Exercise Sheet 8

Discussion on 09.01.2017

Exercise 1 (Trace identity and inequality)
a) Suppose $T := \operatorname{conv}(\{P\} \cup E) \subset \mathbb{R}^2$ is a triangle with node P and opposing edge E. Prove that $f \in H^1(T)$ satisfies the trace identity

$$|E|^{-1} \int_E f \,\mathrm{d}s = |T|^{-1} \int_T f \,\mathrm{d}x + \frac{1}{2}|T|^{-1} \int_T \nabla f(x) \cdot (x - P) \,\mathrm{d}x.$$

b) In the setting of a), show that $f \in H^1(T)$ and $h_T = \operatorname{diam} T$ satisfy the trace inequality

$$\|f\|_{L^2(E)} \lesssim h_T^{-1/2} \|f\|_{L^2(T)} + h_T^{1/2} \|\nabla f\|_{L^2(T)}.$$

Exercise 2 (Equivalences for shape regularity)
For a family of triangulations with the minimum angle condition (cf. Sheet 5, Exercise 4), consider any triangulation $\mathcal{T}$ of this family, any $T \in \mathcal{T}$, $E \in \mathcal{E}(T)$, $h_T = \operatorname{diam} T$ and node $z \in \mathcal{N}$ and define $\mathcal{T}(z) := \{T \in \mathcal{T} \mid z \text{ is a node of } T\}$. Prove

$$|E| \approx |T|^{1/2} \approx h_T \quad \text{and} \quad |\mathcal{T}(z)| \lesssim 1.$$

(Recall that the constants hidden in $\approx$ and $\lesssim$ may depend on the minimum angle ω_0 but not on other properties of the particular triangulation $\mathcal{T}$.)

Exercise 3 (L^2 projection)
For a triangulation $\mathcal{T}$ of the domain $\Omega \subseteq \mathbb{R}^n$, the piecewise constant functions $P_0(\mathcal{T})$, and any $f \in L^2(\Omega)$, define $\Pi_0 f \in P_0(\mathcal{T})$ by

$$\|f - \Pi_0 f\|_{L^2(\Omega)} = \min_{p_0 \in P_0(\mathcal{T})} \|f - p_0\|_{L^2(\Omega)}.$$

a) Prove that any $f \in L^2(\Omega)$ and $T \in \mathcal{T}$ satisfies

$$\Pi_0 f|_T = \fint_T f \,\mathrm{d}x := |T|^{-1} \int_T f \,\mathrm{d}x.$$

b) Prove by utilizing Theorem II.8 and a scaling argument that there exists $C > 0$ such that any $f \in H^1(\Omega)$ and $T \in \mathcal{T}$ satisfies the Poincaré inequality,

$$\|f - \Pi_0 f\|_{L^2(T)} \le C h_T \|\nabla f\|_{L^2(T)}.$$

c) Consider $u \in H^1_0(\Omega)$ resp. $\tilde{u} \in H^1_0(\Omega)$ the solutions to the Poisson model problem with right-hand side f resp. $\Pi_0 f$ and prove

$$\|\nabla(u - \tilde{u})\|_{L^2(\Omega)} \lesssim \operatorname{osc}(f, \mathcal{T}) := \Big(\sum_{T \in \mathcal{T}} h_T^2 \|f - \Pi_0 f\|^2_{L^2(T)} \Big)^{1/2}.$$

Exercise 4 ($H(\mathrm{curl},\Omega)$ functions do not have tangential jumps)
Consider $\Omega \subseteq \mathbb{R}^n$ a bounded domain with piecewise smooth boundary $\partial\Omega$ for $n = 2,3$.
a) For $n = 2$, $v \in C^1(\overline{\Omega})$ and $q \in C^1(\overline{\Omega};\mathbb{R}^2)$, define $\mathrm{Curl}\, v \in C(\overline{\Omega};\mathbb{R}^2)$ and $\mathrm{curl}\, q \in C(\overline{\Omega})$ by

$$\mathrm{Curl}\, v := (\partial v/\partial x_2, -\partial v/\partial x_1)^\top \quad \text{and} \quad \mathrm{curl}\, q := \partial q_2/\partial x_1 - \partial q_1/\partial x_2.$$

Prove that the unit tangential vector $\tau = (-\nu_2, \nu_1)$ satisfies

$$\int_\Omega (v\, \mathrm{curl}\, q - q \cdot \mathrm{Curl}\, v)\, \mathrm{d}x = \int_{\partial\Omega} q v \cdot \tau\, \mathrm{d}s.$$

b) For $n = 3$, $q \in C^1(\overline{\Omega};\mathbb{R}^3)$, define $\mathrm{curl}\, q \in C(\overline{\Omega};\mathbb{R}^3)$ by

$$\mathrm{curl}\, q := (\partial q_3/\partial x_2 - \partial q_2/\partial x_3, \partial q_1/\partial x_3 - \partial q_3/\partial x_1, \partial q_2/\partial x_1 - \partial q_1/\partial x_2)^\top.$$

Prove that $p, q \in C^1(\overline{\Omega};\mathbb{R}^3)$ satisfy

$$\int_\Omega (p \cdot \mathrm{curl}\, q - q \cdot \mathrm{curl}\, p)\, \mathrm{d}x = \int_{\partial\Omega} q \cdot (p \times \nu)\, \mathrm{d}s.$$

c) Let $\mathcal{T}$ a regular triangulation of $\Omega \subseteq \mathbb{R}^2$ and $q \in H^1(\mathcal{T};\mathbb{R}^2)$. Consider the space

$$H(\mathrm{curl},\Omega) := \{q \in L^2(\Omega;\mathbb{R}^2) \mid \exists\, \mathrm{curl}\, q := g \in L^2(\Omega) \text{ with } \int_\Omega q \cdot \mathrm{Curl}\, \varphi\, \mathrm{d}x = \int_\Omega g\varphi\, \mathrm{d}x \text{ for all } \varphi \in \mathcal{D}(\Omega)\}.$$

Prove that $q \in H(\mathrm{curl},\Omega)$ if and only if for any $E = T_+ \cap T_-$ with $T_+, T_- \in \mathcal{T}$ and a unit tangential vector τ_E of E, $[q]_E \cdot \tau_E := (q_{T_+} \cdot \tau_E - q_{T_-} \cdot \tau_E)|_E = 0$ almost everywhere along E.
d) Let $\mathcal{T}$ a regular triangulation of $\Omega \subseteq \mathbb{R}^3$ and $q \in H^1(\mathcal{T};\mathbb{R}^3)$. Consider the space

$$H(\mathrm{curl},\Omega) := \{q \in L^2(\Omega;\mathbb{R}^3) \mid \exists\, \mathrm{curl}\, q := g \in L^2(\Omega;\mathbb{R}^3) \text{ with } \int_\Omega q \cdot \mathrm{curl}\, \varphi\, \mathrm{d}x = \int_\Omega g \cdot \varphi\, \mathrm{d}x \text{ for all } \varphi \in \mathcal{D}(\Omega;\mathbb{R}^3)\}.$$

Prove that $q \in H(\mathrm{curl},\Omega)$ if and only if any $E = T_+ \cap T_-$ with $T_+, T_- \in \mathcal{T}$ satisfies $(q|_{T_+} \times \nu_{T_+} + q|_{T_-} \times \nu_{T_-})|_E = 0$ almost everywhere along E.