For a $K3$ surface $S$, a smooth curve $C \subset S$ and a globally generated linear series $A \in W^d_r(C)$ with $h^0(C, A) = r + 1$, the Lazarsfeld-Mukai vector bundle $E_{C,A}$ is defined via the following elementary modification on $S$

\begin{equation}
0 \to E_{C,A}^I \to H^0(C, A) \otimes \mathcal{O}_S \to A \to 0.
\end{equation}

The bundles $E_{C,A}$ have been introduced more or less simultaneously in the 80’s by Lazarsfeld [L1] and Mukai [M1] and have acquired quite some prominence in algebraic geometry. On one hand, they have been used to show that curves on general $K3$ surfaces verify the Brill-Noether theorem [L1], and this is still the only class of smooth curves known to be general in the sense of Brill-Noether theory in every genus. When $\rho(g, r, d) = 0$, the vector bundle $E_{C,A}$ is rigid and plays a key role in the classification of Fano varieties of coindex 3. For $g = 7, 8, 9$, the corresponding Lazarsfeld-Mukai bundle has been used to coordinatize the moduli space of curves of genus $g$, thus giving rise to a new and more concrete model of $\mathcal{M}_g$, see [M2], [M3], [M4]. Furthermore, Lazarsfeld-Mukai bundles of rank two have led to a characterization of the locus in $\mathcal{M}_g$ of curves lying on $K3$ surfaces in terms of existence of linear series with unexpected syzygies [F], [V]. For a recent survey on this circle of ideas, see [A].

Recently, Lazarsfeld-Mukai bundles have proven to be effective in shedding some light on an interesting conjecture of Mercat in Brill-Noether theory, see [FO1], [FO2], [LMN]. Recall that the Clifford index of a semistable vector bundle $E \in \mathcal{U}_C(n, d)$ on a smooth curve $C$ of genus $g$ is defined as

$$\gamma(E) := \mu(E) - \frac{2}{n} h^0(C, E) + 2 \geq 0.$$ 

Then the higher Clifford indices of the curve $C$ are defined as the quantities

$$\text{Cliff}_n(C) := \min \left\{ \gamma(E) : E \in \mathcal{U}_C(n, d), \ d \leq n(g - 1), \ h^0(C, E) \geq 2n \right\}.$$ 

For any line bundle $L$ on $C$ such that $h^i(C, L) \geq 2$ for $i = 0, 1$, that is, contributing to the Clifford index $\text{Cliff}(C)$, by computing the invariants of the strictly semistable vector bundle $E := L^\otimes n$, one finds $\text{Cliff}_n(C) \leq \text{Cliff}(C)$. Mercat [Me1] predicted that for any smooth curve $C$ of genus $g$, the following equality

$$ (M_n) : \quad \text{Cliff}_n(C) = \text{Cliff}(C).$$ 

should hold. Counterexamples to $(M_2)$ have been found on curves lying on $K3$ surfaces that are special in Noether-Lefschetz sense, see [FO1], [FO2] and [LN2]. However, $(M_2)$ is expected to hold for a general curve of genus $g$, and in fact even for a curve $C$ lying on
a $K3$ surface $S$ such that $\text{Pic}(S) = \mathbb{Z} \cdot C$. For instance, it is known that $(M_2)$ holds on $M_{11}$ outside a certain Koszul divisor (which also admits a Noether-Lefschetz realization), see [FO2] Theorem 1.3. It is also known that $(M_2)$ holds generically on $M_g$ for $g \leq 16$, see [FO1].

It has been proved in [LMN] that rank three restricted Lazarsfeld-Mukai bundles invalidate statement $(M_3)$ in genus $9$ and $11$ respectively, that is, Mercat’s conjecture in rank three fails generically on $M_9$ and $M_{11}$ respectively. This was then extended in [FO2] Theorem 1.4, to show that on a $K3$ surface $S$ with $\text{Pic}(S) = \mathbb{Z} \cdot C$, where $C^2 = 2g - 2$, if $A \in W^2_d(C)$ is a linear system where $d := \lfloor \frac{2g+8}{3} \rfloor$, the restriction to $C$ of the Lazarsfeld-Mukai bundle $E_{C,A}$ is stable and has Clifford index strictly less than $\lfloor \frac{g-1}{2} \rfloor$, in particular, statement $(M_3)$ fails for the curve $C$. For further background on this problem, we also refer to [Me1], [LN1] and [GMN].

The restricted Lazarsfeld-Mukai bundle $E|_C := E_{C,A} \otimes \mathcal{O}_C$ sits in the following exact sequence
\[ 0 \rightarrow Q_A \rightarrow E|_C \rightarrow K_C \otimes A^\vee \rightarrow 0, \]
where $Q_A = M^\vee_A$ is the dual of the kernel bundle defined by the sequence
\[ 0 \rightarrow M_A \rightarrow H^0(C, A) \otimes \mathcal{O}_C \rightarrow A \rightarrow 0. \]

One then shows [V], [FO2] that the sequence (2) is exact on global sections, that is,
\[ h^0(C, E|_C) = h^0(C, K_C \otimes A^\vee) + h^0(C, Q_A) = g - d + 2r + 1. \]

By choosing the degree $d$ minimal such that $W^r_d(C) \neq \emptyset$, precisely $d = r + \lfloor \frac{r(g+1)}{r+1} \rfloor$, it becomes clear that for sufficiently high $g$, one has
\[ \gamma(E|_C) < \text{Cliff}(C), \]
that is, $E|_C$, when semistable, provides a counterexample to Mercat’s conjecture $(M_{r+1})$.

We prove the following result, extending to rank $4$ a picture studied in smaller ranks in the papers [M1], [V], respectively [FO2].

**Theorem 0.1.** Let $S$ be a $K3$ surface with $\text{Pic}(S) = \mathbb{Z} \cdot L$, where $L^2 = 2g - 2$ and write
\[ g = 4i - 4 + \rho \quad \text{and} \quad d = 3i + \rho, \]
with $\rho \geq 0$ and $i \geq 6$. Then for a general curve $C \in |L|$ and a globally generated linear series $A \in W^3_d(C)$ with $h^0(C, A) = 4$, the restriction to $C$ of the Lazarsfeld-Mukai bundle $E_{C,A}$ is stable.

Note that in Theorem 0.1, $\dim W^3_d(C) = \rho$. The rank $3$ version of this result was proved in [FO2]. We record the following consequence of Theorem 0.1:

**Corollary 0.2.** For $C \subset S$ with $g \geq 20$ and $\text{Pic}(S) = \mathbb{Z} \cdot C$, we set $d := \lfloor \frac{4g+14}{3} \rfloor$ and $A \in W^3_d(C)$ with $h^0(C, A) = 4$. Then $E|_C$ is a stable rank $4$ bundle with $\gamma(E|_C) < \lfloor \frac{2g-1}{2} \rfloor$. It follows that the statement $(M_4)$ fails for $C$.

The curves $C$ appearing in Corollary 0.2 are Brill-Noether general, that is, they satisfy $\text{Cliff}(C) = \lfloor \frac{2g-1}{2} \rfloor$, see [L1]. We also show that under mild restrictions, on a very general $K3$ surface, the extension (2) is non-trivial and the restricted Lazarsfeld-Mukai bundle $E|_C$ is simple (see Theorem 1.3). We expect that the bundle $E|_C$ remains stable also for
higher ranks $r + 1 = h^0(C, A)$, at least when $\text{Pic}(S) = \mathbb{Z} \cdot C$. However, our method of proof based on the Bogomolov inequality, seem not to extend easily for $r \geq 4$.

The second topic we discuss in this paper concerns the connection between normal bundles of canonical curves and Mercat’s conjecture. For a smooth canonically embedded curve $C \subset \mathbb{P}^{g-1}$ of genus $g$, we consider the normal bundle $N_C := N_{C/\mathbb{P}^{g-1}}$, and then we define the twist of the conormal bundle $E := N_C^\otimes \otimes K_C^\otimes 2$. By direct calculation

\[ \det(E) = K_C^\otimes (g-5) \quad \text{and} \quad \text{rk}(E) = g - 2. \]

In particular, the vector bundle $E$ contributes to $\text{Cliff}_{g-2}(C)$ if and only if $g \leq 8$. Since $M_{K_C}(-1) = \Omega_{\mathbb{P}^{g-1}|C}$, the bundle $E$ sits in the following exact sequence

\[ 0 \rightarrow E \rightarrow M_{K_C} \otimes K_C \rightarrow K_C^\otimes 3 \rightarrow 0, \]

where $\gamma_{K_C} : H^0(C, M_{K_C} \otimes K_C) \rightarrow H^0(C, K_C^\otimes 3)$ is the Gaussian map of $C$, see [W]. The map $\gamma_{K_C}$ vanishes on symmetric tensors, hence $\text{Ker}(\gamma_{K_C}) = I_2(K_C) \otimes \text{Ker}(\psi_{K_C})$, where

\[ \psi_{K_C} := \gamma_{K_C}|_{\lambda^2 H^0(C, K_C^\otimes 3)} : \bigwedge^2 H^0(C, K_C) \rightarrow H^0(C, K_C^\otimes 3) \]

and $I_2(K_C) = K_{1,1}(C, K_C)$ is the space of quadrics containing the canonical curve $C$. The map $\psi_{K_C}$ has been studied intensely in the context of deformations in $\mathbb{P}^g$ of the cone over the canonical curve $C \subset \mathbb{P}^{g-1}$, see [W]. It is in particular known [CHM], [V] that $\psi_{K_C}$ is surjective for a general curve $C$ of genus $g \geq 12$.

We now specialize to the case $g = 7$, when $E$ contributes to $\text{Cliff}_5(C)$. Then $\text{rk}(E) = 5$ and $\det(E) = K_C^\otimes 2$, therefore $\mu(E) = \frac{24}{5}$. It is easy to show that the Gaussian map $\psi_{K_C}$ is injective for every smooth curve $C$ of genus 7 having maximal Clifford index $\text{Cliff}(C) = 3$. In particular,

\[ H^0(C, E) = I_2(K_C) \]

is a 10-dimensional space, and $\gamma(E) = 2 + \frac{4}{5} < \text{Cliff}(C)$. We establish the following result:

**Theorem 0.3.** The normal bundle $N_{C/\mathbb{P}^6}$ of every canonical curve $C$ of genus 7 with maximal Clifford index is stable. In particular, the Mercat conjecture $(M_5)$ fails for a general curve of genus 7.

The proof of Theorem 0.3 uses an essential way Mukai’s realisation [M3] of a canonical curve $C$ of genus 7 with $\text{Cliff}(C) = 3$ as a linear section of the 10-dimensional spinorial variety $OG(5, 10) \subset \mathbb{P}^{15}$. In particular, the vector bundle $E$ is the restriction to $C$ of the rank 5 spinorial bundle on $OG(5, 10)$, which endows $E$ with an extra structure that only exists in genus 7. Note that the normal bundle of every canonical curve of genus at most 6 is unstable, and more generally, the normal bundle of a tetragonal canonical curve of any genus is unstable (see also Section 3). In particular, we have the following identification of cycles on $M_7$:

\[ \{ |C| \in M_7 : N_C \text{ is unstable} \} = M^4_A, \]

where the right hand side denotes the tetragonal divisor on $M_7$. We make the following conjecture:
Let there exists an elliptic pencil $E \in \mathcal{L}$.

We begin by showing that in rank $h$ of $\text{Pic}(\mathcal{P})$ that if $E < d < g$ of $K$ the linear series $A$ financed by the Sonderforschungsbereich 647 "Raum-Zeit-Materie".

Corollary 1.2. Theorem 1.1. The rank $1$. 1. Simplicity of restricted Lazarsfeld-Mukai bundles

We fix a $K3$ surface $S$, a smooth curve $C \subset S$ of genus $g$ and a globally generated linear series $A \in W^r_d(C)$, with $h^0(C, A) = r + 1$. Using the evaluation sequence (1), we form the vector bundle $F = F_{C,A}$ by dualizing, we obtain an exact sequence for the dual bundle $E = [E_{C,A}]$ of $F_{C,A}$.

\begin{equation}
0 \rightarrow H^0(C, A)^* \otimes O_S \rightarrow E_{C,A} \rightarrow K_C \otimes A^* \rightarrow 0.
\end{equation}

It is well-known [M1], [L1] that $c_1(E) = [C]$ and $c_2(E) = d$; moreover $h^0(S, F) = 0$ and $h^1(S, E) = h^1(S, F) = 0$. Finally, one also has that

\[ \chi(S, E \otimes F) = 2 - 2\rho(g, r, d); \]

in particular, if $E$ is a simple bundle, then $\rho(g, r, d) \geq 0$. Assuming furthermore that $\text{Pic}(S) = \mathbb{Z} \cdot C$, it is also well-known that both $E$ and $F$ are $C$-stable bundles on $S$.

1.1. The rank 2 case. We begin by showing that in rank 2, irrespective of the structure of $\text{Pic}(S)$, a splitting of the restriction $E|_C$ can only be induced by an elliptic pencil on the $K3$ surface.

Theorem 1.1. Let $C \subset S$ be as above and a base point free pencil $A \in W^\lambda_d(C)$ of degree $2 \leq d < g - 1$ with $K_C \otimes A^\lambda$ globally generated. The following conditions are equivalent:

(i) $E|_C \cong A \oplus (K_C \otimes A^\lambda)$;

(ii) There exists an elliptic pencil $N \in \text{Pic}(S)$ such that $N|_C = A$.

(iii) $h^0(S, E \otimes F) < h^0(C, E \otimes F|_C)$.

Corollary 1.2. With notation as above, if $g \leq 2d - 2$ and $A$ is not induced by an elliptic pencil on $S$, then $E|_C$ is simple if and only if $E$ is simple.

Note that it is easy to see that if $E|_C$ is simple, then $E$ is also simple. It is also known that if $E$ is simple, then automatically $g \leq 2d - 2$. 

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Conjecture 0.4. The normal bundle $N_C$ of a general canonical curve $C$ of genus $g \geq 7$ is stable.
Proof. (of Theorem 1.1) (ii)⇒(i). Let $N$ be an elliptic pencil with $N|_C = A$. Consider the exact sequence

$$0 \to N^\vee \to F \to N(-C) \to 0.$$  

Its restriction to $C$ gives a splitting of the dual of the sequence (2) characterizing $E|_C$. Observe that since $d < g - 1$, there is no morphism from $A^\vee$ to $K_C^C \otimes A$.

(i)⇒(ii). Conversely, suppose that $E|_C = A \oplus (K_C \otimes A^\vee)$. Applying $\text{Hom}(K_C \otimes A^\vee, - )$ to the sequence (1), we obtain an exact sequence

$$0 \to \text{Ext}^1(K_C \otimes A^\vee, F) \to \text{Ext}^1(K_C \otimes A^\vee, H^0(C, A) \otimes O_S) \to \text{Ext}^1(K_C \otimes A^\vee, A).$$  

Since the extension class $[E] \in \text{Ext}^1(K_C \otimes A^\vee, H^0(C, A) \otimes O_S)$ maps to the trivial extension in $\text{Ext}^1(K_C \otimes A^\vee, A)$, it follows that there exists a rank 2 bundle $G$ on $S$ which fits into a commutative diagram:

$$(5) \quad \begin{array}{cccccc}
0 & 0 & \downarrow & & & \\
0 & \to F & \to H^0(A) \otimes O_S & \to A & \to 0 \\
0 & \to G & \to E & \to A & \to 0 \\
K_C \otimes A^\vee & \to K_C \otimes A^\vee \\
0 & 0 & & & &
\end{array}$$

Using that $H^0(S, F) = H^1(S, F) = 0$, we obtain $H^0(S, G) \cong H^0(C, K_C \otimes A^\vee)$. Since $h^0(S, E) = h^0(C, A) + h^1(C, A) = h^0(C, A) + h^0(S, G)$, and $h^1(S, E) = 0$, it follows that $H^1(S, G) = 0$. From the second row of (5), we find that $H^0(S, G(-C)) = 0$.

Furthermore, we compute $c_1(G) = 0$ and $c_2(G) = 2d - 2g + 2$. So $c_2(G) < 0 = c_1^2(G)$, that is, $G$ violates Bogomolov’s inequality, and then it sits in an extension

$$(6) \quad 0 \to M \to G \to M^\vee \otimes \mathcal{I}_\Gamma/S \to 0,$$

where $\Gamma$ is a zero-dimensional subscheme of $S$, and $M \in \text{Pic}(S)$ is such that $M^2 > 0$ and $M \cdot H > 0$ for any ample line bundle $H$ on $S$. In particular, $H^0(S, M^\vee) = 0$, and hence $H^0(S, M) \cong H^0(S, G) \cong H^0(C, K_C \otimes A^\vee) \neq 0$. Moreover, since

$$h^0(S, M^\vee \otimes \mathcal{I}_\Gamma/S) = h^1(S, G) = 0,$$

it also follows that $H^1(S, M) = 0$.

On the other hand $H^0(S, F) = 0$, which implies that the composed map

$$M \to G \to K_C \otimes A^\vee$$

is non-zero; in fact, we claim that it is surjective, that is, $M|_C = K_C \otimes A^\vee$. Suppose that $M|_C = K_C \otimes A^\vee(-D')$, with $D' \neq 0$ an effective divisor on $C$. Since $h^0(S, G(-C)) = 0$,
we have \( h^0(S, M(-C)) = 0 \), which implies \( h^0(S, M) \leq h^0(C, M|_C) \). Since we assumed \( K_C \otimes A^\vee \) to be globally generated, we have that
\[
 h^0(S, M) \leq h^0(C, K_C \otimes A^\vee(-D')) < h^0(C, K_C \otimes A^\vee) = h^0(S, M),
\]
a contradiction.

Setting \( N := M^\vee(C) \), we have shown that \( N|_C = A \) and there is an exact sequence
\[
0 \longrightarrow M^\vee \longrightarrow N \longrightarrow A \longrightarrow 0.
\]
Since \( h^0(S, M^\vee) = h^1(S, M^\vee) = 0 \), it follows that \( H^0(S, N) = H^0(C, A) \). To see that \( N \) defines and elliptic pencil, we infer that the exact sequence above and the identity \( h^0(S, M) = h^1(C, A) \) imply \( h^1(S, N) = h^2(S, N) = 0 \) and hence \( N^2 = 0 \) from Riemann-Roch.

\((iii) \Rightarrow (i)\). From the sequence (1) twisted by \( E(-C) \cong F \), we obtain that
\[
H^0(S, E \otimes F(-C)) \subset H^0(C, A) \otimes H^0(S, E(-C)),
\]
and, since \( F \) has no sections, it follows that \( H^0(S, E \otimes F(-C)) = 0 \). We have an exact sequence
\[
0 \longrightarrow H^0(S, E \otimes F) \longrightarrow H^0(S, E \otimes F|_C) \longrightarrow H^1(S, E \otimes F(-C)).
\]
The hypothesis implies that \( H^1(S, E \otimes F(-C)) \neq 0 \). From (1) twisted by \( E(-C) \cong F \), we obtain the exact sequence in cohomology
\[
0 \longrightarrow H^0(C, E|_C \otimes K_C^\vee \otimes A) \longrightarrow H^1(S, E \otimes F(-C)) \longrightarrow H^0(C, A) \otimes H^1(S, E(-C)) = 0,
\]
therefore \( h^0(C, E|_C \otimes K_C^\vee \otimes A) \neq 0 \). The sequence (2) yields to an exact sequence
\[
0 = H^0(C, K_C^\vee \otimes A^{\otimes 2}) \longrightarrow H^0(C, E|_C \otimes K_C^\vee \otimes A) \longrightarrow H^0(C, O_C) \rightarrow H^1(C, K_C^\vee \otimes A^{\otimes 2}).
\]
Then \( H^0(C, E|_C \otimes K_C^\vee \otimes A) \rightarrow H^0(C, O_C) \) is an isomorphism and under the coboundary map
\[
H^0(C, O_C) \ni 1 \mapsto 0 \in H^1(C, K_C^\vee \otimes A^{\otimes 2}),
\]
that is, the sequence (2) is split.

Note that we also have \( h^1(S, E \otimes F(-C)) = 1 \) and \( h^0(C, E \otimes F|_C) = h^0(S, E \otimes F) + 1 \).

\((i) \Rightarrow (iii)\). From the hypothesis and from the sequence (2), we find
\[
h^0(C, E|_C \otimes A^\vee) = h^0(C, K_C \otimes A^{\otimes(-2)}) + 1.
\]
Furthermore, \( h^0(S, E \otimes F) = h^0(C, E|_C \otimes A^\vee); \) twist (4) by \( F \) and use the vanishing of \( h^0(F) \) and that of \( h^1(F) \).

On the other hand, since \( E|_C \cong A \oplus K_C \otimes A^\vee \), we have
\[
h^0(C, E \otimes F|_C) = 2 + h^0(C, K_C \otimes A^{\otimes(-2)}),
\]
hence \( h^0(C, E \otimes F|_C) = h^0(S, E \otimes F) + 1 \). \( \square \)
1.2. Lazarsfeld-Mukai bundles of higher rank. We study when the restriction \( E|_C \) is a simple vector bundle. Our main tool is a variant of the Bogomolov instability theorem.

**Theorem 1.3.** Let \( S \) be a K3 surface and \( C \subset S \) a smooth curve of genus \( g \geq 4 \) such that \( \text{Pic}(S) = \mathbb{Z} \cdot C \). We fix positive integers \( r \) and \( d \) such that
\[
\rho(g, r, d) \geq 0, \quad g \geq 2r + 4 \quad \text{and} \quad d \leq \frac{3r(g-1)}{2r+2}.
\]
Then for any linear series \( A \in W^*_d(C) \) such that \( h^0(C, A) = r+1 \) and \( K_C \otimes A^\vee \) is globally generated, the restricted Lazarsfeld-Mukai bundle \( E|_C \) is simple.

Note that in the special case \( \rho(g, r, d) = 0 \), the constraints from the previous statement give rise to the bound \( g > 2r + 5 \).

**Proof.** Step 1. We first establish that the natural extension (2), that is,
\[
0 \longrightarrow Q_A \longrightarrow E|_C \longrightarrow K_C \otimes A^\vee \longrightarrow 0
\]
is non-trivial. Assuming that (2) is trivial. Then there is an injective morphism from \( K_C \otimes A^\vee \) to \( E|_C \) and hence a surjective map \( F(C) \rightarrow A \). Then
\[
G := \text{Ker}\{F(C) \rightarrow A\}
\]
is a vector bundle of rank \( r+1 \) with Chern classes \( c_1(G) = (r-1)[C] \) and
\[
c_2(G) = c_2(F(C)) - c_1(F(C)) \cdot C + \deg(A) = 2d + r(r-3)(g-1).
\]
We compute the discriminant of \( G \)
\[
\Delta(G) = 2\text{rk}(G)c_2(G) - (\text{rk}(G) - 1)c_1^2(G) = 4d(r+1) - 8r(g-1) < 0,
\]
hence \( G \) is unstable. Applying [HL] Theorem 7.3.4, there exists a subsheaf \( M \subset G \) with
\[
\xi^2_{M,G} \geq -\frac{\Delta(G)}{r(r+1)^2},
\]
where \( \xi_{M,G} = c_1(M)/\text{rk}(M) - c_1(G)/\text{rk}(G) \). Setting \( c_1(M) = k \cdot [C] \) and \( s := \text{rk}(M) \), the previous inequality becomes
\[
\left( \frac{k}{s} - \frac{r-1}{r+1} \right)^2 (2g-2) \geq \frac{8r(g-1) - 4d(r+1)}{r(r+1)^2}.
\]
Note that \( M \) destabilizes \( G \), which coupled with the stability of \( F(C) \) yields
\[
\frac{r - 1}{r+1} \leq \frac{k}{s} < \frac{r}{r+1},
\]
implying after manipulations \( 2d(r+1) > 6(g-1)r \), thus contradicting the hypothesis.

Step 2. Assuming that \( E|_C \) is non-simple, we deduce that the extension (2) splits. We consider the exact sequence
\[
H^0(S, E \otimes F) \longrightarrow H^0(C, E \otimes F|_C) \longrightarrow H^1(S, E \otimes F(-C)).
\]
and it suffices to show that \( H^1(S, E \otimes F(-C)) = 0 \). Assuming this not to be the case, twisting (1) by \( E(-C) \) induces the exact sequence
\[
H^0(C, A \otimes E|_C \otimes K|_C) \longrightarrow H^1(S, E \otimes F(-C)) \longrightarrow H^0(C, A) \otimes H^1(S, E(-C)).
\]
Since $H^1(S, E(-C)) = 0$, we obtain that $H^0(C, A \otimes E|_{C} \otimes K_C^\vee) \neq 0$. Furthermore, $Q_A$ is a stable bundle and since $\mu(Q_A \otimes A \otimes K_C^\vee) < 0$, we find that

$$H^0(C, Q_A \otimes A \otimes K_C^\vee) = 0,$$

hence we also have the sequence induced from (2) after twisting with $A \otimes K_C^\vee$

$$0 \rightarrow H^0(C, E|_{C} \otimes K_C^\vee \otimes A) \rightarrow H^0(C, \mathcal{O}_C) \rightarrow H^1(C, K_C^\vee \otimes A \otimes Q_A),$$

We conclude that the coboundary map $H^0(C, \mathcal{O}_C) \rightarrow H^1(C, Q_A \otimes A \otimes K_C^\vee)$ is trivial, that is, $E|_{C} \cong Q_A \oplus (K_C \otimes A^\vee)$, which completes the proof.

2. Stability of restricted Lazarsfeld-Mukai bundles

2.1. The rank 2 case. If $C \subset S$ is an ample curve, then with one exception ($g = 10$ and $C$ a smooth plane sextic), $\text{Cliff}(C)$ is computed by a pencil, see [CP] Proposition 3.3. We show that in rank 2 the semistability of the LM bundle is preserved under restriction.

**Theorem 2.1.** Let $S$ be a $K3$ surface, $C \subset S$ an ample curve of genus $g \geq 4$ and $A \in W_1^4(C)$ a pencil computing $\text{Cliff}(C)$. If $E_{C,A}$ is $C$-semistable on $S$, then $E|_{C}$ is also semistable on $C$. Moreover, if $E_{C,A}$ is $C$-stable on $S$, then $E|_{C}$ is stable on $C$.

**Proof.** The proof of the stability is similar, and hence we discuss the semistability part only. We write $A = \mathcal{O}_C(D)$, where $D$ is an effective divisor on $C$. Suppose $E|_{C}$ is unstable and consider an exact sequence

$$0 \rightarrow L_1 \rightarrow E|_{C} \rightarrow K_C \otimes L_1^\vee \rightarrow 0,$$

with $\deg(L_1) \geq g$. Since $L_1 \notin A$, the composed map $L_1 \rightarrow E|_{C} \rightarrow K_C \otimes A^\vee$ must be non-zero, that is, $L_1 = K_C(-D - D_1)$, where $D_1$ is an effective divisor on $C$. Set $d_1 := \deg(D_1)$. Consider the elementary modification

$$0 \rightarrow V \rightarrow E \rightarrow A(D_1) \rightarrow 0$$

induced by the composition $E \rightarrow E|_{C} \rightarrow A(D_1)$. Then

$$c_1(V) = 0 \text{ and } c_2(V) = 2d + d_1 - 2g + 2 < 0,$$

hence $V$ is unstable with respect to any polarization and fits in an exact sequence

$$0 \rightarrow M \rightarrow V \rightarrow M^\vee \otimes \mathcal{I}_{\Gamma/S} \rightarrow 0,$$

where $\Gamma \subset S$ is a 0-dimensional subscheme and $M$ is a divisor class that intersects positively any ample class on $S$ and with $M^2 > 0$. From (7) and (8) we find that $H^0(S, M) \cong H^0(S, V)$ and $H^0(S, M(-C)) = 0$. Dualizing (7), we obtain the sequence

$$0 \rightarrow F \rightarrow V^\vee \rightarrow K_C(-D - D_1) \rightarrow 0,$$

from which, using that $V \cong V^\vee$, we obtain $H^0(S, V) = H^0(C, K_C(-D - D_1))$.

We claim that $\text{Cliff}(A(D_1)) = \text{Cliff}(C)$. Recall that $h^0(S, E) = h^0(C, A) + h^1(C, A)$, and, from the sequence (7) we write

$$h^0(S, E) \leq h^0(C, A(D_1)) + h^1(C, A(D_1)).$$

By assumption, the pencil $A$ computes $\text{Cliff}(C)$, which implies

$$\text{Cliff}(C) = g + 1 - h^0(A) - h^1(A) \geq g + 1 - h^0(A(D_1)) - h^1(A(D_1)) = \text{Cliff}(A(D_1)).$$
It follows that $\text{Cliff}(A(D_1)) = \text{Cliff}(C)$, in particular $K_C(-D-D_1)$ is globally generated.

Clearly, $M \not\subseteq F$, hence the composition $\varphi : M \to V \to K_C(-D-D_1)$ is non-zero and one writes $\text{Im}(\varphi) = K_C(-D - D_1 - D_2)$, where $D_2$ is an effective divisor on $C$. If $D_2 \neq 0$, then one has the sequence of inequalities

$$h^0(S, M) \leq h^0(C, K_C(-D - D_1 - D_2)) < h^0(C, K_C(-D - D_1)) = h^0(S, M),$$

a contradiction. Therefore $M|_C = K_C(-D - D_1)$, Viewing $M$ as a subsheaf of $E$, we find $\mu(M) = M \cdot C = \deg(L_1) > \mu(E)$, thus bringing the proof to an end. \hfill $\square$

**Remark 2.2.** If $E_{C,A}$ is stable, then it is simple and hence $d = \lfloor \frac{g+3}{2} \rfloor$, see [L1]. Conversely, if $C' \subset S$ is an ample curve of genus $g$ and gonality $\lfloor \frac{g+3}{2} \rfloor$, then it was shown in [LC] that the LM bundle $E_{C,A}$ corresponding to a general curve $C \in |O_S(C')|$ and a pencil $A \in W_3^{\lfloor \frac{g+3}{2} \rfloor}(C)$ is $C$-semistable (even stable when $g$ is odd).

### 2.2. Stability of Lazarsfeld-Mukai bundles of rank four.

We show that restrictions of LM bundles of rank 4 on very general $K3$ surfaces of genus $g \geq 20$ are stable. Similar results were established in [V] and [FO2] for rank 2 and 3 respectively. We fix integers $i \geq 6$ and $\rho \geq 0$ and write

$$g := 4i - 4 + \rho \quad \text{and} \quad d := 3i + \rho,$$

so that $\rho(g, 3, d) = \rho$. Let $S$ be a $K3$ surface and $C \subset S$ a curve of genus $g$ such that $\text{Pic}(S) = \mathbb{Z} \cdot C$, and pick a globally generated linear series $A \in W_3^{\lfloor \frac{g+3}{2} \rfloor}(C)$ with $h^0(C, A) = 4$.

**Proof of Theorem 0.1.** Our previous results show that $E|_C$ is simple, hence indecomposable. Suppose $E|_C$ is not stable and fix a maximal destabilizing sequence

$$0 \to M \to E|_C \to N \to 0.$$

Put $d_N := \deg(N)$ and $d_M := \deg(M) = 2g - 2 - d_N$. Since $M$ is destabilizing,

$$\frac{d_M}{\text{rk}(M)} \geq \frac{g - 1}{2}, \quad \frac{d_N}{\text{rk}(N)} \leq \frac{g - 1}{2}. \tag{9}$$

The bundle $N$, being a quotient of $E$, is globally generated. Since $H^0(C, E|_C) = 0$, clearly $N \neq O_C$, therefore $h^0(C, N) \geq 2$. From the inequalities (9) it follows that $\text{rk}(N) > 1$, because $C$ has maximal gonality.

**Step 1.** We prove that $M$ is a line bundle. Assume that, on the contrary,

$$\text{rk}(M) = \text{rk}(N) = 2$$

and consider the elementary modification $G := \text{Ker}(E \to N)$. Its Chern classes are given as follows:

$$c_1(G) = -[C], \quad c_2(G) = d + d_N - 2(g - 1),$$

and its discriminant equals $\Delta(G) = -64i + 110 + 8d_N - 14\rho < 0$, because of (9). In particular, there exists a saturated subsheaf $F \subset G$ which verifies the inequalities

$$\mu(G) \leq \mu(F) < \mu(E), \tag{10}$$

$$\zeta_{F,G}^2 \geq -\frac{\Delta(G)}{48}. \tag{11}$$
Write $c_1(F) = \alpha \cdot [C]$ and $\text{rk}(F) = \beta \leq 3$. The above inequality (11) becomes

$$\left(\frac{\alpha}{\beta} + \frac{1}{4}\right)^2 (2g - 2) \geq -\frac{\Delta(G)}{48}.$$  

We apply (10) for $\mu(F) = \alpha(2g - 2)/\beta$ and obtain

$$-\frac{1}{4} \leq \frac{\alpha}{\beta} < \frac{1}{4},$$

hence $\alpha = 0$, and the inequality (11) reads in this case $d_N \geq 5i - 10 + \rho$. Recalling that $d_N \leq g - 1 = 4i + 5 + \rho$, we obtain a contradiction whenever $i \geq 6$.

**Step 2.** We construct an elementary modification, in order to reach a contradiction.

From (9), we have $d_M \geq \frac{q+1}{2}$. The composite map $M \to E_C \to K_C \otimes A^v$ is not zero, for else $M \to Q_A$ and since $\mu(Q_A \otimes M^v) < 0$, one contradicts the semistability of $Q_A$. We set $A_1 := K_C \otimes A^v \otimes M^v$ and obtain a surjection $F(C)_C \to A \otimes A_1$ inducing, as before, an elementary modification

$$V := \text{Ker}(F(C) \to A \otimes A_1).$$

By direct computation we show that $\Delta(V) < 0$. Indeed, we compute

$$c_1(V) = 2 \cdot [C], \quad c_2(V) = d + 2g - 2 - d_M,$$

hence $\Delta(V) = 8c_2(V) - 3c_1^2(V) = 8(d - d_M - g + 1) = 8(5 - d_M - i) < 0$.

We obtain a destabilizing sheaf $P \subset V$, with $\text{rk}(P) = b \leq 3$ and $c_1(P) := a \cdot [C]$, such that the following inequalities are both satisfied

$$(12) \quad \left(\frac{a}{b} - \frac{1}{2}\right)^2 (2g - 2) \geq -\frac{\Delta(V)}{48} \quad \text{and} \quad \mu(V) \leq \mu(P) < \mu(F(C)).$$

The second inequality gives $\frac{1}{2} \leq \frac{a}{b} < \frac{3}{4}$, which leaves two possibilities: either $a = 1$ and $b = 2$, when via (12) one finds that $\Delta(V) \geq 0$, a contradiction, or else $a = 2$ and $b = 3$, when inequalities (12) and (9) clash.

3. Normal Bundle of Canonical Curves of Genus 7

The aim of this section is to prove Theorem 0.3 and we begin by recalling Mukai’s results [M3] on canonical curves of genus 7. We choose a vector space $U := \mathbb{C}^{10}$ and a non-degenerate quadratic form $q : U \to \mathbb{C}$, defining a smooth 8-dimensional quadric $Q \subset \mathbb{P}(U) = \mathbb{P}^9$.

The algebraic group $\text{Spin}(U)$ corresponding to the Dynkin diagram $D_5$ admits two 16-dimensional half-spin representations $S^+$ and $S^-$, which correspond to maximal weights $\alpha^+$ and $\alpha^-$ respectively. The homogeneous spaces $V^\pm := \text{Spin}(U)/P(\alpha^\pm)$ are both 10-dimensional and can be realized as the two irreducible components of the Grassmannian $G_4(5, U)$ of projective 4-planes inside $\mathbb{P}(U)$ which are isotropic with respect to the quadratic form $q$. From now on, we set

$$V := V^+ \subset \mathbb{P}(S^+) = \mathbb{P}^{15}. $$
Note that $\text{Aut}(V) = SO(10)$. If $E$ is the restriction to $V$ of the tautological bundle on $G(5, 10)$, one has an exact sequence of vector bundles on $V$:

$$0 \rightarrow \mathcal{E}^\vee \rightarrow U \otimes \mathcal{O}_V \rightarrow \mathcal{E} \rightarrow 0.$$  

By the adjunction formula, smooth curvilinear sections of $V$ are canonical curves of genus 7 and Mukai [M3] showed that each curve $[C] \in \mathcal{M}_7$ with $\text{Cliff}(C) = 3$ appears in this way. Precisely, there is a birational map

$$\alpha : G(7, 16) \rightarrow \mathcal{M}_7, \quad \alpha(\Lambda) := [\Lambda \cap V],$$

where $\Lambda \cong \mathbb{P}^6$. Given a curve $[C] \in \mathcal{M}_7$, the inverse $\alpha^{-1}([C])$ is constructed precisely via the twist of the conormal bundle on $C$ mentioned in the introduction.

Let $C \subset \mathbb{P}^6$ be a smooth canonical curve with $\text{Cliff}(C) = 3$, and set $E := N_C^\vee \cong \mathbb{P}^6(2)$. One has an identification $H^0(C, E) = I_2(K_C)$ and $E$ is a globally generated bundle. The tautological map

$$\phi_E : C \rightarrow G(5, H^0(C, E))$$

is easily shown to be injective and its image lies on $V$. In particular, the vector bundle $E$ is the restricted spinorial bundle, that is, $E = \mathcal{E}_C$ and one has an exact sequence:

$$0 \rightarrow \mathcal{E}^\vee \rightarrow H^0(C, E) \otimes C \rightarrow E \rightarrow 0.$$  

Note that $W^1_4(C) = \emptyset$, while $W^1_5(C)$ is a curve. We are going to make essential use of the following fact:

**Lemma 3.1.** Let $C$ as above and $A \in W^1_5(C)$. Then there are no surjections $E \twoheadrightarrow A$.

**Proof.** We proceed by contradiction. Assume that there is such a pencil $A \in W^1_5(C)$, then use the base point free pencil trick to write the following diagram:

$$
\begin{array}{c}
0 \rightarrow E^\vee \rightarrow H^0(C, E) \otimes C \rightarrow E \rightarrow 0 \\
0 \rightarrow A^\vee \rightarrow H^0(C, A) \otimes C \rightarrow A \rightarrow 0 \\
\end{array}
$$

In particular, $H^0(C, E \otimes A^\vee) \neq 0$. Via the identification $H^0(C, E) = I_2(K_C)$, this implies that if $L := K_C \otimes A^\vee \in W^2_7(C)$, then the multiplication map

$$\text{Sym}^2 H^0(C, L) \rightarrow H^0(C, L^{\otimes 2})$$

is not injective. This is possible only if $L$ is not birationally very ample, in particular, $C$ must be trigonal, which is not the case. \hfill \Box

We are now in a position to prove that $E$ is a stable vector bundle.

**Proof of Theorem 0.3.** Suppose that $0 \rightarrow F \rightarrow E \rightarrow M \rightarrow 0$ is a destabilizing sequence for the vector bundle $E$, that is, with $\mu(F) \geq \mu(E) = \frac{24}{7}$. Since $E$ is globally generated, so is any of its quotient, in particular $M$ too. We distinguish several possibilities, depending on the ranks that appear:
(i) \( \text{rk}(F) = 4 \) and \( M \) is line bundle. Then \( \deg(F) \geq 20 \), hence \( \deg(M) \leq 4 \). Since \( C \) is not tetragonal, \( h^0(C, M) \leq 1 \). Note that \( M \neq \mathcal{O}_C \), for \( H^0(C, E^\vee) = 0 \). It follows that \( M \) is not globally generated, a contradiction.

(ii) \( \text{rk}(F) = 1 \) and we may assume that \( \deg(F) = 5 \). Suppose first that \( h^0(C, F) = 0 \), therefore \( h^0(C, K_C \otimes F^\vee) = 1 \), and hence \( K_C \otimes F^\vee \) is not globally generated. Since one has a surjection \( E^\vee(1) \to K_C \otimes F^\vee \), we reach a contradiction by observing that \( E^\vee(1) \) is globally generated. Indeed, via Serre duality, this last statement is equivalent to the equality \( h^0(C, E(p)) = h^0(C, E) = 10 \), for every point \( p \in C \). From the exact sequence

\[ 0 \to E(p) \to M_{K_C} \otimes K_C(p) \to K_C^{\otimes 3}(p) \to 0, \]

we obtain that \( \text{deg}(\text{Sym}^2 H^0(C, M_{K_C} \otimes K_C(p))) = \text{deg}(K_C^{\otimes 3}(p)) \). The conclusion follows, since \( H^0(C, M_{K_C} \otimes K_C) = H^0(C, M_{K_C} \otimes K_C(p)) \).

Suppose now that \( h^0(C, F) \geq 1 \). The case \( h^0(C, F) \geq 2 \) having been discarded in the course of proving Lemma 3, we assume that \( h^0(C, F) = 1 \), hence \( h^0(C, K_C \otimes F^\vee) = 2 \). We obtain that the map \( \text{Sym}^2 H^0(C, K_C \otimes F^\vee) \to H^0(C, K_C^{\otimes 3} \otimes F^\vee(-2)) \) is not injective, which contradicts the base point free pencil trick.

(iii) \( \text{rk}(F) = 3 \), and then \( \deg(F) \geq 15 \), hence \( \deg(M) \leq 9 \). This time we may assume that \( F \) is stable. If \( M \) is not stable, we choose a line subbundle \( A \subset M \) of maximal degree, which we pull-back under the surjection \( E \to M \), to obtain the exact sequence

\[ 0 \to G \to E \to M/A \to 0. \]

We obtain that \( \deg(M/A) \leq \deg(M)/2 \leq 9/2 \), that is, \( \deg(M/A) \leq 4 \). In particular, \( M/A \) is not globally generated, which is again a contradiction, so we can assume that both \( F \) and \( M \) are stable vector bundles. Since \( h^0(C, M) + h^0(C, F) \geq h^0(C, E) = 10 \), the strategy is to use the fact that the Mercat statements \( (M_2) \) and \( (M_3) \) have been established for curves \( C \) of genus 7 with maximal Clifford index, that is,

\[ \text{Cliff}_3(C) = \text{Cliff}_2(C) = 3, \]

see [LN3] Theorem 4.5. In particular, if both \( F \) and \( M \) contribute to their respective Clifford indices, that is, \( h^0(C, F) \geq 6 \) and \( h^0(C, M) \geq 4 \) respectively, then we write

\[ \frac{9}{2} + 3 \leq \frac{3}{2} \gamma(F) + \gamma(M) = \frac{1}{2} \left( \deg(F) + \deg(M) \right) - h^0(C, F) - h^0(C, M) + 5, \]

that is, \( h^0(C, F) + h^0(C, M) \leq \frac{19}{2} \), a contradiction.

Assume now that one of the bundles \( F \) or \( M \) does not contribute to its Clifford index. Since \( M \) is globally generated, \( h^0(C, M) \geq 2 \). We can have \( h^0(C, M) = 2 \), only when \( M = \mathcal{O}_C^{\oplus 2} \), which is impossible, for \( \mathcal{O}_C^{\oplus 2} \) is not a direct summand of \( E \). If \( h^0(C, M) = 3 \), then \( \deg(M) \geq 7 \), and one has equality if and only if \( M = Q_L \), where \( L \in W^2_2(C) \). Assuming this to be the case, we choose two points \( p, q \in C \) that correspond to a node in the plane model \( \phi_L : C \to \mathbf{P}^2 \), that is, \( A := L(-p-q) \in W^1_3(C) \). Then there is a surjection \( Q_L \to A \), which by composition gives rise to a surjection \( E \to A \). This contradicts Lemma 3. Thus we may assume that \( \deg(M) \geq 8 \), and accordingly, \( \deg(F) \leq 16 \). Then we compute

\[ \gamma(F) = \mu(F) - \frac{2}{3} h^0(C, F) + 2 \leq \frac{16}{3} - \frac{14}{3} + 2 < \text{Cliff}(C), \]
which again contradicts the equality \( \text{Cliff}_3(C) = 3 \).

(iv) \( \text{rk}(F) = 2 \), and then \( \deg(F) \geq 10 \) and \( \deg(M) \leq 14 \). We may assume this time that \( M \) is stable. If \( F \) is not stable, then it has a line subbundle \( A \rightarrow F \) with \( \deg(A) \geq 5 \), and we are back to case (ii). Thus both \( M \) and \( F \) are stable bundles, and we proceed precisely like in case (iii).

It is instructive to remark that the normal of a canonical curve of genus \( g < 7 \) is never stable. More generally we have the following:

**Proposition 3.2.** The normal bundle of a tetragonal canonical curve of genus \( g \) is unstable.

**Proof.** More generally, we begin with a \( k:1 \) covering \( f: C \rightarrow \mathbb{P}^1 \), and consider the rank \((k - 1)\)-vector bundle \( \mathcal{F}^\vee := f_* \mathcal{O}_C / \mathcal{O}_{\mathbb{P}^1} \) on the projective line. Then \( \pi: X = \mathbb{P}(\mathcal{F}) \rightarrow \mathbb{P}^1 \) is a scroll of dimension \( k - 1 \), which contains the canonical curve \( C \) and which can be embedded by the tautological bundle \( \mathcal{O}_X(1) \) in \( \mathbb{P}^{g-1} \) as a variety of degree \( g - k + 1 \). Denoting by \( H, R \in \text{Pic}(X) \) the class of the hyperplane section and that of the ruling respectively, we have

\[
K_X \equiv -(k-1)H + (g-k-1)R,
\]

whereas obviously \( C \cdot H = 2g - 2 \) and \( C \cdot R = k \). We compute the degree of the normal bundle \( N_{C/X} \) and find:

\[
\deg(N_{C/X}) = \deg(T_{X|C}) + \deg(K_C) = k(g + k - 1).
\]

We write the usual exact sequence relating normal bundles

\[
0 \rightarrow N_{C/X} \rightarrow N_{C/\mathbb{P}^{g-1}} \rightarrow N_{X/\mathbb{P}^{g-1}} \otimes \mathcal{O}_C \rightarrow 0,
\]

and compare the slopes

\[
\mu(N_{C/X}) = \frac{k(g + k - 1)}{k - 2} \quad \text{and} \quad \mu(N_{C/\mathbb{P}^{g-1}}) = \frac{2(g-1)(g+1)}{g-2}.
\]

We conclude that for \( k = 4 \) and \( g \geq 6 \), the normal bundle \( N_{C/X} \) is a destabilizing subbundle of \( N_{C/\mathbb{P}^{g-1}} \). For \( g \) at most 5, every canonical curve of genus \( g \) is a complete intersection which obviously produces a destabilizing line subbundle.

\[ \square \]

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