

# Exercise Sheet 9

Discussion on 19.01.2017

## Exercise 1 (Instable $P_1$ - $P_0$ element for the Stokes equations)

Consider the Stokes equations on the domain  $\Omega := (0, 1)^2$ . Prove that the discretization  $X_h := S_0^1(\mathcal{T}; \mathbb{R}^2)$  and  $Y_h := P_0(\mathcal{T}) \cap L_0^2(\Omega)$  for a criss triangulation  $\mathcal{T}$  of  $\Omega$  (i.e. a set of congruent squares which are bisected from the lower left to the upper right corner, see Figure 1) lead to an unstable saddle point problem.

Hint: Therefore show that the inf-sup conditions is not satisfied or show that  $\{v_h \in X_h \mid \forall q_h \in Y_h : \int_{\Omega} q_h \operatorname{div} v_h \, dx = 0\} = \{0\}$ .

## Exercise 2 (Fortin operator for $P_2$ - $P_0$ element for the Stokes equations – Compatibility)

Recall the averaging operator  $J_1 : P_1(\mathcal{T}) \rightarrow S_0^1(\mathcal{T})$  with

$$(J_1 g)(z) := \sum_{T \in \mathcal{T}(z)} (g|_T)(z) / |\mathcal{T}(z)| \quad \text{for all } z \in \mathcal{N}(\Omega) \text{ with } \mathcal{T}(z) := \{T \in \mathcal{T} \mid z \in T\}.$$

For any interior edge  $E \in \mathcal{E}(\Omega)$ , define the  $P_2$  basis function  $\varphi_E \in S_0^2(\mathcal{T})$  on  $E$  by piecewise quadratic interpolation of the values

$$\varphi_E(z) = 0 = \varphi_E(\operatorname{mid}(F)) \quad \text{for all } z \in \mathcal{N} \text{ and } F \in \mathcal{E}(\Omega) \setminus \{E\} \quad \text{and} \quad \varphi_E(\operatorname{mid}(E)) = 1.$$

Given  $v \in H_0^1(\Omega; \mathbb{R}^2)$  and some coefficients  $\{\alpha_E \in \mathbb{R}^2 \mid E \in \mathcal{E}(\Omega)\}$ , set

$$v_h := J_1(\Pi_0 v) + \sum_{E \in \mathcal{E}(\Omega)} \alpha_E \varphi_E \in S_0^2(\mathcal{T}; \mathbb{R}^2).$$

Show, that there exists a specific choice of the coefficients  $\alpha_E$  such that

$$\int_{\Omega} q_h \operatorname{div}(v - v_h) \, dx = 0 \quad \text{for all } q_h \in P_0(\mathcal{T}).$$

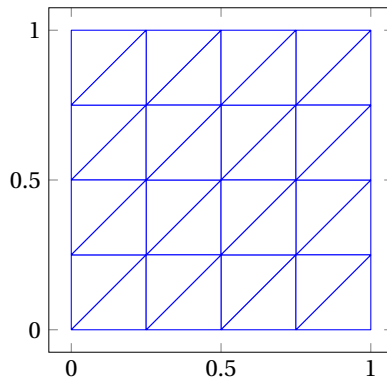


Figure 1: Example of a criss triangulation

**Exercise 3 (Fortin operator for  $P_2$ - $P_0$  element for the Stokes equations – Stability)**

Prove that there exists an  $h$ -independent generic constant  $C > 0$  such that, for any  $v \in H_0^1(\Omega; \mathbb{R}^2)$ , the construction  $v_h \in S_0^2(\mathcal{T}; \mathbb{R}^2)$  from Exercise 2 satisfies

$$\|\nabla v_h\|_{L^2(\Omega)} \leq C \|\nabla v\|_{L^2(\Omega)}.$$

**Exercise 4 (Computing boundary sides)**

Given an admissible triangulation by the data structure `c4n` and `n4e`, the following lines of code produce an array `s4e`. The rows of `s4e` contain the numbers of the three sides of each triangle according to Figure 2.

```

1 allSides = [n4e(:, [2 3]); n4e(:, [3 1]); n4e(:, [1 2])];
2 [~, FirstInd, Back] = unique(sort(allSides, 2), 'rows', 'first');
3 [~, sortInd] = sort(FirstInd);
4 S = length(sortInd);
5 sideNo(sortInd) = 1:S;
6 s4e = reshape(sideNo(Back), size(n4e));

```

Extend the code to compute a list, which contains the numbers of the boundary sides.

Hint: Use `accumarray(s4e(:, 1), [S 1])` to count the adjacent triangles for each side.

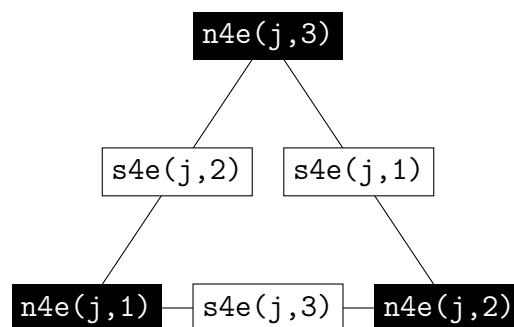


Figure 2: Local numbering of the sides of the triangle  $T_j$  for each  $j = 1, \dots, |\mathcal{T}|$