Exercise Sheet 3

Discussion on 14.11.2016

Exercise 1 (Regularity of solutions)
Let $\gamma \in (\pi, 2\pi)$ and $\Omega := \{(r\cos\varphi, r\sin\varphi) \mid 0 < r < 1, 0 < \varphi < \gamma\}$ and $u(r,\varphi) = r^{\pi/\gamma}\sin(\varphi\pi/\gamma)$ in polar coordinates on Ω. Prove that $u : \Omega \to \mathbb{R}$ solves the Poisson model problem on Ω with right-hand side $f \equiv 0$ and respective boundary data but $u \notin C^1(\overline{\Omega})$ for $\gamma \in (\pi, 2\pi)$.
Hint: You may use the formula for the Laplacian in polar coordinates,

$$\Delta u = \partial^2 u/\partial r^2 + r^{-1}\partial u/\partial r + r^{-2}\partial^2 u/\partial\varphi^2.$$

Exercise 2 (Weak derivatives are necessary)
Let $\Omega = (-1,1)^n \subseteq \mathbb{R}^n$ and $v_\varepsilon, \operatorname{sgn} : \overline{\Omega} \to \mathbb{R}$ with $v_\varepsilon(x) := \sqrt{|x|^2 + \varepsilon^2} - \varepsilon$ and

$$\operatorname{sgn}(x) := \begin{cases} \frac{x}{|x|} & \text{for } x \neq 0, \\ 0 & \text{for } x = 0. \end{cases}$$

Prove that v_ε and its gradient converge to $v = |\bullet|$ and sgn in $L^2(\Omega)$ for $\varepsilon \searrow 0$ and use that to show that $C^1(\overline{\Omega})$, equipped with the scalar product $(\bullet,\bullet)_{L^2(\Omega)} + (\nabla\bullet, \nabla\bullet)_{L^2(\Omega)}$, is *not* a Hilbert space.

Exercise 3 (Globally continuous, piecewise differentiable functions)
Let $\Omega \subseteq \mathbb{R}^n$ open, bounded with piecewise smooth boundary, $(\Omega_j)_{j=1,\dots,J}$ open, bounded and disjoint subsets of Ω with piecewise smooth boundary with $\overline{\Omega} = \overline{\Omega}_1 \cup \dots \cup \overline{\Omega}_J$. Show that any $u \in C(\overline{\Omega})$ with $u|_{\Omega_j} \in C^1(\overline{\Omega}_j)$ is weakly differentiable.

Exercise 4 (Solutions to PMP)
In the setting of the Poisson model problem, let $V = H^1_D(\Omega)$, $a : V \times V \to \mathbb{R}$ defined by $a(u,v) = \int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x$ and $F = \int_\Omega f v \,\mathrm{d}x + \int_{\Gamma_N} g v \,\mathrm{d}s$. Prove that there exists a unique solution $u \in V$ to

$$a(u,v) = F(v) \quad \text{for any } v \in V,$$

and that this solution satisfies

$$\|u\|_{H^1[\Omega)} \leq (1 + C_P^2)\max\{1, C_\gamma\}\big(\|f\|_{L^2(\Omega)} + \|g\|_{L^2(\Gamma_N)}\big).$$