Exercise Sheet 1

Discussion on 31.10.2016

Exercise 1 (Errors of difference quotients)
Let $J \in \mathbb{N}$, $\Delta x := 1/J$ and $x_j := j\Delta x$ for $j = 0,\dots,J$. Show that $u \in C^4([0,1])$ and any suitable $j \in \{0,\dots,J\}$ satisfies

$$|\partial^{\pm} u(x_j) - u'(x_j)| \le \frac{\Delta x}{2} \|u''\|_{C([0,1])}$$
$$|\hat{\partial} u(x_j) - u'(x_j)| \le \frac{(\Delta x)^2}{6} \|u'''\|_{C([0,1])}$$
$$|\partial^{+}\partial^{-} u(x_j) - u''(x_j)| \le \frac{(\Delta x)^2}{12} \|u^{(4)}\|_{C([0,1])}.$$

Exercise 2 (Error estimate for implicit Euler scheme)
Additionally to the notation of Exercise 1.1, let $T > 0$, $K \in \mathbb{N}$, $\Delta t := T/K$ and $t_k := k\Delta t$ for $k = 0,\dots,K$. Prove that $u \in C^4([0,T]\times[0,1])$, any $k = 0,\dots,K$ and the $(U_j^k)_{jk}$ from the implicit Euler scheme satisfy

$$\sup_{j=0,\dots,J} |u(t_k,x_j) - U_j^k| \le \frac{t_k}{2}(\Delta t + (\Delta x)^2)\big(\|\partial_x^4 u\|_{C([0,T]\times[0,1])} + \|\partial_t^2 u\|_{C([0,T]\times[0,1])}\big).$$

Exercise 3 (Integration by parts)
a) Prove that $\Delta x > 0$, $(V_j)_{j=0,\dots,J} \in \mathbb{R}^{J+1}$, and $(W_j)_{j=0,\dots,J} \in \mathbb{R}^{J+1}$ with $V_0 = V_J = W_0 = W_J = 0$ satisfy

$$\sum_{j=1}^{J-1} \Delta x \left(\frac{V_{j+1} - 2V_j + V_{j-1}}{(\Delta x)^2}\right) W_j = -\sum_{j=0}^{J-1} \Delta x \left(\frac{V_{j+1} - V_j}{\Delta x}\right)\left(\frac{W_{j+1} - W_j}{\Delta x}\right).$$

b) Let $\Omega \subset \mathbb{R}^n$, $n = 1,2,3$ be a bounded domain with piecewise smooth boundary $\partial\Omega$, where $\operatorname{div} = \nabla$ for $n = 1$, and outward normal ν along $\partial\Omega$. For $v \in C^1(\overline{\Omega})$, $q \in C^1(\overline{\Omega};\mathbb{R}^n)$, show

$$\int_\Omega (v \operatorname{div} q + \nabla v \cdot q)\,\mathrm{d}x = \int_{\partial\Omega} v q \cdot \nu \,\mathrm{d}s.$$

Hint: You may assume that Gauss's divergence theorem holds for bounded domains with piecewise smooth boundary.

Exercise 4 (Numerical comparison of Euler schemes)
Implement and compare the explicit and implicit Euler scheme for the heat equation (functions `explicit_euler` and `implicit_euler` from the book of S. Bartels, available at `http://extras.springer.com/2016/978-3-319-32353-4`). Study the influence of the parameters Δt and Δx and determine cases where one of the schemes is favorable to the other. How can this be explained?