Exercise Sheet 11

Discussion on 30.01.2017

Exercise 1 (Euler formulas)
Let $\mathcal{T}$ denote a regular triangulation of the simply connected bounded domain Ω with nodes $\mathcal{N}$, edges $\mathcal{E}$ and interior edges $\mathcal{E}(\Omega)$. Prove that

(a) $|\mathcal{N}| + |\mathcal{T}| = 1 + |\mathcal{E}|$,

(b) $2|\mathcal{T}| + 1 = |\mathcal{N}| + |\mathcal{E}(\Omega)|$.

How can these formulas be generalized for multiply connected domains?

Exercise 2 (Divergence free Crouzeix-Raviart basis functions)
Let $\mathcal{T}$ denote a regular triangulation of the bounded Lipschitz domain $\Omega \subset \mathbb{R}^2$. Specify a basis of the space Z_{CR} of the divergence free Crouzeix-Raviart functions with vanishing boundary data, i.e.,

$$Z_{\mathrm{CR}} := \{v_{\mathrm{CR}} \in CR_0^1(\mathcal{T};\mathbb{R}^2) \mid \forall T \in \mathcal{T}, \operatorname{div} v_{\mathrm{CR}}|_T = 0\}.$$

Hint: Find a set of divergence free linearly independent functions an use a dimension argument. For simplicity, assume that Ω is simply connected.

Exercise 3 (Comparison results)
Let $\mathcal{T}$ denote a regular triangulation of the polygonal Lipschitz domain $\Omega \subset \mathbb{R}^2$ with mesh size function $h \in P_0(\mathcal{T})$, i.e., $h|_T := \operatorname{diam}(T)$ for all $T \in \mathcal{T}$. For the piecewise constant right-hand side function $f \in P_0(\mathcal{T}) \subset L^2(\Omega)$, let $u \in H_0^1(\Omega)$ solve $-\Delta u = f$ and $u_{\mathrm{C}} \in S_0^1(\mathcal{T})$, $u_{\mathrm{CR}} \in CR_0^1(\mathcal{T})$, and $p_{\mathrm{RT}} \in RT_0(\mathcal{T})$ the corresponding solutions of the conforming P_1-FEM, the non-conforming Crouzeix-Raviart-FEM, and the mixed Raviart-Thomas-FEM.

(a) Prove with Proposition III.17 and Proposition IV.1 that

$$\|\nabla(u - u_{\mathrm{C}})\|_{L^2(\Omega)} \lesssim \|\nabla_{\mathrm{NC}}(u - u_{\mathrm{CR}})\|_{L^2(\Omega)}.$$

(b) Prove with the Marini identity that

$$\|\nabla_{\mathrm{NC}}(u - u_{\mathrm{CR}})\|_{L^2(\Omega)} \lesssim \|\nabla u - p_{\mathrm{RT}}\|_{L^2(\Omega)} + \|hf\|_{L^2(\Omega)}.$$

(c) Prove that

$$\|\nabla u - p_{\mathrm{RT}}\|_{L^2(\Omega)} \lesssim \|\nabla_{\mathrm{NC}}(u - u_{\mathrm{CR}})\|_{L^2(\Omega)}.$$

(d) Prove that

$$\|\nabla_{\mathrm{NC}}(u - u_{\mathrm{CR}})\|_{L^2(\Omega)} \lesssim \|\nabla(u - u_{\mathrm{C}})\|_{L^2(\Omega)}$$

Exercise 4 (Numerical experiments for comparison results)

(a) Let $(\Omega_j \mid j \in \mathbb{N})$ denote a sequence of two-dimensional regular polygons with 2^j edges and boundary nodes $\mathcal{N}(\partial\Omega) \subset B_1(0)$ on the unit sphere. Write a Matlab function that computes the geometry `c4n`, `n4e`, and `n4sDb` of this domain for arbitrary $j \in \mathbb{N}$. Use a projection of the boundary nodes on the unit sphere after each uniform red-refinement.

(b) Solve $-\Delta u = 1$ on the polygonal domains Ω_j, $j = 1, 2, \ldots$ from exercise 4 (a) with the non-conforming Crouzeix-Raviart-FEM and the mixed Raviart-Thomas-FEM. Create the convergence history plots using the error estimators `estimateCREtaSides` and `estimateRT0EtaSides` from the AFEM package.