_0,\zeta_0)+\int_0^T\frac{\partial\mathcal{E}}{\partial t}(t,\underline{u}_\tau(t),\underline{\pi}_\tau(t),\underline{\zeta}_\tau(t))\,\mathrm{d}t.$$

The first term is easy by w-l.s.c. Further note that

$$\frac{\partial\mathcal{E}}{\partial t}(t,u,\pi,\zeta)=\int_\Omega\mathbb{C}(\zeta)\big(e(u{+}u_{\rm Dir}(t)){-}\pi\big):e\Big(\frac{\partial u_{\rm Dir}}{\partial t}(t)\Big) \\ +\mathbb{H}\nabla\big(e(u{+}u_{\rm Dir}(t)){-}\pi\big)\vdots\nabla e\Big(\frac{\partial u_{\rm Dir}}{\partial t}(t)\Big)-\frac{\partial g}{\partial t}(t){\cdot}u\,\mathrm{d}x-\int_{\Gamma_{\rm Neu}}\frac{\partial f}{\partial t}(t){\cdot}u\,\mathrm{d}S,$$

thus the limit in the external-power-term is easy
(continuity + Lebesque theorem).

Thus the only difficult term is the dissipation.

we have at disposal the estimate

$$\Big\|\big(\sigma_{\rm Y}(\underline{\zeta}_\tau)-\sigma_{\rm Y}(\zeta)\big)\Big|\frac{\partial\pi_\tau}{\partial t}\Big|\Big\|_{{\rm Meas}(\bar Q)}\le\ell_{\sigma_{\rm Y}}\big\|\underline{\zeta}_\tau-\zeta\big\|_{L^\infty(Q)}\Big\|\frac{\partial\pi_\tau}{\partial t}\Big\|_{{\rm Meas}(\bar Q)}\to 0$$

with $\ell_{\sigma_{\rm Y}}$ the modulus of Lipschitz continuity of $\sigma_{\rm Y}$ on $[0,1]$.

Then, using also $\zeta_\tau\to\zeta$ in $C(\bar Q)$ already proved, we obtain

$$\begin{aligned}\liminf_{\tau\to0}\int_0^T\widehat{\mathcal R}\Big(\underline{\zeta}_\tau;\frac{{\rm d}\pi_\tau}{{\rm d}t},\frac{{\rm d}\zeta_\tau}{{\rm d}t}\Big)\,{\rm d}t&=\liminf_{\tau\to0}\int_{\bar Q}\sigma_{\rm Y}(\underline{\zeta}_\tau)\Big|\frac{\partial\pi_\tau}{\partial t}\Big|({\rm d}x{\rm d}t)\\&=\lim_{\tau\to0}\int_{\bar Q}\big(\sigma_{\rm Y}(\underline{\zeta}_\tau)-\sigma_{\rm Y}(\zeta)\big)\Big|\frac{\partial\pi_\tau}{\partial t}\Big|({\rm d}x{\rm d}t)+\liminf_{\tau\to0}\int_{\bar Q}\sigma_{\rm Y}(\zeta)\Big|\frac{\partial\pi_\tau}{\partial t}\Big|({\rm d}x{\rm d}t)\\&\ge\ 0\ +\int_{\bar Q}\sigma_{\rm Y}(\zeta)\Big|\frac{\partial\pi}{\partial t}\Big|({\rm d}x{\rm d}t);\end{aligned}$$

for the used weak* lower semicontinuity of $\displaystyle\frac{\partial\pi}{\partial t}\mapsto\int_{\bar Q}\sigma_{\rm Y}(\zeta)\Big|\frac{\partial\pi}{\partial t}\Big|({\rm d}x{\rm d}t)$.

L. Ambrosio, N. Fusco, D. Pallara (2000), E. Giusti (2003)

Proof: Step 3: Limit passage in the semi-stability.

$\exists$? a mutual recovery sequence $\{(\widehat{u}_\tau, \widehat{\pi}_\tau)\}_{\tau>0}$ in the sense that

$$\limsup_{\tau\to 0} \Big(\mathcal{E}(t, \widetilde{u}_\tau, \widetilde{\pi}_\tau, \underline{\zeta}_\tau(t)) + \mathcal{R}(\underline{\zeta}_\tau(t); \widetilde{\pi}_\tau - \bar{\pi}_\tau(t), 0) - \mathcal{E}(t, \bar{u}_\tau(t), \bar{\pi}_\tau(t), \underline{\zeta}_\tau(t))\Big)$$
$$\leq \mathcal{E}(t, \widetilde{u}, \widetilde{\pi}, \zeta(t)) + \mathcal{R}(\underline{\zeta}_\tau(t); \widetilde{\pi} - \pi(t), 0) - \mathcal{E}(t, u(t), \pi(t), \zeta(t)).$$

We choose

$$\widetilde{u}_\tau = \bar{u}_\tau(t) + \widetilde{u} - u(t) \quad \text{and} \quad \widetilde{\pi}_\tau = \bar{\pi}_\tau(t) + \widetilde{\pi} - \pi(t).$$

Then:

$$
\begin{aligned}
&\lim_{\tau\to0}\mathcal{E}(t,\tilde u_\tau,\tilde\pi_\tau,\underline{\zeta}_\tau(t))+\mathcal{R}(\underline{\zeta}_\tau(t);\tilde\pi_\tau-\bar\pi_\tau(t),0)-\mathcal{E}(t,\bar u_\tau(t),\bar\pi_\tau(t),\underline{\zeta}_\tau(t))\\
&=\lim_{\tau\to0}\bigg(\int_\Omega\frac12\mathbb{C}(\underline{\zeta}_\tau(t))\big(e(\tilde u_\tau+\bar u_\tau(t)+2u_{\rm Dir}(t))-\tilde\pi_\tau-\bar\pi_\tau(t)\big):\big(e(\tilde u_\tau-\bar u_\tau(t))-\tilde\pi_\tau+\bar\pi_\tau(t)\big)\\
&\quad+\frac12\mathbb{H}\nabla\big(e(\tilde u_\tau+\bar u_\tau(t)+2u_{\rm Dir}(t))-\tilde\pi_\tau-\bar\pi_\tau(t)\big)\,\vdots\,\nabla\big(e(\tilde u_\tau-\bar u_\tau(t))-\tilde\pi_\tau+\bar\pi_\tau(t)\big)\,{\rm d}x\\
&\quad+\int_{\bar\Omega}\big[\sigma_{\rm Y}(\underline{\zeta}_\tau(t))\big|\tilde\pi_\tau-\bar\pi_\tau(t)\big|({\rm d}x)-\int_\Omega g(t)\cdot(\tilde u_\tau-\bar u_\tau(t))\,{\rm d}x-\int_{\Gamma_{\rm Neu}}f(t)\cdot(\tilde u_\tau-\bar u_\tau(t))\,{\rm d}S\bigg)\\
&=\lim_{\tau\to0}\bigg(\int_\Omega\frac12\mathbb{C}(\underline{\zeta}_\tau(t))\big(e(\tilde u_\tau+\bar u_\tau(t)+2u_{\rm Dir}(t))-\tilde\pi_\tau-\bar\pi_\tau(t)\big):\big(e(\tilde u-u(t))-\tilde\pi+\pi(t)\big)\\
&\quad+\frac12\mathbb{H}\nabla\big(e(\tilde u_\tau+\bar u_\tau(t)+2u_{\rm Dir}(t))-\tilde\pi_\tau-\bar\pi_\tau(t)\big)\,\vdots\,\nabla\big(e(\tilde u-u(t))-\tilde\pi+\pi(t)\big)\,{\rm d}x\\
&\quad+\int_{\bar\Omega}\sigma_{\rm Y}(\underline{\zeta}_\tau(t))\big|\tilde\pi-\pi(t)\big|({\rm d}x)\bigg)-\int_\Omega g(t)\cdot(\tilde u-u(t))\,{\rm d}x-\int_{\Gamma_{\rm Neu}}f(t)\cdot(\tilde u-u(t))\,{\rm d}S\\
&=\int_\Omega\frac12\mathbb{C}(\underline{\zeta}_\tau(t))\big(e(\tilde u_\tau+\bar u_\tau(t)+2u_{\rm Dir}(t))-\tilde\pi_\tau-\bar\pi_\tau(t)\big):\big(e(\tilde u-u(t))-\tilde\pi+\pi(t)\big)\\
&\quad+\frac12\mathbb{H}\nabla\big(e(\tilde u_\tau+\bar u_\tau(t)+2u_{\rm Dir}(t))-\tilde\pi_\tau-\bar\pi_\tau(t)\big)\,\vdots\,\nabla\big(e(\tilde u-u(t))-\tilde\pi+\pi(t)\big)\,{\rm d}x\\
&\quad+\int_{\bar\Omega}\sigma_{\rm Y}(\zeta)\big|\tilde\pi-\pi(t)\big|({\rm d}x)-\int_\Omega g(t)\cdot(\tilde u-u(t))\,{\rm d}x-\int_{\Gamma_{\rm Neu}}f(t)\cdot(\tilde u-u(t))\,{\rm d}S\\
&=\mathcal{E}(t,\tilde u,\tilde\pi,\zeta(t))+\mathcal{R}(\zeta(t);\tilde\pi-\pi(t),0)-\mathcal{E}(t,u(t),\pi(t),\zeta(t)).
\end{aligned}
$$

Note that we used also $\sigma_{\rm Y}(\underline{\zeta}_\tau(t))|\tilde\pi-\pi(t)|\to\sigma_{\rm Y}(\zeta)|\tilde\pi-\pi(t)|$ in ${\rm Meas}(\bar\Omega)$.

Proof: Step 4: Limit passage in the damage flow rule.

$$\int_Q a(v) + \Big(\frac{1}{2}\mathbb{C}'(\underline{\zeta}_\tau)\bar{e}_{\mathrm{el},\tau} : \bar{e}_{\mathrm{el},\tau} - \kappa\,\mathrm{div}\big(|\nabla\bar{\zeta}_\tau|^{r-2}\nabla\bar{\zeta}_\tau\big) - b'(\bar{\zeta}_\tau) + \bar{\xi}_\tau\Big)\Big(v - \frac{\partial\zeta_\tau}{\partial t}\Big)\,\mathrm{d}x\mathrm{d}t \ge \int_Q a\Big(\frac{\partial\zeta_\tau}{\partial t}\Big)\,\mathrm{d}x\mathrm{d}t$$

We need $\bar{e}_{\mathrm{el},\tau} \to e_{\mathrm{el}}$ strongly in $L^2(Q;\mathbb{R}^{d\times d}_{\mathrm{sym}})$.

We know that $\nabla\bar{e}_{\mathrm{el},\tau}(t) \to \nabla e_{\mathrm{el}}(t)$ weakly in $L^2(\Omega;\mathbb{R}^{d\times d\times d})$

– here uniqueness of stresses is used!
(G.Dal Maso, A.DeSimone, M.G.Mora 2006)
for simple materials without damage.)

Thus $\bar{e}_{\mathrm{el},\tau}(t) \to e_{\mathrm{el}}(t)$ strongly in $L^{6-\epsilon}(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}})$ if $d \le 3$.

Then, by the uniform bounds in time and by Lebesgue's theorem, $\bar{e}_{\mathrm{el},\tau} \to e_{\mathrm{el}}$ strongly even in $L^{1/\epsilon}(0,T;L^{3-\epsilon}(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}}))$, $\epsilon > 0$.

Then the only difficult terms are $\kappa\int_Q \mathrm{div}(|\nabla\bar{\zeta}_\tau|^{r-2}\nabla\bar{\zeta}_\tau)\frac{\partial\zeta_\tau}{\partial t}\,\mathrm{d}x\mathrm{d}t$ and $\int_Q \bar{\xi}_\tau(-\frac{\partial\zeta_\tau}{\partial t})\,\mathrm{d}x\mathrm{d}t$ because so far we know only the weak convergence of $\mathrm{div}(|\nabla\bar{\zeta}_\tau|^{r-2}\nabla\bar{\zeta}_\tau)$ and of $\frac{\partial\zeta_\tau}{\partial t}$ in $L^2(Q)$.

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$$\limsup_{\tau\to0}\int_Q \mathrm{div}(|\nabla\bar\zeta_\tau|^{r-2}\nabla\bar\zeta_\tau)\frac{\partial\zeta_\tau}{\partial t}\,\mathrm{d}x\mathrm{d}t = -\liminf_{\tau\to0}\int_Q |\nabla\bar\zeta_\tau|^{r-2}\nabla\bar\zeta_\tau\cdot\nabla\frac{\partial\zeta_\tau}{\partial t}\,\mathrm{d}x\mathrm{d}t$$
$$\le \limsup_{\tau\to0}\int_\Omega \frac{1}{r}\big|\nabla\zeta_0\big|^r - \frac{1}{r}\big|\nabla\zeta_\tau(T)\big|^r\,\mathrm{d}x$$
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where we used $\nabla\zeta_\tau(T)\to\nabla\zeta(T)$ weakly in $L^r(\Omega;\mathbb{R}^d)$ and where the last equality relies on the regularity property $\mathrm{div}(|\nabla\zeta|^{r-2}\nabla\zeta)\in L^2(Q)$ and can be proved either by a mollification in time by a time-difference technique (G. Grün, 1995) or in space.

The convergence in the inclusion $\bar\xi_\tau\in N_{[0,1]}(\bar\zeta_\tau)$ is easy due to the maximal monotonicity of $N_{[0,1]}(\cdot)$. Then

$$\limsup_{\tau\to0}\int_Q \bar\xi_\tau\Big(-\frac{\partial\zeta_\tau}{\partial t}\Big)\,\mathrm{d}x\mathrm{d}t = \limsup_{\tau\to0}\bigg(\int_\Omega \delta_{[0,1]}(\zeta_0)\,\mathrm{d}x - \int_\Omega \delta_{[0,1]}(\zeta_\tau(T))\,\mathrm{d}x\bigg)$$
$$\le \int_\Omega \delta_{[0,1]}(\zeta_0)\,\mathrm{d}x - \int_\Omega \delta_{[0,1]}(\zeta(T))\,\mathrm{d}x = \int_Q \xi\Big(-\frac{\partial\zeta}{\partial t}\Big)\,\mathrm{d}x\mathrm{d}t.$$

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To the uniqueness of the stresses:

absolute continuity valid like in the undamageable simple-material case due to viscosity in damage flow rule and the argumentation is to be used for the hyperstresses which are not explicitly subjected to damage:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big(\big\langle\mathbb{H}\nabla(e_{\rm el}^{(1)}-e_{\rm el}^{(2)}),\nabla(e_{\rm el}^{(1)}-e_{\rm el}^{(2)})\big\rangle+\big\langle\mathbb{C}(\zeta)(e_{\rm el}^{(1)}-e_{\rm el}^{(2)}),e_{\rm el}^{(1)}-e_{\rm el}^{(2)}\big\rangle\Big)$$
$$=-\frac{1}{2}\Big\langle\mathbb{C}'(\zeta)\frac{\partial\zeta}{\partial t}(e_{\rm el}^{(1)}-e_{\rm el}^{(2)}),e_{\rm el}^{(1)}-e_{\rm el}^{(2)}\Big\rangle$$
$$\le\max_{0\le z\le 1}|\mathbb{C}'(z)|\Big\|\frac{\partial\zeta}{\partial t}\Big\|_{L^2(\Omega)}\big\|e_{\rm el}^{(1)}-e_{\rm el}^{(2)}\big\|^2_{L^4(\Omega;\mathbb{R}^{d\times d})}$$

from which $e_{\rm el}^{(1)}=e_{\rm el}^{(2)}$ follows by Gronwall's inequality when used positive-definiteness of $\mathbb{C}(\cdot)$ and of $\mathbb{H}$ after integrated over $[0,t]$.
which, for $\mathbb{H}=0$ and $\mathbb{C}'=0$, reduces to the simple inequality for the undamageable simple material as in (G.A.Maugin, 1992)
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Proof: Step 5: Energy equality.

1) The damage flow rule (which holds a.e. on Q) can be tested by $\frac{\partial\zeta}{\partial t}\in L^2(Q)$.

We again use $\int_\Omega \frac{1}{r}\big|\nabla\zeta_0\big|^r - \frac{1}{r}\big|\nabla\zeta(T)\big|^r\,\mathrm{d}x = \int_Q \mathrm{div}(|\nabla\zeta|^{r-2}\nabla\zeta)\frac{\partial\zeta}{\partial t}\,\mathrm{d}x\mathrm{d}t$.

Moreover, as $\xi\in\partial\delta_{[0,1]}(\frac{\partial\zeta}{\partial t})$, we have
$\int_Q \xi\frac{\partial\zeta}{\partial t}\,\mathrm{d}x\mathrm{d}t = \int_\Omega \delta(\zeta(T)) - \delta(\zeta(0))\mathrm{d}x = 0-0=0$.

We thus obtain

$$\int_\Omega \frac{\kappa}{r}\big|\nabla\zeta(T)\big|^r - b(\zeta(T))\,\mathrm{d}x + \int_Q \frac{1}{2}\mathbb{C}'(\zeta)e_{\rm el}:e_{\rm el} + \widehat{a}\Big(\frac{\mathrm{d}\zeta}{\mathrm{d}t}\Big)\,\mathrm{d}x\mathrm{d}t$$
$$= \int_\Omega \frac{\kappa}{r}\big|\nabla\zeta_0\big|^r - b(\zeta_0)\,\mathrm{d}x.$$

2) We test formally the momentum equilibrium by $\frac{\partial u}{\partial t}$ and plastic flow rule by $\frac{\partial \pi}{\partial t}$.
Approximation of Lebesgue integrals by Riemann's sums (an idea of H.HAHN (Sitzungber.Math.Phys.Kl.K.Akad.Wiss.Wien,1914) used in the context of R.I.P. by

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here modified for Stieltjes-type integral with the fixed L^2-weight $\frac{\partial \varepsilon}{\partial t}$ and the above semistability.

Define $\mathfrak{S}_1 \in L^1([0,T])$ and $\mathfrak{S}_2 \in L^2([0,T]; L^2(\Omega; \mathbb{R}^{d\times d}_{\rm sym}))$ defined by

$$\mathfrak{S}_1 : t \mapsto \|\mathfrak{s}(t)\|^2_{L^2(\Omega;\mathbb{R}^{d\times d}_{\rm sym})} : [0,T] \to \mathbb{R} \qquad \text{and}$$

$$\mathfrak{S}_2 : t \mapsto \mathfrak{s}(t) : [0,T] \to L^2(\Omega; \mathbb{R}^{d\times d}_{\rm sym}), \qquad \mathfrak{s}(t) = \Big[\mathbb{D}\frac{\partial \varepsilon}{\partial t} - \mathcal{B}(\vartheta) + \sigma_{\rm Dir}\Big](t,\cdot),$$

$\forall \eta > 0$: a partition $0 = t_0^\eta < t_1^\eta < \ldots < t_{N_\eta}^\eta = T$ with $\max_{i=1,\ldots,N_\eta} t_i^\eta - t_{i-1}^\eta \le \eta$ so that

$$\sum_{i=1}^{N_\eta} \int_{t_{i-1}^\eta}^{t_i^\eta} |\mathfrak{S}_1(t_{i-1}^\eta) - \mathfrak{S}_1(t)| \mathrm{d}t \to 0$$

$$\text{and} \qquad \sum_{i=1}^{N_\eta} \int_{t_{i-1}^\eta}^{t_i^\eta} \big\|\mathfrak{S}_2(t_{i-1}^\eta) - \mathfrak{S}_2(t)\big\|_{L^2(\Omega;\mathbb{R}^{d\times d}_{\rm sym})} \mathrm{d}t \to 0.$$

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$$\text{and} \quad \sum_{i=1}^{N_\eta}\int_{t_{i-1}^\eta}^{t_i^\eta} \big\|\mathfrak{S}_2(t_{i-1}^\eta) - \mathfrak{S}_2(t)\big\|_{L^2(\Omega;\mathbb{R}^{d\times d}_{\rm sym})}{\rm d}t \to 0.$$

Define the piece-wise constant functions

$$\mathfrak{S}_{\ell,\eta}(t) = \mathfrak{S}_\ell(t^\eta_{i-1}) \qquad \text{for} \quad t \in (t^\eta_i, t^\eta_{i-1}), \quad \ell = 1, 2.$$

We have

$$\mathfrak{S}_{1,\eta}(t) = \big\|\mathfrak{S}_{2,\eta}(t)\big\|^2_{L^2(\Omega;\mathbb{R}^{d\times d}_{\rm sym})} \quad \text{for a.a. } t,$$
$$\mathfrak{S}_{1,\eta} \to \mathfrak{S}_1 \quad \text{in } L^1(0,T),$$
$$\mathfrak{S}_{2,\eta} \to \mathfrak{S}_2 \quad \text{in } L^1(0,T;L^2(\Omega;\mathbb{R}^{d\times d}_{\rm sym})).$$

In particular, $\{\mathfrak{S}_{1,\eta}\}_{\eta>0}$ is bounded in $L^1(0,T)$, so that
$\{\mathfrak{S}_{2,\eta}\}_{\eta>0}$ is bounded in $L^2(0,T;L^2(\Omega;\mathbb{R}^{d\times d}_{\rm sym}))$.
$\Longrightarrow$ $\exists$ a subsequence such that

$$\mathfrak{S}_{2,\eta} \to \mathfrak{S}_2 \quad \text{weakly in } L^2(Q;\mathbb{R}^{d\times d}_{\rm sym}),$$

and, in particular, the Lebesque-Stieltjes integral is approximated:

$$\int_Q \mathfrak{S}_{2,\eta}{:}\frac{\partial\varepsilon}{\partial t}\,\mathrm{d}x\mathrm{d}t \to \int_Q \mathfrak{S}_2{:}\frac{\partial\varepsilon}{\partial t}\,\mathrm{d}x\mathrm{d}t.$$

Now we assume the partitions chosen so that the semistability holds at all $0<t_1^\eta<\ldots<t_{N_\eta-1}^\eta<T$
(possibly not in $t_{N_\eta}^\eta<T$, while for $t_0^\eta=0$ it is to assume).

The semistability at t_{i-1}^η tested by (u,π) at t_i^η, and summed up gives:

$$0\le\int_\Omega\frac12\mathbb{C}\varepsilon(t_{N_\eta}^\eta):\varepsilon(t_{N_\eta}^\eta)-\frac12\mathbb{C}\varepsilon(t_0^\eta):\varepsilon(t_0^\eta)\mathrm{d}x$$
$$+\sum_{i=1}^{N_\eta}\int_\Omega\mathfrak{s}(t_{i-1}^\eta):\big(\varepsilon(t_i^\eta)-\varepsilon(t_{i-1}^\eta)\big)\mathrm{d}x+\sum_{i=1}^{N_\eta}\int_{\bar\Omega}\delta_S^*(\cdot)\big[\pi(t_i^\eta)-\pi(t_{i-1}^\eta)\big)\big](\mathrm{d}x).$$

For limitting $\eta\to0$, we use that $t_{N_\eta}^\eta=T$ and $t_0^\eta=0$ are fixed, and

$$\sum_{i=1}^{N_\eta}\int_\Omega\mathfrak{s}(t_{i-1}^\eta):\big(\varepsilon(t_i^\eta)-\varepsilon(t_{i-1}^\eta)\big)\mathrm{d}x=\int_Q\mathfrak{S}_{2,\eta}:\frac{\partial\varepsilon}{\partial t}\,\mathrm{d}x\mathrm{d}t$$
$$\to\int_Q\mathfrak{S}_2:\frac{\partial\varepsilon}{\partial t}\,\mathrm{d}x\mathrm{d}t=\int_Q\mathfrak{s}:\frac{\partial\varepsilon}{\partial t}\,\mathrm{d}x\mathrm{d}t=\int_Q\Big(\mathbb{D}\frac{\partial\varepsilon}{\partial t}-\mathcal{B}(\vartheta)+\sigma_{\rm Dir}\Big):\frac{\partial\varepsilon}{\partial t}\,\mathrm{d}x\mathrm{d}t$$

and $\displaystyle\sum_{i=1}^{N_\eta}\int_{\bar\Omega}\delta_S^*(\cdot)\big[\pi(t_i^\eta)-\pi(t_{i-1}^\eta)\big)\big](\mathrm{d}x)\le\mathrm{Var}_S(\pi;0,T)$ by definition.

Now we assume the partitions chosen so that the semistability holds at all $0<t_1^\eta<...<t_{N_\eta-1}^\eta<T$
(possibly not in $t_{N_\eta}^\eta<T$, while for $t_0^\eta=0$ it is to assume).
The semistability at t_{i-1}^η tested by (u,π) at t_i^η, and summed up gives:

$$0\le\int_\Omega\frac12\mathbb{C}\varepsilon(t_{N_\eta}^\eta):\varepsilon(t_{N_\eta}^\eta)-\frac12\mathbb{C}\varepsilon(t_0^\eta):\varepsilon(t_0^\eta)\mathrm{d}x$$
$$+\sum_{i=1}^{N_\eta}\int_\Omega\mathfrak{s}(t_{i-1}^\eta):\big(\varepsilon(t_i^\eta)-\varepsilon(t_{i-1}^\eta)\big)\mathrm{d}x+\sum_{i=1}^{N_\eta}\int_{\bar\Omega}\delta_S^*(\cdot)\big[\pi(t_i^\eta)-\pi(t_{i-1}^\eta)\big)\big](\mathrm{d}x).$$

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$$\sum_{i=1}^{N_\eta}\int_\Omega\mathfrak{s}(t_{i-1}^\eta):\big(\varepsilon(t_i^\eta)-\varepsilon(t_{i-1}^\eta)\big)\mathrm{d}x=\int_Q\mathfrak{S}_{2,\eta}:\frac{\partial\varepsilon}{\partial t}\,\mathrm{d}x\mathrm{d}t$$
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The semistability at t_{i-1}^η tested by (u,π) at t_i^η, and summed up gives:

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Altogether, for $\eta \to 0$, we obtain the "inverse" energy inequality:

$$\int_\Omega \frac{1}{2}\mathbb{C}\varepsilon(T){:}\varepsilon(T) - \frac{1}{2}\mathbb{C}\varepsilon(0){:}\varepsilon(0)\mathrm{d}x + \mathrm{Var}_S(\pi;0,T) + \int_Q \sigma_{\mathrm{Dir}}{:}\frac{\partial\varepsilon}{\partial t}\,\mathrm{d}x\mathrm{d}t \geq 0.$$

This is ultimately used for the limit passage with $\tau \to 0$:

$$\begin{aligned}
&\int_{\bar{Q}} \mathfrak{h}_\pi(\mathrm{d}x\mathrm{d}t) + \int_Q \mathbb{D}\frac{\partial\varepsilon}{\partial t}{:}\frac{\partial\varepsilon}{\partial t}\,\mathrm{d}x\mathrm{d}t = \mathrm{Var}_S(\pi;0,T) + \int_Q \mathbb{D}\frac{\partial\varepsilon}{\partial t}{:}\frac{\partial\varepsilon}{\partial t}\,\mathrm{d}x\mathrm{d}t \\
&\leq \liminf_{\tau\downarrow 0}\int_Q \delta_S^*\Big(\frac{\partial\pi_\tau}{\partial t}\Big) + \mathbb{D}\frac{\partial\varepsilon_\tau}{\partial t}{:}\frac{\partial\varepsilon_\tau}{\partial t}\,\mathrm{d}x\mathrm{d}t \leq \limsup_{\tau\downarrow 0}\int_Q \delta_S^*\Big(\frac{\partial\pi_\tau}{\partial t}\Big) + \mathbb{D}\frac{\partial\varepsilon_\tau}{\partial t}{:}\frac{\partial\varepsilon_\tau}{\partial t}\,\mathrm{d}x\mathrm{d}t \\
&\leq \limsup_{\tau\downarrow 0}\bigg(\int_\Omega \frac{1}{2}\mathbb{C}\varepsilon_{0,\tau}{:}\varepsilon_{0,\tau} + \tau|\pi_{0,\tau}|^2 - \frac{1}{2}\mathbb{C}\varepsilon_\tau(T){:}\varepsilon_\tau(T)\,\mathrm{d}x \\
&\qquad + \int_Q \big(\mathcal{B}(\bar{\vartheta}_\tau) - \overline{(\sigma_{\mathrm{Dir}})}_\tau\big){:}\frac{\partial\varepsilon_\tau}{\partial t}\,\mathrm{d}x\mathrm{d}t\bigg) \\
&\leq \int_\Omega \frac{1}{2}\mathbb{C}\varepsilon_0{:}\varepsilon_0 - \frac{1}{2}\mathbb{C}\varepsilon(T){:}\varepsilon(T)\,\mathrm{d}x + \int_Q \big(\mathcal{B}(\vartheta) - \sigma_{\mathrm{Dir}}\big){:}\frac{\partial\varepsilon}{\partial t}\,\mathrm{d}x\mathrm{d}t \\
&\leq \mathrm{Var}_S(\pi;0,T) + \int_Q \mathbb{D}\frac{\partial\varepsilon}{\partial t}{:}\frac{\partial\varepsilon}{\partial t}\,\mathrm{d}x\mathrm{d}t.
\end{aligned}$$

Altogether, for $\eta \to 0$, we obtain the "inverse" energy inequality:

$$\int_\Omega \frac{1}{2}\mathbb{C}\varepsilon(T){:}\varepsilon(T) - \frac{1}{2}\mathbb{C}\varepsilon(0){:}\varepsilon(0)\mathrm{d}x + \mathrm{Var}_S(\pi;0,T) + \int_Q \sigma_{\mathrm{Dir}}{:}\frac{\partial\varepsilon}{\partial t}\,\mathrm{d}x\mathrm{d}t \geq 0.$$

This is ultimately used for the limit passage with $\tau \to 0$:

$$\begin{aligned}
&\int_{\bar{Q}} \mathfrak{h}_\pi(\mathrm{d}x\mathrm{d}t) + \int_Q \mathbb{D}\frac{\partial\varepsilon}{\partial t}{:}\frac{\partial\varepsilon}{\partial t}\,\mathrm{d}x\mathrm{d}t = \mathrm{Var}_S(\pi;0,T) + \int_Q \mathbb{D}\frac{\partial\varepsilon}{\partial t}{:}\frac{\partial\varepsilon}{\partial t}\,\mathrm{d}x\mathrm{d}t \\
&\leq \liminf_{\tau\downarrow 0}\int_Q \delta_S^*\Big(\frac{\partial\pi_\tau}{\partial t}\Big) + \mathbb{D}\frac{\partial\varepsilon_\tau}{\partial t}{:}\frac{\partial\varepsilon_\tau}{\partial t}\,\mathrm{d}x\mathrm{d}t \leq \limsup_{\tau\downarrow 0}\int_Q \delta_S^*\Big(\frac{\partial\pi_\tau}{\partial t}\Big) + \mathbb{D}\frac{\partial\varepsilon_\tau}{\partial t}{:}\frac{\partial\varepsilon_\tau}{\partial t}\,\mathrm{d}x\mathrm{d}t \\
&\leq \limsup_{\tau\downarrow 0}\bigg(\int_\Omega \frac{1}{2}\mathbb{C}\varepsilon_{0,\tau}{:}\varepsilon_{0,\tau} + \tau|\pi_{0,\tau}|^2 - \frac{1}{2}\mathbb{C}\varepsilon_\tau(T){:}\varepsilon_\tau(T)\,\mathrm{d}x \\
&\qquad\qquad + \int_Q \big(\mathcal{B}(\bar\vartheta_\tau) - \overline{(\sigma_{\mathrm{Dir}})}_\tau\big){:}\frac{\partial\varepsilon_\tau}{\partial t}\,\mathrm{d}x\mathrm{d}t\bigg) \\
&\leq \int_\Omega \frac{1}{2}\mathbb{C}\varepsilon_0{:}\varepsilon_0 - \frac{1}{2}\mathbb{C}\varepsilon(T){:}\varepsilon(T)\,\mathrm{d}x + \int_Q \big(\mathcal{B}(\vartheta) - \sigma_{\mathrm{Dir}}\big){:}\frac{\partial\varepsilon}{\partial t}\,\mathrm{d}x\mathrm{d}t \\
&\leq \mathrm{Var}_S(\pi;0,T) + \int_Q \mathbb{D}\frac{\partial\varepsilon}{\partial t}{:}\frac{\partial\varepsilon}{\partial t}\,\mathrm{d}x\mathrm{d}t.
\end{aligned}$$

Altogether:

$$\liminf_{\tau\downarrow 0}\int_Q \delta_S^*\Big(\frac{\partial\pi_\tau}{\partial t}\Big) + \mathbb{D}\frac{\partial\varepsilon_\tau}{\partial t}:\frac{\partial\varepsilon_\tau}{\partial t}\,\mathrm{d}x\mathrm{d}t = \int_{\bar{Q}} \mathfrak{h}_\pi(\mathrm{d}x\mathrm{d}t) + \int_Q \mathbb{D}\frac{\partial\varepsilon}{\partial t}:\frac{\partial\varepsilon}{\partial t}\,\mathrm{d}x\mathrm{d}t$$

so that

$$\delta_S^*\Big(\frac{\partial\pi_\tau}{\partial t}\Big) \to \mathfrak{h}_\pi = \text{the measure "}\delta_S^*\Big(\frac{\partial\pi}{\partial t}\Big)\text{" weakly* in } \mathrm{Meas}(\bar{Q}),$$

and

$$\mathbb{D}\frac{\partial\varepsilon_\tau}{\partial t}:\frac{\partial\varepsilon_\tau}{\partial t} \to \mathbb{D}\frac{\partial\varepsilon}{\partial t}:\frac{\partial\varepsilon}{\partial t} \text{ strongly in } L^1(Q).$$

The limit passage in the heat equation accomplished.

The equilibrium equation and the “upper” energy inequality simple.

Altogether:

$$\liminf_{\tau\downarrow 0}\int_Q \delta_S^*\Big(\frac{\partial\pi_\tau}{\partial t}\Big) + \mathbb{D}\frac{\partial\varepsilon_\tau}{\partial t}:\frac{\partial\varepsilon_\tau}{\partial t}\,\mathrm{d}x\mathrm{d}t = \int_{\bar Q}\mathfrak{h}_\pi(\mathrm{d}x\mathrm{d}t) + \int_Q \mathbb{D}\frac{\partial\varepsilon}{\partial t}:\frac{\partial\varepsilon}{\partial t}\,\mathrm{d}x\mathrm{d}t$$

so that

$$\delta_S^*\Big(\frac{\partial\pi_\tau}{\partial t}\Big) \to \mathfrak{h}_\pi = \text{the measure "}\delta_S^*\Big(\frac{\partial\pi}{\partial t}\Big)\text{" weakly* in } \mathrm{Meas}(\bar Q),$$

and

$$\mathbb{D}\frac{\partial\varepsilon_\tau}{\partial t}:\frac{\partial\varepsilon_\tau}{\partial t} \to \mathbb{D}\frac{\partial\varepsilon}{\partial t}:\frac{\partial\varepsilon}{\partial t} \text{ strongly in } L^1(Q).$$

The limit passage in the heat equation accomplished.

The equilibrium equation and the “upper” energy inequality simple.

Here, as $\mathbb{C}$ is not constant, we will still see the term $\big(\frac{1}{2}\mathbb{C}'(\zeta)e_{\rm el}{:}e_{\rm el}\big)\frac{\partial\zeta}{\partial t}$ which results by the formal substitution

$$\mathbb{C}(\zeta)e_{\rm el}{:}\frac{\partial}{\partial t}e_{\rm el} = \frac{\partial}{\partial t}\frac{1}{2}\mathbb{C}(\zeta)e_{\rm el}{:}e_{\rm el} - \big(\frac{1}{2}\mathbb{C}'(\zeta)e_{\rm el}{:}e_{\rm el}\big)\frac{\partial\zeta}{\partial t};$$

note that $\mathbb{C}(\zeta)e_{\rm el}{:}\frac{\partial}{\partial t}e_{\rm el}$ is not well defined since $\frac{\partial}{\partial t}e_{\rm el}$ is not well controlled.
Thus we obtain

$$\int_\Omega \frac{1}{2}\mathbb{C}(\zeta(T))e_{\rm el}(T){:}e_{\rm el}(T) + \frac{1}{2}\mathbb{H}\nabla e_{\rm el}(T)\,\vdots\,\nabla e_{\rm el}(T)\,{\rm d}x$$
$$+\int_{[0,T]\times\bar\Omega}\sigma_{\rm Y}(\zeta)\Big|\frac{\partial\pi}{\partial t}\Big|({\rm d}x{\rm d}t) = \int_Q \Big(\frac{1}{2}\mathbb{C}'(\zeta)e_{\rm el}{:}e_{\rm el}\Big)\frac{\partial\zeta}{\partial t}\,{\rm d}x{\rm d}t$$
$$+\int_\Omega \frac{1}{2}\mathbb{C}(\zeta_0)e_{\rm el}(0){:}e_{\rm el}(0) + \frac{1}{2}\mathbb{H}\nabla e_{\rm el}(0)\,\vdots\,\nabla e_{\rm el}(0)\,{\rm d}x.$$

Summing it with the previous contribution from damage then gives the energy balance.

Numerics: the lowest-order spatial discretisation by the conformal finite-element method (FEM). In view of the used regularity $\mathrm{div}(|\nabla\zeta|^{r-2}\nabla\zeta) \in L^2$, the straightforward discretisation therefore employs P2-elements for u and ζ and P1-elements for π.

Rigorously speaking, due to the assumed smoothness of Ω, one should consider FEM on a nonpolyhedral, curved domain. The two minimization problems are then to be restricted on the corresponding finite-dimensional subspaces, and the solution thus obtained is denoted by $u_{\tau h}^k$, $\pi_{\tau h}^k$, and $\zeta_{\tau h}^k$, with $h > 0$ denoting the mesh size.

Convergence for $h \to 0$ and $\tau \to 0$ just a modification of the above proof.

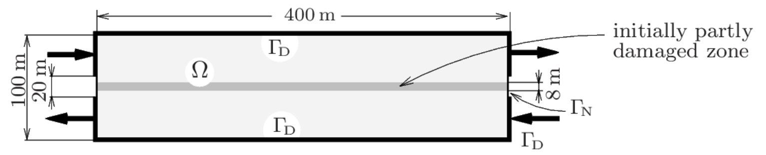
The explicit construction of the mutual recovery sequence takes additionally a finite-element approximation:

$$\widetilde{u}_{\tau h} = \bar{u}_{\tau h}(t) + \Pi_h^{(2)}(\widetilde{u} - u(t)) \quad \text{and} \quad \widetilde{\pi}_{\tau h} = \bar{\pi}_{\tau h}(t) + \Pi_h^{(1)}(\widetilde{\pi} - \pi(t))$$

with $\Pi_h^{(k)}$ a projector onto the Pk-FEM space.

(S.Bartels, A.Mielke, T.R 2012)

Computational simulations.



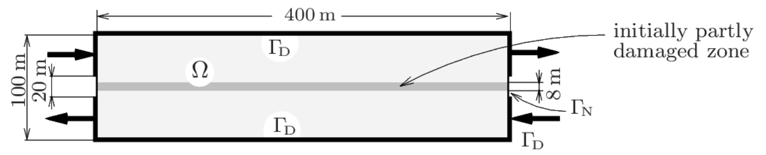
Geometry used for the computational experiment. The Dirichlet conditions have been prescribed on $\Gamma_{\rm Dir}$ moving horizontally in opposite directions with the constant velocity $\pm 10^{-8}$m/s.

Isotropic material: $\mathbb{C}(1) \sim E_{\rm Young} = 27\,$GPa, Poisson' ratio $\nu = 0.2$, $\mathbb{C}(0) = \mathbb{C}(1)/10$, the elastic domain $\Sigma(\zeta) := \{\sigma \in \mathbb{R}^{d\times d}_{\rm dev};\ |\sigma| \le \zeta\sigma_{\rm y}\}$ with $\sigma_{\rm y} = 2\,$MPa, the dissipation potential $a(\frac{\partial z}{\partial t}) := a_1\frac{\partial z}{\partial t}^{-} + a_2(\frac{\partial z}{\partial t}^{-})^2 + cb(\frac{\partial z}{\partial t}^{+})^2$ with $a_1 = 10\,$Pa, $a_2 = 0.1\,$Pa s, and $c = 100\,$kPa s, while the damage stored energy $b(\zeta) = b_0\zeta$ used $b_0 = 10^{-3}$Pa, and the damage length-scale coefficient $\kappa = 10^{-6}\,$J/m.

The initial conditions: $\pi_0 = 0$, $\zeta_0 = 1$ (or $\zeta_0 = 1/2$ in a middle narrow horizontal stripe)

(Shortcuts in implementation: $\mathbb{H} = 0$ and P1-FEM for ζ.)

Computational simulations.



Geometry used for the computational experiment. The Dirichlet conditions have been prescribed on $\Gamma_{\rm Dir}$ moving horizontally in opposite directions with the constant velocity $\pm 10^{-8}$m/s.

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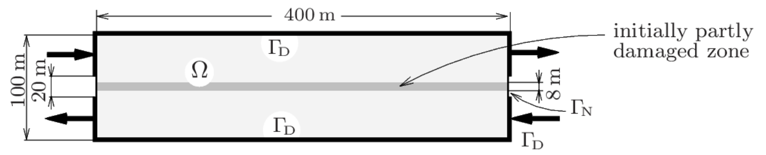
Computational simulations.



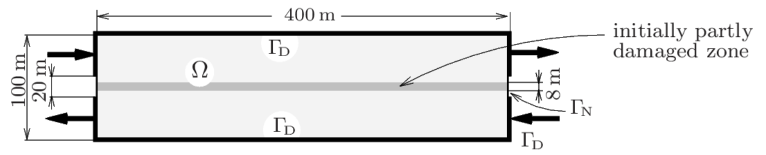
Geometry used for the computational experiment. The Dirichlet conditions have been prescribed on $\Gamma_{\rm Dir}$ moving horizontally in opposite directions with the constant velocity $\pm 10^{-8}$m/s.

Isotropic material: $\mathbb{C}(1) \sim E_{\rm Young} = 27\,$GPa, Poisson' ratio $\nu = 0.2$, $\mathbb{C}(0) = \mathbb{C}(1)/10$, the elastic domain $\Sigma(\zeta) := \{\sigma \in \mathbb{R}^{d\times d}_{\rm dev};\ |\sigma| \le \zeta\sigma_{\rm y}\}$ with $\sigma_{\rm y} = 2\,$MPa, the dissipation potential $a(\frac{\partial z}{\partial t}) := a_1 \frac{\partial z}{\partial t}^{-} + a_2(\frac{\partial z}{\partial t}^{-})^2 + cb(\frac{\partial z}{\partial t}^{+})^2$ with $a_1 = 10\,$Pa, $a_2 = 0.1\,$Pa s, and $c = 100\,$kPa s, while the damage stored energy $b(\zeta) = b_0\zeta$ used $b_0 = 10^{-3}$Pa, and the damage length-scale coefficient $\kappa = 10^{-6}$ J/m.

The initial conditions: $\pi_0 = 0$, $\zeta_0 = 1$ (or $\zeta_0 = 1/2$ in a middle narrow horizontal stripe)

(Shortcuts in implementation: $\mathbb{H} = 0$ and P1-FEM for ζ.)

Computational simulations.



Geometry used for the computational experiment. The Dirichlet conditions have been prescribed on $\Gamma_{\rm Dir}$ moving horizontally in opposite directions with the constant velocity $\pm 10^{-8}$m/s.

Isotropic material: $\mathbb{C}(1) \sim E_{\rm Young} = 27\,$GPa, Poisson' ratio $\nu = 0.2$, $\mathbb{C}(0) = \mathbb{C}(1)/10$, the elastic domain $\Sigma(\zeta) := \{\sigma \in \mathbb{R}^{d\times d}_{\rm dev};\ |\sigma| \le \zeta\sigma_{\rm y}\}$ with $\sigma_{\rm y} = 2\,$MPa, the dissipation potential $a(\frac{\partial z}{\partial t}) := a_1\frac{\partial z}{\partial t}^{-} + a_2(\frac{\partial z}{\partial t}^{-})^2 + cb(\frac{\partial z}{\partial t}^{+})^2$ with $a_1 = 10\,$Pa, $a_2 = 0.1\,$Pa s, and $c = 100\,$kPa s, while the damage stored energy $b(\zeta) = b_0\zeta$ used $b_0 = 10^{-3}$Pa, and the damage length-scale coefficient $\kappa = 10^{-6}$ J/m.

The initial conditions: $\pi_0 = 0$, $\zeta_0 = 1$ (or $\zeta_0 = 1/2$ in a middle narrow horizontal stripe)

(Shortcuts in implementation: $\mathbb{H} = 0$ and P1-FEM for ζ.)

Experiment of the horizontally shifted plates: first stress increases, then rupture starts propagating towards the center, and eventually everything goes into sliding regime.

damage ζ **plastic strain** $|\pi|$ **von-Mises stress** $|\mathrm{dev}\,\sigma|$

at time $t = 1$

Calculations and visualization: courtesy of Jan Valdman (Czech Acad. Sci.).

Deformation of the specimen depicted by displacement u magnified 25000 $\times$.

Experiment of the horizontally shifted plates: first stress increases, then rupture starts propagating towards the center, and eventually everything goes into sliding regime.

damage ζ **plastic strain** $|\pi|$ **von-Mises stress** $|\mathrm{dev}\,\sigma|$

at time $t = 2$

Calculations and visualization: courtesy of Jan Valdman (Czech Acad. Sci.).

Deformation of the specimen depicted by displacement u magnified 25000 $\times$.

Experiment of the horizontally shifted plates: first stress increases, then rupture starts propagating towards the center, and eventually everything goes into sliding regime.

damage ζ **plastic strain** $|\pi|$ **von-Mises stress** $|\mathrm{dev}\,\sigma|$

at time $t = 3$

Calculations and visualization: courtesy of Jan Valdman (Czech Acad. Sci.).

Deformation of the specimen depicted by displacement u magnified 25000 $\times$.

Experiment of the horizontally shifted plates: first stress increases, then rupture starts propagating towards the center, and eventually everything goes into sliding regime.

damage ζ **plastic strain** $|\pi|$ **von-Mises stress** $|\mathrm{dev}\,\sigma|$

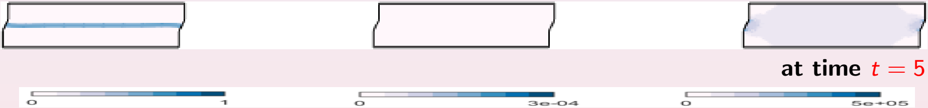
at time $t = 4$

Calculations and visualization: courtesy of Jan Valdman (Czech Acad. Sci.).

Deformation of the specimen depicted by displacement u magnified $25000\times$.

Experiment of the horizontally shifted plates: first stress increases, then rupture starts propagating towards the center, and eventually everything goes into sliding regime.

damage ζ **plastic strain** $|\pi|$ **von-Mises stress** $|\mathrm{dev}\,\sigma|$



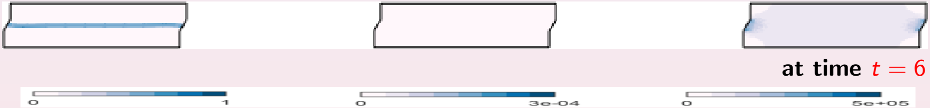
at time $t = 5$

Calculations and visualization: courtesy of Jan Valdman (Czech Acad. Sci.).

Deformation of the specimen depicted by displacement u magnified 25000 $\times$.

Experiment of the horizontally shifted plates: first stress increases, then rupture starts propagating towards the center, and eventually everything goes into sliding regime.

damage ζ **plastic strain** $|\pi|$ **von-Mises stress** $|\mathrm{dev}\,\sigma|$



at time $t = 6$

Calculations and visualization: courtesy of Jan Valdman (Czech Acad. Sci.).

Deformation of the specimen depicted by displacement u magnified $25000\times$.

Experiment of the horizontally shifted plates: first stress increases, then rupture starts propagating towards the center, and eventually everything goes into sliding regime.

damage ζ **plastic strain** $|\pi|$ **von-Mises stress** $|\mathrm{dev}\,\sigma|$

at time $t = 7$

Calculations and visualization: courtesy of Jan Valdman (Czech Acad. Sci.).

Deformation of the specimen depicted by displacement u magnified $25000\times$.

Experiment of the horizontally shifted plates: first stress increases, then rupture starts propagating towards the center, and eventually everything goes into sliding regime.

damage ζ **plastic strain** $|\pi|$ **von-Mises stress** $|\mathrm{dev}\,\sigma|$

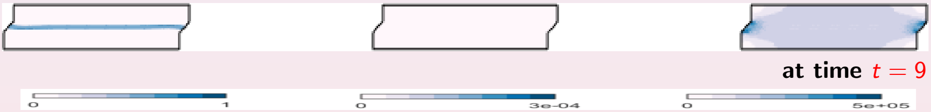
at time $t = 8$

Calculations and visualization: courtesy of Jan Valdman (Czech Acad. Sci.).

Deformation of the specimen depicted by displacement u magnified $25000\times$.

Experiment of the horizontally shifted plates: first stress increases, then rupture starts propagating towards the center, and eventually everything goes into sliding regime.

damage ζ **plastic strain** $|\pi|$ **von-Mises stress** $|\mathrm{dev}\,\sigma|$



Calculations and visualization: courtesy of Jan Valdman (Czech Acad. Sci.).

Deformation of the specimen depicted by displacement u magnified 25000 $\times$.

Experiment of the horizontally shifted plates: first stress increases, then rupture starts propagating towards the center, and eventually everything goes into sliding regime.

damage ζ **plastic strain** $|\pi|$ **von-Mises stress** $|\mathrm{dev}\,\sigma|$

at time $t = 10$

Calculations and visualization: courtesy of Jan Valdman (Czech Acad. Sci.).

Deformation of the specimen depicted by displacement u magnified 25000 $\times$.

Experiment of the horizontally shifted plates: first stress increases, then rupture starts propagating towards the center, and eventually everything goes into sliding regime.

damage ζ **plastic strain** $|\pi|$ **von-Mises stress** $|\mathrm{dev}\,\sigma|$

at time $t = 11$

Calculations and visualization: courtesy of Jan Valdman (Czech Acad. Sci.).

Deformation of the specimen depicted by displacement u magnified 25000 $\times$.

Experiment of the horizontally shifted plates: first stress increases, then rupture starts propagating towards the center, and eventually everything goes into sliding regime.

damage ζ **plastic strain** $|\pi|$ **von-Mises stress** $|\mathrm{dev}\,\sigma|$

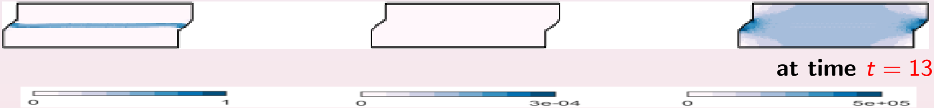
at time $t = 12$

Calculations and visualization: courtesy of Jan Valdman (Czech Acad. Sci.).

Deformation of the specimen depicted by displacement u magnified $25000\times$.

Experiment of the horizontally shifted plates: first stress increases, then rupture starts propagating towards the center, and eventually everything goes into sliding regime.

damage ζ **plastic strain** $|\pi|$ **von-Mises stress** $|\mathrm{dev}\,\sigma|$



at time $t = 13$

Calculations and visualization: courtesy of Jan Valdman (Czech Acad. Sci.).

Deformation of the specimen depicted by displacement u magnified 25000 $\times$.

Experiment of the horizontally shifted plates: first stress increases, then rupture starts propagating towards the center, and eventually everything goes into sliding regime.

damage ζ **plastic strain** $|\pi|$ **von-Mises stress** $|\mathrm{dev}\,\sigma|$

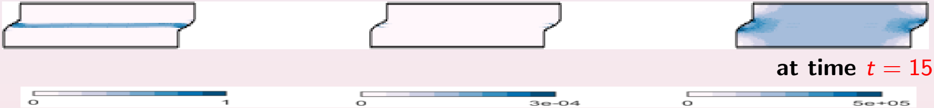
at time $t = 14$

Calculations and visualization: courtesy of Jan Valdman (Czech Acad. Sci.).

Deformation of the specimen depicted by displacement u magnified 25000 $\times$.

Experiment of the horizontally shifted plates: first stress increases, then rupture starts propagating towards the center, and eventually everything goes into sliding regime.

damage ζ **plastic strain** $|\pi|$ **von-Mises stress** $|\mathrm{dev}\,\sigma|$



at time $t = 15$

Calculations and visualization: courtesy of Jan Valdman (Czech Acad. Sci.).

Deformation of the specimen depicted by displacement u magnified 25000 $\times$.

Experiment of the horizontally shifted plates: first stress increases, then rupture starts propagating towards the center, and eventually everything goes into sliding regime.

damage ζ **plastic strain** $|\pi|$ **von-Mises stress** $|\mathrm{dev}\,\sigma|$

at time $t = 16$

Calculations and visualization: courtesy of Jan Valdman (Czech Acad. Sci.).

Deformation of the specimen depicted by displacement u magnified 25000 $\times$.

Experiment of the horizontally shifted plates: first stress increases, then rupture starts propagating towards the center, and eventually everything goes into sliding regime.

damage ζ **plastic strain** $|\pi|$ **von-Mises stress** $|\mathrm{dev}\,\sigma|$

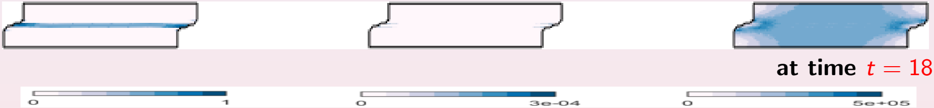
at time $t = 17$

Calculations and visualization: courtesy of Jan Valdman (Czech Acad. Sci.).

Deformation of the specimen depicted by displacement u magnified 25000 $\times$.

Experiment of the horizontally shifted plates: first stress increases, then rupture starts propagating towards the center, and eventually everything goes into sliding regime.

damage ζ **plastic strain** $|\pi|$ **von-Mises stress** $|\mathrm{dev}\,\sigma|$



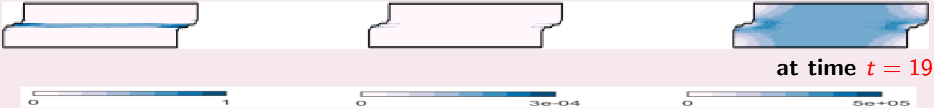
at time $t = 18$

Calculations and visualization: courtesy of Jan Valdman (Czech Acad. Sci.).

Deformation of the specimen depicted by displacement u magnified 25000 $\times$.

Experiment of the horizontally shifted plates: first stress increases, then rupture starts propagating towards the center, and eventually everything goes into sliding regime.

damage ζ **plastic strain** $|\pi|$ **von-Mises stress** $|\mathrm{dev}\,\sigma|$



at time $t = 19$

Calculations and visualization: courtesy of Jan Valdman (Czech Acad. Sci.).

Deformation of the specimen depicted by displacement u magnified 25000 $\times$.

Experiment of the horizontally shifted plates: first stress increases, then rupture starts propagating towards the center, and eventually everything goes into sliding regime.

damage ζ **plastic strain** $|\pi|$ **von-Mises stress** $|\mathrm{dev}\,\sigma|$

at time $t = 20$

Calculations and visualization: courtesy of Jan Valdman (Czech Acad. Sci.).

Deformation of the specimen depicted by displacement u magnified 25000 $\times$.

Experiment of the horizontally shifted plates: first stress increases, then rupture starts propagating towards the center, and eventually everything goes into sliding regime.

damage ζ **plastic strain** $|\pi|$ **von-Mises stress** $|\mathrm{dev}\,\sigma|$

at time $t = 21$

Calculations and visualization: courtesy of Jan Valdman (Czech Acad. Sci.).

Deformation of the specimen depicted by displacement u magnified 25000 $\times$.

Experiment of the horizontally shifted plates: first stress increases, then rupture starts propagating towards the center, and eventually everything goes into sliding regime.

damage ζ **plastic strain** $|\pi|$ **von-Mises stress** $|\mathrm{dev}\,\sigma|$

at time $t = 22$

Calculations and visualization: courtesy of Jan Valdman (Czech Acad. Sci.).

Deformation of the specimen depicted by displacement u magnified 25000 $\times$.

Experiment of the horizontally shifted plates: first stress increases, then rupture starts propagating towards the center, and eventually everything goes into sliding regime.

damage ζ **plastic strain** $|\pi|$ **von-Mises stress** $|\mathrm{dev}\,\sigma|$

at time $t = 23$

Calculations and visualization: courtesy of Jan Valdman (Czech Acad. Sci.).

Deformation of the specimen depicted by displacement u magnified 25000 $\times$.

Experiment of the horizontally shifted plates: first stress increases, then rupture starts propagating towards the center, and eventually everything goes into sliding regime.

damage ζ **plastic strain** $|\pi|$ **von-Mises stress** $|\mathrm{dev}\,\sigma|$

at time $t = 24$

Calculations and visualization: courtesy of Jan Valdman (Czech Acad. Sci.).

Deformation of the specimen depicted by displacement u magnified 25000 $\times$.

Experiment of the horizontally shifted plates: first stress increases, then rupture starts propagating towards the center, and eventually everything goes into sliding regime.

damage ζ **plastic strain** $|\pi|$ **von-Mises stress** $|\mathrm{dev}\,\sigma|$

at time $t = 25$

Calculations and visualization: courtesy of Jan Valdman (Czech Acad. Sci.).

Deformation of the specimen depicted by displacement u magnified 25000 $\times$.

Experiment of the horizontally shifted plates: first stress increases, then rupture starts propagating towards the center, and eventually everything goes into sliding regime.

damage ζ **plastic strain** $|\pi|$ **von-Mises stress** $|\mathrm{dev}\,\sigma|$

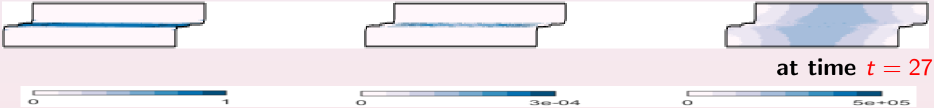
at time $t = 26$

Calculations and visualization: courtesy of Jan Valdman (Czech Acad. Sci.).

Deformation of the specimen depicted by displacement u magnified $25000\times$.

Experiment of the horizontally shifted plates: first stress increases, then rupture starts propagating towards the center, and eventually everything goes into sliding regime.

damage ζ **plastic strain** $|\pi|$ **von-Mises stress** $|\mathrm{dev}\,\sigma|$



at time $t = 27$

Calculations and visualization: courtesy of Jan Valdman (Czech Acad. Sci.).

Deformation of the specimen depicted by displacement u magnified 25000 $\times$.

Experiment of the horizontally shifted plates: first stress increases, then rupture starts propagating towards the center, and eventually everything goes into sliding regime.

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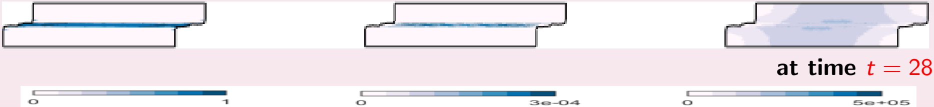


at time $t = 28$

Calculations and visualization: courtesy of Jan Valdman (Czech Acad. Sci.).

Deformation of the specimen depicted by displacement u magnified $25000\,\times$.

Experiment of the horizontally shifted plates: first stress increases, then rupture starts propagating towards the center, and eventually everything goes into sliding regime.

damage ζ **plastic strain** $|\pi|$ **von-Mises stress** $|\mathrm{dev}\,\sigma|$

at time $t = 29$

Calculations and visualization: courtesy of Jan Valdman (Czech Acad. Sci.).

Deformation of the specimen depicted by displacement u magnified 25000 $\times$.

Experiment of the horizontally shifted plates: first stress increases, then rupture starts propagating towards the center, and eventually everything goes into sliding regime.

damage ζ **plastic strain** $|\pi|$ **von-Mises stress** $|\mathrm{dev}\,\sigma|$

at time $t = 30$

Calculations and visualization: courtesy of Jan Valdman (Czech Acad. Sci.).

Deformation of the specimen depicted by displacement u magnified 25000 $\times$.

Experiment of the horizontally shifted plates: first stress increases, then rupture starts propagating towards the center, and eventually everything goes into sliding regime.

damage ζ **plastic strain** $|\pi|$ **von-Mises stress** $|\mathrm{dev}\,\sigma|$

at time $t = 31$

Calculations and visualization: courtesy of Jan Valdman (Czech Acad. Sci.).

Deformation of the specimen depicted by displacement u magnified 25000 ×.

Experiment of the horizontally shifted plates: first stress increases, then rupture starts propagating towards the center, and eventually everything goes into sliding regime.

damage ζ **plastic strain** $|\pi|$ **von-Mises stress** $|\mathrm{dev}\,\sigma|$

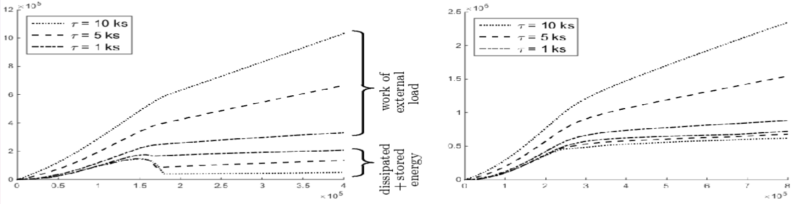
at time $t = 32$

Calculations and visualization: courtesy of Jan Valdman (Czech Acad. Sci.).

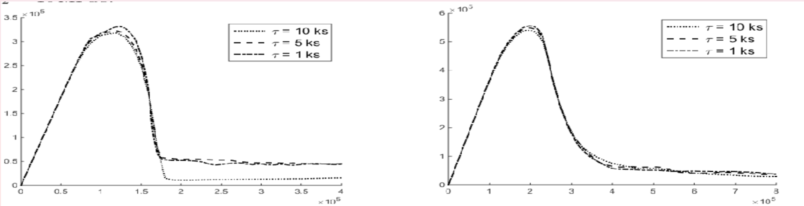
Deformation of the specimen depicted by displacement u magnified 25000 $\times$.

Energy inequality (in current time t) – convergence within two refinements

$$\mathcal{E}(t,u_\tau(t),\pi_\tau(t),\zeta_\tau(t))+\int_0^t\widehat{\mathcal{R}}\Big(\hat\zeta_\tau;\frac{\partial\pi_\tau}{\partial t},\frac{\partial\zeta_\tau}{\partial t}\Big)\,\mathrm{d}t\le\mathcal{E}(0,u_0,\pi_0,\zeta_0)+\int_0^t\frac{\partial\mathcal{E}}{\partial t}(\cdot,\underline{u}_\tau,\underline{\pi}_\tau,\underline{\zeta}_\tau)\,\mathrm{d}t.$$

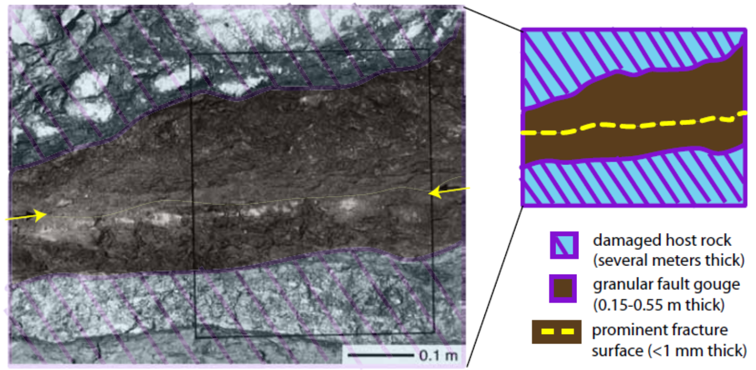


The left-hand and the right-hand sides for three different time steps $\tau=10,\ 5,\ 1\,\mathrm{ks}$.
Less viscous damage ⇒ slower convergence of the energy residuum to 0: the left figure for $a_2=0.1\,\mathrm{MPa\,s}$ vs the right one for $a_2=10\,\mathrm{MPa\,s}$.



Reaction-force evolution - converged already when energetics has a substantial gap.

Applications in modelling of lithospheric faults
– a very narrow core vs. a wider damage zone around:



Field observations from an exhumed lithospheric fault.
F.M.Chester, J.S.Chester in *Tectonophysics* 295 (1998) 199-221.
(reprinted also in E.G.Daub, J.M.Carlson: Friction, Fracture, and Earthquakes)

Yet, instead of $(e_{\rm el},\zeta)\mapsto\mathbb{C}(\zeta)e_{\rm el}{:}e_{\rm el}=\frac12\lambda(\zeta)I_1^2+\mu(\zeta)I_2$, with $I_1={\rm tr}\,e_{\rm el}$, $I_2=|e_{\rm el}|^2$

one considers $(e_{\rm el},\zeta)\mapsto\frac12\lambda(\zeta)I_1^2+\mu(\zeta)I_2-\gamma(\zeta)I_1\sqrt{I_2}$.

V. Lyakhovsky & V.P. Myasnikov (1984)

later e.g. Y. Ben Zion, V. Lyakhovsky, Y. Hamiel, Z. Reches, etc. etc.

Typically, $\lambda(\zeta)=\lambda_0$,
$\mu(\zeta)=\mu_0-\mu_1\zeta$,
$\gamma(\zeta)=\gamma_1\zeta$.

The elastic stress is then $\Big(\lambda(\zeta)-\gamma(\zeta)\sqrt{I_2}\Big){\rm tr}\,e_{\rm el}+\Big(2\mu(\zeta)e_{\rm el}-\gamma(\zeta)\dfrac{I_1}{\sqrt{I_2}}\Big)e_{\rm el}$.

The driving stress for damage $\sigma_{\rm dam}=\frac12\lambda'(\zeta)I_1^2+\mu'(\zeta)I_2-\gamma'(\zeta)I_1/\sqrt{I_2}$
can now be positive even without the contribution of the b-term

$\Rightarrow$ healing mechanism (even dominant)!

To preserve coercivity, one should modify it as softening under very large strain

$$(e_{\rm el},\zeta)\mapsto\frac{\lambda(\zeta)I_1^2+2\mu(\zeta)I_2-2\gamma(\zeta)I_1\sqrt{I_2}}{\sqrt{4+\epsilon I_2}}\qquad\text{with } I_1={\rm tr}\,e_{\rm el}\text{ and } I_2=|e_{\rm el}|^2$$

with $\epsilon>0$ presumably small.

Yet, instead of $(e_{\rm el},\zeta)\mapsto\mathbb{C}(\zeta)e_{\rm el}{:}e_{\rm el}=\frac{1}{2}\lambda(\zeta)I_1^2+\mu(\zeta)I_2$, with $I_1=\mathrm{tr}\,e_{\rm el}$, $I_2=|e_{\rm el}|^2$

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Some open problems:

Avoiding the concept of nonsimple materials seems nonsimple indeed.

Again complete damage does not seem to be investigated with plasticity yet rate-dependent complete damage with diffusion is by
(C. Heinemann, C. Kraus, WIAS Preprint 2012)

Homework (for tutorial):
Rate-independent damage without gradient (compensated by the nonsimple-material regularization).

Some references:

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More on: www.karlin.mff.cuni.cz/~roubicek/trpublic.htm
or: https://www.researchgate.net/profile/Tomas_Roubicek2

Thanks a lot for your attention.

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Vielen Dank für Ihre Aufmerksamkeit.