

Exercise Sheet 7

Discussion on 12.12.2016

Exercise 1 (Best-approximation with side condition)

Suppose the closed subspaces X_h and M_h of the Hilbert spaces X and M satisfy the inf-sup condition and consider $g \in X^*$ and the spaces

$$V(g) := \{v \in X \mid b(v, \mu) = \langle g, \mu \rangle_{X^*, X} \text{ for any } \mu \in M\} \text{ and}$$

$$V_h(g) := \{v_h \in X_h \mid b(v_h, \mu_h) = \langle g, \mu_h \rangle_{X^*, X} \text{ for any } \mu_h \in M_h\}.$$

Show that there exists $C > 0$ such that any $u \in V(g)$ satisfies

$$\inf_{v_h \in V_h(g)} \|u - v_h\|_X \leq C \inf_{w_h \in X_h} \|u - w_h\|_X.$$

Hint: Utilize the Fortin interpolation operator (Theorem III.10).

Exercise 2 (Gradients and divergence)

a) Suppose that $u \in L^2(\Omega)$ and $p \in L^2(\Omega; \mathbb{R}^n)$. Show that $p = -\nabla u$ and $u \in H_0^1(\Omega)$ is equivalent to

$$\int_{\Omega} u \operatorname{div} q \, dx = \int_{\Omega} p \cdot q \, dx \quad \text{for any } q \in H(\operatorname{div}, \Omega).$$

b) Suppose that $g \in L^2(\Omega)$ and $p \in L^2(\Omega; \mathbb{R}^n)$. Show that $g = -\operatorname{div} p$ and $p \in H(\operatorname{div}, \Omega)$ with $p \cdot \nu = 0$ on $\partial\Omega$ is equivalent to

$$\int_{\Omega} v g \, dx = \int_{\Omega} p \cdot \nabla v \, dx \quad \text{for any } v \in H^1(\Omega).$$

Exercise 3 ($H(\operatorname{div}, \Omega)$ functions do not have normal jumps)

a) Let \mathcal{T} a regular triangulation of $\Omega \subseteq \mathbb{R}^2$ and $q \in H^1(\mathcal{T}; \mathbb{R}^2) := \{q \in L^2(\Omega; \mathbb{R}^2) \mid q|_T \in H^1(T) \text{ for any } T \in \mathcal{T}\}$. Consider the space

$$H(\operatorname{div}, \Omega) := \{v \in L^2(\Omega; \mathbb{R}^2) \mid \exists g \in L^2(\Omega) \text{ with } \int_{\Omega} v \cdot \nabla \varphi \, dx = \int_{\Omega} g \varphi \, dx \text{ for all } \varphi \in \mathcal{D}(\Omega)\}.$$

Prove that $q \in H(\operatorname{div}, \Omega)$ if and only if for any $E = T_+ \cap T_-$ with $T_+, T_- \in \mathcal{T}$, $[q]_E \cdot \nu_E := (q_{T_+} \cdot \nu_{T_+} + q_{T_-} \cdot \nu_{T_-})|_E = 0$ almost everywhere along E .

b) Let $f \in L^2(\Omega) \setminus \{0\}$ and $u \in H_0^1(\Omega)$ the exact weak solution to $-\Delta u = f$. For an arbitrary regular triangulation \mathcal{T} of Ω , let $u_h \in S_0^1(\mathcal{T})$ be the P_1 finite element solution. Prove that $u_h \neq u$.

Exercise 4 (Dirichlet boundary data)

Consider the Poisson model problem with inhomogeneous Dirichlet boundary data: Given $u_D \in C(\Gamma_D)$, $g \in C(\Gamma_N)$ and $f \in L^2(\Omega)$, seek $u: \Omega \rightarrow \mathbb{R}$ with

$$-\Delta u = f \text{ in } \Omega, \quad u = u_D \text{ on } \Gamma_D, \quad \text{and} \quad \nabla u \cdot n = g \text{ on } \Gamma_N.$$

a) Modify the weak formulation of the Poisson model problem with the space $V_D := \{u \in H^1(\Omega) \mid u|_{\Gamma_D} = u_D\}$ to include inhomogeneous Dirichlet boundary data.

b) Utilize the split $u = u_0 + \tilde{u}_D$, where $u_0 \in H_D^1(\Omega)$ and $\tilde{u}_D \in V_D$ to incorporate the boundary data in the right-hand side of the formulation.

c) Study the MATLAB function FEM10 below and modify it to include inhomogeneous Dirichlet boundary data. As an approximation of \tilde{u}_D , utilize the nodal interpolation on boundary nodes and zero on the inner nodes.

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1 function [x,A,nrDoFs] = FEM10(c4n,n4e,n4sDb,f)
2     N=size(c4n,1); d=size(c4n,2);
3     A=sparse(N,N); b=zeros(N,1); x=zeros(N,1);
4     for j=1:size(n4e,1)
5         area=abs(det([ones(1,d+1);c4n(n4e(j,:),:)])/factorial(d));
6         grads=[ones(1,d+1);c4n(n4e(j,:),:)]\[zeros(1,d);eye(d)];
7         A(n4e(j,:),n4e(j,:))=A(n4e(j,:),n4e(j,:))+area*(grads*grads
8             ');
9         b(n4e(j,:))=b(n4e(j,:))+f(sum(c4n(n4e(j,:),:))/(d+1))*ones(
10             d+1,1)*area/(d+1);
11     end
12     freeNodes=setdiff(1:N,unique(n4sDb)); nrDoFs=length(freeNodes
13         );
14     x(freeNodes)=A(freeNodes,freeNodes)\b(freeNodes);
15 end

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If you want to test your code, a start routine, a plot routine and mesh data is available at the course homepage https://www.mathematik.hu-berlin.de/~ccafm/lehre_BZQ_Numerik/CPDE/tutorials.shtml.