# Exercise Sheet 2

Discussion on 07.11.2016

**Exercise 1 (Discrete inverse inequality)**
Prove that $\Delta x > 0$ and $(V_j)_{j=0,\dots,J} \in \mathbb{R}^{J+1}$ with $V_0 = V_J = 0$ satisfy

$$\sum_{j=0}^{J-1} \Delta x \left( \frac{V_{j+1} - V_j}{\Delta x} \right)^2 \le \frac{4}{(\Delta x)^2} \sum_{j=0}^{J} \Delta x \, V_j^2.$$

**Exercise 2 (Discrete version of Friedrichs inequality)**
**a)** Let $J \in \mathbb{N}$, $J \ge 2$ and $A \in \mathbb{R}^{(J-1)\times(J-1)}$ given by

$$A = \begin{pmatrix} 2 & -1 & & 0 \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ 0 & & -1 & 2 \end{pmatrix} \in \mathbb{R}^{(J-1)\times(J-1)}.$$

Prove that for any $k = 1,\dots,J-1$, the vector $x^k \in \mathbb{R}^{J-1}$ with components $x_j^k = \sin(kj\pi/J)$ is an eigenvector of $A$ with eigenvalue $\lambda_k := 2(1-\cos(k\pi/J)) > 0$.
**b)** Show that $\pi^2/(2J^2) \le \lambda_1$.
**c)** Use a), b) and the estimate $\lambda_{\min}(A)|v|^2 \le v^\top A v$ for any $v \in \mathbb{R}^{J-1}$ to prove that there exists $C > 0$ such that any $J \in \mathbb{N}$, $\Delta x := 1/J$ and any $(V_j)_{j=0,\dots,J} \in \mathbb{R}^{J+1}$ with $V_0 = V_J = 0$ satisfy

$$\sum_{j=0}^{J-1} \Delta x \, V_j^2 \le C \sum_{j=0}^{J} \Delta x \left( \frac{V_{j+1} - V_j}{\Delta x} \right)^2.$$

**Exercise 3 (Conservation of Energy)**
Prove that for the exact solution $u \in C^2([0,T]\times[0,1])$ to the wave equation, the energy

$$\frac{1}{2}\int_0^1 (\partial_t u(t,x))^2 + c^2(\partial_x u(t,x))^2 \,\mathrm{d}x$$

is constant in $t \in [0,T]$.

**Exercise 4 (Conservation of energy in a naive implicit scheme)**
Implement the implicit scheme

$$\partial_t^+ \partial_t^- U_j^k - c^2 \partial_x^+ \partial_x^- U_j^{k+1} = 0$$

in Matlab, possibly by modification of the function `wave_explicit.m` from the book of S. Bartels (available at `http://extras.springer.com/2016/978-3-319-32353-4`). Illustrate by numerical experiments and explain why the explicit scheme is preferable.