Exercise Sheet 13

Discussion on 13.02.2017

Exercise 1 (Arbitrarily bad convergence for uniform refinements)
Let $\mathcal{T}_\ell, \ell \in \mathbb{N}$, denote a sequence of consecutive red-refinements of the initial regular triangulation $\mathcal{T}_0$ of Ω. Given an arbitrary monotonically decreasing sequence $\varepsilon_k, k \in \mathbb{N}$, show that there exists a right-hand side $f \in H^{-1}(\Omega) := (H_0^1(\Omega))^*$ such that the exact weak solution $u \in H_0^1(\Omega)$ to the Poisson model problem $-\Delta u = f$ and the corresponding finite element approximations $u_\ell \in S_0^1(\mathcal{T}_\ell)$ satisfy

$$\|\nabla(u - u_\ell)\|_{L^2(\Omega)} = \varepsilon_\ell.$$

Exercise 2 (Convergence for any refinement)
Let Ω denote a polygonal Lipschitz domain, $f \in L^2(\Omega)$, $a : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ a scalar product and $\mathcal{T}_0$ an initial regular triangulation of Ω. Let $\mathcal{T}_\ell, \ell \in \mathbb{N}$, be a sequence of arbitrary consecutive refinements, i.e., $\mathcal{T}_\ell$ is (arbitrary) refinement of $\mathcal{T}_{\ell-1}$, with associated finite element solutions $u_\ell \in S_0^1(\mathcal{T}_\ell) \subset H_0^1(\Omega)$ with

$$a(u_\ell, v) = \int_\Omega f\, v\, dx \qquad \text{for all } v \in S_0^1(\mathcal{T}_\ell).$$

(a) Prove that the sequence $u_\ell, \ell \in \mathbb{N}$, converges to a function $u_\infty \in H_0^1(\Omega)$.

(b) Does the limit u_∞ solve the weak formulation

$$a(u, v) = \int_\Omega f\, v\, dx \qquad \text{for all } v \in H_0^1(\Omega)\,? \tag{1}$$

(c) State a sufficient criterion on the sequence $\mathcal{T}_\ell, \ell \in \mathbb{N}$, such that $u_\infty = u$ holds with the solution $u \in H_0^1(\Omega)$ to the weak formulation (1).

Exercise 3 (Two-energy principle)
Let $\mathcal{T}$ denote a regular triangulation of the polygonal Lipschitz domain $\Omega \subset \mathbb{R}^2$. Define the midpoint function mid $\in P_0(\mathcal{T};\mathbb{R})$ by $\mathrm{mid}(\mathcal{T})|_T := \mathrm{mid}(T) := \sum_{z \in \mathcal{N}(T)} z/3 \in \mathbb{R}^2$. Let $u_\mathrm{C} \in S_0^1(\mathcal{T})$ solve the conforming P_1-FEM and $p_\mathrm{RT} \in RT_0(\mathcal{T})$ the mixed Raviart-Thomas-FEM of the Poisson model problem with right-hand side $f \in L^2(\Omega)$.

(a) For any $T \in \mathcal{T}$, prove that

$$p_\mathrm{RT}|_T = \Pi_0 p_\mathrm{RT} + c_T(\bullet - \mathrm{mid}(T)) \quad \text{with } c_T = \operatorname{div} p_\mathrm{RT}$$

and that $(\bullet - \mathrm{mid}(\mathcal{T}))$ is $L^2(\Omega;\mathbb{R}^2)$-orthogonal on $P_0(\mathcal{T};\mathbb{R}^2)$.

(b) Compute $\| \bullet - \mathrm{mid}(T) \|_{L^2(T)}$ in terms of $|T|$ and $\mathcal{N}(T) = \{P_1, P_2, P_3\}$.

(c) Prove that

$$\| p_{\mathrm{RT}} - \nabla u_{\mathrm{C}} \|^2_{L^2(\Omega)} = \| \Pi_0 p_{\mathrm{RT}} - \nabla u_{\mathrm{C}} \|^2_{L^2(\Omega)} + \sum_{T \in \mathcal{T}} c_T^2 \| \bullet - \mathrm{mid}(T) \|^2_{L^2(T)}.$$

Exercise 4 (Two-energy principle – Numerical experiment)
Use Exercise 3 (b)–(c) and the functions from the AFEM package to compute the a posteriori error estimator $\mu(\mathcal{T}) := \left(\sum_{T \in \mathcal{T}} \mu^2(\mathcal{T}, T) \right)^{1/2}$ with

$$\mu^2(\mathcal{T}, T) := \| p_{\mathrm{RT}} - \nabla u_{\mathrm{C}} \|^2_{L^2(T)}.$$

Compare the values of this estimator with the usual edge-oriented error estimator $\eta(\mathcal{T}) := \left(\sum_{E \in \mathcal{E}} \eta(\mathcal{T}, E) \right)^{1/2}$ from `estimateRT0EtaSides` in one convergence history plot.

Hint: The function `solveRT0Poisson` returns an array `p` which contains the three coefficients with respect to the local Raviart-Thomas basis functions

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \text{and} \quad x - \mathrm{mid}(T)$$

on each triangle $T \in \mathcal{T}$.