The Fermat cubic and special Hurwitz loci in $\overline{\mathcal{M}}_g$

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Abstract

We compute the class of the compactified Hurwitz divisor $\overline{\mathfrak{TR}}_d$ in $\overline{\mathcal{M}}_{2d-3}$ consisting of curves of genus $g = 2d-3$ having a pencil $\mathfrak{g}^1_d$ with two unspecified triple ramification points. This is the first explicit example of a geometric divisor on $\overline{\mathcal{M}}_g$ which is not pulled-back form the moduli space of pseudo-stable curves. We show that the intersection of $\overline{\mathfrak{TR}}_d$ with the boundary divisor Δ_1 in $\overline{\mathcal{M}}_g$ picks-up the locus of Fermat cubic tails.

1 Introduction

Hurwitz loci have played a basic role in the study of the moduli space of curves at least since 1872 when Clebsch, and later Hurwitz, proved that $\mathcal{M}_g$ is irreducible by showing that a certain Hurwitz space parameterizing coverings of $\mathbf{P}^1$ is connected (see [Hu], or [Fu2] for a modern proof). Hurwitz cycles on $\overline{\mathcal{M}}_g$ are essential in the work of Harris and Mumford [HM] on the Kodaira dimension of $\overline{\mathcal{M}}_g$ and are expected to govern the length of minimal affine stratifications of $\mathcal{M}_g$. Faber and Pandharipande have proved that the class of any Hurwitz cycle on $\overline{\mathcal{M}}_{g,n}$ is tautological (cf. [FP]). Very few explicit formulas for the classes of such cycles are known.

We define a *Hurwitz divisor in $\overline{\mathcal{M}}_g$ with n degrees of freedom* as follows: We fix integers $k_1, \ldots, k_n \geq 3$ and positive integers d, g such that

$$k_1 + k_2 + \cdots + k_n = 2d - g + n - 1.$$

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Then $\mathcal{H}_{g:\,k_1,\dots,k_n}$ is the locus of curves $[C] \in \mathcal{M}_g$ having a degree d morphism $f : C \to \mathbf{P}^1$ together with n distinct points $p_1, \dots, p_n \in C$ such that $\mathrm{mult}_{p_i}(f) \geq k_i$ for $i = 1, \dots, n$. When $n = 0$ and $g = 2d - 1$, we recover the Brill-Noether divisor of d-gonal curves studied extensively in [HM]. For $n = 1$ we obtain Harris' divisor $\mathcal{H}_{g:\,k}$ of curves having a linear series $C \stackrel{d:1}{\to} \mathbf{P}^1$ with a $k = (2d - g + 1)$-fold point, cf. [H]. If $n = 1$ and $d = g - 1$ then $\mathcal{H}_{g:\,g-1}$ specializes to S. Diaz's divisor of curves $[C] \in \mathcal{M}_g$ having an exceptional Weierstrass point $p \in C$ with $h^0(C, \mathcal{O}_C((g-1)p)) \geq 1$ (cf. [Di]).

Since $\mathcal{H}_{g:k_1,\dots,k_n}$ is the push-forward of a cycle of codimension $n+1$ in $\mathcal{M}_{g,n}$, as n increases the problem of calculating the class of $\overline{\mathcal{H}}_{g:k_1,\dots,k_n}$ becomes more and more difficult. In this paper we carry out the first study of a Hurwitz locus having at least 2 degrees of freedom, and we treat the simplest non-trivial case, when $n = 2$, $k_1 = k_2 = 3$ and $g = 2d - 3$. Our main result is the calculation of the class of $\overline{\mathfrak{TR}}_d := \overline{\mathcal{H}}_{2d-3:\,3,3}$. As usual we denote by $\lambda \in \mathrm{Pic}(\overline{\mathcal{M}}_g)$ the Hodge class and by $\delta_0, \dots, \delta_{[g/2]} \in \mathrm{Pic}(\overline{\mathcal{M}}_g)$ the codimension 1 classes on the moduli stack corresponding to the boundary divisors of $\overline{\mathcal{M}}_g$:

Theorem 1.1. *We fix $d \geq 3$ and denote by $\mathfrak{TR}_d$ the locus of curves $[C] \in \mathcal{M}_{2d-3}$ having a covering $C \stackrel{d:1}{\to} \boldsymbol{P}^1$ with two unspecified triple ramification points. Then $\mathfrak{TR}_d$ is an effective divisor on $\mathcal{M}_{2d-3}$ and the class of its compactification $\overline{\mathfrak{TR}}_d$ inside $\overline{\mathcal{M}}_{2d-3}$ is given by the formula:*

$$\overline{\mathfrak{TR}}_d \equiv 2\frac{(2d-6)!}{d!\,(d-3)!}\big(a\,\lambda - b_0\;\delta_0 - b_1\,\delta_1 - \cdots - b_{d-2}\,\delta_{d-2}\big) \in \mathrm{Pic}(\overline{\mathcal{M}}_{2d-3}),$$

where

$$a = 24(36d^4 - 36d^3 - 640d^2 + 1885 - 1475),$$
$$b_0 = 144d^4 - 528d^3 - 298d^2 + 3049d - 2940,$$
$$\text{and}\quad b_i = 12i(2d-3-i)(36d^3 - 156d^2 + 180d - 5),\ \ \text{for}\ \ 1 \leq i \leq d-2.$$

The divisor $\overline{\mathfrak{TR}}_d$ is also the first example of a geometric divisor in $\overline{\mathcal{M}}_g$ which is not a pull-back of an effective divisor from the space $\overline{\mathcal{M}}_g^{\mathrm{ps}}$ of pseudo-stable curves. Precisely, if we denote by $R \subset \overline{\mathcal{M}}_g$ the extremal ray obtained by attaching to a fixed pointed curve $[C, q]$ of genus $g-1$ a pencil of plane cubics, then $R \cdot \lambda = 1$, $R \cdot \delta_0 = 12$, $R \cdot \delta_1 = -1$ and $R \cdot \delta_\alpha = 0$ for $\alpha \geq 2$. If $\delta := \delta_0 + \cdots + \delta_{[g/2]} \in \mathrm{Pic}(\overline{\mathcal{M}}_g)$ is the total boundary, there exists a divisorial contraction of the extremal ray $R \subset \Delta_1 \subset \overline{\mathcal{M}}_g$ induced by the base point free linear system $|11\lambda - \delta|$ on $\overline{\mathcal{M}}_g$,

$$f : \overline{\mathcal{M}}_g \to \overline{\mathcal{M}}_g^{\mathrm{ps}}.$$

The image is isomorphic to the moduli space of pseudo-stable curves as defined by D. Schubert in [S]. A curve is *pseudo-stable* if it has only nodes and cusps as singularities, and each component of genus 1 (resp. 0) intersects the curve in at least 2 (resp. 3 points). The contraction f is the first step in carrying out the minimal model program for $\overline{\mathcal{M}}_g$, see [HH]. One has an inclusion

$f^*(\mathrm{Eff}(\overline{\mathcal{M}}_g^{\mathrm{ps}})) \subset \mathrm{Eff}(\overline{\mathcal{M}}_g)$. All the geometric divisors on $\overline{\mathcal{M}}_g$ whose class has been computed (e.g. Brill-Noether or Gieseker-Petri divisors [EH], Koszul divisors [Fa1], [Fa2], or loci of curves with an abnormal Weierstrass point [Di]), lie in the subcone $f^*(\mathrm{Eff}(\overline{\mathcal{M}}_g^{\mathrm{ps}}))$. The divisor $\overline{\mathfrak{TR}}_d$ behaves quite differently: If $i : \Delta_1 = \overline{\mathcal{M}}_{1,1} \times \overline{\mathcal{M}}_{g-1,1} \hookrightarrow \overline{\mathcal{M}}_g$ denotes the inclusion, then we have the relation

$$i^*(\overline{\mathfrak{TR}}_d) = \alpha \cdot \{j = 0\} \times \overline{\mathcal{M}}_{g-1,1} + \overline{\mathcal{M}}_{1,1} \times D = \alpha \cdot \{\text{Fermat cubic}\} \times \overline{\mathcal{M}}_{g-1,1} + \overline{\mathcal{M}}_{1,1} \times D,$$

where $\alpha := \frac{3(2d-4)!}{d!\,(d-3)!}$ and $D \subset \overline{\mathcal{M}}_{g-1,1}$ is an explicitly described effective divisor. Hence when restricted to the boundary divisor $\Delta_1 \subset \overline{\mathcal{M}}_g$ of elliptic tails, $\overline{\mathfrak{TR}}_d$ picks-up the locus of *Fermat cubic tails*!

The rich geometry of $\overline{\mathfrak{TR}}_d$ can also be seen at the level of genus 2 curves. We denote by $\chi : \overline{\mathcal{M}}_{2,1} \to \overline{\mathcal{M}}_{2d-3}$ be the map obtained by attaching a fixed tail $[B, q]$ of genus $2d-5$ at the marked point of every curve of genus 2. Then the pull-back under χ of every known geometric divisor on $\overline{\mathcal{M}}_{2,1}$ is a multiple of the Weierstrass divisor $\overline{\mathcal{W}}$ of $\overline{\mathcal{M}}_{2,1}$ (cf. [HM], [EH], [Fa1]). In contrast, for $\overline{\mathfrak{TR}}_d$ we have the following picture:

Theorem 1.2. *If $\chi : \overline{\mathcal{M}}_{2,1} \to \overline{\mathcal{M}}_g$ is as above, we have the following relation in* $\mathrm{Pic}(\overline{\mathcal{M}}_{2,1})$*:*

$$\chi^*(\overline{\mathfrak{TR}}_d) = N_1(d) \cdot \overline{\mathcal{W}} + e(d, 2d-5) \cdot \overline{\mathcal{D}}_1 + a(d-1, 2d-5) \cdot \overline{\mathcal{D}}_2 + a(d, 2d-5) \cdot \overline{\mathcal{D}}_3,$$

where $\mathcal{W} := \{[C, p] \in \mathcal{M}_{2,1} : p \in C \text{ is a Weierstrass point}\}$,

$\mathcal{D}_1 := \{[C, p] \in \mathcal{M}_{2,1} : \exists x \in C - \{p\} \text{ such that } 3x \equiv 3p\}$,

$$\mathcal{D}_2 := \{[C, p] \in \mathcal{M}_{2,1} : \exists l \in G^1_3(C), x \neq y \in C - \{p\} \text{ with } a^l_1(x) \geq 3, a^l_1(y) \geq 3,\ a^l_1(p) \geq 2\},$$

and

$$\mathcal{D}_3 := \{[C, p] \in \mathcal{M}_{2,1} : \exists l \in G^1_4(C), x \neq y \in C - \{p\} \text{ with } a^l_1(p) \geq 4,\ a^l_1(x) \geq 3,\ a^l_1(y) \geq 3\}.$$

The constants $N_1(d), e(d, 2d-5), a(d, 2d-5), a(d-1, 2d-5)$ appearing in the statement are explicitly known and defined in Proposition 2.1. We used the notation $a^l_1(p) := \mathrm{mult}_p(l)$, for the multiplicity of a pencil $l \in G^1_d(C)$ at a point $p \in C$. The classes of the divisors $\overline{\mathcal{D}}_1, \overline{\mathcal{D}}_2, \overline{\mathcal{D}}_3$ on $\overline{\mathcal{M}}_{2,1}$ are determined as well (The class of $\overline{\mathcal{W}}$ is of course well-known, see [EH]):

Theorem 1.3. *One has the following formulas expressed in the basis $\{\psi, \lambda, \delta_0\}$ of* $\mathrm{Pic}(\overline{\mathcal{M}}_{2,1})$*:*

$$\overline{\mathcal{D}}_1 \equiv 80\psi + 10\delta_0 - 120\lambda, \quad \overline{\mathcal{D}}_2 \equiv 160\psi + 17\delta_0 - 200\lambda,$$

$$\text{and} \quad \overline{\mathcal{D}}_3 \equiv 640\psi + 72\delta_0 - 860\lambda.$$

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2 Admissible coverings with two triple points

We begin by recalling a few facts about admissible coverings in the context of points of triple ramification. Let $\mathcal{H}_d^{\mathrm{tr}}$ be the Hurwitz space parameterizing degree d maps $[f : C \to \mathbf{P}^1, q_1, q_2; p_1, \ldots, p_{6d-12}]$, where $[C] \in \mathcal{M}_{2d-3}, q_1, q_2, p_1, \ldots, p_{6d-12}$ are distinct points on $\mathbf{P}^1$ and f has one point of triple ramification over each of q_1 and q_2 and one point of simple ramification over p_i for $1 \le i \le 6d-12$. We denote by $\overline{\mathcal{H}}_d^{\mathrm{tr}}$ the compactification of the Hurwitz space by means of Harris-Mumford admissible coverings (cf. [HM], [ACV] and [Di] Section 5; see also [BR] for a survey on Hurwitz schemes and their compactifications). Thus $\overline{\mathcal{H}}_d^{\mathrm{tr}}$ is the parameter space of degree d maps

$$[f : X \xrightarrow{d:1} R, q_1, q_2; p_1, \ldots, p_{6d-12}],$$

where $[R, q_1, q_2; p_1, \ldots, p_{6d-12}]$ is a nodal rational curve, X is a nodal curve of genus $2d-3$ and f is a finite map which satisfies the following conditions:

- $f^{-1}(R_{\mathrm{reg}}) = X_{\mathrm{reg}}$ and $f^{-1}(R_{\mathrm{sing}}) = X_{\mathrm{sing}}$.
- f has a point of triple ramification over each of q_1 and q_2 and simple ramification over $p_1, \ldots, p_{6d-12}$. Moreover f is étale over each point in $R_{\mathrm{reg}} - \{q_1, q_2, p_1, \ldots, p_{6d-12}\}$.
- If $x \in X_{\mathrm{sing}}$ and $x \in X_1 \cap X_2$ where X_1 and X_2 are irreducible components of X, then $f(X_1)$ and $f(X_2)$ are distinct components of R and

$$\mathrm{mult}_x\{f_{|X_1} : X_1 \to f(X_1)\} = \mathrm{mult}_x\{f_{|X_2} : X_2 \to f(X_2)\}.$$

The group $\mathfrak{S}_2 \times \mathfrak{S}_{6d-12}$ acts on $\overline{\mathcal{H}}_d^{\mathrm{tr}}$ by permuting the triple and the ordinary ramification points of f respectively and we denote by $\mathfrak{H}_d := \overline{\mathcal{H}}_d^{\mathrm{tr}} / \mathfrak{S}_2 \times \mathfrak{S}_{6d-12}$ for the quotient. There exists a stabilization morphism $\sigma : \mathfrak{H}_d \to \overline{\mathcal{M}}_g$ as well as a finite map $\beta : \mathfrak{H}_d \to \overline{\mathcal{M}}_{0,6d-10}$. The description of the local rings of $\overline{\mathcal{H}}_d^{\mathrm{tr}}$ can be found in [HM] pg. 61-62 or [BR] and will be used in the paper. In particular, the scheme $\overline{\mathcal{H}}_d^{\mathrm{tr}}$ is smooth at points $[f : X \to R, q_1, q_2; p_1, \ldots, p_{6d-12}]$ with the property that there are no automorphisms $\phi : X \to X$ with $f \circ \phi = f$.

2.1 The enumerative geometry of pencils on the general curve

We shall determine the intersection multiplicities of $\overline{\mathfrak{TM}}_d$ with standard test curves in $\overline{\mathcal{M}}_g$. For this we need a variety of enumerative results concerning pencils on pointed curves which will be used throughout the paper. For a point $p \in C$ and a linear series $l \in G_d^r(C)$, we denote by

$$a^l(p) : \big(0 < a_0^l(p) < a_1^l(p) < \ldots < a_r^l(p) \le d\big)$$

the *vanishing sequence* of l at p. If $l \in G_d^1(C)$, we say that $p \in C$ is an *n-fold point* if $l(-np) \neq \emptyset$. We first recall the results from [HM] Theorem A and [H] Theorem 2.1.

Proposition 2.1. *Let us fix a general curve $[C,p] \in \mathcal{M}_{g,1}$ and an integer $d \geq 2d-g-1 \geq 0$.*

- *The number of pencils $L \in W^1_d(C)$ satisfying $h^0(L\otimes \mathcal{O}_C(-(2d-g-1)p)) \geq 1$ equals*

$$a(d,g) := (2d-g-1)\frac{g!}{d!\,(g-d+1)!}.$$

- *The number of pairs $(L,x) \in W^1_d(C)\times C$ satisfying $h^0(L\otimes\mathcal{O}_C(-(2d-g)x)) \geq 2$ equals*

$$b(d,g) := (2d-g-1)(2d-g)(2d-g+1)\frac{g!}{d!\,(g-d)!}.$$

- *Fix integers $\alpha,\beta \geq 1$ such that $\alpha+\beta = 2d-g$. The number of pairs $(L,x) \in W^1_d(C)\times C$ satisfying $h^0(L\otimes\mathcal{O}_C(-\beta p-\gamma x)) \geq 1$ equals*

$$c(d,g,\gamma) := \big(\gamma^2(2d-g)-\gamma\big)\binom{g}{d}.$$

- *The number of pairs $(L,x) \in W^1_d(C)\times C$ satisfying the conditions*

$$h^0(L\otimes\mathcal{O}_C(-(2d-g-2)p)) \geq 1 \text{ and } h^0(L\otimes\mathcal{O}_C(-3x)) \geq 1 \text{ equals}$$

$$e(d,g) := 8\frac{g!}{(d-3)!\,(g-d+2)!} - 8\frac{g!}{d!\,(g-d-1)!}.$$

We now prove more specialized results, adapted to our situation of counting pencils with two triple points:

Proposition 2.2. *(1) We fix $d \geq 3$ and a general 2-pointed curve $[C,p,q] \in \mathcal{M}_{2d-6}$. The number of pencils $l \in G^1_d(C)$ having triple points at both p and q equals*

$$F(d) := (2d-6)!\,\Big(\frac{1}{(d-3)!^2} - \frac{1}{d!\,(d-6)!}\Big).$$

(2) For a general curve $[C] \in \mathcal{M}_{2d-4}$, the number of pencils $l \in G^1_d(C)$ having triple ramification at unspecified distinct points $x,y \in C$, equals

$$N(d) := \frac{48(6d^2-28d+35)\,(2d-4)!}{d!\,(d-3)!}\,.$$

(3) We fix a general pointed curve $[C,p] \in \mathcal{M}_{2d-5,1}$. The number of pencils $L \in W^1_d(C)$ satisfying the conditions

$$h^0(L\otimes\mathcal{O}_C(-2p)) \geq 1,\ h^0(L\otimes\mathcal{O}_C(-3x)) \geq 1,\ h^0(L\otimes\mathcal{O}_C(-3y)) \geq 1$$

for unspecified distinct points $x,y \in C$, is equal to

$$N_1(d) := 24(12d^3-92d^2+240d-215)\frac{(2d-4)!}{d!\,(d-2)!}.$$

Remark 2.3. In the formulas for $e(d,g)$ and $F(d)$ we set $1/n! := 0$ for $n < 0$.

Remark 2.4. As a check, for $d = 3$ Proposition 2.2 (2) reads $N(3) = 80$. Thus for a general curve $[C] \in \mathcal{M}_2$ there are $160 = 2 \cdot 80$ pairs of points $(x,y) \in C \times C$, $x \neq y$, such that $3x \equiv 3y$. This can be seen directly by considering the map $\psi : C \times C \longrightarrow \mathrm{Pic}^0(C)$ given by $\psi(x,y) := \mathcal{O}_C(3x - 3y)$. Then $\psi^*(0) = \frac{1}{2}\int_{C\times C}\psi^*(\omega \wedge \omega) = 2 \cdot 3^2 \cdot 3^2 = 162$, where ω is a differential form representing θ. To get the answer to our question we subtract from 162 the contribution of the diagonal $\Delta \subseteq C \times C$. This excess intersection contribution is equal to 2 (cf. [Di]), so in the end we get $160 = 162 - 2$ pairs of distinct points $(x,y) \in C \times C$ with $3x \equiv 3y$.

Proof. **(1)** This is a standard exercise in limit linear series and Schubert calculus in the spirit of [EH]. We let $[C,p,q] \in \mathcal{M}_{2d-6,2}$ degenerate to the stable 2-pointed curve $[C_0 := \mathbf{P}^1 \cup E_1 \cup \ldots \cup E_{2d-6}, p_0, q_0]$, consisting of elliptic tails $\{E_i\}_{i=1}^{2d-6}$ and a rational spine, such that $\{p_i\} = E_i \cap \mathbf{P}^1$, and the marked points p_0, q_0 lie on the spine. We also assume that $p_1, \ldots, p_{2d-6}, p_0, q_0 \in \mathbb{P}^1$ are general points, in particular $p_0, q_0 \in \mathbf{P}^1 - \{p_1, \ldots, p_{2d-6}\}$. Then $F(d)$ is the number of limit $\mathfrak{g}^1_d$'s on C_0 having triple ramification at both p_0 and q_0 and this is the same as the number of $\mathfrak{g}^1_d$'s on $\mathbf{P}^1$ having cusps at $p_1, \ldots, p_{2d-6}$ and triple ramification at p_0 and q_0. This equals the intersection number of Schubert cycles $\sigma^2_{(0,2)}\sigma^{2d-6}_{(0,1)}$ (computed in $H^{\mathrm{top}}(\mathbb{G}(1,d),\mathbb{Z})$). The product can be computed using formula (v) on page 273 in [Fu1] and one finds that

$$\sigma^2_{(0,2)}\,\sigma^{2d-6}_{(0,1)} = (2d-6)!\,\Big(\frac{1}{(d-3)!^2} - \frac{1}{d!\,(d-6)!}\Big).$$

(2) This is more involved. We specialize $[C] \in \mathcal{M}_{2d-4}$ to $[C_0 := \mathbf{P}^1 \cup E_1 \cup \ldots \cup E_{2d-4}]$, where E_i are general elliptic curves, $\{p_i\} = \mathbf{P}^1 \cap E_i$ and $p_1, \ldots, p_{2d-4} \in \mathbf{P}^1$ are general points. Then $N(d)$ is equal to the number of limit $\mathfrak{g}^1_d$'s on C_0 with triple ramification at two distinct points $x, y \in C_0$. Let l be such a limit $\mathfrak{g}^1_d$. We can assume that both x and y are smooth points of C_0 and by the additivity of the Brill-Noether number (see e.g. [EH] pg. 365), we find that x, y must lie on the tails E_i. Since $[E_i, p_i] \in \mathcal{M}_{1,1}$ is general, we assume that $j(E_i) \neq 0$ (that is, none of the E_i's is the Fermat cubic). Then there can be no $l_i \in G^1_3(E_i)$ carrying 3 triple ramification points. There are two cases we consider:
a) There are indices $1 \leq i < j \leq 2d-4$ such that $x \in E_i$ and $y \in E_j$. Then $a^{l_{E_i}}(p_i) = a^{l_{E_j}}(p_j) = (d-3, d)$, hence $3x \equiv 3p_i$ on E_i and $3y \equiv 3p_j$ on E_j. There are 8 choices for $x \in E_i$, 8 choices for $y \in E_j$ and $\binom{2d-4}{2}$ choices for the tails E_i and E_j containing the triple points. On $\mathbf{P}^1$ we count $\mathfrak{g}^1_d$'s with cusps at $\{p_1, \ldots, p_{2d-4}\} - \{p_i, p_j\}$ and triple points at p_i and p_j. This number is again equal to $\sigma^2_{(0,2)}\,\sigma^{2d-6}_{(0,1)} \in H^{\mathrm{top}}(\mathbb{G}(1,d),\mathbb{Z})$ and we get a contribution of

$$64\binom{2d-4}{2}\sigma^2_{(0,2)}\,\sigma^{2d-6}_{(0,1)} = 32(2d-4)!\,\Big(\frac{1}{(d-3)!^2} - \frac{1}{d!\,(d-6)!}\Big). \qquad (1)$$

b) There is $1 \leq i \leq 2d-4$ such that $x, y \in E_i$. We distinguish between two subcases:

b_1) $a^{l_{E_i}}(p_i) = (d-3, d-1)$. On $\mathbb{P}^1$ we count $\mathfrak{g}^1_{d-1}$'s with cusps at $p_1, \ldots, p_{2d-4}$ and this number is $\sigma^{2d-4}_{(0,1)}$ (in $H^{top}(\mathbb{G}(1,d-1),\mathbb{Z})$). On E_i we compute the number of $\mathfrak{g}^1_3$'s having triple ramification at unspecified points $x, y \in E_i - \{p_i\}$ and ordinary ramification at p_i. For simplicity we set $[E_i, p_i] := [E, p]$. If we regard $p \in E$ as the origin of E, then the translation map $(x, y) \mapsto (y - x, -x)$ establishes a bijection between the set of pairs $(x, y) \in E \times E - \Delta$, $x \neq p \neq y \neq x$, such that there is a $\mathfrak{g}^1_3$ in which x, y, p appear with multiplicities $3, 3$ and 2 respectively, and the set of pairs $(u, v) \in E \times E - \Delta$, with $u \neq p \neq v \neq u$ such that there is a $\mathfrak{g}^1_3$ in which u, v, p appear with multiplicities $3, 2$ and 3 respectively. The latter set has cardinality 16, hence the number of pencils $\mathfrak{g}^1_3$ we are counting is $8 = 16/2$. All in all, we find a contribution of

$$8(2d-4)\,\sigma^{2d-4}_{(0,1)} = 16\binom{2d-4}{d-1}. \tag{2}$$

b_2) $a^{l_{E_i}}(p_i) = (d-4, d)$. This time, on $\mathbf{P}^1$ we look at $\mathfrak{g}^1_d$'s with cusps at $\{p_1, \ldots, p_{2d-4}\} - \{p_i\}$ and a 4-fold point at p_1. Their number is $\sigma_{(0,3)}\,\sigma^{2d-5}_{(0,1)} \in H^{\mathrm{top}}(\mathbb{G}(1,d),\mathbb{Z}))$. On E_i we compute the number of $\mathfrak{g}^1_4$'s for which there are distinct points $x, y \in E_i - \{p_i\}$ such that p_i, x, y appear with multiplicities $4, 3$ and 3 respectively. Again we set $[E_i, p_i] := [E, p]$ and denote by Σ the closure in $E \times E$ of the locus

$$\{(u,v) \in E \times E - \Delta : \exists l \in G^1_4(E) \text{ such that } a^l_1(p) = 4,\ a^l_1(u) \geq 3,\ a^l_1(v) \geq 2\}.$$

The class of the curve Σ can be computed easily. If F_i denotes the numerical equivalence class of a fibre of the projection $\pi_i : E \times E \to E$ for $i = 1, 2$, then

$$\Sigma \equiv 10F_1 + 5F_2 - 2\Delta. \tag{3}$$

The coefficients in this expression are determined by intersecting Σ with Δ and the fibres of π_i. First, one has that $\Sigma \cap \Delta = \{(x,x) \in E \times E : x \neq p,\ 4p \equiv 4x\}$ and then $\Sigma \cap \pi_2^{-1}(p) = \{(y,p) \in E \times E : y \neq p,\ 3p \equiv 3y\}$. These intersections are all transversal, hence $\Sigma \cdot \Delta = 15, \Sigma \cdot F_2 = 8$, whereas obviously $\Sigma \cdot F_1 = 3$. This proves (3).

The number of pencils $l \subseteq |\mathcal{O}_E(4p)|$ having two extra triple points will then be equal to $1/2$ #(ramification points of $\pi_2 : \Sigma \to E$) $= \Sigma^2/2 = 20$. We have obtained in this case a contribution of

$$20(2d-4)\,\sigma_{(0,3)}\,\sigma^{2d-5}_{(0,1)} = 80\binom{2d-4}{d}. \tag{4}$$

Adding together (1),(2) and (4), we obtain the stated formula for $N(d)$.

(3) We relate $N_1(d)$ to $N(d)$ by specializing the general curve from $\mathcal{M}_{2d-4}$ to $[C \cup_p E] \in \Delta_1 \subset \overline{\mathcal{M}}_{2d-4}$, where $[C, p] \in \mathcal{M}_{2d-5,1}$ and $[E, p] \in \overline{\mathcal{M}}_{1,1}$. Under this degeneration $N(d)$ becomes the number of admissible coverings $f : X \stackrel{d:1}{\to} R$ having as source a nodal curve X stably equivalent to $C \cup_p E$ and as target a genus 0 nodal curve R. Moreover, f possesses distinct unspecified triple ramification points $x, y \in X_{\mathrm{reg}}$. There are a number of cases depending on the position of x and y.

(3_a) $x, y \in C - \{p\}$. In this case $\deg(f_C) = d$ and because of the generality of $[C, p]$, f_C has to be one of the finitely many $\mathfrak{g}^1_d$'s having two distinct triple points and a simple ramification point at $p \in C$. The number of such coverings is precisely $N_1(d)$. By the compatibility condition on ramification indices at p, we find that $\deg(f_E) = 2$ and the E-aspect of f is induced by $|\mathcal{O}_E(2p)|$. The curve X is obtained from $C \cup_p E$ by inserting $d-2$ copies of $\mathbf{P}^1$ at the points in $f_C^{-1}(f(p)) - \{p\}$. We then map these rational curves isomorphically to $f(E)$. This admissible cover has no automorphisms and it should be counted with multiplicity 1.
(3_b) $x, y \in E - \{p\}$. The curve $[C] \in \mathcal{M}_{2d-5}$ being Brill-Noether general, it carries no linear series $\mathfrak{g}^1_{d-2}$, hence $\deg(f_C) \geq d-1$. We distinguish two subcases:

If $\deg(f_C) = d-1$, then f_C is one of the $a(d-1, 2d-5)$ linear series $\mathfrak{g}^1_{d-1}$ on C having p as an ordinary ramification point. Since C and E meet only at p, we have that $\deg(f_E) = 3$, and f_E corresponds to a $\mathfrak{g}^1_3$ on E having two unspecified triple points and a simple ramification point at p. There are 8 such $\mathfrak{g}^1_3$'s on E (see the proof of Proposition 2.2). To obtain a degree d admissible covering, we first attach a copy $(\mathbf{P}^1)_1$ of $\mathbf{P}^1$ to E at the point $q \in f_E^{-1}(f(p)) - \{p\}$, then map $(\mathbf{P}^1)_1$ and C map to the same component of R. Then we insert $d-2$ copies of $\mathbf{P}^1$ at the points lying in the same fibre of f_C as p. All these rational curves map to the same copy of R as E. Each of these $8a(d-1, 2d-5)$ admissible coverings is counted with multiplicity 1.

If $\deg(f_C) = d$, then f_C corresponds to one of the $a(d, 2d-5)$ linear series $\mathfrak{g}^1_d$ with a 4-fold point at p. By compatibility, f_E corresponds to a $\mathfrak{g}^1_4$ in which p and two unspecified points $x, y \in E$ appear with multiplicities $4, 3$ and 3 respectively. There are 20 such $\mathfrak{g}^1_4$'s on E, hence $20a(d, 2d-5)$ admissible coverings.
(3_c) $x \in E - \{p\}, y \in C - \{p\}$. In this situation $\deg(f_C) = d$ and f_C corresponds to one of the $e(d, 2d-5)$ coverings $\mathfrak{g}^1_d$ on C having a triple point at p and another unspecified triple point at $y \in C$. Then $\deg(f_E) = 3$ and $3x \equiv 3p$, that is, there are 8 choices of the E-aspect of f. We obtain X by attaching to C copies of $\mathbf{P}^1$ at the $d-3$ points in $f_C^{-1}(f(p)) - \{p\}$, and mapping these curves isomorphically onto $f(C)$.

By degeneration to $[C \cup_p E]$, we have found the relation for $[C, p] \in \mathcal{M}_{2d-5,1}$:

$$N(d) = N_1(d) + 20a(d, 2d-5) + 8a(d-1, 2d-5) + 8e(d, 2d-5).$$

This immediately leads to the claimed expression for $N_1(d)$. ■

3 The class of the divisor $\overline{\mathfrak{TR}}_d$

The strategy to compute the class $[\overline{\mathfrak{TR}}_d]$ is similar to the one employed by Eisenbud and Harris in [EH] to determine the class of the Brill-Noether divisors $[\overline{\mathcal{M}}^r_{g,d}]$ of curves with a $\mathfrak{g}^r_d$ in the case $\rho(g, r, d) = -1$: We determine the restrictions of $\overline{\mathfrak{TR}}_d$ to $\overline{\mathcal{M}}_{0,g}$ and $\overline{\mathcal{M}}_{2,1}$ via obvious flag maps. However, because in the definition of $\overline{\mathfrak{TR}}_d$ we allow 2 degrees of freedom for the triple ramification points, the calculations are much more intricate (and interesting) than in the case of Brill-Noether divisors.

Proposition 3.1. *Consider the flag map $j : \overline{\mathcal{M}}_{0,g} \to \overline{\mathcal{M}}_g$ obtained by attaching g general elliptic tails at the g marked points. Then $j^*(\overline{\mathfrak{TR}}_d) = 0$. If we have a linear relation*

$$\overline{\mathfrak{TR}}_d \equiv a\,\lambda - \sum_{i=0}^{d-2} b_i\,\delta_i \in \mathrm{Pic}(\overline{\mathcal{M}}_g), \text{ then } b_i = \frac{i(g-i)}{g-1} b_1, \text{ for } 1 \le i \le d-2.$$

Proof. The second part of the statement is a consequence of the first: For an effective divisor $D \equiv a\lambda - \sum_{i=0}^{d-2} b_i\delta_i \in \mathrm{Pic}(\overline{\mathcal{M}}_g)$ satisfying the condition $j^*(D) = \emptyset$, we have the relations among its coefficients: $b_i = \frac{i(g-i)}{g-1} b_1$ for $i \ge 1$ (cf. [EH] Theorem 3.1).

Suppose that $[X := R \cup_{x_1} E_1 \cup \ldots \cup_{x_g} E_g] \in j(\overline{\mathcal{M}}_{0,g})$ is a flag curve corresponding to a g-stable rational curve $[R, x_1, \ldots, x_g]$. The elliptic tails $\{E_i\}_{i=1}^g$ are general and we may assume that all the j-invariants are different from 0. In particular, none of the $[E_i, x_i]$'s carries a $\mathfrak{g}^1_3$ with triple ramification points at x_i and at two unspecified points $x, y \in E_i - \{x_i\}$. Assuming that $[X] \in \overline{\mathfrak{TR}}_d$, there exists $l \in \overline{G}^1_d(X)$ a limit $\mathfrak{g}^1_d$, together with distinct ramification points $x \ne y \in X$, such that $a^l_1(x) \ge 3$ and $a^l_1(y) \ge 3$. By blowing-up if necessary the nodes x_i (that is, by inserting chains of $\mathbf{P}^1$'s at the points x_i), we may assume that both x, y are smooth points of X.

We make use of the following facts: On R we have that the inequality

$$\rho(l_R, x_1, \ldots, x_g, z_1, \ldots, z_t) \ge 0,$$

for any choice of distinct points $z_1, \ldots, z_t \in R - \{x_1, \ldots, x_g\}$. On the elliptic tails, we have that $\rho(l_{E_i}, x_i, z) \ge -1$, for any point $z \in E_i - \{x_i\}$, with equality only if $z - x_i \in \mathrm{Pic}^0(E_i)$ is a torsion class. Using these remarks as well as and the additivity of the Brill-Noether number of l, since $\rho(l, x, y) = -3$ it follows that there must exist an index $1 \le i \le g$ such that $x, y \in E_i - \{x_i\}$, and $\rho(l_{E_i}, x_i, x, y) = -3$. This implies that $a^{l_{E_i}}(x_i) = (d-3, d)$ and that $l_{E_i}(-(d-3)x_i) \in G^1_3(E_i)$ has triple ramification points at distinct points x_i, x and y. This can happen only if E_i is isomorphic to the Fermat cubic, a contradiction. ∎

The next result highlights the difference between $\overline{\mathfrak{TR}}$ and all the other geometric divisors in the literature, cf. [HM], [EH], [H], [Fa1], [Fa2]: $\overline{\mathfrak{TR}}$ is the first example of a geometric divisor on $\overline{\mathcal{M}}_g$ not pulled-back from the space $\overline{\mathcal{M}}_g^{\mathrm{ps}}$ of pseudo-stable curves.

Proposition 3.2. *If $\overline{\mathfrak{TR}}_d \equiv a\;\lambda - \sum_{i=0}^{d-2} b_i\;\delta_i \in \mathrm{Pic}(\overline{\mathcal{M}}_g)$, then $a - 12b_0 + b_1 = 4a(d, 2d-4)$.*

Proof. We use a standard test curve in $\overline{\mathcal{M}}_g$ obtained by attaching to the marked point of a general pointed curve $[C, q] \in \mathcal{M}_{2d-4,1}$ a pencil of plane cubics. If $R \subset \overline{\mathcal{M}}_g$ is the family induced by this pencils, then clearly $R \cdot \lambda = 1, R \cdot \delta_0 = 12, R \cdot \delta_1 = -1$ and $R \cdot \delta_j = 0$ for $j \ge 2$.

Set-theoretically, $R \cap \overline{\mathfrak{TR}}_d$ consists of the points corresponding to the elliptic curves $[E, q]$ in the pencil, for which there exists $l \in G^1_3(E)$ as well as two distinct points $x, y \in E - \{q\}$ with $a_1^l(q) = a_1^l(x) = a_1^l(y) = 3$ (It is a standard limit linear series argument to show that the triple points of the limit $\mathfrak{g}_d^1$ must specialize to the elliptic tail). Then E must be isomorphic to the Fermat cubic, (thus $j(E) = 0$, and this curve appears 12 times in the pencil. The pencil $l \in G_3^1(E)$ is of course uniquely determined. Since $\text{Aut}(E, q) = \mathbb{Z}/6\mathbb{Z}$ while a generic element from $\overline{\mathcal{M}}_{1,1}$ has automorphism group $\mathbb{Z}/2\mathbb{Z}$, each point of intersection will contribute $4 = 24/6$ times in the intersection $R \cap \overline{\mathfrak{TR}}_d$. On the side of the genus $2d-4$ component, we count pencils $L \in W_d^1(C)$ with $a_1^L(q) \geq 3$. Using Proposition 2.1 their number is finite and equal to $a(d, 2d-4)$, hence $R \cdot \overline{\mathfrak{TR}}_d = 4a(d, 2d-4)$. ∎

Next we describe the restriction of $\overline{\mathfrak{TR}}_d$ under the map $\chi : \overline{\mathcal{M}}_{2,1} \to \overline{\mathcal{M}}_{2d-3}$ obtained by attaching a fixed tail B of genus $2d-5$ to each pointed curve $[C, p] \in \mathcal{M}_{2,1}$. It is revealing to compare Theorem 1.2 to Propositions 4.1 and 5.5 in [EH]: When $\rho(g, r, d) = -1$, the pull-back of the Brill-Noether divisor $\chi^*(\overline{\mathcal{M}}_{g,d}^r)$ is irreducible and supported on $\overline{\mathcal{W}}$. By contrast, $\overline{\mathfrak{TR}}_d$ displays a much richer geometry.

Proof of Theorem 1.2. We fix a general pointed curve $[B, p] \in \mathcal{M}_{2d-5,1}$. For each $[C, p] \in \mathcal{M}_{2,1}$, we study degree d admissible coverings $[f : X \to R, q_1, q_2; p_1, \ldots, p_{6d-12}] \in \overline{\mathcal{H}}_d^{\text{tr}}$ with source curve X stably equivalent to $C \cup_p B$, and target R a nodal curve of genus 0. Moreover, f is assumed to have distinct points of triple ramification $x, y \in X_{\text{reg}}$, where $f(x) = q_1$ and $f(y) = q_2$. It is easy to check that both x and y must lie either on C or on B (and not on rational components of X we may insert). Depending on their position we distinguish four cases:
(i) $x, y \in B$. A parameter count shows that $\deg(f_B) = d$ and $p \in B$ must be a simple ramification point for f_B. By compatibility of ramification sequences at p, then f_C must also be simply ramified at p, that is, $p \in C$ is a Weierstrass point and f_C is induced by $|\mathcal{O}_C(2p)|$. There is a canonical way of completing $\{f_C, f_B\}$ to an element in $\mathfrak{H}_d$, by attaching rational curves to B at the points in $f_B^{-1}(f(p)) - \{p\}$. For a fixed $[C, p] \in \overline{\mathcal{W}}$, the Hurwitz scheme is smooth at each of the points $t \in \overline{\mathcal{H}}_d^{\text{tr}}$ corresponding to an admissible coverings $\{f_C, f_B\}$ of the type described above. Since t has no automorphisms permuting some of the branch points, it follows that $\mathfrak{H}_d = \overline{\mathcal{H}}_d^{\text{tr}} / \mathfrak{S}_2 \times \mathfrak{S}_{6d-12}$ is also smooth at each of the $N_1(d)$ points in the fibre $\sigma^{-1}([C \cup_p B])$. This implies that $N_1(d) \cdot \overline{\mathcal{W}}$ appears as an irreducible component in the pull-back divisor $\chi^*(\overline{\mathfrak{TR}}_d)$.
(ii) $x, y \in C$, $\deg(f_B) = d$. Clearly $\deg(f_C) \geq 4$ and the B-aspect of the covering must have a 4-fold point at p. There are $a(d, 2d-5)$ choices for f_B, whereas f_C corresponds to a linear series $l_C \in G_4^1(C)$ with $a_1^{l_C}(p) = 4$ and which has two other points of triple ramification. To obtain the domain of an admissible covering, we attach to B rational curves at the $(d-4)$ points in $f_B^{-1}(f(p)) - \{p\}$. We map these curves isomorphically onto $f_C(C)$. The divisor $a(d, 2d-5) \cdot \overline{\mathcal{D}}_3$ is an irreducible component of $\chi^*(\overline{\mathfrak{TR}}_d)$.
(iii) $x, y \in C$, $\deg(f_B) = d-1$. In this case the B-aspect corresponds to one of the $a(d-1, 2d-5)$ linear series $l_B \in G_{d-1}^1(B)$ with simple ramification at p, while f_C is a degree 3 covering having two unspecified points of triple ramification and simple ramification at $p \in C$. To obtain a point in $\mathfrak{H}_d$, we attach a rational curve T' to C at the remaining point in $f_C^{-1}(f(p) - \{p\}$. We then map

T' isomorphically onto $f_B(B)$. Next, we attach $d-3$ rational curves to B at the points $f_B^{-1}(f(p)) - \{p\}$, which we map isomorphically onto $f_C(C)$. Each resulting admissible covering has no automorphisms and is a smooth point of $\mathfrak{H}_d$. Thus $a(d-1, 2d-5) \cdot \overline{\mathcal{D}}_2$ is a component of $\chi^*(\overline{\mathfrak{TR}}_d)$.

(iv) $x \in C, y \in B$. After a moment of reflection we conclude that $\deg(f_B) = d$, that is, f_B corresponds to one of the $e(d, 2d-5)$ coverings $l_B \in G^1_d(B)$ with $a_1^{l_B}(p) = 3$ and $a_1^{l_B}(y) = 3$ at some unspecified point $y \in B - \{p\}$. The C-aspect of f is determined by the choice of a point $x \in C - \{p\}$ such that $3x \equiv 3p$. Hence $e(d, 2d-5) \cdot \overline{\mathcal{D}}_1$ is the final irreducible component of $\chi^*(\overline{\mathfrak{TR}}_d)$. ∎

As a consequence of Proposition 3.1 and Theorem 1.2 we are in a position to determine all the δ_i-coefficients ($i \geq 1$) in the expansion of $\overline{\mathfrak{TR}}_d$ in the basis of $\mathrm{Pic}(\overline{\mathcal{M}}_g)$:

Theorem 3.3. *If* $\overline{\mathfrak{TR}}_d \equiv a\,\lambda - \sum_{i=0}^{d-2} b_i\,\delta_i \in \mathrm{Pic}(\overline{\mathcal{M}}_g)$, *then we have that*

$$b_i = \frac{(2d-6)!}{2\,d!(d-3)!}\, i(2d-3-i)(36d^3 - 156d^2 + 180d - 5), \quad \text{for all } 1 \leq i \leq d-2.$$

Proof. We use the obvious relations $\chi^*(\delta_2) = -\psi$, $\chi^*(\lambda) = \lambda, \chi^*(\delta_0) = \delta_0$, $\chi^*(\delta_1) = \delta_1$. If for a class $E \in \mathrm{Pic}(\overline{\mathcal{M}}_{2,1})$ we denote by $(E)_\psi$ the coefficient of ψ in its expansion in the basis $\{\psi, \lambda, \delta_0\}$ of $\mathrm{Pic}(\overline{\mathcal{M}}_{2,1})$ (see also the next section for details on the divisor theory of $\overline{\mathcal{M}}_{2,1}$), then, using Proposition 3.2, we can write the following relation:

$$b_2 = \frac{2(g-2)}{g-1} b_1 = N_1(d)(\overline{\mathcal{W}})_\psi + e(d, 2d-5)(\overline{\mathcal{D}}_1)_\psi + \\ a(d-1, 2d-5)(\overline{\mathcal{D}}_2)_\psi + a(d, 2d-5)(\overline{\mathcal{D}}_3)_\psi.$$

We determine the coefficients $(\overline{\mathcal{D}}_i)_\psi$ for $1 \leq i \leq 3$ by intersecting each of these divisors with a general fibral curve $F := \{[C, p]\}_{p \in C} \subset \overline{\mathcal{M}}_{2,1}$ of the projection $\pi : \overline{\mathcal{M}}_{2,1} \to \overline{\mathcal{M}}_2$. (Note that $(\overline{\mathcal{W}})_\psi = 3$).

It is useful to recall that if $[C, q] \in \mathcal{M}_{2,1}$ is a fixed general pointed curve and $a \geq b \geq 0$ are integers, then the number of pairs $(p, x) \in C \times C, p \neq x$ satisfying a linear equivalence relation $a \cdot x \equiv b \cdot p + (a-b) \cdot q$ in $\mathrm{Pic}^a(C)$, equals

$$r(a, b) := 2(a^2b^2 - 1). \tag{5}$$

We start with $\overline{\mathcal{D}}_1$ and note that $F \cdot \overline{\mathcal{D}}_1$ is the number of pairs $(x, p) \in C \times C$ with $x \neq p$, such that $3x \equiv 3p$, which is equal to $r(3,3) = 160$ and then $(\overline{\mathcal{D}}_1)_\psi = r(3,3)/(2g-2) = 80$. To compute $F \cdot \overline{\mathcal{D}}_2$ we note that there are $80 = r(3,3)/2$ pencils $L \in W^1_3(C)$ with two distinct triple ramification points. From the Hurwitz-Zeuthen formula, each such pencil has 4 more simple ramification points, thus $(\overline{\mathcal{D}}_2)_\psi = 4 \times 80/(2g-2) = 160$. Finally, $F \cdot \overline{\mathcal{D}}_3 = n_0/2$, where by n_0 we denote the number of pencils $l \in W^1_4(C)$ having one unspecified point of total ramification and two further points of triple ramification, that is there exist mutually distinct points $x, y, p \in C$ with $a_1^l(p) = 4$ and $a_1^l(x) = a_1^l(y) = 3$.

We compute n_0 by letting C specialize to a curve of compact type $[C_0 := C_1 \cup_q C_2]$, where $[C_1, q], [C_2, q] \in \mathcal{M}_{1,1}$. Then n_0 is the number of admissible coverings $f : X \stackrel{4:1}{\to} R$, where R is of genus 0 and X is stably equivalent to C_0 and has a 4-fold ramification point $p \in X_{\mathrm{reg}}$ and triple ramification points $x, y \in X_{\mathrm{reg}}$. We distinguish three cases:
(i) $x, y \in C_2$ and $p \in C_1$ (Or $x, y \in C_1$ and $p \in C_2$). In this case $\deg(f_{C_1}) = \deg(f_{C_2}) = 4$ and we have the linear equivalence $4p \equiv 4q$ on C_1. This yields 15 choices for $p \neq q$. On C_2 we count $\mathfrak{g}^1_4$'s with total ramification at q, and two unspecified triple points. This number is equal to 20 (see the proof of Proposition 2.2). Reversing the role of C_1 and C_2 we double the number of coverings and we find $600 = 2 \cdot 15 \cdot 20$ admissible $\mathfrak{g}^1_4$'s.
(ii) $x, p \in C_2$ and $y \in C_1$ (Or $x, p \in C_1$ and $y \in C_2$). In this situation $\deg(f_{C_1}) = 3$ and $\deg(f_{C_2}) = 4$ and on C_1 we have the linear equivalence $3y \equiv 3q$, which gives 8 choices for y. On C_2 we count $l_{C_2} \in G^1_4(C_2)$ in which two unspecified points $p, x \in C_2$ appear with multiplicities 4 and 3 respectively, while $a_1^{l_{C_2}}(q) = 3$. By translation, this is the same as the number of pairs of distinct points $(u, v) \in C_2 - \{q\} \times C_2 - \{q\}$ such that there exists $l_2 \in G^1_4(C_2)$ with $a_1^{l_2}(q) = 4, a_1^{l_2}(x) = a_1^{l_2}(y) = 3$. This number equals 40 (again, see the proof of Proposition 2.2). By reversing the role of C_1 and C_2 the total number of coverings in case (ii) is $640 = 2 \cdot 8 \cdot 40$.
(iii) $x, y, p \in C_1$ (or $x, y, p \in C_2$). A quick parameter count shows that $\deg(f_{C_2}) = 2$ and $\mathrm{mult}_q(f_{C_2}) = \mathrm{mult}_q(f_{C_1}) = 2$. Hence f_{C_2} is induced by $|\mathcal{O}_{C_2}(2q)|$. On C_1 we count $\mathfrak{g}^1_4$'s in which the points p, x, y, q appear with multiplicities $4, 3, 3$ and 2 respectively. The translation on C_2 from p to q shows that we are yet again in the situation of Proposition 2.2 and this last number is 20. We interchange C_1 and C_2 and we find 40 admissible $\mathfrak{g}^1_4$'s on $C_1 \cup C_2$ with all the non-ordinary ramification concentrated on a single component.

By adding (i), (ii) and (iii) together, we obtain $n_0 = 600 + 640 + 40 = 1280$. This determines $(\overline{\mathcal{D}}_3)_\psi = n_0/(2g-2) = 640$ and completes the proof. ∎

4 The divisor theory of $\overline{\mathcal{M}}_{2,1}$

The remaining part of the calculation of $[\overline{\mathfrak{TR}}_d]$ has been reduced to the problem of determining the divisor classes $[\overline{\mathcal{D}}_i]$ $(i = 1, 2, 3)$ on $\overline{\mathcal{M}}_{2,1}$. We recall some things about divisor theory on this space (see also [EH]). There are two boundary divisor classes:

- δ_0, whose generic point is an irreducible 1-pointed nodal curve of genus 2.
- δ_1, with generic point being a transversal union of two elliptic curves with the marked point lying on one of the components.

If $\pi : \overline{\mathcal{M}}_{2,1} \to \overline{\mathcal{M}}_2$ is the universal curve then $\psi := c_1(\omega_\pi) \in \mathrm{Pic}(\overline{\mathcal{M}}_{2,1})$ denotes the tautological class and $\lambda = \pi^*(\lambda) \in \mathrm{Pic}(\overline{\mathcal{M}}_{2,1})$ is the Hodge class. Unlike the case $g \geq 3$, λ is a boundary class on $\overline{\mathcal{M}}_2$, and we have Mumford's genus 2 relation:

$$\lambda = \frac{1}{10}\delta_0 + \frac{1}{5}\delta_1.$$

The classes ψ, λ and δ_1 form a basis of $\mathrm{Pic}(\overline{\mathcal{M}}_{2,1}) \otimes \mathbb{Q}$. The class of the Weierstrass divisor has been computed in [EH] Theorem 2:

$$\overline{\mathcal{W}} \equiv 3\psi - \lambda - \delta_1. \tag{6}$$

We start by determining the class of $\overline{\mathcal{D}}_1$ of 3-torsion points:

Proposition 4.1. *The class of the closure in $\overline{\mathcal{M}}_{2,1}$ of the effective divisor*

$$\mathcal{D}_1 = \{[C,p] \in \mathcal{M}_{2,1} : \exists x \in C - \{p\} \text{ such that } 3x \equiv 3p\}$$

is given by $[\overline{\mathcal{D}}_1] = 80\psi + 10\delta_0 - 120\lambda \in \mathrm{Pic}(\overline{\mathcal{M}}_{2,1})$.

Proof. We introduce the map $\chi : \overline{\mathcal{M}}_{2,1} \to \overline{\mathcal{M}}_4$ given by $\chi([C,p]) := [B \cup_p C]$, where $[B,p]$ is a general 1-pointed curve of genus 2. On $\overline{\mathcal{M}}_4$ we have the divisor of curves with an exceptional Weierstrass point $\mathfrak{Di} := \{[C] \in \mathcal{M}_4 : \exists x \in C \text{ such that } h^0(C,3x) \geq 2\}$. Its class has been computed by Diaz [Di]: $\overline{\mathfrak{Di}} \equiv 264\lambda - 30\delta_0 - 96\delta_1 - 128\delta_2 \in \mathrm{Pic}(\overline{\mathcal{M}}_4)$.

We claim that $\chi^*(\overline{\mathfrak{Di}}) = \overline{\mathcal{D}}_1 + 16 \cdot \overline{\mathcal{W}}$. Indeed, let $[C,p] \in \mathcal{M}_{2,1}$ be such that $\chi([C,p]) \in \overline{\mathfrak{Di}}$. Then there is a limit $\mathfrak{g}^1_3$ on $X := B \cup_p C$, say $l = \{l_B, l_C\}$, which has a point of total ramification at some $x \in X_{\mathrm{reg}}$. There are two possibilities:
(i) If $x \in C$, then $a^{l_B}(p) = (0,3)$, hence $l_B = |\mathcal{O}_B(3p)|$, while on C we have the linear equivalence $3p \equiv 3x$, that is, $[C,p] \in \overline{\mathcal{D}}_1$.
(ii) If $x \in B$, then $a^{l_C}(p) = (1,3)$, that is, $p \in B$ is a Weierstrass point and moreover $l_C = p + |\mathcal{O}_C(2p)|$. On B we have that $a^{l_B}(p) = (0,2)$ and $a^{l_B}(x) = (0,3)$, that is, $3x \equiv 2p + y$ for some $y \in B - \{p, y\}$. There are $r(3,1) = 16$ such pairs (x,y).

Thus we have proved that $\chi^*(\overline{\mathfrak{Di}}) = \overline{\mathcal{D}}_1 + 16 \cdot \overline{\mathcal{W}}$ (We would have obtained the same conclusion using admissible coverings instead of limit $\mathfrak{g}^1_3$'s). We find the formula for $[\overline{\mathcal{D}}_1]$ if we remember that $\chi^*(\delta_0) = \delta_0$, $\chi^*(\delta_1) = \delta_1$, $\chi^*(\delta_2) = -\psi$ and $\chi^*(\lambda) = \lambda$. ∎

4.1 The divisor $\overline{\mathfrak{TR}}_3$ and the class of $\overline{\mathcal{D}}_2$

We compute the class of the divisor $\overline{\mathcal{D}}_2$ on $\overline{\mathcal{M}}_{2,1}$ by determining directly the class of $\overline{\mathfrak{TR}}_3$ in genus 3 (In this case $\overline{\mathcal{D}}_3 = \emptyset$). Much of the set-up we develop here is valid for arbitrary $d \geq 3$ and will be used in the next section when we compute the class $[\overline{\mathfrak{TR}}_4]$ on $\overline{\mathcal{M}}_5$. We fix a general $[C,p] \in \mathcal{M}_{2d-4,1}$ and introduce the following enumerative invariant:

$$N_2(d) := \#\{l \in G^1_d(C) : \exists x \neq y \in C - \{p\} \text{ such that } l(-3x) \neq \emptyset \text{ and } l(-p-2y) \neq \emptyset\}.$$

For instance, $N_2(3)$ is the number of pairs $(x,y) \in C \times C$, $x \neq p \neq y$ such that $3x \equiv p + 2y$, hence $N_2(3) = r(3,2) = 70$ (cf. formula (5)).
For each $d \geq 4$ we fix a general pointed curve $[B,q] \in \mathcal{M}_{2d-5,1}$ and define the invariant:

$$N_3(d) := \#\{l \in G^1_d(B) : \exists x \neq y \in B - \{q\} \text{ such that } l(-3x) \neq \emptyset \text{ and } l(-2q-2y) \neq \emptyset\}.$$

Theorem 4.2. *The closure of the divisor* $\mathfrak{TR}_3 := \{[C] \in \mathcal{M}_3 : \exists x \neq p \in C$ *with* $3x \equiv 3x\}$ *is linearly equivalent to the class*

$$\overline{\mathfrak{TR}}_3 \equiv 2912\lambda - 311\delta_0 - 824\delta_1 \in \mathrm{Pic}(\overline{\mathcal{M}}_3).$$

It follows that $\overline{\mathcal{D}}_2 \equiv -200\lambda + 160\psi + 17\delta_0 \in \mathrm{Pic}(\overline{\mathcal{M}}_{2,1})$.

Proof. For most of this proof we assume $d \geq 3$ and we specialize to the case of $\overline{\mathcal{M}}_3$ only at the very end. We write $\overline{\mathfrak{TR}}_d \equiv a\,\lambda - b_0\,\delta_0 - \cdots - b_{d-2}\,\delta_{d-2} \in \mathrm{Pic}(\overline{\mathcal{M}}_g)$ and we have already determined $b_1, \ldots, b_{d-2}$ (cf. Theorem 3.3) while we know that $a - 12b_0 + b_1 = 4a(d, 2d-4)$ (cf. Proposition 3.2). We need one more relation involving a, b_0 and b_1, which we obtain by intersecting $\overline{\mathfrak{TR}}_d$ with the test curve

$$C^0 := \{\frac{C}{q \sim p}\}_{p \in C} \subset \Delta_0 \subset \overline{\mathcal{M}}_g$$

obtained from a general curve $[C, q] \in \mathcal{M}_{2d-4,1}$. The number $C^0 \cdot \overline{\mathfrak{TR}}_d$ counts (with appropriate multiplicities) admissible coverings

$$t := [f : X \overset{d:1}{\to} R,\ q_1, q_2 : p_1, \ldots, p_{6d-12}] \bmod \mathfrak{S}_2 \times \mathfrak{S}_{6d-12} \in \mathfrak{H}_d,$$

where the source X is stably equivalent to the curve $C \cup_{\{p,q\}} T$ ($q \in C$) obtained by "blowing-up" $\frac{C}{q \sim p}$ at the node and inserting a rational curve T. These covers should possess two points of triple ramification $x, y \in X_{\mathrm{reg}}$ such that $f(x) = q_1, f(y) = q_2$. Suppose $t \in C^0 \cdot \overline{\mathfrak{TR}}$ and again we distinguish a number of possibilities:
(i) $x, y \in C$. Then $\deg(f_C) = d$ and f_C corresponds to one of the $N(d)$ linear series $l \in G^1_d(C)$ with two points of triple ramification. The point $q \in C$ is such that $l(-p-q) \neq \emptyset$, which, after having fixed l, gives $d-1$ choices. Clearly $\mathrm{mult}_q(f_C) = \mathrm{mult}_q(f_T) = 1$. This implies that $\deg(f_T) = 2$ and f_T is given by $|\mathcal{O}_T(p+q)|$. To obtain out of $\{f_C, f_B\}$ a point $t \in \overline{\mathcal{H}}^{\mathrm{tr}}_d$, we attach rational curves to C at the points in $f_C^{-1}(f(p)) - \{p, q\}$ and map these isomorphically onto the component $f_T(T)$ of R. Each such cover has an automorphism $\phi : X \to X$ of order 2 such that $\phi_C = \mathrm{id}_C$, $\phi_{T'} = \mathrm{id}_{T'}$, for every rational component $T' \neq T$ of X, but ϕ_T interchanges the 2 branch points of T. Even though $t \in \overline{\mathcal{H}}^{\mathrm{tr}}_d$ is a smooth point (because there is no automorphism of X preserving *all* the ramification points of f), if $\tau \in \mathfrak{S}_{6d-12}$ is the involution exchanging the marked points lying on $f_T(T)$, then $\tau \cdot t = t$. Therefore $\overline{\mathcal{H}}^{\mathrm{tr}}_d / \mathfrak{S}_2 \to \overline{\mathcal{M}}_g$ is simply ramified at t. In a general deformation $[\mathcal{X} \to \mathcal{R}]$ of $[f : X \to R]$ in $\overline{\mathcal{H}}^{\mathrm{tr}}_d$ we blow-down T and obtain a rational double point, hence the image of $\mathcal{R}$ in $\overline{\mathcal{M}}_g$ meets Δ_0 with multiplicity 2. Since $\overline{\mathcal{H}}^{\mathrm{tr}}_d / \mathfrak{S}_2 \to \overline{\mathcal{M}}_g$ is ramified anyway, it follows that each of the $(d-1)N(d)$ admissible coverings found at this step is to be counted with multiplicity 1.
(ii) $x \in C, y \in T$. Since C has only finitely many $\mathfrak{g}^1_{d-1}$'s, all simply ramified and having no ramification in the fibre over q, we must have that $\deg(f_C) = d$ and $\deg(f_T) = 3$. Moreover, C and T map via f onto the two components of the target R in such a way that $f_C(p) = f_C(q) = f_T(p) = f_C(q)$. In particular, both f_C and f_T are simply ramified at either p or q. If f_C is ramified at $q \in C$, then f_C is induced

by one of the $e(d, 2d-4)$ linear series $l \in G^1_d(C)$ with one unassigned point of triple ramification and one assigned point of simple ramification. Having fixed l, there are $d-2$ choices for $p \in C$ such that $l(-2q-p) \neq \emptyset$. On T there is a unique $\mathfrak{g}^1_3$ corresponding to a map $f_T : T \to \mathbf{P}^1$ such that $f_T^*(0) = 2q+p$ and $f_T^*(\infty) = 3y$, for some $y \in T - \{q, p\}$. Finally, we attach $d-3$ rational curves to C at the points in $f_C^{-1}(f(q)) - \{p, q\}$ and we map these components isomorphically onto $f_T(T)$.

The other possibility is that f_C is unramified at q and ramified at p. The number of such $\mathfrak{g}^1_d$'s is $N_2(d)$. On the side of T, there is a unique way of choosing $f_T : T \stackrel{3:1}{\to} \mathbf{P}^1$ such that $f_T^*(0) = q+2p$ and $f_T^*(\infty) = 3y$. Because the map $\sigma : \mathfrak{H}_d \to \overline{\mathcal{M}}_g$ blows-down the component T, if $[\mathcal{X} \to \mathcal{R}]$ is a general deformation of $[f : X \to R]$ then $\sigma(\mathcal{R})$ meets Δ_0 with multiplicity 3 (see also [Di], pg. 47-52). Thus $\overline{\mathfrak{TR}}_d \cdot \Delta_0$ has multiplicity 3 at the point $[C/p \sim q]$. The admissible coverings constructed at this step have no automorphisms, hence they each must be counted with multiplicity 3. This yields a total contribution of $3(d-2)e(d, 2d-4) + 3N_2(d)$.

(iii) $x, y \in T - \{p, q\}$. Here there are two subcases. First, we assume that $\deg(f_C) = d-1$, that is, f_C is induced by one of the $\frac{(2d-4)!}{(d-1)!(d-2)!}$ linear series $l \in G^1_{d-1}(C)$. For each such l, there are $d-2$ possibilities for p such that $l(-q-p) \neq \emptyset$. Clearly $\deg(f_T) = 3$ and the admissible covering f is constructed as follows: Choose $f_T : T \to \mathbf{P}^1$ such that $f_T^*(0) = 3x$, $f_T^*(\infty) = 3y$ and $f_T^*(1) = p+q+q'$. We map C to the component of R other than $f_T(T)$ by using $l \in G^1_{d-1}(C)$ and $f_C(p) = f_T(p)$ and $f_C(q) = f_T(q)$. We attach to T a rational curve T' at the point q' and map T' isomorphically onto $f(C)$. Finally we attach $d-3$ rational curves to C at the points in $f_C^{-1}(f(q)) - \{q, p\}$. Each of these $\binom{2d-4}{d-1}$ elements of $\mathfrak{h}_d$ is counted with multiplicity 2.

We finally deal with the case $\deg(f_C) = d$. Since a $\mathfrak{g}^1_3$ on $\mathbf{P}^1$ with two points of total ramification must be unramified everywhere else, it follows that $\deg(f_T) \geq 4$. The generality assumption on $[C, q]$ implies that $\deg(f_T) = 4$. The C-aspect of f is induced by $l \in G^1_d(C)$ for which there are integers $\beta, \gamma \geq 1$ with $\beta + \gamma = 4$ and a point $p \in C$ such that $l(-\beta p - \gamma q) \neq \emptyset$. Proposition 2.1 gives the number $c(d, 2d-4, \gamma)$ of such $l \in G^1_d(C)$. On the side of T, we choose $f_T : T \stackrel{4:1}{\to} \mathbf{P}^1$ such that $f_T^*(0) = 3x$, $f_T^*(\infty) = 3y$ and $f_T^*(1) = \beta p + \gamma q$. When $\gamma \in \{1, 3\}$, up to isomorphism there is a unique such f_T having 3 triple ramification points. By direct computation we have the formula:

$$f_T : T \to \mathbf{P}^1, \ f_T(t) := \frac{2t^3(t-2)}{2t-1},$$

which has the properties that $f_T^{(i)}(0) = f_T^{(i)}(\infty) = f_T^{(i)}(1) = 0$, for $i = 1, 2$. When $\gamma = 2$, there are two $\mathfrak{g}^1_4$'s with 2 points of triple ramification and 2 points of simple ramification lying in the same fibre. It is important to point out that f_T (and hence the admissible covering f as well), has an automorphism of order 2 which preserves the points of attachment $p, q \in T$ but interchanges x and y (In coordinates, if $x = 0, y = \infty \in T$, check that $f_T(1/t) = 1/f_T(t)$). This implies that $\overline{\mathcal{H}}_d^{\mathrm{tr}} \to \overline{\mathcal{M}}_d$ is (simply) ramified at $[X \to R]$. Furthermore, a calculation similar to [Di] pg. 47-50, shows that the image in $\overline{\mathcal{M}}_g$ of a generic deformation in $\overline{\mathcal{H}}_d^{\mathrm{tr}}$ of $[X \to T]$ meets the divisor Δ_0 with multiplicity $4 = \beta + \gamma$. It follows that $\overline{\mathfrak{TR}}_d \cdot \Delta_0$ has multiplicity $4/2 = 2$ in a neighbourhood of $[C/p \sim q]$, that is, each covering found at this step gets counted with multiplicity 2 in the product $C^0 \cdot \overline{\mathfrak{TR}}$. Coverings of this type give a contribution of

$$2c(d, 2d-4, 1) + 2c(d, 2d-4, 3) + 4c(d, 2d-4, 2) = 128\binom{2d-4}{d}.$$

Thus we can write the following equation:

$$(2g-2)b_0 - b_1 = C^0 \cdot \overline{\mathfrak{TR}}_d = \tag{7}$$

$$= (d-1)N(d) + 3N_2(d) + 3(d-2)e(d, 2d-4) + 128\binom{2d-4}{d} + 2\binom{2d-4}{d-1}.$$

For $d = 3$, when $N_2(d) = 70$, all terms in (7) are known and this finishes the proof. ∎

5 The divisor $\overline{\mathfrak{TR}}_5$ and the class of $\overline{\mathcal{D}}_3$

In this section we finish the computation of $[\overline{\mathfrak{TR}}_d]$ (and implicitly compute $[\overline{\mathcal{D}}_3] \in \mathrm{Pic}(\overline{\mathcal{M}}_{2,1})$ and determine $N_2(d)$ for all $d \geq 3$ as well). According to (7) it suffices to compute $N_2(4)$ to determine $[\overline{\mathfrak{TR}}_4] \in \mathrm{Pic}(\overline{\mathcal{M}}_5)$. Then applying Theorem 1.2 we obtain $[\overline{\mathcal{D}}_3]$ which will finish the calculation of $[\overline{\mathfrak{TR}}_d]$ for $g = 2d-3$. We summarize some of the enumerative results needed in this section:

Proposition 5.1. *We fix a general* 2*-pointed elliptic curve* $[E, p, q] \in \mathcal{M}_{1,2}$.
(a) There are 11 *pencils* $l \in G^1_3(E)$ *such that there exist distinct points* $x, y \in E - \{p, q\}$ *with* $a^l_1(x) = 3$, $a^l_1(q) = 2$ *and* $l(-p-2y) \neq \emptyset$.
(b) There are 38 *pencils* $l \in G^1_4(E)$ *such that there exist distinct points* $x, y \in E - \{p, q\}$ *with* $a^l_1(p) = 4$, $a^l_1(x) = 3$ *and* $l(-q-2y) \neq \emptyset$.

Proof. **(a)** We denote by $\mathcal{U}$ the closure in $E \times E$ of the locus

$$\{(u, v) \in E \times E - \Delta : \exists l \in G^1_3(E) \text{ such that } a^l_1(q) = 3,\ a^l_1(u) \geq 2,\ a^l_1(v) \geq 2\}$$

and denote by F_i the (numerical class of the) fibre of the projection $\pi_i : E \times E \to E$ for $i = 1, 2$. Using that $\mathcal{U} \cap \Delta = \{(u, u) : u \neq q,\ 3u \equiv 3q\}$ (this intersection is transversal!), it follows that $\mathcal{U} \equiv 4(F_1 + F_2) - \Delta$. If $q \in E$ is viewed as the origin of E, then the isomorphism $E \times E \ni (x, y) \mapsto (-x, y - x) \in E \times E$ shows that the number of $l \in G^1_3(E)$ we are computing, equals the intersection number $\mathcal{U} \cdot \mathcal{V}$ on $E \times E$, where

$$\mathcal{V} := \{(u, v) \in E \times E : 2v + u \equiv 4q - p\}.$$

Since $\mathcal{V} \equiv 3F_1 + 6F_2 - 2\Delta$, we reach the stated answer by direct calculation.
(b) We specialize $[E, p, q] \in \mathcal{M}_{1,2}$ to the stable curve $[E \cup_r T, p, q] \in \overline{\mathcal{M}}_{1,2}$, where $[T, r, p, q] \in \overline{\mathcal{M}}_{0,3}$. We count admissible coverings $[f : X \xrightarrow{4:1} R, \tilde{p}, \tilde{q}]$, where $\tilde{p}, \tilde{q} \in X_{\mathrm{reg}}$, R is a nodal curve of genus 0 and there exist points $x, y \in X_{\mathrm{reg}}$ with the property that the divisors $4\tilde{p}, 3x, \tilde{q} + 2y$ on X all appear in distinct fibres of f.

Moreover $[X, \tilde{p}, \tilde{q}]$ is a pointed curve stably equivalent to $[E \cup_r T, p, q]$. There are three possibilities:

(1) $x, y \in E$. Then $f_T : T \stackrel{4:1}{\to} (\mathbf{P}^1)_1$ is uniquely determined by the properties $f_T^*(0) = 4p$ and $f_T^*(\infty) = 3r + q$, while $f_E : E \stackrel{3:1}{\to} (\mathbf{P}^1)_2$ is such that r and some point $x \in E - \{r\}$ appear as points of total ramification. In particular, $3x \equiv 3r$ on E, which gives 8 choices for x. Each such f_E has 2 remaining points of simple ramification, say $y_1, y_2 \in E$ and we take a rational curve T' which we attach to T at q and map isomorphically onto $(\mathbf{P}^1)_2$. Choose $\tilde{q} \in T'$ with the property that $f(\tilde{q}) = f_E(y_i)$ for $i \in \{1, 2\}$ and obviously $\tilde{p} = p \in T$. This procedure produces $16 = 8 \cdot 2$ admissible $\mathfrak{g}^1_4$'s.

(2) $x \in T$, $y \in E$. Now $f_T : T \stackrel{4:1}{\to} (\mathbf{P}^1)_1$ has the properties $f_T^*(0) = 4p, f_T^*(1) \geq 2r + q$ and $f_T^*(\infty) \geq 3x$ for some $x \in T$ (Up to isomorphism, there are 2 choices for f_T). Then $f_E : E \stackrel{2:1}{\to} (\mathbf{P}^1)_2$ is ramified at r and at some point $y \in E - \{r\}$ such that $2y \equiv 2r$. This gives 3 choices for f_E. We attach two rational curve T' and T'' to T at the points q and $q' \in f_T^{-1}(f(q)) - \{r, q\}$ respectively. We then map T' and T'' isomorphically onto $(\mathbf{P}^1)_2$. Finally we choose $\tilde{p} = p \in T$ and $\tilde{q} \in T'$ uniquely determined by the condition $f_{T'}(\tilde{q}) = f_E(y)$. We have produced $6 = 2 \cdot 3$ coverings.

(3) $x \in E, y \in T$. Counting ramification points on T we quickly see that $\deg(f_E) = 3$ and $f_E : E \to (\mathbf{P}^1)_2$ is such that $f_E^*(0) = 3x$ and $f_E^*(\infty) = 3r$, which gives 8 choices for f_E. Moreover $f_T : T \stackrel{4:1}{\to} (\mathbf{P}^1)_1$ must satisfy the properties $f_T^*(0) = 4p$, $f_T^*(1) \geq q + 2y$ and $f_T^*(\infty) = 3r + r'$ for some $r' \in T$. If $[T, p, q, r] = [\mathbf{P}^1, 0, 1, \infty] \in \overline{\mathcal{M}}_{0,3}$, then

$$f_T(t) = \frac{t^4}{t - r'}, \text{ where } r' \in \{\frac{1 + \sqrt{-2}}{4}, \frac{1 - \sqrt{-2}}{4}\}.$$

Thus we obtain another $16 = 8 \cdot 2$ admissible $\mathfrak{g}^1_4$'s in this case. Adding (1), (2) and (3), we found $38 = 16 + 6 + 16$ admissible coverings $\mathfrak{g}^1_4$ on $E \cup_r T$ and this finishes the proof. ∎

Proposition 5.2. *We fix a general pointed curve* $[C, p] \in \mathcal{M}_{3,1}$. *Then there are* 210 *pencils* $l = \mathcal{O}_C(2p + 2x) \in G^1_4(C)$, $x \in C$, *having an unspecified triple point.*

Proof. We define the map $\phi : C \times C \to \mathrm{Pic}^1(C)$ given by

$$\phi(x, y) := \mathcal{O}_C\big(2p + 2x - 3y\big).$$

A standard calculation shows that $\phi^*(W_1(C)) = g(g-1) \cdot 2^2 \cdot 3^2 = 216$ (Use Poincaré's formula $[W_1(C)] = \theta^2/2$). Set-theoretically it is clear that $\phi^*(W_1(C)) \cap \Delta = \{(p, p)\}$. A local calculation similar to [Di] pg. 34-36, shows that the intersection multiplicity at the point (p, p) is equal to $6 = g(g-1)$, hence the answer to our question. ∎

5.1 The invariant $N_2(d)$

We have reached the final step of our calculation and we now compute $N_2(d)$. We denote by $\overline{\mathcal{A}}_d$ the Hurwitz stack parameterizing admissible coverings of degree d

$$t := [f : (X,p) \stackrel{d:1}{\longrightarrow} R,\ q_0; p_0;\ p_1, \ldots, p_{6d-13}],$$

where $[X,p]$ is a pointed nodal curve of genus $2d-4$, $[R, q_0; p_0 : p_1, \ldots, p_{6d-13}]$ is a pointed nodal curve of genus 0, and f is an admissible covering in the sense of [HM] having a point of triple ramification $x \in f^{-1}(q_0)$, a point of simple ramification $y \in X - \{p\}$ such that $f(y) = f(p) = p_0$ and points of simple ramification in the fibres over $p_1, \ldots, p_{6d-13}$. The symmetric group $\mathfrak{S}_{6d-13}$ acts on $\overline{\mathcal{A}}_d$ by permuting the branch points $p_1, \ldots, p_{6d-13}$ and the stabilization map

$$\phi : \overline{\mathcal{A}}_d / \mathfrak{S}_{6d-13} \to \overline{\mathcal{M}}_{2d-4,1},\ \ \phi(t) := [X,p]$$

is generically finite of degree $N_2(d)$.

We completely describe the fibre $\phi^{-1}([C \cup_q E, p])$, where $[C,q] \in \mathcal{M}_{2d-5,1}$ and $[E,q,p] \in \mathcal{M}_{1,2}$ are general pointed curves. We count admissible covers $f : (X, \tilde{p}) \to R$ as above, where $[X, \tilde{p}]$ is stably equivalent to $[C \cup_q E, p]$. Depending on the position of the ramification points $x, y \in X$ we distinguish between the following cases:

(i) $x \in C, y \in E$. From Brill-Noether theory, we know that $\deg(f_C) \in \{d-1, d\}$. If $\deg(f_C) = d$, then one possibility is that both f_C and f_E are triply ramified at q. In this case f_C is induced by one of the $e(d, 2d-5)$ linear series $l \in G^1_d(C)$ with $l(-3q) \neq \emptyset$ and $l(-3x) \neq \emptyset$, for some $x \in C - \{q\}$. The covering f_E is of degree 3 and it induces a linear equivalence $3q \equiv 2y + p$ on E which has 4 solutions $y \in E$. To obtain X we attach to C rational curves at the $d-3$ points in $f_C^{-1}(f(q)) - \{q\}$. We have exhibited in this way $4e(d, 2d-5)$ automorphism-free points in $\phi^{-1}([C \cup_q E, p])$ which are counted with multiplicity 1. Another possibility is that both f_C and f_E are simply ramified at q and the fibre $f_C^{-1}(f(q))$ contains a second point $z \neq q$ of simple ramification. The number of such $l \in G^1_d(C)$ has been denoted by $N_3(d)$. Having chosen f_C, then $f_E : E \stackrel{2:1}{\to} (\mathbf{P}^1)_2$ is induced by $|\mathcal{O}_E(2q)|$. Then we attach a rational curve T to C at z, and we map $T \stackrel{2:1}{\to} (\mathbf{P}^1)_2$ using the linear system $|\mathcal{O}_T(2q)|$ in such a way that the remaining ramification point of f_T maps to $f_E(p)$. We produce $N_3(d)$ smooth points of $\overline{\mathcal{A}}_d / \mathfrak{S}_{6d-13}$ via this construction. In both these cases $\tilde{p} = p \in C \cup E$.

(ii) $x, y \in C$. Now $\deg(f_C) = d-1$ and f_C is induced by one of the $b(d-1, 2d-5) = e(d-1, 2d-5)$ linear series $l \in G^1_{d-1}(C)$ with $l(-3x) \neq \emptyset$ for some $x \in C - \{p\}$. Moreover, $f_C(q)$ is not a branch point of f_C which implies that $\deg(f_E) = 2$ and that f_E is induced by $|\mathcal{O}_E(p+q)|$. Obviously, f_C and f_E map to different components of R. To obtain the source $(X, \tilde{p})$ of our covering, we first attach $d-2$ rational curves to C at all the points in $f_C^{-1}(f(q)) - \{q\}$ and map these curves $1:1$ onto $f_E(E)$. Then we attach a curve $T' \cong \mathbf{P}^1$, this time to E at the point q and map T' isomorphically onto $f_C(C)$. The point $\tilde{q} \in X$ lies on the tail T' and is characterized by the property $f_{T'}(\tilde{p}) = f_C(y)$, where $y \in C$ is one of the $6d-16$

simple ramification points of l. This procedure produces $(6d-16)b(d-1,2d-5)$ admissible coverings in $\phi^{-1}([C\cup_q E, p])$.
(iii) $x\in E, y\in E$. If $\deg(f_C)=d$, then $\deg(f_E)\geq 4$ and f_C is given by one of the $a(d,2d-5)$ linear series $l\in G^1_d(C)$ such that $l(-4q)\neq\emptyset$. Then $f_E: E\stackrel{4:1}{\to}\mathbf{P}^1$ has the properties that (up to an automorphism of the base) $f_E^*(0)=4q$, $f_E^*(1)\geq p+2y$ and $f^*(\infty)\geq 3x$, for some points $x,y\in E-\{p,q\}$. The number of such $\mathfrak{g}^1_4$'s has been computed in Proposition 5.1 (b) and it is equal to 38. Therefore this case produces $38a(d,2d-5)$ coverings. If on the contrary, $\deg(f_C)=d-1$, then f_C is induced by one of the $a(d-1,2d-5)$ linear series $l\in G^1_{d-1}(C)$ such that $l(-2q)\neq\emptyset$, while $f_E: E\stackrel{3:1}{\to}\mathbf{P}^1$ is such that (up to an automorphism of the base) $f_E^*(0)\geq 2q$, $f_E^*(1)=p+2y$, $f_E^*(\infty)=3x$ for some $x,y\in E-\{p,q\}$. After making these choices, we attach $d-3$ rational curves to C at the point $\{q'\}=f_C^{-1}(f(q))-\{q\}$ and we map these isomorphically onto $f_E(E)$. Furthermore, we attach a rational curve T' to E at the point $\{q'\}=f_E^{-1}(f(q))-\{q\}$ and map T' isomorphically onto $f_C(C)$. Using Proposition 5.1 (a), we obtain $11a(d-1,2d-5)$ admissible coverings. Altogether part (iii) provides $38a(d-1,2d-5)+11a(d-1,2d-5)$ points in $\overline{\mathcal{A}}_d/\mathfrak{S}_{6d-13}$.
(iv) $x\in E, y\in C$. In this case, since p and y lie in different components, we know that we have to "blow-up" the point p and insert a rational curve which is mapped to the component $f_C(C)$ of R. Thus $\deg(f_C)\leq d-1$, and by Brill-Noether theory it follows that $\deg(f_C)=d-1$. Precisely, f_C is induced by one of the $a(d-1,2d-5)$ linear series $l\in G^1_{d-1}(C)$ such that $l(-2q)\neq\emptyset$. Furthermore, $f_E: E\stackrel{3:1}{\to}\mathbf{P}^1$ can be chosen such that $f_E^*(0)=p+2q$ and $f_E^*(\infty)=3x$ for some $x\in E$. This gives the linear equivalence $3x\equiv p+2q$ on E which has 9 solutions. We attach $d-3$ rational curves at the points in $f_C^{-1}(f(q))-\{q\}$ and map these $1:1$ onto $f_E(E)$. Finally, we attach a rational curve T' to E at the point p and map T' such that $f(T')=f(C)$. We pick $\tilde{p}\in T'$ with the property that $f_{T'}(\tilde{p})=f_C(y)$, where $y\in C$ is one of the $6d-15$ ramification points of f_C. We have obtained $9(6d-15)a(d-1,2d-5)$ admissible coverings in this way.

We have completely described $\phi^{-1}([C\cup_q E,p])$ and it is easy to check that all these coverings have no automorphisms, hence they give rise to smooth points in $\overline{\mathcal{A}}_d$ and that the map ϕ is unramified at each of these points. Thus

$$N_2(d)=\deg(\phi)=4e(d,2d-5)+(6d-16)b(d-1,2d-5)+38a(d,2d-5)+$$

$$+11a(d-1,2d-5)+9(6d-15)\,a(d-1,2d-5)+N_3(d).$$

For $d=4$, we know that $N_3(4)=210$ (cf. Proposition 5.2), which determines $N_2(4)$ and the class $[\overline{\mathcal{D}}_3]$. We record these results:

Theorem 5.3. *The locus $\mathcal{D}_3$ of pointed curves $[C,p]\in\mathcal{M}_{2,1}$ with a pencil $l\in G^1_4(C)$ totally ramified at p and having two points of triple ramification, is a divisor on $\mathcal{M}_{2,1}$. The class of its compactification in $\overline{\mathcal{M}}_{2,1}$ is given by the formula:*

$$\overline{\mathcal{D}}_3\equiv 640\psi-860\lambda+72\delta_0\in \mathrm{Pic}(\overline{\mathcal{M}}_{2,1}).$$

Theorem 5.4. *For a general pointed curve $[C, p] \in \mathcal{M}_{2d-4,1}$ the number of pencils $L \in W^1_d(C)$ satisfying the conditions*

$$h^0(L \otimes \mathcal{O}_C(-3x)) \geq 1 \text{ and } h^0(L \otimes \mathcal{O}_C(-p-2y)) \geq 1$$

for some points $x, y \in C - \{p\}$, is equal to

$$N_2(d) = \frac{6(40d^2 - 179d + 212)\,(2d-4)!}{d!\,(d-3)!}.$$

Remark 5.5. As a check, for $d = 3$, the number $N_2(3)$ computes the number of pairs $(x, y) \in C \times C$ such that $p \neq x \neq y \neq p$ and $3x \equiv p + 2y$. This number is equal to $r(3, 2) = 70$ which matches Theorem 5.4.

Theorem 5.6. *We fix an integer $d \geq 4$. For a general pointed curve $[C, p] \in \mathcal{M}_{2d-5,1}$, the number of pencils $L \in W^1_d(C)$ satisfying the conditions*

$$h^0(L \otimes \mathcal{O}_C(-3x)) \geq 1 \text{ and } h^0(L \otimes \mathcal{O}_C(-2p-2y)) \geq 1$$

for some points $x, y \in C - \{p\}$, is equal to

$$N_3(d) = \frac{84(d-3)(2d^2 - 10d + 13)\,(2d-4)!}{d!\,(d-2)!}.$$

Remark 5.7. For $d = 4$, Theorem 5.6 specializes to Proposition 5.2 and we find again that $N_3(4) = 210$.

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