GREEN'S CONJECTURE FOR GENERAL COVERS

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M. Green's Conjecture on syzygies of canonical curves $\phi_{K_C}: C \to \mathbf{P}^{g-1}$, asserting the following vanishing of Koszul cohomology groups [G]

$$K_{p,2}(C, K_C) = 0 \Leftrightarrow p < \mathrm{Cliff}(C),$$

has been one of the most investigated problems in the last decades in the theory of algebraic curves. Based on the principle that all non-trivial syzygies are generated by secants to the canonical curve $C \subset \mathbf{P}^{g-1}$, the conjecture is appealing because it predicts that one can read off the Clifford index of the curve (measuring the complexity of C in its moduli space) from the graded Betti diagram of the canonical embedding. Voisin [V1], [V2] established Green's conjecture for general curves $[C] \in \mathcal{M}_g$ of any genus.

Building on the work of Voisin, the first author [A3] has found a Brill-Noether theoretic sufficient condition for a curve to satisfy Green's Conjecture. If $[C] \in \mathcal{M}_g$ is a d-gonal curve with $2 \le d \le \frac{g}{2} + 1$ satisfying the *linear growth condition*

$$\dim W^1_{g-d+2}(C) = \rho(g, 1, g-d+2) = g - 2d + 2, \tag{1}$$

then C satisfies both Green's Conjecture and the Gonality Conjecture [GL2].

Condition (1) is equivalent to $\dim W^1_{d+n}(C) \le n$ for all $0 \le n \le g-2d+2$. In particular, it implies that C has a finite number of pencils of minimal degree. The case of odd genus and maximal gonality treated by [V2] is automatically excluded from condition (1). One aim of this paper is to establish Green's conjecture for classes of curves where condition (1) manifestly fails, in particular for curves having an infinite number of minimal pencils. Typical examples are curves whose Clifford indices are not computed by pencils, and their covers. Precisely, if X is a curve of Clifford dimension $r(X) := r \ge 2$, then $\mathrm{gon}(X) = \mathrm{Cliff}(X) + 3$ and X carries an infinite number of pencils of minimal degree [CM]. If $f: C \to X$ is a branched covering of X of sufficiently high genus, then $\mathrm{gon}(C) = \deg(f) \cdot \mathrm{gon}(X)$ and C carries infinitely many pencils of minimal degree, all pulled-back from X. In particular, condition (1) fails for C.

Theorem 0.1.
(i) Set $d \ge 3,\ g \ge d^2 + 1$ and let $C \to \Gamma \subset \boldsymbol{P}^2$ be a general genus g double covering of a smooth plane curve of degree d. Then $K_{2d-5,2}(C, K_C) = 0$ and C satisfies Green's Conjecture.

(ii) Let $g \ge 2d^2 + 1$ and $C \to \Gamma \subset \boldsymbol{P}^2$ be a general genus g fourfold cover of a smooth plane curve of degree d. Then C satisfies Green's Conjecture.

In a similar vein, we have a result about triple coverings of elliptic curves.

Theorem 0.2. *Let $C \to E$ be a general triple covering of genus $g \ge 13$ of an elliptic curve. Then $K_{3,2}(C, K_C) = 0$ and C satisfies Green's Conjecture.*

Curves with Clifford dimension 3 have been classified in [ELMS]. If $[X] \in \mathcal{M}_g$ is such that $r(X) = 3$, then $g = 10$ and X is the complete intersection of two cubic surfaces in $\mathbf{P}^3$. The very ample $\mathfrak{g}^3_9$ computes $\mathrm{Cliff}(X) = 3$, whereas dim $W^1_6(C) = 1$; each minimal pencil of X is induced by planes through a trisecant line to $X \subset \mathbf{P}^3$. We prove the following result:

Theorem 0.3. *Let $C \to X$ be a general double covering of genus $g \geq 28$ of a smooth curve X with $r(X) = 3$. Then $K_{9,2}(C, K_C) = 0$ and C satisfies Green's Conjecture.*

The second aim of this paper is to study syzygies of curves with a fixed point free involution. We denote by $\mathcal{R}_g$ the moduli space of pairs $[C, \eta]$ where $[C] \in \mathcal{M}_g$ and $\eta \in \mathrm{Pic}^0(C) - \{\mathcal{O}_C\}$ is a root of the trivial bundle, that is, $\eta^{\otimes 2} = \mathcal{O}_C$. Equivalently, $\mathcal{R}_g$ parametrizes étale double covers $f : \widetilde{C} \to C$, where $g(\widetilde{C}) = 2g-1$ and $f_*(\mathcal{O}_{\widetilde{C}}) = \mathcal{O}_C \oplus \eta$. The moduli space $\mathcal{R}_g$ admits a Deligne-Mumford compactification $\overline{\mathcal{R}}_g$ by means of stable Prym curves, that comes equipped with two morphisms

$$\pi : \overline{\mathcal{R}}_g \to \overline{\mathcal{M}}_g \quad \text{and} \quad \chi : \overline{\mathcal{R}}_g \to \overline{\mathcal{M}}_{2g-1},$$

obtained by forgetting $\widetilde{C}$ and C respectively. We refer to [FL] for a detailed study of the birational geometry and intersection theory of $\overline{\mathcal{R}}_g$.

One may ask whether Green's conjecture holds for a curve $[\widetilde{C}] \in \mathcal{M}_{2g-1}$ corresponding to a general point $[\widetilde{C} \xrightarrow{f} C] \in \mathcal{R}_g$. Note that since $\widetilde{C}$ does not satisfy Petri's theorem [1] the question is a little delicate. In spite of that we have the following answer:

Theorem 0.4. *Let us fix a general étale double cover $[f : \widetilde{C} \to C] \in \mathcal{R}_g$.*

(i) If $g \equiv 1 \bmod 2$, then $\widetilde{C}$ is of maximal gonality, that is, $\mathrm{gon}(\widetilde{C}) = g+1$. In particular $\widetilde{C}$ satisfies Green's Conjecture.

(ii) If $g \equiv 0 \bmod 2$, then $\mathrm{gon}(\widetilde{C}) = g$ and $\dim W^1_{g+1}(\widetilde{C}) = 1$. It follows that $\widetilde{C}$ satisfies Green's Conjecture.

We provide two proofs of this result. The statement concerning Green's conjecture follows via condition (1). The first proof is by specialization to a boundary divisor of $\overline{\mathcal{R}}_g$ and uses limit linear series. The second proof, which we now briefly explain can be viewed as a counterpart of Voisin's result [V1] and has the advantage of singling out an explicit locus in $\mathcal{R}_g$ where Green's conjecture holds.

Let $\mathcal{F}^{\mathfrak{N}}_g$ be the 11-dimensional moduli space of genus g Nikulin surfaces. A very general point of $\mathcal{F}^{\mathfrak{N}}_g$ corresponds to a double cover $f : \widetilde{S} \to S$ of a $K3$ surface, branched along a set $R_1 + \cdots + R_8$ of eight mutually disjoint (-2)-curves, as well as a linear system $L \in \mathrm{Pic}(S)$, where $L^2 = 2g-2$ and $L \cdot R_i = 0$ for $i = 1, \ldots, 8$. We choose a smooth curve $C \in |L|$, set $\widetilde{C} := f^{-1}(C) \subset \widetilde{S}$. Then the restriction $f_C : \widetilde{C} \to C$ defines an element

[1] Choose an odd theta-characteristic $\epsilon \in \mathrm{Pic}^{g-1}(C)$ such that $h^0(C, \eta \otimes \epsilon) \geq 1$. Then $f^*(\epsilon)$ is a theta-characteristic on $\widetilde{C}$ with $h^0(\widetilde{C}, f^*(\epsilon)) = h^0(C, \epsilon) + h^0(C, \epsilon \otimes \eta) \geq 2$, that is, $\widetilde{C}$ possesses a vanishing theta-null.

of $\mathcal{R}_g$. We show that the canonical bundle of $\widetilde{C}$ has minimal syzygies and when the lattice $\mathrm{Pic}(S)$ is minimal, that is, of rank 9.

Theorem 0.5. *Let $f_C : \widetilde{C} \to C$ be a double cover corresponding to a very general Nikulin surface of genus g.*

(i) If $g \equiv 1 \bmod 2$, then $\mathrm{gon}(\widetilde{C}) = g + 1$.

(ii) If $g \equiv 0 \bmod 2$, then $\mathrm{gon}(\widetilde{C}) = g$.

In both cases, the curve $\widetilde{C}$ verifies Green's Conjecture.

We point out that in this situation both C and $\widetilde{C}$ are sections of (different) $K3$ surfaces, hence by [AF2] they verify Green's Conjecture. The significance of Theorem 0.5 lies in showing that the Brill-Noether theory of C and $\widetilde{C}$ is the one expected from a general element of $\mathcal{R}_g$.

1. Koszul cohomology

We fix a smooth algebraic curve C, a line bundle L on C and a space of sections $W \subset H^0(C, L)$. Given two integers p and q, the Koszul cohmology group $K_{p,q}(C, L, W)$ is the cohomology at the middle of the complex

$$\wedge^{p+1} W \otimes H^0(C, L^{\otimes(q-1)}) \longrightarrow \wedge^p W \otimes H^0(C, L^{\otimes q}) \longrightarrow \wedge^{p-1} W \otimes H^0(C, L^{\otimes(q+1)})$$

If $W = H^0(C, L)$ we denote the corresponding Koszul cohomology group by $K_{p,q}(C, L)$.

For a globally generated line bundle L, Lazarsfeld [L2] provided a description of Koszul cohomology in terms of kernel bundles. If $W \subset H^0(C, L)$ generates L one defines $M_W := \mathrm{Ker}\{W \otimes \mathcal{O}_C \to L\}$. When $W = H^0(C, L)$, we write $M_W := M_L$. The kernel of the Koszul differential coincides with

$$H^0(C, \wedge^p M_W \otimes L^q) \subset \wedge^p W \otimes H^0(C, L^{\otimes q})$$

and hence one has the following isomorphism:

$$K_{p,q}(C, L, W) \cong \mathrm{Coker}\Big\{\wedge^{p+1} W \otimes H^0(C, L^{q-1}) \to H^0(C, \wedge^p M_W \otimes L^q)\Big\}.$$

Note that for $q = 1$ the hypothesis of being globally generated is no longer necessary, and we do have a similar description for $K_{p,1}$ with values in any line bundle. Indeed, if $\mathrm{Bs}|L| = B$, and M_L is the kernel of the evaluation map on global sections, then $M_L \cong M_{L(-B)}$. Applying the definition, the identification $H^0(C, L(-B)) \cong H^0(C, L)$ and the inclusion $H^0(C, L(-B)^{\otimes 2}) \subset H^0(C, L^{\otimes 2})$ induce an isomorphism, for any p, between $H^0(C, \wedge^p M_{L(-B)} \otimes L(-B))$ and $H^0(C, \wedge^p M_L \otimes L)$. In particular, $K_{p,1}(C, L(-B)) \cong K_{p,1}(C, L)$ and

$$K_{p,1}(C, L) \cong \mathrm{Coker}\Big\{\wedge^{p+1} H^0(C, L) \to H^0(C, \wedge^p M_L \otimes L)\Big\}$$

as claimed.

1.1. **Projections of syzygies.** Let L be a line bundle on C and assume that $x \in C$ is not a base point of L. Setting $W_x := H^0(C, L(-x))$, we have an induced short exact sequence

$$0 \longrightarrow W_x \longrightarrow H^0(C, L) \longrightarrow \mathbb{C}_x \longrightarrow 0.$$

From the restricted Euler sequences corresponding to L and $L(-x)$ respectively, we obtain an exact sequence

$$0 \longrightarrow M_{L(-x)} \longrightarrow M_L \longrightarrow \mathcal{O}_C(-x) \longrightarrow 0,$$

and further, for any integer $p \geq 0$,

$$0 \longrightarrow \wedge^{p+1} M_{L(-x)} \otimes L \longrightarrow \wedge^{p+1} M_L \otimes L \longrightarrow \wedge^p M_{L(-x)} \otimes L(-x).$$

The exact sequence of global sections, together with the natural sequence

$$0 \longrightarrow \wedge^{p+2} W_x \longrightarrow \wedge^{p+2} H^0(C, L) \longrightarrow \wedge^{p+1} W_x \longrightarrow 0,$$

induce an exact sequence

$$0 \to K_{p+1,1}(C, L, W_x) \longrightarrow K_{p+1,1}(C, L) \xrightarrow{\mathrm{pr_x}} K_{p,1}(C, L(-x)),$$

where the induced map $\mathrm{pr}_x : K_{p+1,1}(C, L) \to K_{p,1}(C, L(-x))$ is the *projection of syzygies* map centered at x. Nonzero Koszul classes survive when they are projected from general points:

Proposition 1.1. *If $0 \neq \alpha \in K_{p+1,1}(C, L)$, then $\mathrm{pr}_x(\alpha) \neq 0 \in K_{p,1}(C, L(-x))$ for a general point $x \in C$.*

We record some immediate consequences and refer to [A1] for complete proofs based on semicontinuity.

Corollary 1.2. *Let L be a line bundle on a curve C and $x \in C$ a point. If $L(-x)$ is nonspecial and $K_{p,1}(C, L(-x)) = 0$ then $K_{p+1,1}(C, L) = 0$.*

Going upwards, it follows from Corollary 1.2 that, for a nonspecial L, the vanishing of $K_{p,1}(C, L)$ implies that $K_{p+e,1}(C, L(E)) = 0$, for any effective divisor E of degree e.

For canonical nodal curves, we have a similar result:

Corollary 1.3. *Let L be a line bundle on a curve C and $x, y \in C$ two points. If $K_{p,1}(C, K_C) = 0$ then $K_{p+1,1}(C, K_C(x + y)) = 0$.*

The proof of Corollary 1.3 follows directly from the Corollary 1.2 for $L = K_C(x+y)$ coupled with isomorphisms $K_{p,1}(C, K_C(y)) \cong K_{p,1}(C, K_C)$. Geometrically, the image of C under the linear system $|K_C(x+y)|$ is a nodal canonical curve, having the two points x and y identified, and the statement corresponds to the projection map from the node.

By induction, from Corollary 1.3 and 1.2 we obtain:

Corollary 1.4. *Let C be a curve and $p \geq 1$ such that $K_{p,1}(C, K_C) = 0$. Then for any effective divisor E of degree e, we have $K_{p+e-1,1}(C, K_C(E)) = 0$.*

1.2. **Koszul vanishing.** Using a secant construction, Green and Lazarsfeld [GL1] have shown that non-trivial geometry (in the forms of existence of special linear series) implies non-trivial syzygies. Precisely, if C is a curve of genus g and $\mathrm{Cliff}(C) = c$, then $K_{g-c-2,1}(C, K_C) \neq 0$, or equivalently, by duality, $K_{c,2}(C, K_C) \neq 0$. Green [G] conjectured in that this should be optimal and the converse should hold:

Conjecture 1.5. *For any curve C of genus g and Clifford index c, one has that*

$$K_{g-c-1,1}(C, K_C) = 0,$$

equivalently, $K_{p,2}(C, K_C) = 0$ for all $p < c$.

In the case of a nonspecial line bundle L on a curve C of gonality d, [GL1] gives us the non-vanishing of $K_{h^0(L)-d-1,1}(C, L) \neq 0$. In the same spirit, one may ask whether this result is optimal. It was conjectured in [GL2] that this should be the case for bundles of large degree.

Conjecture 1.6. *For any curve C of gonality d there exists a nonspecial very ample line bundle L such that $K_{h^0(L)-d,1}(C, L) = 0$.*

1.3. **Curves on $K3$ surfaces.** It was know since the eighties that the locus

$$\mathcal{K}_g := \{[C] \in \mathcal{M}_g : C \text{ lies on a } K3 \text{ surface}\}$$

is transverse to the Brill-Noether strata in $\mathcal{M}_g$. Most notably, curves $[C] \in \mathcal{K}_g$ lying on $K3$ surfaces S with $\mathrm{Pic}(S) = \mathbb{Z} \cdot C$ satisfy the Brill-Noether-Petri theorem, see [L1]. This provides a very elegant solution to the Petri conjecture, and remains to this day, the only explicit example of a *smooth* Brill-Noether general curve of unbounded genus.

Green's hyperplane section theorem [G] asserts that the Koszul cohomology of any $K3$ surface is isomorphic to that of any hyperplane section, that is, $K_{p,q}(S, \mathcal{O}_S(C)) \cong K_{p,q}(C, K_C)$. Voisin has used this fact to find a solution to Green's conjecture for generic curves, see [V1], [V2]:

Theorem 1.7. *Let C be a smooth curve lying on a $K3$ surface S with $\mathrm{Pic}(S) = \mathbb{Z} \cdot C$. Then C satisfies Green's conjecture.*

This result has been extended in [AF2] to cover the case of $K3$ surfaces with arbitrary Picard lattice, in particular curves with arbitrary gonality:

Theorem 1.8. *Green's conjecture is valid for any smooth curve $[C] \in \mathcal{K}_g$ of genus g and gonality $d \leq [\frac{g}{2}] + 1$. The gonality conjecture is valid for smooth curves of Clifford dimension one on a $K3$ surface, general in their linear systems.*

It is natural to ask whether in a linear system whose smooth members are of Clifford dimension one the condition (1) is preserved. The answer in NO, as we shall see in section 4.

2. SYZYGY CONJECTURES FOR GENERAL ÉTALE DOUBLE COVERS

In this section we prove Theorem 0.4 by degeneration. We begin by observing that if $g = 2i$ with $i \in \mathbb{Z}_{>0}$ and $f : \widetilde{C} \to C$ is an étale double cover with $f_*\mathcal{O}_{\widetilde{C}} = \mathcal{O}_C \oplus \eta$, then $\widetilde{C}$ cannot possibly have maximal Clifford index (gonality). The difference variety $C_i - C_i \subset \mathrm{Pic}^0(C)$ covers the Jacobian $\mathrm{Pic}^0(C)$ and there exist effective divisors $D, E \in C_i$ such that $\eta = \mathcal{O}_C(D - E)$. We set $A := f^*(\mathcal{O}_C(E)) \in \mathrm{Pic}^g(\widetilde{C})$ and note that

$$h^0(\widetilde{C}, A) = h^0(C, f_* f^*(\mathcal{O}_C(E)) = h^0(C, \mathcal{O}_C(E)) + h^0(C, \mathcal{O}_C(D)) \geq 2,$$

that is, $A \in W^1_g(C)$. This shows that the image of the map

$$\chi : \overline{\mathcal{R}}_g \to \overline{\mathcal{M}}_{2g-1}, \quad \chi\big([\widetilde{C} \xrightarrow{f} C]\big) := [\widetilde{C}]$$

is contained in the Hurwitz divisor $\overline{\mathcal{M}}^1_{2g-1,g} \subset \overline{\mathcal{M}}_{2g-1}$ of curves with a $\mathfrak{g}^1_g$. For odd g there is no obvious reason why $\widetilde{C}$ should have non-maximal gonality and indeed, we shall show that $\mathrm{gon}(\widetilde{C}) = g + 1$ in this case.

To prove Theorem 0.4 we use the following degeneration. Fix a general pointed curve $[C, p] \in \mathcal{M}_{g-1,1}$ as well as an elliptic curve $[E, p] \in \mathcal{M}_{1,1}$. We fix a non-trivial point $\eta_E \in \mathrm{Pic}^0(E)[2]$, inducing an étale double cover $f_E : \widetilde{E} \to E$, and set $\{x, y\} := f_E^{-1}(p)$. The points $x, y \in \widetilde{E}$ satisfy the linear equivalence $2x \equiv 2y$. We choose two identical copies (C_1, p_1) and (C_2, p_2) of (C, p) and consider the stable curve of genus $2g - 1$

$$X_g := C_1 \cup E \cup C_2 / p_1 \sim x, p_2 \sim y,$$

admitting an admissible double cover $f : X_g \to C \cup_p E$, which can be viewed as a point in the boundary divisor $\pi^*(\Delta_1) \subset \overline{\mathcal{R}}_g$. Note that f maps both copies (C_i, p_i) isomorphically onto (C, p).

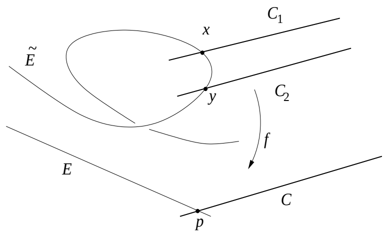

FIGURE 1. The curve X_g.

Theorem 0.4 follows from the following computation coupled with an application of [A3]. The case of even g is revelatory for understanding how the linear growth condition (1) can be verified in order to (non-trivially) establish Green's conjecture for classes of curves of non-maximal Clifford index.

Proposition 2.1. *Let $[X_g \xrightarrow{f} C \cup E] \in \overline{\mathcal{R}}_g$ be the cover constructed above.*

(i) If g is odd then $\mathrm{gon}(X_g) = g + 1$, that is, $[X_g] \notin \overline{\mathcal{M}}^1_{2g-1,g}$.

(ii) If g is even then $\mathrm{gon}(X_g) = g$ and each component of the variety $\overline{G}^1_{g+1}(X_g)$ of limit linear series $\mathfrak{g}^1_{g+1}$ on X_g has dimension 1. In particular X_g satisfies Green's conjecture.

Proof. Throughout the proof we use the notation of [EH] and assume some familiarity with the theory of limit linear series. Suppose first that X_g possesses a limit linear series $l \in \overline{G}^1_g(X_g)$ and denote by l_{C_1}, l_{C_2} and $l_{\widetilde{E}}$ respectively, its aspects on the components of X_g. From the additivity of the adjusted Brill-Noether number we obtain that

$$(2) \qquad -1 = \rho(2g-1,1,g) \geq \rho(l_{C_1},p_1) + \rho(l_{C_2},p_2) + \rho(l_{\widetilde{E}},x,y).$$

Furthermore $\rho(l_{C_i},p_i) \geq 0$, because $[C_i,p_i] \in \mathcal{M}_{g-1,1}$ is general and we apply [EH] Theorem 1.1. It is easy to prove that $\rho(l_{\widetilde{E}},x,y) \geq -1$. This shows that one has equality in (2), that is, l is a *refined limit* $\mathfrak{g}^1_g$ and moreover $\rho(l_{C_i},p_i) = 0$ for $i=1,2$, and $\rho(l_{\widetilde{E}},x,y) = -1$. We denote by (a_0,a_1) (respectively (b_0,b_1)) the vanishing sequence of $l_{\widetilde{C}}$ at the point x (respectively y). From the compatibility of vanishing sequences at the nodes x and y, we find that $a_0+a_1 = g$ and $b_0+b_1 = g$ respectively. On the other hand $l_{\widetilde{E}}$ possesses a section which vanishes at least with order a_0 at x as well as with order b_1 at y (respectively a section which vanishes at least with order a_1 at x and order b_0 at y). Therefore $a_0+b_1 \leq g$ and $a_1+b_0 \leq g$. All in all, since $\rho(l_{\widetilde{E}},x,y) = -1$, this implies that $a_0 = b_0$ and $a_1 = b_1 = g-a_0$, and the following linear equivalence on $\widetilde{E}$ must hold:

$$a_0 \cdot x + (g-a_0)\cdot y \equiv (g-a_0)\cdot x + a_0 \cdot y.$$

Since $x-y \in \mathrm{Pic}^0(\widetilde{E})[2]$, we obtain that $g-2a_0 \equiv \mod 2$. When g is odd this yields a contradiction. On the other when g is even, this argument shows that $\mathrm{gon}(X_g) = g$, in the sense that X_g carries no limit linear series $\mathfrak{g}^1_{g-1}$ and there are a finite number of $\mathfrak{g}^1_g$'s corresponding to the unique choice of an integer $0 \leq a \leq \frac{g}{2}$, a unique $l_{\widetilde{E}} \in G^1_g(\widetilde{E})$ with vanishing sequence $(a_0, g-a_0)$ at both x and y, and to a finite number of $l_{C_i} \in G^1_g(C_i)$ with vanishing sequence $(a_0,g-a_0)$ at $p_i \in C_i$ for $i=1,2$.

We finally show that when g is even, the variety $\overline{G}^1_{g+1}(X_g)$ is of pure dimension 1. Let $l \in \overline{G}^1_{g+1}(X_g)$ be a limit linear series corresponding to a general point in an irreducible component of $\overline{G}^1_{g+1}(X_g)$. Then l is refined and one has the following equality

$$(3) \qquad 1 = \rho(2g-1,1,g+1) = \rho(l_{C_1},p_1) + \rho(l_{C_2},p_2) + \rho(l_{\widetilde{E}},x,y).$$

Components of $\overline{G}^1_{g+1}(X_g)$ correspond to possibilities of choosing the vanishing sequences $a^{l_{\widetilde{E}}}(x)$ and $a^{l_{\widetilde{E}}}(y)$ such that (3) holds. Both curves $[C_i,p_i] \in \mathcal{M}_{g-1,1}$ satisfy the *strong Brill-Noether Theorem*, see [EH] Theorem 1.1, that is, for a Schubert index $\bar{\alpha} := (0 \leq \alpha_0 \leq \alpha_1 \leq g-1)$, the variety

$$G^1_{g+1}\big((C_i,p_i),\bar{\alpha}\big) := \{l_{C_i} \in G^1_{g+1}(C_i) : \alpha^{l_{C_i}}(p_i) \geq \bar{\alpha}\}$$

has expected dimension $\rho(g-1,1,g+1) - \alpha_0 - \alpha_1$. The only possibility that has to be ruled out in order to establish Theorem 0.4 is that when $\rho(l_{C_i},p_i) = 1$ for $i=1,2$ and $\rho(l_{\widetilde{E}},x,y) = -1$, for that would correspond to a 2-dimensional component of $\overline{G}^1_{g+1}(X_g)$. A reasoning very similar to the one above, shows that when g is *even* and $2(x-y) \equiv 0$, there exist no $\mathfrak{g}^1_{g+1}$ on $\widetilde{E}$ with $\rho(l_{\widetilde{E}},x,y) = -1$ which is the aspect of an element from $\overline{G}^1_{g+1}(X_g)$. Hence this case does not occur. It follows that all components of $\overline{G}^1_{g+1}(X_g)$ correspond to the cases $\rho(l_{C_1},p_1) = \rho(l_{\widetilde{E}},x,y) = 0$ and $\rho(l_{C_2},p_2) = 1$, or

$\rho(l_{C_2}, p_2) = \rho(l_{\widetilde{E}}, x, y) = 0$ and $\rho(l_{C_1}, p_1) = 1$. Each such possibility corresponds to a 1-dimensional component of $\overline{G}^1_{g+1}(X_g)$, which finishes the proof. $\square$

3. SYZYGIES OF SECTIONS OF NIKULIN SURFACES

In this section we study syzygies of étale double covers lying on Nikulin $K3$ surfaces. The moduli space $\mathcal{F}_g^{\mathfrak{N}}$ of Nikulin surfaces of genus g has been studied in [FV] which serves as a general reference. Let us recall a few definitions. A *Nikulin involution* on a smooth $K3$ surface Y is a symplectic involution $\iota \in \mathrm{Aut}(Y)$. A Nikulin involution has 8 fixed points [Ni]. The quotient $\bar{Y} := Y/\langle \iota \rangle$ has 8 singularities of type A_1. We denote by $\sigma : \tilde{S} \to Y$ the blow-up of the 8 fixed points, by $E_1, \ldots, E_8 \subset \tilde{S}$ the exceptional divisors, and finally by $\tilde{\iota} \in \mathrm{Aut}(\tilde{S})$ the automorphism induced by ι. Then $S := \tilde{S}/\langle \tilde{\iota} \rangle$ is a smooth $K3$ surface. If $f : \tilde{S} \to S$ is the projection, then $N_i := f(E_i)$ are (-2)-curves on S. The branch divisor of f is equal to $N := \sum_{i=1}^{8} N_i$. We have the following diagram that shall be used for the rest of this section:

$$\begin{array}{ccc} \tilde{S} & \xrightarrow{\ \sigma\ } & Y \\ {\scriptstyle f}\downarrow & & \downarrow \\ S & \longrightarrow & \bar{Y} \end{array} \tag{4}$$

As usual, $H^2(Y, \mathbb{Z}) = U^3 \oplus E_8(-1) \oplus E_8(-1)$ is the unique even unimodular lattice of signature $(3, 19)$, where U is the rank 2 hyperbolic lattice and E_8 is the unique even, negative-definite unimodular lattice of rank 8. As explained in [vGS], the action of the Nikulin involution ι on the group $H^2(Y, \mathbb{Z})$ is given by

$$\iota^*(u, x, y) = (u, y, x),$$

where $u \in U$ and $x, y \in E_8(-1)$. We identify the orthogonal complement

$$\left(H^2(Y, \mathbb{Z})^{\iota}\right)^{\perp} = \{(0, y, -y) : y \in E_8(-1)\} = E_8(-2).$$

Since $\iota^*(x) = -x$ for $x \in \left(H^2(Y, \mathbb{Z})^{\iota}\right)^{\perp}$ whereas $\iota^*(\omega) = \omega$ for $\omega \in H^{2,0}(Y)$, it follows that $x \cdot \omega = 0$, therefore $E_8(-2) \subset \mathrm{Pic}(Y)$. This shows that the Picard number of Y is at least 9.

By construction, the class $\mathcal{O}_S(N_1 + \cdots + N_8)$ is even and we consider the class $e \in \mathrm{Pic}(S)$ such that $e^{\otimes 2} = \mathcal{O}_S(N_1 + \cdots + N_8)$.

Definition 3.1. The *Nikulin lattice* is an even lattice $\mathfrak{N}$ of rank 8 generated by elements $\{\mathfrak{n}_i\}_{i=1}^{8}$ and $\mathfrak{e} := \frac{1}{2}\sum_{i=1}^{8} \mathfrak{n}_i$, with the bilinear form induced by $\mathfrak{n}_i^2 = -2$ for $i = 1, \ldots, 8$ and $\mathfrak{n}_i \cdot \mathfrak{n}_j = 0$ for $i \neq j$.

Note that $\mathfrak{N}$ is the minimal primitive sublattice of $H^2(S, \mathbb{Z})$ containing the classes $N_1, \ldots, N_8$ and e. We fix $g \geq 2$ and consider the lattice

$$\Lambda_g := \mathbb{Z} \cdot \mathfrak{c} \oplus \mathfrak{N},$$

where $\mathfrak{c} \cdot \mathfrak{c} = 2g - 2$. A Nikulin surface of genus g is a $K3$ surface S together with a primitive embedding of lattices $j : \Lambda_g \hookrightarrow \mathrm{Pic}(S)$ such that $C := j(\mathfrak{c})$ is an ample class. The moduli space $\mathcal{F}_g^{\mathfrak{N}}$ of Nikulin surfaces of genus g is an irreducible 11-dimensional variety. Its general point corresponds to a Nikulin surface with $\mathrm{Pic}(S) = \Lambda_g$.

Let $f : \widetilde{S} \to S$ be a Nikulin surface together with a smooth curve $C \subset S$ of genus g such that $C \cdot N = 0$. If $\widetilde{C} := f^{-1}(C)$, then

$$f_C := f_{|\widetilde{C}} : \tilde{C} \to C$$

is an étale double cover induced by the torsion line bundle $e_C := \mathcal{O}_C(e) \in \mathrm{Pic}^0(C)[2]$. Thus $[C, e_C] \in \mathcal{R}_g$.

Since $\widetilde{C}$ is disjoint from the (-1)-curves $E_i \subset \widetilde{S}$, we identify $\widetilde{C}$ with its image $\sigma(\widetilde{C}) \subset Y$. Clearly $\widetilde{C} \in \big(E_8(-2)\big)^{\perp}$ and $(\widetilde{C})^2_Y = 4(g-1)$.

One has the following result, see [vGS] Proposition 2.7 and [GS] Corollary 2.2, based on a description of the map $f^* : H^2(S, \mathbb{Z}) \to H^2(\widetilde{S}, \mathbb{Z})$:

Proposition 3.2. *Let S be a Nikulin surface of genus g such that $j : \Lambda_g \to \mathrm{Pic}(S)$ is an isomorphism. Then $\mathbb{Z} \cdot \widetilde{C} \oplus E_8(-2) \subset \mathrm{Pic}(Y)$ is a sublattice of index 2. Furthermore $E_8(-2)$ is a primitive sublattice of* $\mathrm{Pic}(Y)$.

It follows that $\mathrm{Pic}(Y)$ is generated by $\mathbb{Z} \cdot \widetilde{C} \oplus E_8(-2)$ and an element $\big(\frac{\widetilde{C}}{2}, \frac{v}{2}\big)$, where $v \in E_8(-2)$ is an element such that

$$\frac{\widetilde{C}^2}{2} + \frac{v^2}{4} \equiv 0 \bmod 2.$$

We determine explicitly the Picard lattice of Y when $\mathrm{Pic}(S)$ is minimal hence $[S, j] \in \mathcal{F}_g^{\mathfrak{N}}$ is a general point in moduli. The answer depends on the parity of g.

Proposition 3.3. *Let (S, j) be a Nikulin surface of genus g with* $\mathrm{Pic}(S) = \Lambda_g$.
(i) Suppose g is odd. Then $\mathrm{Pic}(Y)$ *is generated by $\mathbb{Z} \cdot \widetilde{C} \oplus E_8(-2)$ and an element $\big(\frac{\widetilde{C}}{2}, \frac{v}{2}\big)$, where $v^2 = -8$.*
(ii) Suppose g is even. Then $\mathrm{Pic}(Y)$ *is generated by $\mathbb{Z} \cdot \widetilde{C} \oplus E_8(-2)$ and an element $\big(\frac{\widetilde{C}}{2}, \frac{v}{2}\big)$, where $v^2 = -4$.*

Proof. The key point is that the lattice $\mathbb{Z} \cdot \widetilde{C} \subset \mathrm{Pic}(Y)$ is primitive. This implies that if $(\frac{\widetilde{C}}{2}, \frac{v}{2})$ is the generator of $\mathrm{Pic}(Y)$ over $\mathbb{Z} \cdot \widetilde{C} \oplus E_8(-2)$, then $v \neq 0$. The same conclusion follows directly in the case when g is even for parity reasons. $\square$

We are now in a position to prove that a curve $\widetilde{C} \subset Y$ corresponding to a general Nikulin surface $[S, j] \in \mathcal{F}_g^{\mathfrak{N}}$ satisfies Green's conjecture.

Proof of Theorem 0.5. Let us choose an étale double cover $f : \widetilde{C} \to C$, where $\widetilde{C} \subset Y$ lies on a Nikulin surface with minimal Picard lattice and $C \subset S$. Applying [AF2], both $\widetilde{C}$ and C being sections of smooth $K3$ surfaces, satisfy Green's conjecture. It remains to determine the Clifford indices of both curves and for this purpose we resort to [GL3]. First we observe that $\mathrm{Cliff}(C) = [\frac{g-1}{2}]$ and the Clifford index is computed by a pencil, that is, $r(C) = 1$. Indeed, otherwise $\mathrm{Cliff}(C)$ is computed by the restriction to C of a line bundle $\mathcal{O}_S(D)$ on the surface, where $0 < C \cdot D \leq g - 1$. If

$\mathrm{Pic}(S) = \Lambda_g$, then $C \cdot D \equiv 0 \bmod 2g-2$, hence no such line bundle on S can exist, therefore Cliff(C) is maximal.

Assume now that Cliff$(\widetilde{C}) < g-1$. Since $g(\widetilde{C}) = 2g-1$ is odd, it follows automatically that $r(\tilde{C}) = 1$. Applying [GL3], there exists a divisor $D \in \mathrm{Pic}(Y)$ such that $0 \le \widetilde{C} \cdot D \le 2g-2$,

$$h^i(S, \mathcal{O}_S(D)) = h^i(C, \mathcal{O}_{\widetilde{C}}(D)) \ge 2 \ \text{ for } i = 0, 1, \ \text{ and}$$

$$\mathrm{Cliff}(\widetilde{C}) = \mathrm{Cliff}(\mathcal{O}_{\widetilde{C}}(D)) = \widetilde{C} \cdot D - D^2 - 2,$$

where the last formula follows after an application of the Riemann-Roch theorem. Since $\widetilde{C} \in \big(E_8(-2)\big)^{\perp}$, the only class in $D \in \mathrm{Pic}(Y)$ such that $0 \le \widetilde{C} \cdot D \le 2g-2$, is the generator $D := \big(\frac{\widetilde{C}}{2}, \frac{v}{2}\big)$ described in Proposition 3.3. When g is odd we compute that

$$\widetilde{C} \cdot D - D^2 - 2 = \widetilde{C} \cdot \Big(\frac{\widetilde{C}}{2} + \frac{v}{2}\Big) - \Big(\frac{\widetilde{C}}{2} + \frac{v}{2}\Big)^2 - 2 = 2(g-1) - (g-3) - 2 = g-1,$$

which contradicts the assumption Cliff$(\widetilde{C}) < g-1$. Thus $\widetilde{C}$ has maximal Clifford index.

When g is even, then $v^2 = -4$. A similar calculation yields $\widetilde{C} \cdot D - D^2 - 2 = g-2$, hence Cliff$(C) \ge g$. On the other hand, $\mathcal{O}_{\widetilde{C}}(D)$ induces a linear series $\mathfrak{g}^{g/2}_{2g-2}$ on $\widetilde{C}$, which implies that gon$(\widetilde{C}) = $ Cliff$(\widetilde{C}) + 2 = g$.

□

3.1. **The Prym-Green Conjecture and Nikulin surfaces.** An analogue of Green's conjecture for Prym-canonical curves $\phi_{K_C \otimes \eta} : C \to \mathbf{P}^{g-2}$ has been formulated in [FL].

Conjecture 3.4. *Let $[C, \eta] \in \mathcal{R}_{2i+6}$ be a general Prym curve. Then*

$$K_{i,2}(C, K_C \otimes \eta) = 0.$$

It is shown in [FL] that the subvariety in moduli

$$\mathcal{U}_{2i+6,i} := \big\{[C, \eta] \in \mathcal{R}_{2i+6} : K_{i,2}(C, K_C \otimes \eta) \ne 0\big\}$$

is the degeneracy locus of a morphism between two tautological vector bundles of the same rank defined over $\mathcal{R}_{2i+6}$. The statement of the Prym-Green Conjecture is equivalent to the generic non-degeneracy of this morphism. The conjecture, which is true in bounded genus, plays a decisive role in showing that the moduli space $\overline{\mathcal{R}}_{2i+6}$ is a variety of general type when $i \ge 4$. The validity of Conjecture 3.4 for unbounded $i \ge 0$ remains a challenging open problem. In view of Voisin's solution [V1], [V2] of the classical generic Green Conjecture by specialization to curves on $K3$ surfaces, it is an obvious question whether the Prym-Green Conjecture could be proved by specializing to Prym curves on Nikulin surfaces. Unfortunately this is not the case, as it has been already observed in [FV] Theorem 0.6. We give a second, more direct proof of the fact that Prym-canonical curves on Nikulin surfaces have extra syzygies.

Theorem 3.5. *We set $g := 2i + 6$ and let $C \subset S$ be a smooth genus g curve on a Nikulin surface, such that $C \cdot N = 0$. Then $K_{i,2}(C, K_C \otimes e_C) \ne 0$. In particular $[C, e_C] \in \mathcal{U}_{2i+6,i}$ fails to satisfy the Prym-Green conjecture.*

Proof. Since we are in a divisorial case, it is enough to prove the nonvanishing $K_{i+1,1}(C, K_C \otimes e_C) \neq 0$. Keeping the notation of this section, we set $H :\equiv C - e \in \mathrm{Pic}(S)$. By direct calculation $H^2 = 2g-6$, $H \cdot C = C^2 = 2g-2$ and note that $\mathcal{O}_C(H) = K_C \otimes e_C$. The general member $H \in |\mathcal{O}_S(H)|$ is a smooth curve of genus $2i+4$. The Green-Lazarsfeld non-vanishing theorem [GL1] applied to H yields that $K_{i+1,1}(H, K_H) \neq 0$. Since S is a regular surface, one can write an exact sequence

$$0 \longrightarrow H^0(S, \mathcal{O}_S) \longrightarrow H^0(S, \mathcal{O}_S(H)) \longrightarrow H^0(H, K_H) \longrightarrow 0,$$

which induces an isomorphism [G] Theorem (3.b.7)

$$\mathrm{res}_H : K_{i+1,1}(S, \mathcal{O}_S(H)) \cong K_{i+1,1}(H, K_H).$$

Therefore $K_{i+1,1}(S, \mathcal{O}_S(H)) \neq 0$. From [G] Theorem (3.b.1), we write the following exact sequence of Koszul cohomology groups:

$$K_{i+1,1}(S; -C, H) \to K_{i+1,1}(S, H) \to K_{i+1,1}(C, H_C) \to K_{i,2}(S; -C, H) \to \cdots.$$

The group $K_{i+1,1}(S; -C, H)$ is by definition the kernel of the morphism

$$\wedge^{i+1} H^0(S, H) \otimes H^0(S, \mathcal{O}_S(H-C)) \to \wedge^i H^0(S, H) \otimes H^0(S, \mathcal{O}_S(2H-C)).$$

But $H^0(S, \mathcal{O}_S(H-C)) = H^0(S, -e) = 0$, that is, the first map in the exact sequence above is injective, hence $K_{i+1,1}(C, \mathcal{O}_C(H)) \neq 0$. □

4. Green's conjecture for general covers of plane curves

In this section we prove the vanishing of $K_{g-2d+3,1}(C, K_C)$ for general covers of plane curves of degree d. Firstly, we show that the minimal pencils come from the plane curve.

Lemma 4.1. *Let $f : C \to \Gamma$ be a genus g double cover of a plane curve of degree $d \geq 3$. If $g > (d-2)(d+1)$, then C is $(2d-2)$-gonal.*

Proof. Apply the Castelnuovo-Severi inequality, see [ACGH] Chapter VIII. □

Observe that the curves in question carry infinitely many $\mathfrak{g}^1_{2d-2}$ pulled back from Γ, hence they do not verify the linear growth condition (1).

This phenomenon occurs quite often, if the genus is large enough compared to the gonality.

Proposition 4.2. *Let C be a smooth curve of genus g and gonality k such that $g > (k-1)^2$. If C carries two different $\mathfrak{g}^1_k$ then there exists a cover $C \to X$ such that the two $\mathfrak{g}^1_k$ are pullbacks of pencils on X.*

Proof. We apply the Castelnuovo-Severi inequality. The two pencils define a morphism $C \to \mathbf{P}^1 \times \mathbf{P}^1$, and the image is of numerical type (k, k). Then the genus of the normalization X of the image is at most $(k-1)^2$, hence the X cannot be isomorphic to C. The two rulings lifted to X pullback to the original $\mathfrak{g}^1_k$'s on C. □

Theorem 4.3. *Let $C \to \Gamma \subset \boldsymbol{P}^2$ be a general ramified double covering of genus $g \geq d^2+1$ of a smooth plane curve of degree $d \geq 3$. Then C verifies Green's conjecture, that is $K_{2d-5,2}(C, K_C) = 0$.*

Corollary 4.4. *Let $C \to \Gamma \subset \mathbf{P}^2$ be a general ramified double covering of genus $g \geq 17$ of a smooth plane quartic. Then $K_{3,2}(C, K_C) = 0$.*

Remark 4.5. The moduli space of double covers of smooth plane curves of degree d is irreducible, and hence it makes sense to speak about general double covers.

Proof. From the semicontinuity of Koszul cohomology and the irreducibility of the moduli space of double covers over smooth plane curves of degree d, the conclusion follows by exhibiting one example of a double cover C of a plane curve of degree d, for which $K_{2d-5,2} = 0$. The proof goes by induction on the genus g of C, using degenerations.

The first step. Let $S \to \mathbf{P}^2$ be a double cover ramified along a sextic. The inverse image C of a general plane curve Γ of degree d is a $(2d-2)$-gonal smooth curve of genus d^2+1 (the number of ramification points is $6d$). Applying theorem 1.8, it satisfies Green's conjecture, and hence $K_{g-2d+3,1}(C, K_C) = 0$.

The induction step. Suppose that the conclusion is true in genus g. We wish to prove it in genus $g+1$. Consider $f : C \to \Gamma$ a *smooth* genus-g double cover of a plane curve of degree d, for which $K_{g-2d+3,1}(C, K_C) = 0$. Let $x \in \Gamma$ be a general point and $\{x_0, x_1\} = f^{-1}(x) \subset C$ be the fiber over x. Attach a rational curve to C, gluing it over two points $y_0, y_1 \in \mathbf{P}^1$ with C, that is, consider

$$C' := C \cup \mathbf{P}^1 / x_0 \sim y_0,\ x_1 \sim y_1.$$

Observe that there is an admissible double cover $C' \to \Gamma'$, where $\Gamma' = \Gamma \cup \mathbf{P}^1 / x \sim y$, where $y \in \mathbf{P}^1$, see the figure 2.

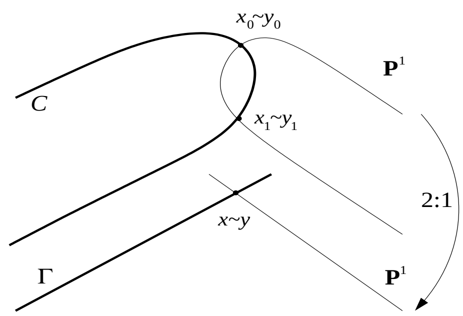

FIGURE 2. The new admissible double cover.

It is clear that the genus of C' equals $g+1$ and $p_a(\Gamma') = p_a(\Gamma)$. Arguing as in [V1], the restriction map provides us with an isomorphism

$$K_{p,1}(C', \omega_{C'}) \cong K_{p,1}(C, K_C(x_0 + x_1)).$$

From the induction hypothesis we know that $K_{g-2d+3,1}(C, K_C) = 0$. Applying Corollary 1.4, it follows that $K_{g-2d+4,1}(C, K_C(x_0 + x_1)) = 0$, hence $K_{(g+1)-2d+3,1}(C', \omega_{C'}) = 0$, the latter being the vanishing we wanted to obtain. $\square$

Proof of the second part of Theorem 0.1. This time we start with a $K3$ surface S which is a cyclic fourfold cover of $\mathbf{P}^2$ branched along a quartic. The inverse image of a general plane curve of degree d is a curve C with $g(C) = 2d^2 + 1$ and $\text{gon}(C) = 4d - 4$. The induction step is similar to the one in Theorem 4.3 see figure 3. □

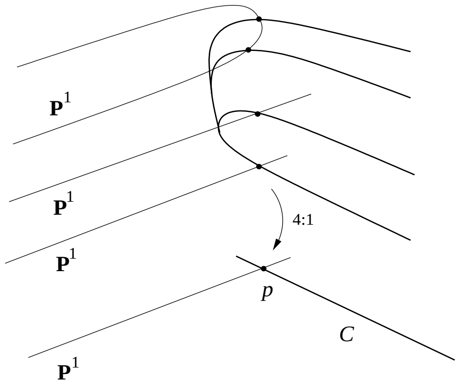

FIGURE 3. The new admissible $4:1$ cover.

The curves on the double plane that we use in the first step of the proof carry infinitely many minimal pencils, and hence they do not verify the linear growth condition (1). They are in fact special in their linear systems. According to [AF2], a general curve in the corresponding linear system does satisfy the linear growth condition. This provides us with an example of a linear system on a $K3$ surface where the dimensions of the Brill-Noether loci jump. However it makes sense to ask the following question:

Question 4.6. Is it true that any smooth curve of even genus and maximal Clifford index on a $K3$ surface carries finitely many minimal pencils?

5. GREEN'S CONJECTURE FOR GENERAL TRIPLE COVERS OF ELLIPTIC CURVES

Applying the Castelnuovo-Severi inequality as in Lemma 4.1, we obtain that if $C \to E$ is a triple cover of an elliptic curve E, then C is 6-gonal as soon as $g(C) \geq 12$.

Theorem 5.1. *Let $C \to E$ be a general triple cover of an elliptic curve, where $g(C) \geq 13$. Then $K_{3,2}(C, K_C) = 0$ and C verifies Green's Conjecture.*

Proof. The proof goes by induction on the genus and is very similar to that of Theorem 4.3. Note that the moduli space of triple covers of elliptic curves is irreducible by e.g. [GHS], hence it suffices to find an example in each genus.

The first step. Let $S \to \mathbf{P}^1 \times \mathbf{P}^1 := Q$ be a cyclic triple cover ramified along a smooth genus 4 curve, which has type $(3,3)$ on Q. It is immediate that S is a $K3$ surface. The inverse image C of a general curve E of type $(2,2)$ is a smooth 6-gonal curve of genus 13, and the induced triple cover $C \to \Gamma$ is ramified over 24 points (the ramification points of a cyclic cover are totally ramified, thus the degree of the ramification divisor is 48). Since S is a $K3$ surface, we apply [AF2], to conclude that $K_{3,2}(C, K_C) = 0$.

The induction step. We suppose that the conclusion is true in genus g and we prove it in genus $g+1$. Consider a triple covering $f : C \to E$, where both C and E are smooth curves, $g(C) = g \geq 13$ and $g(E) = 1$. Assume that $K_{g-5,1}(C, K_C) = 0$. Let $t \in E$ be a non-ramified point and $\{x_0, x_1, x_2\} = f^{-1}(t) \subset C$ be the fiber over t. Attach a rational curve R to C, gluing it along x_0 and x_1, as well as a further rational tail R' meeting C in x_2, that is, consider the (non)-stable curve

$$C' := C \cup R \cup R', \ \ C \cap R = \{x_0, x_1\}, \ \ C \cap R' = \{x_2\}.$$

There exists an admissible triple cover $f' : C' \to E'$, where $E' = \Gamma \cup_t \mathbf{P}^1$, where $f'(R) = f'(R') = \mathbf{P}^1$, $\deg(f'_R) = 2$ and $\deg(f'_{R'}) = 1$.

The genus of C' equals $g+1$ and there is an isomorphism

$$K_{p,1}(C', \omega_{C'}) \cong K_{p,1}(C, K_C(x_0 + x_1)).$$

From the induction hypothesis we know that $K_{g-5,1}(C, K_C) = 0$. Applying projection of syzygies, it follows that $K_{g-4,1}(C, K_C(x_0 + x_1)) = 0$, hence $K_{(g+1)-5,1}(C', \omega_{C'}) = 0$, the latter being the vanishing we were looking for. $\square$

Remark 5.2. A slight modification in the proof shows that Green's Conjecture also holds for general *cyclic triple covers* of elliptic curves with source being a curve of odd genus $g \geq 13$. The modification of the proof appears in the inductive argument. Starting with $f : C \to E$ as above, we can attach a smooth rational curve meeting C at x_0, x_1 and x_2. The resulting curve has genus $g+2$ and smooths to a cyclic cover over an elliptic curve.

6. SYZYGIES OF DOUBLE COVERS OF CURVES OF CLIFFORD DIMENSION 3

We present an inductive proof of Theorem 0.3 and consider a curve $[X] \in \mathcal{M}_{10}$ with $r(X) = 3$, thus $W^3_9(C) \neq \emptyset$ and dim $W^1_6(X) = 1$. If $f : C \to X$ is a genus g double cover, the Castelnuovo-Severi inequality implies that $\mathrm{gon}(C) = 12$ as soon as $g \geq 30$. The critical point in the proof is the starting case, the inductive step is identical to that in the proof of Theorem 0.1.

Proof of Theorem 0.3. We choose a smooth cubic surface $Y = \mathrm{Bl}_6(\mathbf{P}^2)$ and denote by $h \in \mathrm{Pic}(S)$ the class of the pull-back of a line in $\mathbf{P}^2$ and by $E_1, \ldots, E_6$ the exceptional divisors on Y. We choose a general genus 4 curve

$$B \in |-2K_Y| = |\mathcal{O}_Y(6h - 2E_1 - \cdots - 2E_6)|$$

and let $f : S \to Y$ be the double cover branched along B. Then S is a smooth $K3$ surface and let $\iota \in \mathrm{Aut}(S)$ be the covering involution of f. Clearly $H^2(S, \mathbb{Z})^\iota$ can be identified with the pull-back of the Picard lattice of Y, and when $B \in |-2K_Y|$ is general, reasoning along the lines of [AK] Theorem 2.7 we observe that

$$\mathrm{Pic}(S) = H^2(S, \mathbb{Z})^\iota = f^*\mathrm{Pic}(Y) = \mathbb{Z}\langle f^*(h), \mathcal{O}_S(R_1), \ldots, \mathcal{O}_S(R_6)\rangle,$$

where $R_i := f^*(E_i)$ are (-2)-curves. We further choose a general curve $X \in |-3K_Y|$, thus $g(X) = 10$ and $r(X) = 3$. Let $C := f^{-1}(X) \subset S$, hence $g(C) = 28$. As a section of the $K3$ surface S, the curve C satisfies

Green's Conjecture and Theorem 0.3 follows once we show that $\mathrm{gon}(C) = 12$. Assume by contradiction that $\mathrm{gon}(C) < 12$. Applying once more [GL3], there exists a divisor class

$$D \equiv af^*(h) - b_1R_1 - \cdots - b_6R_6 \in \mathrm{Pic}(S),$$

with $a, b_1, \ldots, b_6 \in \mathbb{Z}$, such that $0 \leq C \cdot D \leq g - 1 = 27$, $h^i(S, \mathcal{O}_S(D)) \geq 2$ for $i = 0, 1$ and

$$\mathrm{gon}(C) = \mathrm{Cliff}(\mathcal{O}_C(D)) + 2 = C \cdot D - D^2 =$$

$$= \phi(D) := 18a - 2a^2 - 6(b_1 + \cdots + b_6) + 2(b_1^2 + \cdots + b_6^2) < 12.$$

From the Castelnuovo-Severi inequality, we find that $\phi(D) \geq 9$, hence based on parity $\phi(D) = 10$. Note that $C \cdot D \geq 10$ and is a multiple of 6, hence $C \cdot D \in \{12, 18, 24\}$. We study each of these cases separately. If $C \cdot D = 18$ and $D^2 = 8$, then

$$b_1 + \cdots + b_6 = 3a - 3 \text{ and } b_1^2 + \cdots + b_6^2 = a^2 - 4.$$

By the Cauchy-Schwarz inequality $6(\sum_{i=1}^6 b_i^2) \geq (\sum_{i=1}^6 b_i)^2$, and hence $a^2 - 6a + 11 \leq 0$, which is a contradiction. If $C \cdot D = 24$ and $D^2 = 14$, then

$$b_1 + \cdots + b_6 = 3a - 4 \text{ and } b_1^2 + \cdots + b_6^2 = a^2 - 7,$$

which leads to the contradiction $3a^2 - 24a + 58 \leq 0$. Finally if $C \cdot D = 12$ and $D^2 = 2$, then

$$b_1 + \cdots + b_6 = 3a - 2 \text{ and } b_1^2 + \cdots + b_6^2 = a^2 - 1.$$

Again the Cauchy-Schwarz inequality implies that the only possible case is when $a = 2$ and then $\sum_{i=1}^6 b_i = 4$ and $\sum_{i=1}^6 b_i^2 = 3$. It is obvious (compare the parities) that these diophantine equations have no common solution. We conclude that $\mathrm{gon}(C) = 12$.

$\square$

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