Exercise Sheet 10

Discussion on 23.01.2017

Exercise 1 (Strang-Fix lemma)
Given a Hilbert space V, let $(V_h \,|\, h > 0)$ denote a family of finite dimensional subspaces of V. Recall the norm $\|\bullet\|_{V_h^*}$ on the dual spaces V_h^* of V_h, for $G \in V_h^*$, with

$$\|G\|_{V_h^*} = \sup_{v_h \in V_h \setminus \{0\}} \frac{G(v_h)}{\|v_h\|_V}.$$

Assume that

- $a : V \times V \to \mathbb{R}$ is a bounded bilinear form (i.e., there exists $M > 0$ such that, for all $u, v \in V$, $a(u,v) \le M\|u\|_V\|v\|_V$),
- $a_h : V_h \times V_h \to \mathbb{R}$ are uniformly coercive bilinear forms (i.e., there exists $\alpha > 0$ such that, for all $h > 0$ and $v_h \in V_h$, $a_h(v_h, v_h) \ge \alpha\|v_h\|_V^2$),
- $F \in V^*$,
- $F_h \in V_h^*$,
- $u \in V$ solves $a(u, \bullet) = F$ in V,
- $u_h \in V_h$ solves $a_h(u_h, \bullet) = F_h$ in V_h.

Prove that

$$\|u - u_h\|_V \le \inf_{v_h \in V_h} \Big((1 + M/\alpha)\|u - v_h\|_V + 1/\alpha\, \|a(v_h, \bullet) - a_h(v_h, \bullet)\|_{V_h^*} + 1/\alpha\, \|F - F_h\|_{V_h^*} \Big).$$

Hint: For any $v_h \in V_h$, consider $w_h := u_h - v_h \in V_h$.

Exercise 2 (Crouzeix-Raviart edge-oriented basis functions)
On a triangle $T = \operatorname{conv}(E, P)$ with edge E and opposite node P, let φ_P denote the nodal P_1 basis function associated with P and ψ_E the edge-oriented CR^1 basis function associated with E. Sketch the basis functions φ_P and ψ_E and give an explicit formula of ψ_E (resp. $\nabla\psi_E$) in terms of φ_P (resp. $\nabla\varphi_P$).

Exercise 3 (Integral mean property)
Recall the Crouzeix-Raviart interpolation operator $I_{\mathrm{CR}} : H^1(\Omega) \to CR^1(\mathcal{T})$ with $I_{\mathrm{CR}} v := \sum_{E \in \mathcal{E}} \left(⨍_E v \,\mathrm{d}s \right) \psi_E$. For any $v \in H^1(\Omega)$, prove that $\nabla_{\mathrm{NC}}(I_{\mathrm{CR}} v) = \Pi_0 \nabla v$.

Exercise 4 (CR-FEM implementation)
Recall the `FEM10.m` Matlab function.

```
function [x,A,nrDoFs] = FEM10(c4n,n4e,n4sDb,f)
    N=size(c4n,1); d=size(c4n,2);
    A=sparse(N,N); b=zeros(N,1); x=zeros(N,1);
    for j=1:size(n4e,1)
      area=abs(det([ones(1,d+1);c4n(n4e(j,:),:)']))/factorial(d));
      grads=[ones(1,d+1);c4n(n4e(j,:),:)']\[zeros(1,d);eye(d)];
      A(n4e(j,:),n4e(j,:))=A(n4e(j,:),n4e(j,:))+area*(grads*grads');
      b(n4e(j,:))=b(n4e(j,:))+f(sum(c4n(n4e(j,:),:))/(d+1))*ones(d+1,1)*area/(d+1);
    end
    freeNodes=setdiff(1:N,unique(n4sDb)); nrDoFs=length(freeNodes);
    x(freeNodes)=A(freeNodes,freeNodes)\b(freeNodes);
end
```

Modify the `FEM10.m` function to solve the Crouzeix-Raviart FEM for the PMP with $f \in L^2(\Omega)$, which seeks $u_{\mathrm{CR}} \in CR^1_0(\mathcal{T})$ such that, for all $v_{\mathrm{CR}} \in CR^1_0(\mathcal{T})$,

$$\int_\Omega \nabla_{\mathrm{NC}} u_{\mathrm{CR}} \cdot \nabla_{\mathrm{NC}} v_{\mathrm{CR}} \,\mathrm{d}x = \int_\Omega f\, v_{\mathrm{CR}} \,\mathrm{d}x.$$

Hints:

1. For simplicity, assume that $d = 2$ and $f \equiv 1$.
2. In contrast to the nodal P_1 basis functions, the CR basis functions are edge-oriented, and thus the degrees of freedom are associated with the edges in the triangulation.
3. Use your code from Exercise 4 on Sheet 9 to compute the `s4e` array which provides the global numbers of the three edges of each triangle. The numbers of the boundary sides are necessary to compute the degrees of freedom in `freeSides`.
4. Use the formula from Exercise 2 of this sheet to compute the gradients of the CR basis functions.
5. Use `plotCR` from the homepage to confirm your program.