Exercise Sheet 5

Discussion on 28.11.2016

Exercise 1 (Barycentric coordinates)
Consider a triangle $T = \operatorname{conv}\{P_1, P_2, P_3\}$ and the barycentric coordinates $\lambda_1, \lambda_2, \lambda_3 \in P_1(T)$ defined via $\lambda_j(P_k) = \delta_{jk}$ for $j, k = 1, 2, 3$.
a) Prove that any $\alpha, \beta, \gamma \in \mathbb{N}_0$ satisfy

$$\int_T \lambda_1^\alpha \lambda_2^\beta \lambda_2^\gamma \,\mathrm{d}x = 2|T| \frac{\alpha!\beta!\gamma!}{(2+\alpha+\beta+\gamma)!}.$$

b) For the points $P_{j+3} := (P_j + P_{j+1})/2$, $j = 1, 2$ and $P_6 := (P_3 + P_1)/2$, find the nodal basis functions $\mu_j \in P_2(T)$ with $\mu_j(P_k) = \delta_{jk}$ for $j, k = 1, \dots, 6$.
c) Compute the local mass matrices for $P_1(T)$ and $P_2(T)$, i.e. $M_1 = ((\lambda_i, \lambda_j)_{L^2(\Omega)})_{i,j=1,\dots,3} \in \mathbb{R}^{3\times3}$ and $M_2 = ((\mu_i, \mu_j)_{L^2(\Omega)})_{i,j=1,\dots,6} \in \mathbb{R}^{6\times6}$.

Exercise 2 (P_2 is no C^1 element)
Consider a regular triangulation $\mathcal{T}$ and the P_2 finite element $(T, P_2(T), \mathcal{K}_T)$ for any $T \in \mathcal{T}$. Prove that this finite element is not a C^1 finite element in general.

Exercise 3 (Transformation of finite elements)
Let $(T_{\mathrm{ref}}, \mathcal{P}_{\mathrm{ref}}, \mathcal{K}_{\mathrm{ref}})$ be a finite element and $\Phi_T : T_{\mathrm{ref}} \to T$ an affine diffeomorphism.
a) Show that $(T, \mathcal{P}, \mathcal{K})$ is a finite element, where

$$T = \Phi_T(T_{\mathrm{ref}}), \quad \mathcal{P} = \{q_{\mathrm{ref}} \circ \Phi_T^{-1} \mid q_{\mathrm{ref}} \in \mathcal{P}_{\mathrm{ref}}\}, \quad \mathcal{K} = \{\chi_{\mathrm{ref}} \circ \Phi_T^{-1} \mid \chi_{\mathrm{ref}} \in \mathcal{K}_{\mathrm{ref}}\}.$$

b) Show that the corresponding interpolants I_T and $I_{T_{\mathrm{ref}}}$ and any $v_{\mathrm{ref}} \in W^{m,p}(T_{\mathrm{ref}})$ and $v := v_{\mathrm{ref}} \circ \Phi_T^{-1} \in W^{m,p}(T)$ satisfy

$$(I_T v) \circ \Phi_T = I_{T_{\mathrm{ref}}} v_{\mathrm{ref}}.$$

Exercise 4 (Minimum angle condition)
For any triangle T and node $z \in \mathcal{N}(T)$, denote by $\measuredangle(T, z)$ the interior angle of T at z. Prove that any family $(\mathcal{T}_k)_{k\in\mathbb{N}}$ of regular triangulations with

$$0 < \omega_0 \le \min_{k\in\mathbb{N}} \min_{T\in\mathcal{T}_k} \min_{z\in\mathcal{N}(T)} \measuredangle(T, z) \tag{1}$$

is shape regular. Furthermore, find an example of a family of triangulations that does not satisfy (1) and is not shape regular.