Exercises on Sobolev Spaces

Exercise 1 (Embeddings of $L^p(\Omega)$)
Let $\Omega \subseteq \mathbb{R}^n$ a bounded domain and $1 \leq p \leq q \leq \infty$. Prove that any $f \in L^q(\Omega)$ satisfies $f \in L^p(\Omega)$ and (with the short notation $1/q := 0$ for $q = \infty$),

$$\|f\|_p \leq |\Omega|^{1/p-1/q} \|f\|_q.$$

In particular, it holds $L^q(\Omega) \subseteq L^p(\Omega)$.
Hint: Consider $\tilde{p} := q/p$ and $\tilde{q}$ with $1/\tilde{p} + 1/\tilde{q} = 1$ and use Hölder's inequality.

Exercise 2 (Globally discontinuous, piecewise differentiable functions)
Let $\Omega = (-1,1) \subseteq \mathbb{R}$ and $\operatorname{sgn} : \overline{\Omega} \to \mathbb{R}$ with

$$\operatorname{sgn}(x) := \begin{cases} \frac{x}{|x|} & \text{for } x \neq 0, \\ 0 & \text{for } x = 0. \end{cases}$$

Prove that sgn is *not* weakly differentiable.

Exercise 3 (Functions in $H^1(\Omega)$ don't have jumps)
Let $\Omega \subseteq \mathbb{R}^n$ open, bounded with piecewise smooth boundary, $(\Omega_j)_{j=1,\ldots,J}$ open, bounded and disjoint subsets of Ω with piecewise smooth boundary with $\overline{\Omega} = \overline{\Omega}_1 \cup \cdots \cup \overline{\Omega}_J$. Show that any $u : \Omega \to \mathbb{R}$ with $u|_{\Omega_j} \in C^1(\overline{\Omega}_j)$ satisfies $u \in H^1(\Omega)$ if and only if $u \in C(\overline{\Omega})$.

Exercise 4 ($H^1(\Omega)$ is not embedded in $C^0(\overline{\Omega})$)
Let $\Omega = B_1(0) \subseteq \mathbb{R}^2$ and $u : \Omega \to \mathbb{R}$ with

$$u(x) := \ln\left(|\ln \frac{1}{|x|}|\right).$$

Prove that $u \in H^1(\Omega) \setminus C(\Omega)$.
Hint: Calculate the (classical) partial derivatives on $B_1(0) \setminus \{0\}$ and show that they lie in $L^2(\Omega)$ by transformation to polar coordinates.