Exercise Sheet 4

Discussion on 21.11.2016

Exercise 1 (General second-order problem)
Let $\Omega \subseteq \mathbb{R}^n$ be a bounded Lipschitz domain, $0 < \varepsilon_1 \le \varepsilon_2 < \infty$ and $0 < c_1 \le c_2 < \infty$ fixed. Furthermore, let $\varepsilon \in L^\infty(\Omega;\mathbb{R})$, $b \in L^\infty(\Omega;\mathbb{R}^n)$, and $c \in L^\infty(\Omega;\mathbb{R})$ such that $\varepsilon_1 \le \varepsilon(x) \le \varepsilon_2$, $c_1 \le c(x) \le c_2$ for any $x \in \Omega$, and $\operatorname{div} b = 0$ in the sense that $\int_\Omega b \cdot \nabla\varphi \,\mathrm{d}x = 0$ for all $\varphi \in H^1_0(\Omega)$. Derive a weak formulation of the problem that seeks $u : \Omega \to \mathbb{R}$ with

$$\begin{aligned} -\operatorname{div}\varepsilon\nabla u + b\cdot\nabla u + cu &= f \quad \text{on } \Omega,\\ u &= 0 \quad \text{along } \partial\Omega. \end{aligned}$$

Prove that this formulation has a unique solution.

Exercise 2 (Energy minimization)
Let V be a Hilbert space with scalar product $(\bullet,\bullet)_V$, $a : V\times V \to \mathbb{R}$ a symmetric bilinear form with $0 \le a(v,v)$ for any $v \in V$. Given $f \in V$, define $G(v) := \frac{1}{2}a(v,v) - (f,v)_V$ for any $v \in V$. Prove that the following two statements are equivalent for $u \in V$:

1. $G(u) = \min_{v\in V} G(v)$,
2. $a(u,v) = (f,v)_V$ for any $v \in V$.

Exercise 3 (Q_k-FEM)
Let $T = [0,1]^2 \subseteq \mathbb{R}^2$, $k \in \mathbb{N}$, $0 = t_0 < t_1 < \cdots < t_k = 1$ and define $Q_{ij} := (t_i, t_j) \in T$. Furthermore, let $\mathcal{K} = \{\chi_{ij} \mid i,j = 0,\dots,k\}$ with $\chi_{ij}(v) := v(Q_{ij})$ for $v \in C^\infty(T)$. Prove that $(T, Q_k(T), \mathcal{K})$ is a finite element in the sense of Ciarlet (where $Q_k(T)$ is the space of polynomials of partial degree $\le k$).

Exercise 4 (There is no P_2-NCFEM)
Let $T = \operatorname{conv}\{P_1,P_2,P_3\} \subseteq \mathbb{R}^2$ be a triangle, $0 < \lambda < 1$ and the points $Q_1,\dots,Q_6$ on the edges of T defined by

$$\begin{aligned} Q_1 &:= \lambda P_1 + (1-\lambda)P_2, & Q_2 &:= (1-\lambda)P_1 + \lambda P_2,\\ Q_3 &:= \lambda P_2 + (1-\lambda)P_3, & Q_4 &:= (1-\lambda)P_2 + \lambda P_3,\\ Q_5 &:= \lambda P_3 + (1-\lambda)P_1, & Q_6 &:= (1-\lambda)P_3 + \lambda P_1. \end{aligned}$$

Furthermore, let $\mathcal{K} = \{\chi_0,\dots,\chi_6\}$ with $\chi_j(v) := v(Q_j)$ for $v \in C^\infty(T)$ and $j = 1,\dots,6$. Prove that $(T, P_2(T), \mathcal{K})$ is *not* a finite element in the sense of Ciarlet.
Hint: Use affine transformation on an equilateral triangle to argue that the points $Q_1,\dots,Q_6$ lie on a common ellipse.