Exercise Sheet 6

Discussion on 05.12.2016

Exercise 1 (Convergence history plots)
Suppose that $u \in H^3(\Omega)$ is the exact solution to the Poisson model problem and $\mathcal{T}_0$ a regular triangulation. Consider the sequence $(\mathcal{T}_k)_{k\in\mathbb{N}}$, where $\mathcal{T}_{k+1}$ is the red-refinement of $\mathcal{T}_k$ for any $k \in \mathbb{N}$ (see Figure 1). Furthermore, let $h_k := \max_{T\in\mathcal{T}_k} h_T$ and $\operatorname{ndof}(P_\ell, \mathcal{T}_k)$ the number of global degrees of freedom for the P_ℓ finite element method for $\ell = 1, 2$ on $\mathcal{T}_k$ for $k \in \mathbb{N}$.
a) For a typical situation, sketch the functions $(h_k |u|_{H^2(\Omega)}, \operatorname{ndof}(P_\ell, \mathcal{T}_k))_{k\in\mathbb{N}}$ and $(h_k^2 |u|_{H^3(\Omega)}, \operatorname{ndof}(P_\ell, \mathcal{T}_k))_{k\in\mathbb{N}}$ in a common log-log plot for $\ell = 1, 2$.
(You may utilize the template in Figure 3.)
b) Where in these plots can the functions $(\|\nabla(u - u_k^{(1)})\|_{L^2(\Omega)}, \operatorname{ndof}(P_1, \mathcal{T}_k))_{k\in\mathbb{N}}$ and $(\|\nabla(u - u_k^{(2)})\|_{L^2(\Omega)}, \operatorname{ndof}(P_2, \mathcal{T}_k))_{k\in\mathbb{N}}$ lie for the P_1 (resp. P_2) finite element solutions $u_k^{(1)}$ (resp. $u_k^{(2)}$) on $\mathcal{T}_k$?

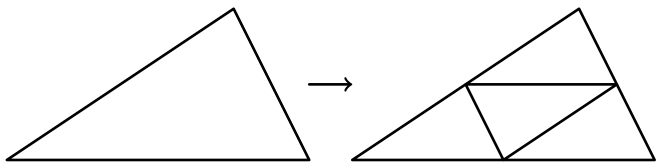

Figure 1: Red refinement of a triangle

Exercise 2 (Error and refinement)
a) Let $(\mathcal{T}_k)_{k\in\mathbb{N}}$ be a sequence of regular triangulations, where $\mathcal{T}_{k+1}$ is a refinement of $\mathcal{T}_k$ for any $k \in \mathbb{N}$. Furthermore, let $u \in H_0^1(\Omega)$ be the exact solution and $u_k \in S_0^1(\Omega)$ the P_1 finite element solution to the Poisson model problem on each level $k \in \mathbb{N}$. Prove that $\|\nabla(u - u_k)\|_{L^2(\Omega)}$ is a monotonically decreasing sequence.
b) Consider the criss-cross triangulation $\mathcal{T}_0$ and its refinement depicted in Figure 2. Prove that the P_1 finite element solutions to the Poisson model problem with $f \equiv 1$ on the triangulations coincide.

Exercise 3 (Inf-sup condition for matrices)
Let U and V be finite-dimensional Hilbert spaces and $b : U \times V \to \mathbb{R}$ a bilinear form. Prove that the inf-sup constant

$$\alpha := \inf_{u\in U} \sup_{v\in V} \frac{b(u, v)}{\|u\|_U \|v\|_V}$$

corresponds to the smallest singular value of a matrix A representing the bilinar form b. Here, for fixed orthonormal bases $(\Phi_j)_{j=1,\ldots,m}$ of U and $(\Psi_k)_{k=1,\ldots,n}$ of V, the matrix A is defined via $A_{jk} = b(\Phi_j, \Psi_k)$ for $j = 1, \ldots, m$ and $k = 1, \ldots, n$.

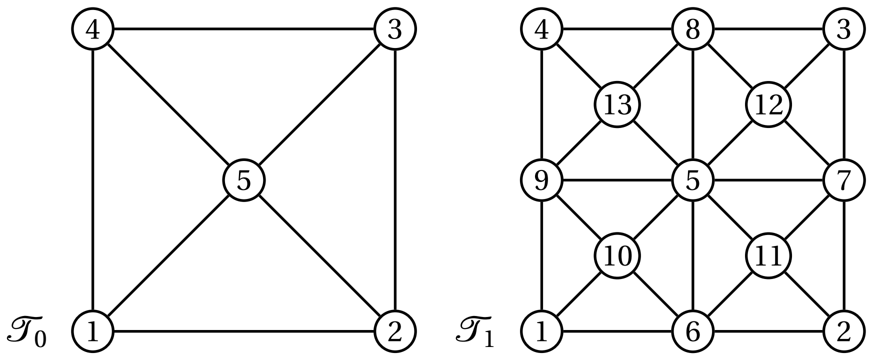

Figure 2: Criss-cross triangulation $\mathcal{T}_0$ (left) and its refinement $\mathcal{T}_1 =$ `bisec(bisec(`$\mathcal{T}_0$`))`

Exercise 4 (`FEM10`)

Study the MATLAB function `FEM10` below, which computes the P_1 finite element solution to the Poisson model problem for homogeneous Dirichlet boundary data and $f \equiv 1$. Modify it to include general right-hand sides $f \in L^2(\Omega)$, given as function handle.

Hint: You may use the midpoint quadrature rule for integration, i.e. for $g \in C(T)$ and $T \in \mathcal{T}$, approximate

$$\int_T g(x)\,\mathrm{d}x \approx |T|\,g(\mathrm{mid}(T)).$$

```
function [x,A,nrDoFs] = FEM10(c4n,n4e,n4sDb)
N=size(c4n,1); d=size(c4n,2);
A=sparse(N,N); b=zeros(N,1); x=zeros(N,1);
for j=1:size(n4e,1)
  area=abs(det([ones(1,d+1);c4n(n4e(j,:),:)']))/factorial(d));
  grads=[ones(1,d+1);c4n(n4e(j,:),:)']\[zeros(1,d);eye(d)];
  A(n4e(j,:),n4e(j,:))=A(n4e(j,:),n4e(j,:))+area*(grads*grads');
  b(n4e(j,:))=b(n4e(j,:))+ones(d+1,1)*area/(d+1); end
freeNodes=setdiff(1:N,unique(n4sDb)); nrDoFs=length(freeNodes);
x(freeNodes)=A(freeNodes,freeNodes)\b(freeNodes); end
```

If you want to test your code, a start routine, a plot routine and mesh data is available at the course homepage `https://www.mathematik.hu-berlin.de/~ccafm/lehre_BZQ_Numerik/CPDE/tutorials.shtml`.

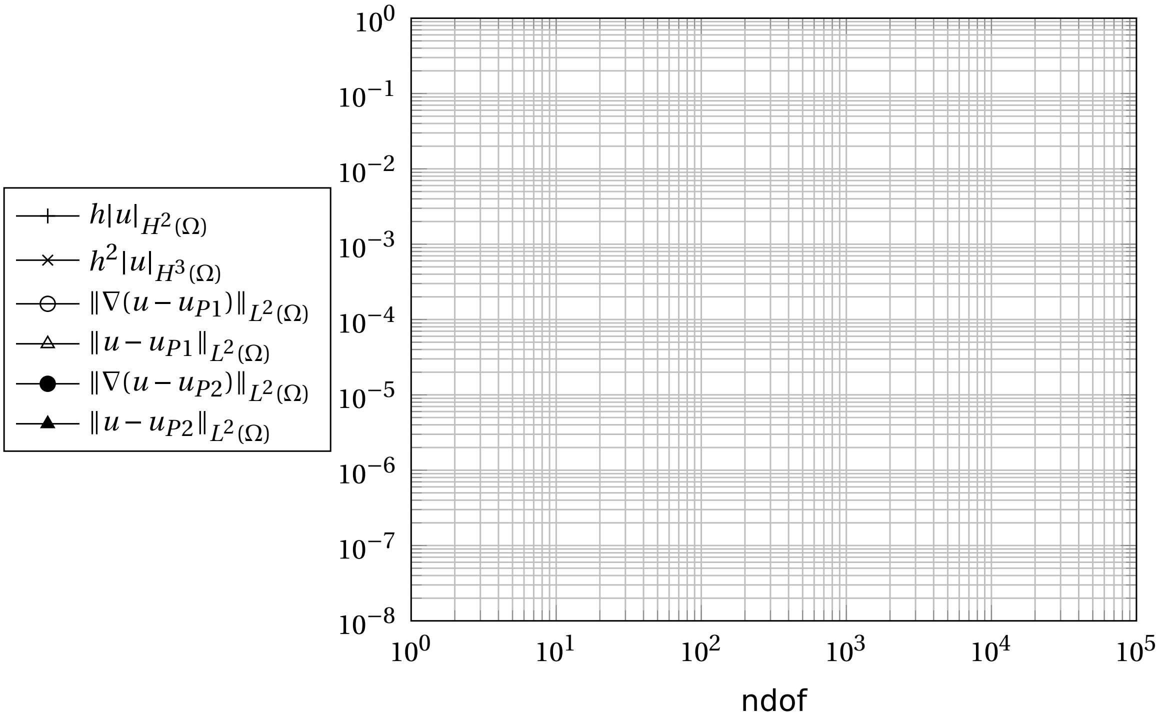

Figure 3: Model convergence history plot for Exercise 1