

Exercise Sheet 12

Discussion on 06.02.2017

Exercise 1 (Korn inequality)

Let $\Omega \subseteq \mathbb{R}^n$ denote a Lipschitz domain. Recall the definition of the symmetric gradient $\varepsilon(v) := (Dv + Dv^\top)/2$ for all $v \in H^1(\Omega; \mathbb{R}^n)$.

- (a) Use integration by parts to prove for all $v \in \mathcal{D}(\Omega; \mathbb{R}^n) \equiv C_c^\infty(\Omega; \mathbb{R}^n)$, that

$$\|Dv\|_{L^2(\Omega)} \lesssim \|\varepsilon(v)\|_{L^2(\Omega)}. \quad (1)$$

- (b) Use a density argument to prove equation (1) for all $v \in H_0^1(\Omega; \mathbb{R}^n)$.

Exercise 2 (Variational formulation of linear elasticity)

Given a right-hand side f and Lamé parameters $\lambda, \mu > 0$, recall that the PDE of linear elasticity seek $u \in C^2(\Omega; \mathbb{R}^n)$ with

$$-\operatorname{div}(\mathbb{C}\varepsilon(u)) = f \text{ in } \Omega \quad \text{and} \quad u \equiv 0 \text{ on } \partial\Omega \quad (2)$$

and the material tensor $\mathbb{C}M := 2\mu M + \lambda \operatorname{tr}(M)I_{n \times n}$ for $M \in \mathbb{R}^{n \times n}$.

- (a) Derive the variational formulation (also: weak formulation) of (2).

- (b) Prove existence and uniqueness of solutions of the variational formulation.

Hint: Assume that the space $H_0^1(\Omega; \mathbb{R}^n)$ of the deformations u is equipped with the usual H^1 energy-norm.

Exercise 3 (Conforming P_1 discretization)

Let \mathcal{T} denote a regular triangulation of the domain Ω . Consider the conforming P_1 discretization $u_h \in S_0^1(\mathcal{T}; \mathbb{R}^n)$ of the deformation $u \in H_0^1(\Omega; \mathbb{R}^n)$ in the variational formulation of Exercise 2 (a). Use Céa's Lemma to prove a best-approximation result. How does the generic constant in the estimate depend on the Lamé parameter λ ?

Exercise 4 (Numerical experiments – Locking phenomenon)

Use the `solveP1P1LinElast.m` Matlab function to solve the problem from Exercise 3 on the unit square `unitSquareData.m`. Compute the exact error `error4eP1Energy.m` (for both components) and compare the values for increasing Lamé parameter $\lambda \rightarrow \infty$. The input data and all necessary functions can be downloaded on the homepage of the lecture.