PLASTICITY AND DAMAGE
— PART III —
rate-independent unidirectional damage with visco-plasticity, thermodynamics, etc.

Tomáš Roubíček

Charles University, Prague & Czech Academy of Sciences

The plot:

Part I: basic scenario: rate-independent plasticity + rate-independent damage

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Part II: perfect plasticity with rate dependent damage with a possible healing

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Part III: rate-independent unidirectional damage with visco-plasticity, thermodynamics, etc.

Part IV: tutorial – further outlooks
(combination with other processes, large strains, etc.)

1. Plain damage-visco-plastic model
   - The governing equation/inclusions
   - The weak formulation
   - Analysis: time discretisation, a-priori estimates, convergence
2. Some modifications and expansions
   - Phenomena like creep or fatique
3. A general thermodynamics and examples
   - A general thermodynamics
   - Example: plasticity with hardening and thermal expnasion

Main features: 1) irreversible (= unidirectional) rate-independent damage

2) visco-elastic material

3) rate-dependent plasticity (allows "cheaply" no hardening)

4) combination with other phenomena or processes (creep, diffusion/swelling)

The classical formulation of the Biot inclusion $\partial_{\frac{dq}{dt}}\mathcal{R}(q;\frac{dq}{dt})+\partial_q\mathcal{E}(t,q)\ni 0$:
"viscosity" in the plastic flow rule + visco-elasticity in Kelvin-Voigt rheology, plastic-dependent damage activation, and
again gradient of π (as in Part I) and damageable yield stress (as in Part II).
no hyper-stresses, no healing force, and no hardening needed (though possible).

The governing equation/inclusions read as:

$$\operatorname{div}\sigma+g=0 \quad \text{with} \quad \sigma=\mathbb{C}(\zeta)e_{\rm el}+\mathbb{D}(\zeta)\frac{\partial e_{\rm el}}{\partial t}, \qquad \text{(momentum equilibrium)}$$

$$\alpha\frac{\partial\pi}{\partial t}+\partial\delta^*_{S(\zeta)}\Big(\frac{\partial\pi}{\partial t}\Big)+\mathbb{H}\pi\ni\operatorname{dev}\sigma+\kappa_1\Delta\pi \qquad \text{(plastic flow rule)}$$

$$\partial\delta^*_{[-a(\pi),\infty)}\Big(\frac{\partial\zeta}{\partial t}\Big)+\frac{1}{2}\mathbb{C}'(\zeta)e_{\rm el}:e_{\rm el}$$
$$+N_{[0,1]}(\zeta)\ni\kappa_2\operatorname{div}\big(|\nabla\zeta|^{r-2}\nabla\zeta\big) \quad \text{with} \quad e_{\rm el}=e(u)-\pi \qquad \text{(damage flow rule)}$$

with the boundary conditions:

$$u=w_{\rm Dir} \qquad \text{on } \Gamma_{\rm Dir},$$
$$\sigma\vec{n}=f \qquad \text{on } \Gamma_{\rm Neu},$$
$$\nabla\zeta\cdot\vec{n}=0 \quad \text{and} \quad \nabla\pi\vec{n}=0 \qquad \text{on } \Gamma.$$

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$$\partial\delta^*_{[-a(\pi),\infty)}\Big(\frac{\partial\zeta}{\partial t}\Big) + \frac{1}{2}\mathbb{C}'(\zeta)e_{\rm el}:e_{\rm el} + N_{[0,1]}(\zeta) \ni \kappa_2\operatorname{div}\big(|\nabla\zeta|^{r-2}\nabla\zeta\big) \quad \text{with} \quad e_{\rm el}=e(u)-\pi \qquad \text{(damage flow rule)}$$

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If $\zeta_0 \le 1$, the normal cone $N_{[0,1]}$ can be replaced by $N_{[0,\infty)}$.

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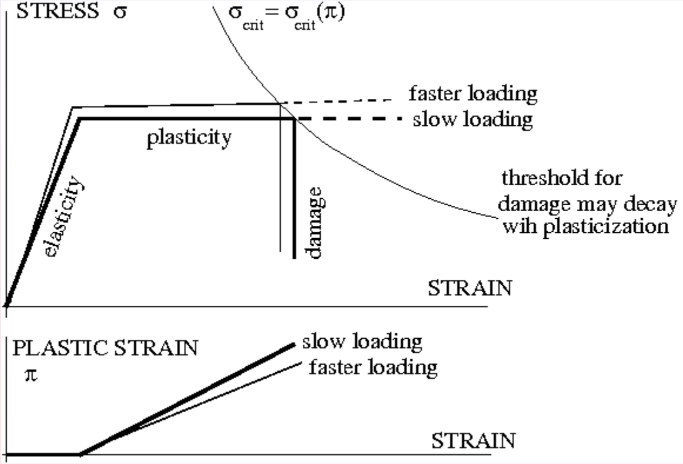
The dependence of a on π may lead to a scenario first plasticizing and then damaging under loading (like in Part I, but) even without any hardening.



This picture is "rate-dependent" due to $\alpha > 0$ and $\mathbb{D} > 0$. For very slow loading, a damage combined with (nearly) perfect plasticity can thus be modelled.

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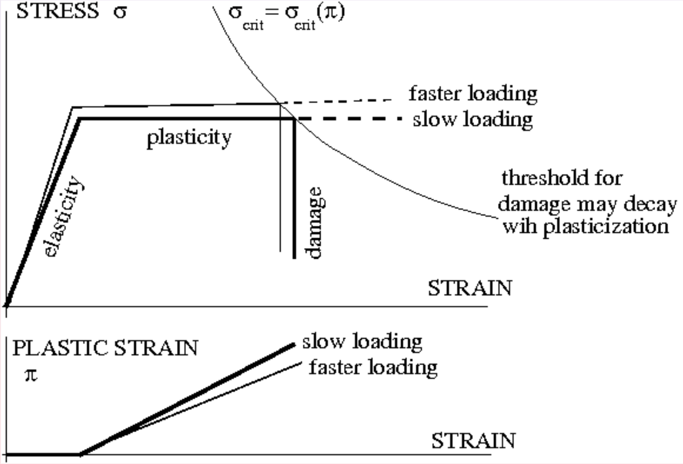

If (even small) kinematic hardening $\mathbb{H} > 0$ is considered, then a "microscopical" interpretation of $a = a(\pi) = \tilde{a}(\mathbb{H}^{-1}\pi)$ depending on hardening.

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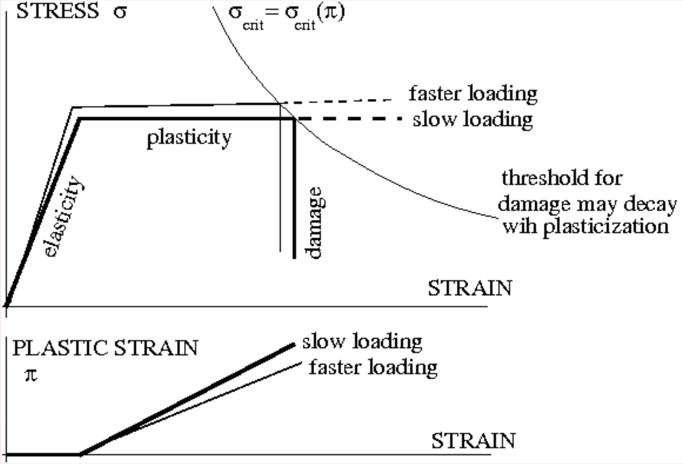

Similarly if (even small) isotropic hardening $b > 0$ is considered (see Part I), then again a "microscopical" interpretation of $a = a(\eta)$ depending on hardening.

A substitution of $u{+}u_{\rm Dir}$ instead of u.

The state space:

$$\big\{(u,\pi,\zeta)\in H^1(\Omega;\mathbb{R}^d)\times H^1(\Omega;\mathbb{R}^{d\times d}_{\rm dev})\times W^{1,r}(\Omega);\ \ u|_{\Gamma_{\rm Dir}}{=}\,0 \text{ on } \Gamma_{\rm Dir}\big\}.$$

The governing functionals:

$$\mathcal{E}(t,u,\pi,\zeta):=\begin{cases}\displaystyle\int_\Omega\frac12\mathbb{C}(\zeta)e_{\rm el}:e_{\rm el}-g(t)\cdot u\\ \displaystyle\quad+\frac{\kappa_1}{2}|\nabla\pi|^2+\frac{\kappa_2}{r}|\nabla\zeta|^r\,{\rm d}x-\int_{\Gamma_{\rm Neu}}f(t)\cdot u\,{\rm d}S & \text{if } \zeta\geq0 \text{ a.e. on } \Omega,\\ \infty & \text{otherwise,}\end{cases}$$

$$\mathcal{R}\Big(\pi,\zeta;\frac{{\rm d}u}{{\rm d}t},\frac{{\rm d}\pi}{{\rm d}t},\frac{{\rm d}\zeta}{{\rm d}t}\Big):=\begin{cases}\displaystyle\int_\Omega\frac{\alpha}{2}\Big|\frac{\partial\pi}{\partial t}\Big|^2+\delta^*_{S(\zeta)}\Big(\frac{\partial\pi}{\partial t}\Big)\\ \displaystyle\quad+a(\pi)\Big|\frac{\partial\zeta}{\partial t}\Big|+\frac12\mathbb{D}(\zeta)\frac{\partial e_{\rm el}}{\partial t}:\frac{\partial e_{\rm el}}{\partial t}\,{\rm d}x & \text{if } \dfrac{\partial\zeta}{\partial t}\leq0 \text{ a.e. on } \Omega,\\ \infty & \text{otherwise.}\end{cases}$$

where now $e_{\rm el}=\big(e(u{+}u_{\rm Dir}(t)\big)-\pi$.

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where now $e_{\rm el}=(e(u{+}u_{\rm Dir}(t))-\pi$.

A weak formulation: main features:

1) the plastic part (u, π): conventional weak formulation, but
$\nabla \frac{\partial \pi}{\partial t}$ is not well controlled
$\Rightarrow$ by-part integration in time needed

2) the damage part: semistability + energy equality (theory of RIS used),
$\frac{\partial \zeta}{\partial t}$ controlled only as a measure (though $a(\pi) \in C(\bar{Q})$)
$\Rightarrow$ by-part integration in time desired.

More specifically, we use:

$$\int_Q \nabla \pi : \nabla \frac{\partial \pi}{\partial t} \,\mathrm{d}x\mathrm{d}t = \int_\Omega \frac{1}{2} |\nabla \pi(T)|^2 \,\mathrm{d}x - \int_\Omega \frac{1}{2} |\nabla \pi(0)|^2 \,\mathrm{d}x \quad \text{and}$$

$$\int_{\bar{Q}} \alpha(\pi) \frac{\partial \zeta}{\partial t} (\mathrm{d}x\mathrm{d}t) = \int_\Omega \alpha(\pi(T))\zeta(T) \,\mathrm{d}x - \int_Q \alpha'(\pi) \frac{\partial \pi}{\partial t} \zeta \,\mathrm{d}x\mathrm{d}t - \int_\Omega \alpha(\pi(0))\zeta(0) \,\mathrm{d}x.$$

The triple (u, π, ζ) with

$$u \in H^1([0,T]; H^1(\Omega; \mathbb{R}^d)),$$
$$\pi \in H^1(0,T; L^2(\Omega; \mathbb{R}^{d\times d}_{\mathrm{dev}})) \cap L^\infty(0,T; H^1(\Omega; \mathbb{R}^{d\times d}_{\mathrm{dev}})),$$
$$\zeta \in \mathrm{B}([0,T]; W^{1,r}(\Omega)) \cap \mathrm{BV}([0,T]; L^1(\Omega))$$

such that also $\Delta \pi \in L^2(Q; \mathbb{R}^{d\times d}_{\mathrm{dev}})$ will be called a weak solution if:

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Momentum equation: $\forall v$:

$$\int_Q \Big(\mathbb{C}(\zeta)e_{\rm el}+\mathbb{D}(\zeta)\frac{\partial e_{\rm el}}{\partial t}\Big):e(v)-g\cdot v\,\mathrm{d}x\mathrm{d}t=\int_{\Gamma_{\rm Neu}} f\cdot v\,\mathrm{d}S\mathrm{d}t.$$

Plastic flow rule: $\forall v$ valued in $\mathbb{R}^{d\times d}_{\rm dev}$:

$$\int_Q \frac{\alpha}{2}|v|^2+\delta^*_{S(\zeta)}(v)-\Big(\mathbb{C}(\zeta)e_{\rm el}+\mathbb{D}(\zeta)\frac{\partial e_{\rm el}}{\partial t}\Big):\Big(v-\frac{\partial \pi}{\partial t}\Big)$$
$$+\kappa_1\nabla\pi\,\vdots\,\Big(\nabla v-\nabla\frac{\partial \pi}{\partial t}\Big)\,\mathrm{d}x\mathrm{d}t\geq\int_Q\frac{\alpha}{2}\Big|\frac{\partial \pi}{\partial t}\Big|^2+\delta^*_{S(\zeta)}\Big(\frac{\partial \pi}{\partial t}\Big)\,\mathrm{d}x\mathrm{d}t$$

Semi-stability: $\forall_{\rm a.a.}t\in[0,T]\ \forall 0\leq\tilde\zeta\leq\zeta(t)$ with $e_{\rm el}(t)=e(u(t)+u_{\rm Dir}(t))-\pi(t)$:

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$$\leq\int_\Omega\frac{1}{2}\mathbb{C}(\tilde\zeta)e_{\rm el}(t):e_{\rm el}(t)+\frac{\kappa_2}{r}|\nabla\tilde\zeta|^r+a(\pi(t))(\tilde\zeta-\zeta(t))\,\mathrm{d}x$$

Energy equality:

$$\int_Q\alpha\Big|\frac{\partial \pi}{\partial t}\Big|^2+\delta^*_{S(\zeta)}\Big(\frac{\partial \pi}{\partial t}\Big)+\mathbb{D}(\zeta)\frac{\partial e_{\rm el}}{\partial t}:\frac{\partial e_{\rm el}}{\partial t}\,\mathrm{d}x\mathrm{d}t+\int_{\bar Q}a(\pi)\Big|\frac{\partial \zeta}{\partial t}\Big|(\mathrm{d}x\mathrm{d}t)$$
$$+\mathcal{E}(T,u(T),\pi(T),\zeta(T))=\mathcal{E}(0,u_0,\pi_0,\zeta_0)+\int_0^T\mathcal{E}'_t(t,u(t),\pi(t),\zeta(t))\,\mathrm{d}t.$$

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$$+\int_\Omega\frac{\kappa_1}{2}|\nabla\pi_0|^2{\rm d}x\ge\int_Q\frac{\alpha}{2}\Big|\frac{\partial\pi}{\partial t}\Big|^2+\delta^*_{S(\zeta)}\Big(\frac{\partial\pi}{\partial t}\Big)\,{\rm d}x{\rm d}t+\int_\Omega\frac{\kappa_1}{2}|\nabla\pi(T)|^2{\rm d}x$$

Semi-stability: $\forall_{\rm a.a.}t\in[0,T]\ \forall 0\le\widetilde\zeta\le\zeta(t)$ with $e_{\rm el}(t)=e(u(t){+}u_{\rm Dir}(t)){-}\pi(t)$:

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$$+\mathcal{E}(T,u(T),\pi(T),\zeta(T))=\mathcal{E}(0,u_0,\pi_0,\zeta_0)+\int_0^T\mathcal{E}_t'(t,u(t),\pi(t),\zeta(t))\,{\rm d}t+\int_\Omega a(\pi_0)\zeta_0{\rm d}x.$$

Time discretisation by fractional-step strategy:

$$\operatorname{div}\sigma_\tau^k + g_\tau^k = 0 \qquad \text{with} \quad \sigma_\tau^k = \mathbb{C}(\zeta_\tau^{k-1})e_{\mathrm{el},\tau}^k + \mathbb{D}(\zeta_\tau^{k-1})\frac{e_{\mathrm{el},\tau}^k - e_{\mathrm{el},\tau}^{k-1}}{\tau},$$

$$\alpha\frac{\pi_\tau^k - \pi_\tau^{k-1}}{\tau} + \partial\delta^*_{S(\zeta_\tau^{k-1})}\Big(\frac{\pi_\tau^k - \pi_\tau^{k-1}}{\tau}\Big) \ni \operatorname{dev}\sigma_\tau^k + \kappa_1\Delta\pi_\tau^k$$

$$\partial\delta^*_{[-a(\pi_\tau^k),\infty)}\Big(\frac{\zeta_\tau^k - \zeta_\tau^{k-1}}{\tau}\Big) + \frac{1}{2}\mathbb{C}'(\zeta_\tau^k)e_{\mathrm{el},\tau}^k : e_{\mathrm{el},\tau}^k$$
$$+ N_{[0,1]}(\zeta_\tau^k) \ni \kappa_2\operatorname{div}\big(|\nabla\zeta_\tau^k|^{r-2}\nabla\zeta\big) \qquad \text{with} \quad e_{\mathrm{el},\tau}^k = e(u_\tau^k + u_{\mathrm{Dir},\tau}^k) - \pi_\tau^k$$

with the boundary conditions:

$$u_\tau^k = 0 \qquad \text{on } \Gamma_{\mathrm{Dir}},$$
$$\sigma_\tau^k\vec{n} = f \qquad \text{on } \Gamma_{\mathrm{Neu}},$$
$$\nabla\zeta_\tau^k\cdot\vec{n} = 0 \quad \text{and} \quad \nabla\pi_\tau^k\vec{n} = 0 \qquad \text{on } \Gamma$$

to be solved first for (u_τ^k, π_τ^k) and then for ζ_τ^k recursively for $k = 1, ..., T/\tau$.

Given $(\pi_\tau^{k-1}, \zeta_\tau^{k-1})$:

A minimization problem to obtain (u_τ^k, π_τ^k):

$$\left.\begin{array}{ll}\text{Minimize} & (u,\pi)\mapsto \mathcal{E}(k\tau, u, \pi, \zeta_\tau^{k-1}) + \mathcal{R}(0, \zeta_\tau^{k-1}; \pi-\pi_\tau^{k-1}, 0)\\ \text{subject to} & u\in H^1(\Omega;\mathbb{R}^d),\quad \pi\in H^1(\Omega;\mathbb{R}^{d\times d}_{\rm dev}),\quad u=0 \text{ on } \Gamma_{\rm Dir},\end{array}\right\}$$

and second minimization problem to obtain ζ_τ^k:

$$\left.\begin{array}{ll}\text{Minimize} & \zeta\mapsto \mathcal{E}(k\tau, u_\tau^k, \pi_\tau^k, \zeta) + \tau\mathcal{R}\Big(\pi_\tau^{k-1}, 0; 0, \dfrac{\zeta-\zeta_\tau^{k-1}}{\tau}\Big)\\ \text{subject to} & \zeta\in W^{1,r}(\Omega),\quad 0\le\zeta\le\zeta_\tau^{k-1} \text{ on } \Omega,\end{array}\right\}$$

Solutions exist by coercivity, convexity, and lower semicontinuity arguments.

If $\mathbb{C}'$ is nondecreasing (again with respect to the Löwner's ordering), these problems are convex.

Given $(\pi_\tau^{k-1},\zeta_\tau^{k-1})$:

A minimization problem to obtain (u_τ^k,π_τ^k):

$$\left.\begin{array}{ll}\text{Minimize} & (u,\pi)\mapsto\mathcal{E}(k\tau,u,\pi,\zeta_\tau^{k-1})+\mathcal{R}(0,\zeta_\tau^{k-1};\pi-\pi_\tau^{k-1},0)\\ \text{subject to} & u\in H^1(\Omega;\mathbb{R}^d),\quad \pi\in H^1(\Omega;\mathbb{R}^{d\times d}_{\rm dev}),\quad u=0 \text{ on } \Gamma_{\rm Dir},\end{array}\right\}$$

and second minimization problem to obtain ζ_τ^k:

$$\left.\begin{array}{ll}\text{Minimize} & \zeta\mapsto\mathcal{E}(k\tau,u_\tau^k,\pi_\tau^k,\zeta)+\tau\mathcal{R}\Big(\pi_\tau^{k-1},0;0,\dfrac{\zeta-\zeta_\tau^{k-1}}{\tau}\Big)\\ \text{subject to} & \zeta\in W^{1,r}(\Omega),\quad 0\le\zeta\le\zeta_\tau^{k-1}\ \text{ on } \Omega,\end{array}\right\}$$

Solutions exist by coercivity, convexity, and lower semicontinuity arguments.

If $\mathbb{C}'$ is nondecreasing (again with respect to the Löwner's ordering), these problems are convex.

We test the discrete inclusions respectively by $u_\tau^k - u_\tau^{k-1}$, $\pi_\tau^k - \pi_\tau^{k-1}$, and $\zeta_\tau^k - \zeta_\tau^{k-1}$. Relying on the convexity of $\mathcal{E}(k\tau,\cdot,\cdot,\zeta_\tau^{k-1})$ and of $\mathcal{E}(k\tau,u_\tau^k,\pi_\tau^k,\cdot)$, we obtain the estimates

$$\mathcal{E}(k\tau,u_\tau^k,\pi_\tau^k,\zeta_\tau^{k-1}) + \tau\int_\Omega \alpha\Big|\frac{\pi_\tau^k-\pi_\tau^{k-1}}{\tau}\Big|^2 + \delta^*_{S(\zeta_\tau^{k-1})}\Big(\frac{\pi_\tau^k-\pi_\tau^{k-1}}{\tau}\Big)$$

$$+\mathbb{D}(\zeta_\tau^{k-1})\frac{e_{\mathrm{el},\tau}^k-e_{\mathrm{el},\tau}^{k-1}}{\tau}:\frac{e_{\mathrm{el},\tau}^k-e_{\mathrm{el},\tau}^{k-1}}{\tau}\,\mathrm{d}x \le \mathcal{E}(k\tau,u_\tau^{k-1},\pi_\tau^{k-1},\zeta_\tau^{k-1}),$$

$$\mathcal{E}(k\tau,u_\tau^k,\pi_\tau^k,\zeta_\tau^k) + \int_\Omega a(\pi_\tau^k)(\zeta_\tau^k-\zeta_\tau^{k-1})\,\mathrm{d}x \le \mathcal{E}(k\tau,u_\tau^k,\pi_\tau^k,\zeta_\tau^{k-1}).$$

By summing these estimates, we can again enjoy the cancellation of the terms $\pm\mathcal{E}(k\tau,u_\tau^k,\pi_\tau^k,\zeta_\tau^{k-1})$, and thus obtain

$$\mathcal{E}(k\tau,u_\tau^k,\pi_\tau^k,\zeta_\tau^k)+\tau\widehat{\mathcal{R}}\Big(\pi_\tau^k,\zeta_\tau^{k-1};\frac{u_\tau^k-u_\tau^{k-1}}{\tau},\frac{\pi_\tau^k-\pi_\tau^{k-1}}{\tau},\frac{\zeta_\tau^k-\zeta_\tau^{k-1}}{\tau}\Big) \le \mathcal{E}(k\tau,u_\tau^{k-1},\pi_\tau^{k-1},\zeta_\tau^{k-1})$$

$$= \mathcal{E}((k-1)\tau,u_\tau^{k-1},\pi_\tau^{k-1},\zeta_\tau^{k-1}) + \int_{(k-1)\tau}^{k\tau}\frac{\partial\mathcal{E}}{\partial t}(t,u_\tau^{k-1},\pi_\tau^{k-1},\zeta_\tau^{k-1})\,\mathrm{d}t$$

with the dissipation rate $\widehat{\mathcal{R}}$ defined as

$$\widehat{\mathcal{R}}(\pi,\zeta;\dot u,\dot\pi,\dot\zeta) := \int_\Omega \delta^*_{S(\zeta)}(\dot\pi) + a(\pi)|\dot\zeta| + \mathbb{D}(\zeta)\dot e_{\mathrm{el}}:\dot e_{\mathrm{el}}\,\mathrm{d}x \quad \text{with } \dot e_{\mathrm{el}} = e(\dot u)-\dot\pi.$$

We test the discrete inclusions respectively by $u_\tau^k-u_\tau^{k-1}$, $\pi_\tau^k-\pi_\tau^{k-1}$, and $\zeta_\tau^k-\zeta_\tau^{k-1}$.
Relying on the convexity of $\mathcal{E}(k\tau,\cdot,\cdot,\zeta_\tau^{k-1})$ and of $\mathcal{E}(k\tau,u_\tau^k,\pi_\tau^k,\cdot)$, we obtain the estimates

$$\mathcal{E}(k\tau,u_\tau^k,\pi_\tau^k,\zeta_\tau^{k-1})+\tau\int_\Omega\alpha\Big|\frac{\pi_\tau^k-\pi_\tau^{k-1}}{\tau}\Big|^2+\delta^*_{S(\zeta_\tau^{k-1})}\Big(\frac{\pi_\tau^k-\pi_\tau^{k-1}}{\tau}\Big)$$
$$+\mathbb{D}(\zeta_\tau^{k-1})\frac{e_{\mathrm{el},\tau}^k-e_{\mathrm{el},\tau}^{k-1}}{\tau}:\frac{e_{\mathrm{el},\tau}^k-e_{\mathrm{el},\tau}^{k-1}}{\tau}\,\mathrm{d}x\le\mathcal{E}(k\tau,u_\tau^{k-1},\pi_\tau^{k-1},\zeta_\tau^{k-1}),$$
$$\mathcal{E}(k\tau,u_\tau^k,\pi_\tau^k,\zeta_\tau^k)+\int_\Omega a(\pi_\tau^k)(\zeta_\tau^k-\zeta_\tau^{k-1})\,\mathrm{d}x\le\mathcal{E}(k\tau,u_\tau^k,\pi_\tau^k,\zeta_\tau^{k-1}).$$

By summing these estimates, we can again enjoy the cancellation of the terms $\pm\mathcal{E}(k\tau,u_\tau^k,\pi_\tau^k,\zeta_\tau^{k-1})$, and thus obtain

$$\mathcal{E}(k\tau,u_\tau^k,\pi_\tau^k,\zeta_\tau^k)+\tau\widehat{\mathcal{R}}\Big(\pi_\tau^k,\zeta_\tau^{k-1};\frac{u_\tau^k-u_\tau^{k-1}}{\tau},\frac{\pi_\tau^k-\pi_\tau^{k-1}}{\tau},\frac{\zeta_\tau^k-\zeta_\tau^{k-1}}{\tau}\Big)\le\mathcal{E}(k\tau,u_\tau^{k-1},\pi_\tau^{k-1},\zeta_\tau^{k-1})$$
$$=\mathcal{E}((k-1)\tau,u_\tau^{k-1},\pi_\tau^{k-1},\zeta_\tau^{k-1})+\int_{(k-1)\tau}^{k\tau}\frac{\partial\mathcal{E}}{\partial t}(t,u_\tau^{k-1},\pi_\tau^{k-1},\zeta_\tau^{k-1})\,\mathrm{d}t$$

with the dissipation rate $\widehat{\mathcal{R}}$ defined as

$$\widehat{\mathcal{R}}(\pi,\zeta;\dot u,\dot\pi,\dot\zeta):=\int_\Omega\delta^*_{S(\zeta)}(\dot\pi)+a(\pi)|\dot\zeta|+\mathbb{D}(\zeta)\dot e_{\mathrm{el}}:\dot e_{\mathrm{el}}\,\mathrm{d}x\quad\text{with }\dot e_{\mathrm{el}}=e(\dot u)-\dot\pi.$$

By the discrete Gronwall inequality, we obtain boundedness of $\sup_{t\in[0,T]}\mathcal{E}_\tau(t,\bar{u}_\tau,\bar{\pi}_\tau,\bar{\zeta}_\tau)$ and $\int_0^T\widehat{\mathcal{R}}(\bar{\pi}_\tau,\underline{\zeta}_\tau;\frac{\mathrm{d}\pi_\tau}{\mathrm{d}t},\frac{\mathrm{d}\zeta_\tau}{\mathrm{d}t})\,\mathrm{d}t$.

From the coercivity of $\mathcal{E}$ and $\mathcal{R}$, we thus obtain the a-priori estimates:

$$\big\|\bar{e}_{\mathrm{el},\tau}\big\|_{L^\infty(0,T;L^2(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}}))}\le C,\qquad \big\|e_{\mathrm{el},\tau}\big\|_{H^1(0,T;L^2(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}}))}\le C,$$
$$\big\|\bar{\pi}_\tau\big\|_{L^\infty(0,T;H^1(\Omega;\mathbb{R}^{d\times d}_{\mathrm{dev}}))}\le C,\qquad \big\|\pi_\tau\big\|_{H^1(0,T;L^2(\Omega;\mathbb{R}^{d\times d}_{\mathrm{dev}}))}\le C,$$
$$\big\|\bar{\zeta}_\tau\big\|_{\mathrm{B}([0,T];W^{1,r}(\Omega))\cap\mathrm{BV}([0,T];L^1(\Omega))\cap L^\infty(Q)}\le C,$$

so that, by Korn's inequality, using $e(u_\tau)=e_{\mathrm{el},\tau}+\pi_\tau$, also

$$\big\|\bar{u}_\tau\big\|_{L^\infty(0,T;H^1(\Omega;\mathbb{R}^d))}\le C,\qquad \big\|u_\tau\big\|_{H^1(0,T;H^1(\Omega;\mathbb{R}^d))}\le C,$$

and by comparison also

$$\big\|\Delta\bar{\pi}_\tau\big\|_{L^2(Q;\mathbb{R}^{d\times d}_{\mathrm{dev}})}=\frac{1}{\kappa_1}\Big\|\alpha\frac{\partial\pi_\tau}{\partial t}+\partial\delta^*_{S(\underline{\zeta}_\tau)}\Big(\frac{\partial\pi_\tau}{\partial t}\Big)-\mathrm{dev}\,\bar{\sigma}_\tau\Big\|_{L^2(Q;\mathbb{R}^{d\times d}_{\mathrm{dev}})}\le C.$$

The same estimate as for $\bar{\zeta}_\tau$ also holds for $\underline{\zeta}_\tau$.

By the discrete Gronwall inequality, we obtain boundedness of $\sup_{t\in[0,T]}\mathcal{E}_\tau(t,\bar{u}_\tau,\bar{\pi}_\tau,\bar{\zeta}_\tau)$ and $\int_0^T\widehat{\mathcal{R}}(\bar{\pi}_\tau,\underline{\zeta}_\tau;\frac{\mathrm{d}\pi_\tau}{\mathrm{d}t},\frac{\mathrm{d}\zeta_\tau}{\mathrm{d}t})\,\mathrm{d}t$.

From the coercivity of $\mathcal{E}$ and $\mathcal{R}$, we thus obtain the a-priori estimates:

$$\|\bar{e}_{\mathrm{el},\tau}\|_{L^\infty(0,T;L^2(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}}))}\le C,\qquad \|e_{\mathrm{el},\tau}\|_{H^1(0,T;L^2(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}}))}\le C,$$
$$\|\bar{\pi}_\tau\|_{L^\infty(0,T;H^1(\Omega;\mathbb{R}^{d\times d}_{\mathrm{dev}}))}\le C,\qquad \|\pi_\tau\|_{H^1(0,T;L^2(\Omega;\mathbb{R}^{d\times d}_{\mathrm{dev}}))}\le C,$$
$$\|\underline{\zeta}_\tau\|_{\mathrm{B}([0,T];W^{1,r}(\Omega))\cap\mathrm{BV}([0,T];L^1(\Omega))\cap L^\infty(Q)}\le C,$$

so that, by Korn's inequality, using $e(u_\tau)=e_{\mathrm{el},\tau}+\pi_\tau$, also

$$\|\bar{u}_\tau\|_{L^\infty(0,T;H^1(\Omega;\mathbb{R}^d))}\le C,\qquad \|u_\tau\|_{H^1(0,T;H^1(\Omega;\mathbb{R}^d))}\le C,$$

and by comparison also

$$\|\Delta\bar{\pi}_\tau\|_{L^2(Q;\mathbb{R}^{d\times d}_{\mathrm{dev}})}=\frac{1}{\kappa_1}\Big\|\alpha\frac{\partial\pi_\tau}{\partial t}+\partial\delta^*_{S(\underline{\zeta}_\tau)}\Big(\frac{\partial\pi_\tau}{\partial t}\Big)-\mathrm{dev}\,\bar{\sigma}_\tau\Big\|_{L^2(Q;\mathbb{R}^{d\times d}_{\mathrm{dev}})}\le C.$$

The same estimate as for $\bar{\zeta}_\tau$ also holds for $\underline{\zeta}_\tau$.

Convergence:

1) Banach selection principle:

$$\begin{aligned} &u_\tau \to u && \text{weakly in } H^1(0,T;H^1(\Omega;\mathbb{R}^d)),\\ &\pi_\tau \to \pi && \text{weakly in } H^1(0,T;L^2(\Omega;\mathbb{R}^{d\times d}_{\rm dev})),\\ &\bar\zeta_\tau \to \zeta,\ \ \underline{\zeta}_\tau \to \zeta && \text{weakly* in } L^\infty(Q), \end{aligned}$$

2) the limit passage in the discrete momentum equilibrium

$$\mathrm{div}\Big(\mathbb{C}(\underline{\zeta}_\tau)\bar{e}_{\mathrm{el},\tau}+\mathbb{D}\frac{\partial e_{\mathrm{el},\tau}}{\partial t}\Big)+\bar{g}_\tau=0$$

by weak continuity (+ compactness in ζ)

3) strong convergence of $\bar{e}_{\mathrm{el},\tau}$ in $L^2([0,T]\times\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}}))$ (like in Part I):
...it needs $\mathbb{D}$ not depending on ζ, however!

the discrete momentum equilibrium $\mathrm{div}(\mathbb{C}(\underline{\zeta}_\tau)\bar{e}_{\mathrm{el},\tau}+\mathbb{D}\frac{\partial}{\partial t}e_{\mathrm{el},\tau})+\bar{g}_\tau=0$

the discrete plastic flow-rule $\alpha\bar{\xi}_\tau-\mathrm{dev}\,\bar{\sigma}_\tau=\kappa_1\Delta\bar{\pi}_\tau$ with
$\bar{\sigma}_\tau=\mathbb{C}(\underline{\zeta}_\tau)\bar{e}_{\mathrm{el},\tau}+\mathbb{D}\frac{\partial}{\partial t}e_{\mathrm{el},\tau}$ and $\bar{\xi}_\tau\in\partial\delta_S^*(\frac{\partial}{\partial t}\pi_\tau)$ and
$\bar{e}_{\mathrm{el},\tau}=e(\bar{u}_\tau-\bar{u}_{\mathrm{Dir},\tau})-\bar{\pi}_\tau$ with B.C. considered in the weak sense and
tested respectively by $\bar{u}_\tau(t)-u(t)$ and $\bar{\pi}_\tau(t)-\pi(t)$ and integrated over $[0,T]$.

$$\int_Q \mathbb{C}(\underline{\zeta}_\tau)(\bar{e}_{\mathrm{el},\tau}-e_{\mathrm{el}}):(\bar{e}_{\mathrm{el},\tau}-e_{\mathrm{el}})+\frac{\kappa_1}{2}\big|\nabla\bar{\pi}_\tau-\nabla\pi\big|^2\,\mathrm{d}x\mathrm{d}t$$
$$\le\int_Q -\mathbb{C}(\underline{\zeta}_\tau)e_{\mathrm{el}}:(\bar{e}_{\mathrm{el},\tau}-e_{\mathrm{el}})+\bar{\xi}_\tau:(\bar{\pi}_\tau-\pi)+\frac{\kappa_1}{2}\nabla\pi\,\vdots\,\nabla(\bar{\pi}_\tau-\pi)$$
$$-\mathbb{D}\frac{\partial e_{\mathrm{el},\tau}}{\partial t}:(\bar{e}_{\mathrm{el},\tau}-e_{\mathrm{el}})-\alpha\frac{\partial\pi_\tau}{\partial t}:(\bar{\pi}_\tau-\pi)-\bar{f}_\tau\cdot(\bar{u}_\tau-u)\,\mathrm{d}x\mathrm{d}t$$
$$-\int_{\Sigma_{\mathrm{Neu}}}\bar{g}_\tau\cdot(\bar{u}_\tau-u)\mathrm{d}S\mathrm{d}t\to 0.$$

3) strong convergence of $\bar{e}_{\mathrm{el},\tau}$ in $L^2([0,T]\times\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}}))$ (like in Part I):
...it needs $\mathbb{D}$ not depending on ζ, however!

the discrete momentum equilibrium $\mathrm{div}(\mathbb{C}(\underline{\zeta}_\tau)\bar{e}_{\mathrm{el},\tau}+\mathbb{D}\frac{\partial}{\partial t}e_{\mathrm{el},\tau})+\bar{g}_\tau=0$

the discrete plastic flow-rule $\alpha\bar{\xi}_\tau-\mathrm{dev}\,\bar{\sigma}_\tau=\kappa_1\Delta\bar{\pi}_\tau$ with
$\bar{\sigma}_\tau=\mathbb{C}(\underline{\zeta}_\tau)\bar{e}_{\mathrm{el},\tau}+\mathbb{D}\frac{\partial}{\partial t}e_{\mathrm{el},\tau}$ and $\bar{\xi}_\tau\in\partial\delta^*_S(\frac{\partial}{\partial t}\pi_\tau)$ and
$\bar{e}_{\mathrm{el},\tau}=e(\bar{u}_\tau-\bar{u}_{\mathrm{Dir},\tau})-\bar{\pi}_\tau$ with B.C. considered in the weak sense and
tested respectively by $\bar{u}_\tau(t)-u(t)$ and $\bar{\pi}_\tau(t)-\pi(t)$ and integrated over $[0,T]$.

$$
\begin{aligned}
&\int_Q\mathbb{C}(\underline{\zeta}_\tau)(\bar{e}_{\mathrm{el},\tau}-e_{\mathrm{el}}):(\bar{e}_{\mathrm{el},\tau}-e_{\mathrm{el}})+\frac{\kappa_1}{2}\big|\nabla\bar{\pi}_\tau-\nabla\pi\big|^2\,\mathrm{d}x\mathrm{d}t\\
&\le\int_Q-\mathbb{C}(\underline{\zeta}_\tau)e_{\mathrm{el}}:(\bar{e}_{\mathrm{el},\tau}-e_{\mathrm{el}})+\bar{\xi}_\tau:(\bar{\pi}_\tau-\pi)+\frac{\kappa_1}{2}\nabla\pi\,\vdots\,\nabla(\bar{\pi}_\tau-\pi)\\
&\qquad-\mathbb{D}\frac{\partial e_{\mathrm{el},\tau}}{\partial t}:(\bar{e}_{\mathrm{el},\tau}-e_{\mathrm{el}})-\alpha\frac{\partial\pi_\tau}{\partial t}:(\bar{\pi}_\tau-\pi)-\bar{f}_\tau\cdot(\bar{u}_\tau-u)\,\mathrm{d}x\mathrm{d}t\\
&\qquad\qquad-\int_{\Sigma_{\mathrm{Neu}}}\bar{g}_\tau\cdot(\bar{u}_\tau-u)\mathrm{d}S\mathrm{d}t\to0.
\end{aligned}
$$

3) strong convergence of $\bar{e}_{\mathrm{el},\tau}$ in $L^2([0,T]\times\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}})$ (like in Part I):

...it needs $\mathbb{D}$ not depending on ζ, however!

the discrete momentum equilibrium $\mathrm{div}(\mathbb{C}(\underline{\zeta}_\tau)\bar{e}_{\mathrm{el},\tau}+\mathbb{D}\frac{\partial}{\partial t}e_{\mathrm{el},\tau})+\bar{g}_\tau=0$

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$\bar{e}_{\mathrm{el},\tau}=e(\bar{u}_\tau-\bar{u}_{\mathrm{Dir},\tau})-\bar{\pi}_\tau$ with B.C. considered in the weak sense and
tested respectively by $\bar{u}_\tau(t)-u(t)$ and $\bar{\pi}_\tau(t)-\pi(t)$ and integrated over $[0,T]$.

$$\int_Q\mathbb{C}(\underline{\zeta}_\tau)\big(\bar{e}_{\mathrm{el},\tau}-e_{\mathrm{el}}\big):\big(\bar{e}_{\mathrm{el},\tau}-e_{\mathrm{el}}\big)+\frac{\kappa_1}{2}\big|\nabla\bar{\pi}_\tau-\nabla\pi\big|^2\,\mathrm{d}x\mathrm{d}t$$
$$\le\int_Q-\mathbb{C}(\underline{\zeta}_\tau)e_{\mathrm{el}}:\big(\bar{e}_{\mathrm{el},\tau}-e_{\mathrm{el}}\big)+\bar{\xi}_\tau:\big(\bar{\pi}_\tau-\pi\big)+\frac{\kappa_1}{2}\nabla\pi\,\vdots\,\nabla\big(\bar{\pi}_\tau-\pi\big)$$
$$-\mathbb{D}\frac{\partial e_{\mathrm{el},\tau}}{\partial t}:\big(\bar{e}_{\mathrm{el},\tau}-e_{\mathrm{el}}\big)-\alpha\frac{\partial\pi_\tau}{\partial t}:\big(\bar{\pi}_\tau-\pi\big)-\bar{f}_\tau\cdot\big(\bar{u}_\tau-u\big)\,\mathrm{d}x\mathrm{d}t$$
$$-\int_{\Sigma_{\mathrm{Neu}}}\bar{g}_\tau\cdot(\bar{u}_\tau-u)\mathrm{d}S\mathrm{d}t\to0.$$

$\Rightarrow\quad\bar{e}_{\mathrm{el},\tau}(t)\to e_{\mathrm{el}}(t)\quad\&\quad\bar{\pi}_\tau(t)\to\pi(t)\quad$ strongly in $H^1(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}})$

$\Rightarrow\quad e(\bar{u}_\tau(t))=e(u_{\mathrm{Dir},\tau}(t))+\bar{\pi}_\tau(t)+\bar{e}_{\mathrm{el},\tau}(t)\to e(u(t))$ strongly in $L^2(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}})$

$\Rightarrow\quad\bar{u}_\tau(t)\to u(t)$ strongly in $H^1(\Omega;\mathbb{R}^d)$.

3) strong convergence of $\bar{e}_{\mathrm{el},\tau}$ in $L^2([0,T]\times\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}})$ (like in Part I):
...it needs $\mathbb{D}$ not depending on ζ, however!

the discrete momentum equilibrium $\mathrm{div}(\mathbb{C}(\underline{\zeta}_\tau)\bar{e}_{\mathrm{el},\tau}+\mathbb{D}\frac{\partial}{\partial t}e_{\mathrm{el},\tau})+\bar{g}_\tau=0$

the discrete plastic flow-rule $\alpha\bar{\xi}_\tau-\mathrm{dev}\,\bar{\sigma}_\tau=\kappa_1\Delta\bar{\pi}_\tau$ with
$\bar{\sigma}_\tau=\mathbb{C}(\underline{\zeta}_\tau)\bar{e}_{\mathrm{el},\tau}+\mathbb{D}\frac{\partial}{\partial t}e_{\mathrm{el},\tau}$ and $\bar{\xi}_\tau\in\partial\delta^*_S(\frac{\partial}{\partial t}\pi_\tau)$ and
$\bar{e}_{\mathrm{el},\tau}=e(\bar{u}_\tau-\bar{u}_{\mathrm{Dir},\tau})-\bar{\pi}_\tau$ with B.C. considered in the weak sense and
tested respectively by $\bar{u}_\tau(t)-u(t)$ and $\bar{\pi}_\tau(t)-\pi(t)$ and integrated over $[0,T]$.

$$
\begin{aligned}
&\int_Q \mathbb{C}(\underline{\zeta}_\tau)\big(\bar{e}_{\mathrm{el},\tau}-e_{\mathrm{el}}\big):\big(\bar{e}_{\mathrm{el},\tau}-e_{\mathrm{el}}\big)+\frac{\kappa_1}{2}\big|\nabla\bar{\pi}_\tau-\nabla\pi\big|^2\,\mathrm{d}x\mathrm{d}t\\
&\le\int_Q -\mathbb{C}(\underline{\zeta}_\tau)e_{\mathrm{el}}:\big(\bar{e}_{\mathrm{el},\tau}-e_{\mathrm{el}}\big)+\bar{\xi}_\tau:\big(\bar{\pi}_\tau-\pi\big)+\frac{\kappa_1}{2}\nabla\pi\,\vdots\,\nabla\big(\bar{\pi}_\tau-\pi\big)\\
&\qquad-\mathbb{D}\frac{\partial e_{\mathrm{el},\tau}}{\partial t}:\big(\bar{e}_{\mathrm{el},\tau}-e_{\mathrm{el}}\big)-\alpha\frac{\partial\pi_\tau}{\partial t}:\big(\bar{\pi}_\tau-\pi\big)-\bar{f}_\tau\cdot\big(\bar{u}_\tau-u\big)\,\mathrm{d}x\mathrm{d}t\\
&\qquad\qquad-\int_{\Sigma_{\mathrm{Neu}}}\bar{g}_\tau\cdot(\bar{u}_\tau-u)\,\mathrm{d}S\mathrm{d}t\to 0.
\end{aligned}
$$

Important note: $S\subset\mathbb{R}^{d\times d}_{\mathrm{dev}}$ bounded $\Rightarrow$ $(\bar{\xi}_\tau)_{\tau>0}\subset L^\infty(Q;\mathbb{R}^{d\times d}_{\mathrm{dev}})$ bounded
$\Rightarrow$ relatively compact in $L^2(0,T;H^1(\Omega;\mathbb{R}^{d\times d}_{\mathrm{dev}})^*)$ (here $\nabla\pi$ needed!)
$\Rightarrow$ $\int_Q\bar{\xi}_\tau(t):(\bar{\pi}_\tau(t)-\pi(t))\,\mathrm{d}x\mathrm{d}t\to 0$

3) strong convergence of $\bar{e}_{\mathrm{el},\tau}$ in $L^2([0,T]\times\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}}))$ (like in Part I):
...it needs $\mathbb{D}$ not depending on ζ, however!

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$\bar{e}_{\mathrm{el},\tau}=e(\bar{u}_\tau-\bar{u}_{\mathrm{Dir},\tau})-\bar{\pi}_\tau$ with B.C. considered in the weak sense and tested respectively by $\bar{u}_\tau(t)-u(t)$ and $\bar{\pi}_\tau(t)-\pi(t)$ and integrated over $[0,T]$.

$$\int_Q\mathbb{C}(\underline{\zeta}_\tau)(\bar{e}_{\mathrm{el},\tau}-e_{\mathrm{el}}):(\bar{e}_{\mathrm{el},\tau}-e_{\mathrm{el}})+\frac{\kappa_1}{2}\big|\nabla\bar{\pi}_\tau-\nabla\pi\big|^2\,\mathrm{d}x\mathrm{d}t$$
$$\le\int_Q-\mathbb{C}(\underline{\zeta}_\tau)e_{\mathrm{el}}:(\bar{e}_{\mathrm{el},\tau}-e_{\mathrm{el}})+\bar{\xi}_\tau:(\bar{\pi}_\tau-\pi)+\frac{\kappa_1}{2}\nabla\pi\,\vdots\,\nabla(\bar{\pi}_\tau-\pi)$$
$$-\mathbb{D}\frac{\partial e_{\mathrm{el},\tau}}{\partial t}:(\bar{e}_{\mathrm{el},\tau}-e_{\mathrm{el}})-\alpha\frac{\partial\pi_\tau}{\partial t}:(\bar{\pi}_\tau-\pi)-\bar{f}_\tau\cdot(\bar{u}_\tau-u)\,\mathrm{d}x\mathrm{d}t$$
$$-\int_{\Sigma_{\mathrm{Neu}}}\bar{g}_\tau\cdot(\bar{u}_\tau-u)\mathrm{d}S\mathrm{d}t\to0.$$

Another note: $\limsup_{\tau\to0}\int_Q-\mathbb{D}\frac{\partial e_{\mathrm{el},\tau}}{\partial t}:(\bar{e}_{\mathrm{el},\tau}-e_{\mathrm{el}})\,\mathrm{d}x\mathrm{d}t\le\int_\Omega\frac{1}{2}\mathbb{D}e_{\mathrm{el},\tau}(0):e_{\mathrm{el},\tau}(0)\,\mathrm{d}x$
$-\limsup_{\tau\to0}\int_\Omega\frac{1}{2}\mathbb{D}e_{\mathrm{el},\tau}(T):e_{\mathrm{el},\tau}(T)\,\mathrm{d}x+\lim_{\tau\to0}\int_Q\mathbb{D}\frac{\partial e_{\mathrm{el},\tau}}{\partial t}:e_{\mathrm{el}}\,\mathrm{d}x\mathrm{d}t=0$
(here we needed $\mathbb{D}$ constant)

3) strong convergence of $\bar{e}_{\mathrm{el},\tau}$ in $L^2([0,T]\times\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}}))$ (like in Part I):

...it needs $\mathbb{D}$ not depending on ζ, however!

the discrete momentum equilibrium $\mathrm{div}(\mathbb{C}(\underline{\zeta}_\tau)\bar{e}_{\mathrm{el},\tau}+\mathbb{D}\frac{\partial}{\partial t}e_{\mathrm{el},\tau})+\bar{g}_\tau=0$

the discrete plastic flow-rule $\alpha\bar{\xi}_\tau-\mathrm{dev}\,\bar{\sigma}_\tau=\kappa_1\Delta\bar{\pi}_\tau$ with

$\bar{\sigma}_\tau=\mathbb{C}(\underline{\zeta}_\tau)\bar{e}_{\mathrm{el},\tau}+\mathbb{D}\frac{\partial}{\partial t}e_{\mathrm{el},\tau}$ and $\bar{\xi}_\tau\in\partial\delta^*_S(\frac{\partial}{\partial t}\pi_\tau)$ and

$\bar{e}_{\mathrm{el},\tau}=e(\bar{u}_\tau-\bar{u}_{\mathrm{Dir},\tau})-\bar{\pi}_\tau$ with B.C. considered in the weak sense and tested respectively by $\bar{u}_\tau(t)-u(t)$ and $\bar{\pi}_\tau(t)-\pi(t)$ and integrated over $[0,T]$.

$$
\begin{aligned}
&\int_Q \mathbb{C}(\underline{\zeta}_\tau)\big(\bar{e}_{\mathrm{el},\tau}-e_{\mathrm{el}}\big):\big(\bar{e}_{\mathrm{el},\tau}-e_{\mathrm{el}}\big)+\frac{\kappa_1}{2}\big|\nabla\bar{\pi}_\tau-\nabla\pi\big|^2\,\mathrm{d}x\mathrm{d}t\\
&\le\int_Q -\mathbb{C}(\underline{\zeta}_\tau)e_{\mathrm{el}}:\big(\bar{e}_{\mathrm{el},\tau}-e_{\mathrm{el}}\big)+\bar{\xi}_\tau:\big(\bar{\pi}_\tau-\pi\big)+\frac{\kappa_1}{2}\nabla\pi\,\vdots\,\nabla\big(\bar{\pi}_\tau-\pi\big)\\
&\qquad-\mathbb{D}\frac{\partial e_{\mathrm{el},\tau}}{\partial t}:\big(\bar{e}_{\mathrm{el},\tau}-e_{\mathrm{el}}\big)-\alpha\frac{\partial\pi_\tau}{\partial t}:\big(\bar{\pi}_\tau-\pi\big)-\bar{f}_\tau\cdot\big(\bar{u}_\tau-u\big)\,\mathrm{d}x\mathrm{d}t\\
&\qquad\qquad-\int_{\Sigma_{\mathrm{Neu}}}\bar{g}_\tau\cdot(\bar{u}_\tau-u)\mathrm{d}S\mathrm{d}t\to 0.
\end{aligned}
$$

Similarly also $\limsup_{\tau\to 0}\int_Q -\alpha\frac{\partial\pi_\tau}{\partial t}:\big(\bar{\pi}_\tau-\pi\big)\,\mathrm{d}x\mathrm{d}t\le 0.$

Note also the differences from a similar estimate in Part I

(with now $\mathbb{H}=0$ here).

3) strong convergence of $\bar{e}_{\mathrm{el},\tau}$ in $L^2([0,T]\times\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}}))$ (like in Part I):
...it needs $\mathbb{D}$ not depending on ζ, however!

the discrete momentum equilibrium $\mathrm{div}(\mathbb{C}(\underline{\zeta}_\tau)\bar{e}_{\mathrm{el},\tau}+\mathbb{D}\frac{\partial}{\partial t}e_{\mathrm{el},\tau})+\bar{g}_\tau=0$

the discrete plastic flow-rule $\alpha\bar{\xi}_\tau-\mathrm{dev}\,\bar{\sigma}_\tau=\kappa_1\Delta\bar{\pi}_\tau$ with
$\bar{\sigma}_\tau=\mathbb{C}(\underline{\zeta}_\tau)\bar{e}_{\mathrm{el},\tau}+\mathbb{D}\frac{\partial}{\partial t}e_{\mathrm{el},\tau}$ and $\bar{\xi}_\tau\in\partial\delta^*_S(\frac{\partial}{\partial t}\pi_\tau)$ and
$\bar{e}_{\mathrm{el},\tau}=e(\bar{u}_\tau-\bar{u}_{\mathrm{Dir},\tau})-\bar{\pi}_\tau$ with B.C. considered in the weak sense and
tested respectively by $\bar{u}_\tau(t)-u(t)$ and $\bar{\pi}_\tau(t)-\pi(t)$ and integrated over $[0,T]$.

$$\begin{aligned}\int_Q\mathbb{C}(\underline{\zeta}_\tau)(\bar{e}_{\mathrm{el},\tau}-e_{\mathrm{el}}):(\bar{e}_{\mathrm{el},\tau}-e_{\mathrm{el}})+\frac{\kappa_1}{2}\big|\nabla\bar{\pi}_\tau-\nabla\pi\big|^2\,\mathrm{d}x\mathrm{d}t\\
\le\int_Q-\mathbb{C}(\underline{\zeta}_\tau)e_{\mathrm{el}}:(\bar{e}_{\mathrm{el},\tau}-e_{\mathrm{el}})+\bar{\xi}_\tau:(\bar{\pi}_\tau-\pi)+\frac{\kappa_1}{2}\nabla\pi\,\vdots\,\nabla(\bar{\pi}_\tau-\pi)\\
-\mathbb{D}\frac{\partial e_{\mathrm{el},\tau}}{\partial t}:(\bar{e}_{\mathrm{el},\tau}-e_{\mathrm{el}})-\alpha\frac{\partial\pi_\tau}{\partial t}:(\bar{\pi}_\tau-\pi)-\bar{f}_\tau\cdot(\bar{u}_\tau-u)\,\mathrm{d}x\mathrm{d}t\\
-\int_{\Sigma_{\mathrm{Neu}}}\bar{g}_\tau\cdot(\bar{u}_\tau-u)\mathrm{d}S\mathrm{d}t\to0.\end{aligned}$$

Similarly also $\limsup_{\tau\to0}\int_Q-\alpha\frac{\partial\pi_\tau}{\partial t}:(\bar{\pi}_\tau-\pi)\,\mathrm{d}x\mathrm{d}t\le0.$

Note also the differences from a similar estimate in Part I
(with now $\mathbb{H}=0$ here).

4) limit passage in the discrete plastic flow rule (after by-part summation):

$$\int_Q \frac{\alpha}{2}|v|^2 + \delta^*_{S(\underline{\zeta}_\tau)}(v) - \Big(\mathbb{C}(\underline{\zeta}_\tau)\bar{e}_{\mathrm{el},\tau} + \mathbb{D}\frac{\partial e_{\mathrm{el},\tau}}{\partial t}\Big):\Big(v - \frac{\partial \pi_\tau}{\partial t}\Big) + \kappa_1 \nabla\bar{\pi}_\tau \vdots \nabla v \,\mathrm{d}x\mathrm{d}t$$
$$+ \int_\Omega \frac{\kappa_1}{2}|\nabla\pi_0|^2 \mathrm{d}x \geq \int_Q \frac{\alpha}{2}\Big|\frac{\partial \pi_\tau}{\partial t}\Big|^2 + \delta^*_{S(\underline{\zeta}_\tau)}\Big(\frac{\partial \pi_\tau}{\partial t}\Big)\,\mathrm{d}x\mathrm{d}t + \int_\Omega \frac{\kappa_1}{2}|\nabla\pi_\tau(T)|^2\,\mathrm{d}x$$

is simply by weak (lower semi-)continuity.

4) limit passage in the discrete plastic flow rule (after by-part summation):

$$\int_Q \frac{\alpha}{2}|v|^2 + \delta^*_{S(\underline{\zeta}_\tau)}(v) - \Big(\mathbb{C}(\underline{\zeta}_\tau)\bar{e}_{\mathrm{el},\tau} + \mathbb{D}\frac{\partial e_{\mathrm{el},\tau}}{\partial t}\Big):\Big(v - \frac{\partial \pi_\tau}{\partial t}\Big) + \kappa_1 \nabla\bar{\pi}_\tau \,\vdots\, \nabla v \,\mathrm{d}x\mathrm{d}t$$
$$+ \int_\Omega \frac{\kappa_1}{2}|\nabla\pi_0|^2 \mathrm{d}x \geq \int_Q \frac{\alpha}{2}\Big|\frac{\partial \pi_\tau}{\partial t}\Big|^2 + \delta^*_{S(\underline{\zeta}_\tau)}\Big(\frac{\partial \pi_\tau}{\partial t}\Big)\,\mathrm{d}x\mathrm{d}t + \int_\Omega \frac{\kappa_1}{2}|\nabla\pi_\tau(T)|^2\,\mathrm{d}x$$

is simply by weak (lower semi-)continuity. The only difficult term is

$$\limsup_{\tau\to 0}\int_Q \Big(\mathbb{C}(\underline{\zeta}_\tau)\bar{e}_{\mathrm{el},\tau} + \mathbb{D}\frac{\partial e_{\mathrm{el},\tau}}{\partial t}\Big):\frac{\partial \pi_\tau}{\partial t}\,\mathrm{d}x\mathrm{d}t$$
$$= \limsup_{\tau\to 0}\int_Q \Big(\mathbb{C}(\underline{\zeta}_\tau)\bar{e}_{\mathrm{el},\tau} + \mathbb{D}\frac{\partial e_{\mathrm{el},\tau}}{\partial t}\Big):\Big(e\Big(\frac{\partial u_\tau}{\partial t}\Big) - \frac{\partial e_{\mathrm{el},\tau}}{\partial t}\Big)\,\mathrm{d}x\mathrm{d}t$$
$$= \limsup_{\tau\to 0}\int_Q -\mathbb{D}\frac{\partial e_{\mathrm{el},\tau}}{\partial t}:\frac{\partial e_{\mathrm{el},\tau}}{\partial t}\,\mathrm{d}x\mathrm{d}t + \lim_{\tau\to 0}\Big(\int_Q \bar{f}_\tau\cdot\frac{\partial u_\tau}{\partial t}\,\mathrm{d}x\mathrm{d}t + \int_{\Sigma_{\mathrm{Neu}}} \bar{g}_\tau\cdot\frac{\partial u_\tau}{\partial t}\,\mathrm{d}S\mathrm{d}t\Big)$$
$$\leq \limsup_{\tau\to 0}\int_Q -\mathbb{D}\frac{\partial e_{\mathrm{el}}}{\partial t}:\frac{\partial e_{\mathrm{el}}}{\partial t}\,\mathrm{d}x\mathrm{d}t + \lim_{\tau\to 0}\Big(\int_Q f\cdot\frac{\partial u}{\partial t}\,\mathrm{d}x\mathrm{d}t + \int_{\Sigma_{\mathrm{Neu}}} g\cdot\frac{\partial u}{\partial t}\,\mathrm{d}S\mathrm{d}t\Big)$$
$$= \int_Q \Big(\mathbb{C}(\zeta)e_{\mathrm{el}} + \mathbb{D}\frac{\partial e_{\mathrm{el}}}{\partial t}\Big):\frac{\partial \pi}{\partial t}\,\mathrm{d}x\mathrm{d}t$$

(because we already passed to the limit in the momentum equation)

4) limit passage in the discrete plastic flow rule (after by-part summation):

$$\int_Q \frac{\alpha}{2}|v|^2 + \delta^*_{S(\underline{\zeta}_\tau)}(v) - \Big(\mathbb{C}(\underline{\zeta}_\tau)\bar{e}_{\mathrm{el},\tau} + \mathbb{D}\frac{\partial e_{\mathrm{el},\tau}}{\partial t}\Big) : \Big(v - \frac{\partial \pi_\tau}{\partial t}\Big) + \kappa_1 \nabla\bar{\pi}_\tau \,\vdots\, \nabla v \,\mathrm{d}x\mathrm{d}t$$
$$+ \int_\Omega \frac{\kappa_1}{2}|\nabla\pi_0|^2 \mathrm{d}x \geq \int_Q \frac{\alpha}{2}\Big|\frac{\partial \pi_\tau}{\partial t}\Big|^2 + \delta^*_{S(\underline{\zeta}_\tau)}\Big(\frac{\partial \pi_\tau}{\partial t}\Big)\,\mathrm{d}x\mathrm{d}t + \int_\Omega \frac{\kappa_1}{2}|\nabla\pi_\tau(T)|^2\,\mathrm{d}x$$

is simply by weak (lower semi-)continuity. Therefore, in the limit we obtain

$$\int_Q \frac{\alpha}{2}|v|^2 + \delta^*_{S(\zeta)}(v) - \Big(\mathbb{C}(\zeta)e_{\mathrm{el}} + \mathbb{D}\frac{\partial e_{\mathrm{el}}}{\partial t}\Big) : \Big(v - \frac{\partial \pi}{\partial t}\Big) + \kappa_1 \nabla\pi \,\vdots\, \nabla v \,\mathrm{d}x\mathrm{d}t$$
$$+ \int_\Omega \frac{\kappa_1}{2}|\nabla\pi_0|^2 \mathrm{d}x \geq \int_Q \frac{\alpha}{2}\Big|\frac{\partial \pi}{\partial t}\Big|^2 + \delta^*_{S(\zeta)}\Big(\frac{\partial \pi}{\partial t}\Big)\,\mathrm{d}x\mathrm{d}t + \int_\Omega \frac{\kappa_1}{2}|\nabla\pi(T)|^2\mathrm{d}x$$

which is the weak formulation of the plastic flow rule we saw above.

5) limit passage in the discrete semi-stability (integrated over $[0,T]$):
$\forall 0\le\widetilde\zeta\le\zeta$ on Q with $\bar e_{\mathrm{el},\tau}=e(\bar u_\tau{+}\bar u_{\mathrm{Dir},\tau}){-}\bar\pi_\tau$:

$$\int_Q \frac12\mathbb{C}(\bar\zeta_\tau)\bar e_{\mathrm{el},\tau}:\bar e_{\mathrm{el},\tau}+\frac{\kappa_2}{r}|\nabla\bar\zeta_\tau|^r\,\mathrm{d}x\mathrm{d}t \\ \le\int_Q \frac12\mathbb{C}(\widetilde\zeta)\bar e_{\mathrm{el},\tau}:\bar e_{\mathrm{el},\tau}+\frac{\kappa_2}{r}|\nabla\widetilde\zeta|^r+a(\bar\pi_\tau)(\widetilde\zeta-\bar\zeta_\tau)\,\mathrm{d}x\mathrm{d}t$$

is simple since we have already proved the strong convergence of $\bar e_{\mathrm{el},\tau}$.
In the limit, after desintegration, we obtain for a.a. $t\in[0,T]$:

$$\int_\Omega \frac12\mathbb{C}(\zeta)e_{\mathrm{el}}(t):e_{\mathrm{el}}(t)+\frac{\kappa_2}{r}|\nabla\zeta(t)|^r\,\mathrm{d}x \\ \le\int_\Omega \frac12\mathbb{C}(\widetilde\zeta)e_{\mathrm{el}}(t):e_{\mathrm{el}}(t)+\frac{\kappa_2}{r}|\nabla\widetilde\zeta|^r+a(\pi(t))(\widetilde\zeta-\zeta(t))\,\mathrm{d}x,$$

which is the semi-stability we saw above (but here only for a.a. t).

6) limit passage in the energy equality:

$$\int_Q \alpha\Big|\frac{\partial\pi}{\partial t}\Big|^2 + \delta^*_{S(\zeta)}\Big(\frac{\partial\pi}{\partial t}\Big) + \mathbb{D}\frac{\partial e_{\rm el}}{\partial t}:\frac{\partial e_{\rm el}}{\partial t} + a'(\pi)\zeta\frac{\partial\pi}{\partial t}\,{\rm d}x{\rm d}t + \int_\Omega a(\pi(T))\zeta(T)\,{\rm d}x$$
$$+\mathcal{E}(T,u(T),\pi(T),\zeta(T))=\mathcal{E}(0,u_0,\pi_0,\zeta_0)+\int_0^T\mathcal{E}_t'(t,u(t),\pi(t),\zeta(t))\,{\rm d}t + \int_\Omega a(\pi_0)\zeta_0{\rm d}x.$$

relies on

1) the identity

$$\int_Q \Delta\pi:\frac{\partial\pi}{\partial t}\,{\rm d}x{\rm d}t = \frac{1}{2}\int_\Omega |\nabla\pi_0|^2 - |\nabla\pi(T)|^2,$$

which exploits here the regularity $\Delta\pi\in L^2(Q;\mathbb{R}^{d\times d}_{\rm dev})$ and can be proved either by a mollification in time by a time-difference technique (G. Grün, 1995) or in space (as used already in Part II but for ζ instead of π).

2) the Riemann-sum argument

Remark: $\mathbb{D} = 0$ possible

(C. Heinemann, C. Kraus, WIAS Preprint 2012)
(E.Bonetti, C.Heinemann, C.Kraus, A.Segatti, WIAS Preprint 2013)

then we would loose e.g. the estimate

$$\|\bar{e}_{\mathrm{el},\tau} - \underline{e}_{\mathrm{el},\tau}\|_{L^2(Q;\mathbb{R}^{d\times d})} \le \tau \|\tfrac{\partial}{\partial t} e_{\mathrm{el},\tau}\|_{L^2(Q;\mathbb{R}^{d\times d})} \to 0$$

but we did not need it anyhow

(as if we were consider e.g. $S = S(\zeta, \varepsilon)$
and used a semi-implicit discretisation)

One simplification: plasticity $\rightarrow$ creep (Maxwell rheology) $\Leftarrow$ $S(\zeta)\equiv\{0\}$
in combination with the Kelvin-Voigt rheology $\Rightarrow$ Jeffreys' rheology

---

One modification: strain controlled viscosity and plasticity
(instead of stress controlled)

Instead of the former governing equation/inclusions:

$$\operatorname{div}\sigma+g=0 \quad \text{with} \quad \sigma=\mathbb{C}(\zeta)e_{\rm el}+\mathbb{D}(\zeta)\frac{\partial e_{\rm el}}{\partial t}, \qquad \text{(momentum equilibrium)}$$

$$\alpha\frac{\partial\pi}{\partial t}+\partial\delta^*_{S(\zeta)}\Big(\frac{\partial\pi}{\partial t}\Big)\ni\operatorname{dev}\sigma+\kappa_1\Delta\pi \qquad \text{(plastic flow rule)}$$

$$\partial\delta^*_{[-a(\pi),\infty)}\Big(\frac{\partial\zeta}{\partial t}\Big)+\frac12\mathbb{C}'(\zeta)e_{\rm el}:e_{\rm el}+N_{[0,1]}(\zeta)\ni\kappa_2\operatorname{div}\big(|\nabla\zeta|^{r-2}\nabla\zeta\big) \quad \text{with} \quad e_{\rm el}=e(u)-\pi \qquad \text{(damage flow rule)}$$

The modified governing equation/inclusions read as:

$$\operatorname{div}\Big(\mathbb{C}(\zeta)(e(u)-\pi)+\mathbb{D}(\zeta)e\Big(\frac{\partial u}{\partial t}\Big)\Big)+g=0, \qquad \text{(momentum equilibrium)}$$

$$\alpha\frac{\partial\pi}{\partial t}+\partial\delta^*_{S(\zeta)}\Big(\frac{\partial\pi}{\partial t}\Big)\ni\operatorname{dev}\big(\mathbb{C}(\zeta)(e(u)-\pi)\big)+\kappa_1\Delta\pi \qquad \text{(plastic flow rule)}$$

$$\partial\delta^*_{[-a(\pi),\infty)}\Big(\frac{\partial\zeta}{\partial t}\Big)+\frac12\mathbb{C}'(\zeta)(e(u)-\pi):(e(u)-\pi)+N_{[0,1]}(\zeta)\ni\kappa_2\operatorname{div}\big(|\nabla\zeta|^{r-2}\nabla\zeta\big) \qquad \text{(damage flow rule)}$$

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$$\partial\delta^*_{[-a(\pi),\infty)}\Big(\frac{\partial\zeta}{\partial t}\Big) + \frac{1}{2}\mathbb{C}'(\zeta)e_{\rm el} : e_{\rm el} + N_{[0,1]}(\zeta) \ni \kappa_2\operatorname{div}\big(|\nabla\zeta|^{r-2}\nabla\zeta\big) \quad \text{with} \quad e_{\rm el} = e(u){-}\pi \qquad \text{(damage flow rule)}$$

The modified governing equation/inclusions read as:

$$\operatorname{div}\Big(\mathbb{C}(\zeta)(e(u){-}\pi) + \mathbb{D}(\zeta)e\Big(\frac{\partial u}{\partial t}\Big)\Big) + g = 0, \qquad \text{(momentum equilibrium)}$$

$$\alpha\frac{\partial\pi}{\partial t} + \partial\delta^*_{S(\zeta)}\Big(\frac{\partial\pi}{\partial t}\Big) \ni \operatorname{dev}\big(\mathbb{C}(\zeta)(e(u){-}\pi)\big) + \kappa_1\Delta\pi \qquad \text{(plastic flow rule)}$$

$$\partial\delta^*_{[-a(\pi),\infty)}\Big(\frac{\partial\zeta}{\partial t}\Big) + \frac{1}{2}\mathbb{C}'(\zeta)(e(u){-}\pi) : (e(u){-}\pi) + N_{[0,1]}(\zeta) \ni \kappa_2\operatorname{div}\big(|\nabla\zeta|^{r-2}\nabla\zeta\big) \qquad \text{(damage flow rule)}$$

$$\mathcal{R}\Big(\pi,\zeta;\frac{\mathrm{d}u}{\mathrm{d}t},\frac{\mathrm{d}\pi}{\mathrm{d}t},\frac{\mathrm{d}\zeta}{\mathrm{d}t}\Big) := \begin{cases} \displaystyle\int_\Omega \alpha\Big|\frac{\partial\pi}{\partial t}\Big|^2 + \delta^*_{S(\zeta)}\Big(\frac{\partial\pi}{\partial t}\Big) \\ \displaystyle\quad + a(\pi)\Big|\frac{\partial\zeta}{\partial t}\Big| + \frac{1}{2}\mathbb{D}e\big(\frac{\partial u}{\partial t}\big) : e\big(\frac{\partial u}{\partial t}\big)\,\mathrm{d}x & \text{if } \displaystyle\frac{\partial\zeta}{\partial t}\le 0 \text{ a.e. on } \Omega, \\ \infty & \text{otherwise.} \end{cases}$$

or

$$\mathcal{R}\Big(\pi,\zeta;\frac{\mathrm{d}u}{\mathrm{d}t},\frac{\mathrm{d}\pi}{\mathrm{d}t},\frac{\mathrm{d}\zeta}{\mathrm{d}t}\Big) := \begin{cases} \displaystyle\int_\Omega \alpha\Big|\frac{\partial\pi}{\partial t}\Big|^2 + \delta^*_{S(\zeta)}\Big(\frac{\partial\pi}{\partial t}\Big) \\ \displaystyle\quad + a(\pi)\Big|\frac{\partial\zeta}{\partial t}\Big| + \frac{1}{2}\mathbb{D}\frac{\partial e_{\rm el}}{\partial t} : \frac{\partial e_{\rm el}}{\partial t}\,\mathrm{d}x & \text{if } \displaystyle\frac{\partial\zeta}{\partial t}\le 0 \text{ a.e. on } \Omega, \\ \infty & \text{otherwise.} \end{cases}$$

Kelvin-Voigt rheology
in combination with the Maxwell rheology $\Rightarrow$ Jeffreys' rheology

Under cyclical loading, the damage threshold (and possibly also the yield stress) is to depend not on π but rather on the number of cycles – the total dissipated energy at a current spot. This models the phenomenon of a fatique:

$$a = a(d) \qquad \text{with} \qquad d(t,x) = \int_0^t \delta^*_{S(\zeta(t',x))}\Big(\frac{\partial \pi}{\partial t}(t',x)\Big)\,\mathrm{d}t'.$$

An example of a rate-independent relation $\frac{\partial d}{\partial t} = \delta^*_{S(\zeta)}\big(\frac{\partial \pi}{\partial t}\big)$ which is not in the Biot-equation form.

If gradient viscosity of the type $\alpha\nabla\frac{\mathrm{d}\pi}{\mathrm{d}t}$ is considered, then compactness in d by the Aubin-Lions theorem: if $S(\zeta) = \{\sigma;\ |\sigma| \le \sigma_{\rm Y}(\zeta)\}$,

$$\nabla d = \int_0^t \nabla\Big(\sigma_{\rm Y}(\zeta)\Big|\frac{\partial \pi}{\partial t}\Big|\Big)\,\mathrm{d}t = \int_0^t \sigma'_{\rm Y}(\zeta)\nabla\zeta\Big|\frac{\partial \pi}{\partial t}\Big| + \sigma_{\rm Y}(\zeta){\rm Dir}\Big(\frac{\partial \pi}{\partial t}\Big)\nabla\frac{\partial \pi}{\partial t}\,\mathrm{d}t.$$

Then $\nabla\zeta \in L^\infty(L^2)$ and $\frac{\partial \pi}{\partial t} \in L^2(L^6)$ and $\nabla\frac{\partial \pi}{\partial t} \in L^2(L^2)$ implies $\nabla d \in L^\infty(L^3)$. Thus strong convergence of approximate d's in $L^1(L^1)$ follows.

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A general thermodynamics:

$\psi = \psi(u, z, \theta)$ a specific free energy
$s = -\psi'_\theta(u, z, \theta)$ a specific enthalpy
$w(u, \theta, s) := \psi(u, \theta) + \theta s$ a specific internal energy (GIBBS' relation)
The entropy equation reads as

$$\theta \frac{\partial s}{\partial t} + \operatorname{div} j = r \qquad \text{with } \overset{\text{heat flux}}{j} = -K(u, z, \theta)\nabla\theta \qquad \leftarrow \text{FOURIER's law}$$

where $K = K(u, z, \theta)$ is a heat-transfer coefficient (matrix),
and r the dissipation (i.e. heat production) rate.

Differentiating $s := -\psi'_\theta(u, z, \theta)$ in time $\Rightarrow$

$$\frac{\partial s}{\partial t} = -\psi''_{\theta u}(u, z, \theta)\frac{\partial u}{\partial t} - \psi''_{\theta z}(u, z, \theta)\frac{\partial z}{\partial t} - \psi''_{\theta\theta}(u, z, \theta)\frac{\partial \theta}{\partial t}.$$

the specific heat capacity $c_{\rm v} = c_{\rm v}(u, z, \theta) := -\theta s'_\theta(u, z, \theta) = -\theta\psi''_{\theta\theta}(u, z, \theta)$.
The heat-transfer equation

$$c_{\rm v}(u, z, \theta)\frac{\partial \theta}{\partial t} - \operatorname{div}\big(K(u, z, \theta)\nabla\theta\big) = r + \theta\psi''_{\theta u}(u, z, \theta)\frac{\partial u}{\partial t} + \theta\psi''_{\theta z}(u, z, \theta)\frac{\partial z}{\partial t}.$$

An enthalpy-like transformation (assuming, for simplicity, $c_{\rm v} = c_{\rm v}(u,\theta)$ only):

$$C_{\rm v}(u,\theta) := \int_0^1 \theta c_{\rm v}(u,t\theta)\,{\rm d}t.$$

We use the calculus:

$$\frac{\partial}{\partial t}\big[C_{\rm v}(u,\theta)\big] = \int_0^1 c_{\rm v}(u,t\theta)\frac{\partial\theta}{\partial t} + \theta[c_{\rm v}]'_\theta(u,t\theta)t\frac{\partial\theta}{\partial t} + \theta[c_{\rm v}]'_u(u,t\theta)\frac{\partial u}{\partial t}\,{\rm d}t$$
$$= \frac{\partial\theta}{\partial t}\Big(\int_0^1 c_{\rm v}(u,t\theta) + [c_{\rm v}]'_\theta(u,t\theta)t\theta\,{\rm d}t\Big) + \Big(\int_0^1 \theta[c_{\rm v}]'_u(u,t\theta)\,{\rm d}t\Big)\frac{\partial u}{\partial t}$$
$$= \frac{\partial\theta}{\partial t}\int_0^1 \frac{\rm d}{{\rm d}t}\big(c_{\rm v}(u,t\theta)t\big)\,{\rm d}t + [C_{\rm v}]'_u(u,\theta)\frac{\partial u}{\partial t} = \frac{\partial\theta}{\partial t}c_{\rm v}(u,\theta) + [C_{\rm v}]'_u(u,\theta)\frac{\partial u}{\partial t}$$

with $[C_{\rm v}]'_u(u,\theta) = \int_0^1 \theta[c_{\rm v}]'_u(u,t\theta)\,{\rm d}t$.

The heat-transfer equation:

$$\frac{\partial\vartheta}{\partial t} - {\rm div}\big(K(u,z,\theta)\nabla\theta\big) = r + \Big(\theta\psi''_{\theta u}(u,\theta) + [C_{\rm v}]'_u(u,\theta)\Big)\frac{\partial u}{\partial t}$$

needs to be controlled

together with $\vartheta = C_{\rm v}(u,\theta)$.

A more general situation $c_{\rm v} = c_{\rm v}(u,z,\theta)$:

$$\frac{\partial\vartheta}{\partial t} - {\rm div}\big(K(u,z,\theta)\nabla\theta\big) = r + \Big(\theta\psi''_{\theta u}(u,z,\theta) + [C_{\rm v}]'_u(u,z,\theta)\Big)\frac{\partial u}{\partial t}$$
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Stored energy used in Part I (here for simplicity without isotropic hardening):

$$E(t,u,\pi,\theta)=\frac{1}{2}\mathbb{C}(e(u)-\pi-\mathbb{E}\theta){:}(e(u)-\pi-\mathbb{E}\theta)+\frac{1}{2}\mathbb{H}\pi{:}\pi-g(t)\cdot u.$$

An important feature in isotropic (e.g. polycrystalic) materials: $\mathrm{dev}\,\mathbb{E}=0$.

Thermodynamics of the plasticity with hardening:

$u=$ displacement,

$z=(\pi,\eta)=$ the plastic deformation and the hardening parameter,

viscosity, inertia, thermal expansion, $\theta=$ temperature, heat equation

$$\varrho\frac{\partial^2 u}{\partial t^2}-\mathrm{div}\Big(\mathbb{D}\frac{\partial e(u)}{\partial t}\Big)-\mathrm{div}\big(\mathbb{C}(e(u)-\pi-\mathbb{E}\theta)\big)=f,$$

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Analysis bears several peculiarities:

- Fully implicit time discretization does not yield an incremental problem with a variational structure (existence by Schauder fixed point only, calculation by Newton iterative method converged)
- energetic-solution concept important
  (weak convergence of the dissipative heat source)
- fine a-priori estimates: test the force equilibrium by $\frac{\partial u}{\partial t}$,
  test the flow rule $\frac{\partial \pi}{\partial t}$, $\frac{\partial \eta}{\partial t}$,
  test the heat equation by 1
- positivity of temperature,
- test the heat equation by $1-1/(1+\theta)^\epsilon$, $\epsilon>0$,
  L^1-theory for heat equation (Boccardo, Galouët, et al.) and interpolation of the adiabatic-heat term (Gagliardo,Nirenberg): $\nabla\theta\in L^{5/4-\epsilon}$
- numerics: FEM discretization, regularization, subsequent convergence (positivity of temperature likely difficult even on accute meshes).

T.R. (in SIAM J.Math.Anal. 2010), numerics S.Bartels+T.R. (in M2AN 2011)

Some left aspects:

anisothermal models with diffusion or
dynamical models –elastic waves
(some are T.R. & G.Tomassetti, arXiv no.1412.4949)

Some open problems:

Again complete damage does not seem to be investigated with visco-plasticity.

convergence if damageable viscosity, i.e. $\mathbb{D} = \mathbb{D}(\zeta)$

Some references:

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More on: www.karlin.mff.cuni.cz/~roubicek/trpublic.htm
or: https://www.researchgate.net/profile/Tomas_Roubicek2

Thanks a lot for your attention.

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Vielen Dank für Ihre Aufmerksamkeit.