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Gaussian maps, Gieseker-Petri loci and large theta-characteristics

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1. Introduction

For an integer $g \geqq 1$ we consider the moduli space $\mathcal{S}_g$ of smooth spin curves parametrizing pairs (C, L), where C is a smooth curve of genus g and L is a theta-characteristic, that is, a line bundle on C such that $L^2 \cong K_C$. It has been known classically that the natural map $\pi : \mathcal{S}_g \to \mathcal{M}_g$ is finite of degree 2^{2g} and that $\mathcal{S}_g$ is a disjoint union of two components $\mathcal{S}_g^{\text{even}}$ and $\mathcal{S}_g^{\text{odd}}$ corresponding to even and odd theta-characteristics. A geometrically meaningful compactification $\overline{\mathcal{S}}_g$ of $\mathcal{S}_g$ has been constructed by Cornalba by means of stable spin curves of genus g (cf. [C]). The space $\overline{\mathcal{S}}_g$ and more generally the moduli spaces $\overline{\mathcal{S}}_{g,n}^{1/r}$ of stable n-pointed r-spin curves of genus g, parametrizing pointed curves with r-roots of the canonical bundle, have attracted a lot of attention in recent years, partly due to a conjecture of Witten relating intersection theory on $\overline{\mathcal{S}}_{g,n}^{1/r}$ to generalized KdV hierarchies (see e.g. [JKV]).

For each $g, r \geqq 0$ one can define the locus

$$\mathcal{S}_g^r := \{(C, L) \in \mathcal{S}_g : h^0(L) \geqq r + 1 \text{ and } h^0(L) \equiv r + 1 \bmod 2\}.$$

We also set $\mathcal{M}_g^r := \pi(\mathcal{S}_g^r)$. It has been proved by Harris that each component of $\mathcal{S}_g^r$ has dimension $\geqq 3g - 3 - \binom{r+1}{2}$ (cf. [H]). This bound is known to be sharp when r is very small: it is a classical result that $\mathcal{S}_g^1$ is a divisor in $\mathcal{S}_g$, while for $r = 2, 3$ we have that $\mathcal{S}_g^r$ has pure codimension $r(r+1)/2$ in $\mathcal{S}_g$ for all $g \geqq 8$ (cf. [T1]). On the other hand clearly the bound is far from optimal when r is relatively large with respect to g in the sense that there are examples when $\mathcal{S}_g^r \neq \emptyset$ although $3g - 3 - \binom{r+1}{2}$ is very negative: the hyperelliptic locus $\mathcal{H}_g \subset \mathcal{M}_g$ is contained in $\mathcal{M}_g^{[(g-1)/2]}$ and there are Castelnuovo extremal curves $C \subset \mathbf{P}^r$ of genus $3r$ such that $K_C = \mathcal{O}_C(2)$, which gives that $\mathcal{S}_{3r}^r \neq \emptyset$ for all $r \geqq 3$ (see e.g. [CdC]). It is thus natural to ask to what extent Harris' bound is sharp. We give a partial answer to this question by proving the following:

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Theorem 1.1. *For* $1 \leqq r \leqq 11$, $r \neq 10$, *there exists an explicit integer* $g(r)$ *such that for all* $g \geqq g(r)$ *the moduli space* $\mathcal{S}_g^r$ *has at least one component of codimension* $\binom{r+1}{2}$ *in* $\mathcal{S}_g$. *The general point* $[C, L]$ *of such a component corresponds to a smooth curve* $C \subset \mathbf{P}^r$, *with* $L = \mathcal{O}_C(1)$ *and* $K_C = \mathcal{O}_C(2)$.

For a precise formula for $g(r)$ we refer to Section 3. We conjecture the existence of a component of $\mathcal{S}_g^r$ of codimension $\binom{r+1}{2}$ for any $r \geqq 1$ and $g \geqq \binom{r+2}{2}$ and we indicate a way to construct such a component (see Conjecture 3.4). Theorem 1.1 is proved inductively using the following result:

Theorem 1.2. *We fix integers* $r, g_0 \geqq 1$. *If* $\mathcal{S}_{g_0}^r$ *has a component of codimension* $\binom{r+1}{2}$ *in* $\mathcal{S}_{g_0}$, *then for every* $g \geqq g_0$, *the space* $\mathcal{S}_g^r$ *has a component of codimension* $\binom{r+1}{2}$ *in* $\mathcal{S}_g$.

To apply Theorem 1.2 however, one must have a starting case for the inductive argument. This is achieved by carrying out an infinitesimal study of the loci $\mathcal{S}_g^r$ which will relate theta-characteristics to Gaussian maps on curves. Recall that for a smooth curve C and a line bundle L on C, the *Gaussian* or *Wahl* map $\psi_L : \bigwedge^2 H^0(L) \to H^0(K_C \otimes L^2)$ is defined essentially by

$$\psi_L(s \wedge t) := s\,dt - t\,ds.$$

The map ψ_L has attracted considerable interest being studied especially in the context of deformation theory (see [W1] and the references therein). Wahl proved the remarkable fact that if C sits on a K3 surface then ψ_{K_C} cannot be surjective, which should be contrasted with the result of Ciliberto, Harris and Miranda saying that ψ_{K_C} is surjective for the general curve C of genus $g = 10$ or $g \geqq 12$ (cf. [CHM]). In a completely different direction, in a previous work we made essential use of the Gaussian map ψ_{K_C} for $g = 10$ to construct a counterexample to the Harris-Morrison Slope Conjecture on effective divisors on $\bar{\mathcal{M}}_g$ (cf. [FP]).

There are several powerful criteria in the literature ensuring the surjectivity of ψ_L when L has large degree (see e.g. [Pa], Theorem G), but very little seems to be known about when is the map ψ_L injective, or more generally, what is the behaviour of ψ_L when the line bundle L is special (cf. [W1], Question 5.8.1). In Section 5 we go some way towards answering this question by showing the following:

Theorem 1.3. *For the general curve* C *of genus* g *and for any line bundle* L *on* C *of degree* $d \leqq g + 2$, *the Gaussian map* ψ_L *is injective.*

We refer to Theorem 5.4 for a more general statement that bounds the dimension of $\mathrm{Ker}(\psi_L)$ even when $d > g + 2$. In the case when L is a very ample line bundle giving an embedding $C \subset \mathbf{P}^n$, Theorem 1.3 can be interpreted as saying that the associated curve $C \to \mathbf{P}^N$ obtained by composing the Gauss map $C \to G(2, n+1)$, $C \ni p \mapsto \mathbb{T}_p(C)$, with the Plücker embedding of the Grassmannian of lines, is nondegenerate. Alternatively one can read this result in terms of (absence of) certain self-correspondences on the general curve C (see Proposition 5.7).

In Section 4 we relate the Gieseker-Petri loci on $\bar{\mathscr{M}}_g$ to the moduli spaces $\mathscr{S}^r_{g,n}$ of n-pointed spin curves consisting of collections $(C, p_1, \dots, p_n, L)$, where $(C, p_1, \dots, p_n) \in \mathscr{M}_{g,n}$ and L is a degree k line bundle on C such that $L^2 \otimes \mathscr{O}_C(p_1 + \cdots + p_n) = K_C$ and $h^0(L) \geqq r+1$. Here of course we assume that $2k + n = 2g - 2$.

We recall that the Giseker-Petri Theorem asserts that for a general curve C of genus g and for any line bundle L on C, the map $\mu_0(L) : H^0(L) \otimes H^0(K_C \otimes L^{-1}) \to H^0(K_C)$ is injective (see e.g. [EH2]). It is straightforward to see that if $\mu_0(L)$ is not injective then $h^0(L), h^0(K_C \otimes L^{-1}) \geqq 2$ and it is an old problem to describe the locus in $\mathscr{M}_g$ where the Gieseker-Petri Theorem fails, in particular to determine its components and their dimensions.

We fix integers $r, d \geqq 1$ such that $\rho(g, r, d) = g - (r+1)(g - d + r) \geqq 0$. As usual, $G^r_d(C)$ is the variety of linear systems $\mathfrak{g}^r_d$ on C, and if $(L, V) \in G^r_d(C)$, we denote by $\mu_0(V) : V \otimes H^0(K_C \otimes L^{-1}) \to H^0(K_C)$ the multiplication map. We define the Gieseker-Petri locus of type (r, d)

$$GP^r_{g,d} := \{[C] \in \mathscr{M}_g : \exists \text{ a base point free } (L, V) \in G^r_d(C) \text{ with } \mu_0(V) \text{ not injective}\}.$$

There are only two instances when this locus is well understood. First, $GP^1_{g,g-1}$ can be identified with the above introduced locus $\mathscr{M}^1_g$ of curves with a vanishing theta-null which is known to be an irreducible divisor (cf. [T3]). Then for even $g \geqq 4$, $GP^1_{g,(g+2)/2}$ is a divisor on $\mathscr{M}_g$ which has an alternate description as the branch locus of the natural map $H_{g,(g+2)/2} \to \mathscr{M}_g$ from the Hurwitz scheme of coverings of $\mathbf{P}^1$ of degree $(g+2)/2$ with source curve of genus g. This last divisor played a crucial role in the proof that $\mathscr{M}_g$ is of general type for even $g \geqq 24$ (cf. [EH3]). It is natural to ask whether more generally, all loci $GP^r_{g,d}$ are divisors and we give a partial affirmative answer to this question:

Theorem 1.4. *For integers* $g \geqq 4$ *and* $(g+2)/2 \leqq k \leqq g-1$, *the Giseker-Petri locus* $GP^1_{g,k}$ *has a divisorial component.*

As an easy consequence we mention the following:

Corollary 1.5. *For* $g \geqq 4$ *and* $0 \leqq n \leqq g-4$, *the moduli space* $\mathscr{S}^1_{g,n}$ *has at least one component of dimension* $3g-4$.

This last statement can be compared to Polishchuk's recent result that the moduli space $\mathscr{S}^0_{g,n}$ is of pure dimension $3g - 3 + n/2$ (cf. [Po], Theorem 1.1).

2. Limit theta-characteristics

In this section, after briefly recalling some basic facts about stable spin curves, we characterize limit theta-characteristics on certain stable curves of compact type after which we prove Theorem 1.2.

We review a few things about the moduli space $\overline{\mathscr{S}}_g$ (see [C] for more details). If X is a nodal curve, a smooth rational component R of X is called *exceptional* if $\#\big(R \cap \overline{(X - R)}\big) = 2$. The curve X is called *quasistable* if every two exceptional components are disjoint. Every quasistable curve is obtained by blowing-up some of the nodes of a stable curve.

A *stable spin curve* consists of a triple (X, L, α), where X is a quasistable curve with $p_a(X) = g$, L is a line bundle on X of degree $g-1$ with $L_R = \mathcal{O}_R(1)$ for each exceptional component R and $\alpha : L^2 \to \omega_X$ is a homomorphism such that $\alpha_C \neq 0$ for any non-exceptional component C of X. A *family of stable spin curves* is a triple $(f : \mathscr{C} \to T, \mathscr{L}, \alpha)$, where $f : \mathscr{C} \to T$ is a flat family of quasistable curves, $\mathscr{L}$ is a line bundle on $\mathscr{C}$ and $\alpha : \mathscr{L}^2 \to \omega_f$ is a homomorphism such that α_{C_t} gives a spin structure on each fibre $C_t = f^{-1}(t)$.

The stack $\overline{\mathscr{S}}_g$ of stable spin curves of genus g has been constructed in [C] where it is also proved that there exists a finite map $\pi : \overline{\mathscr{S}}_g \to \overline{\mathscr{M}}_g$ whose fibre over $[C] \in \overline{\mathscr{M}}_g$ is the set of stable spin structures on quasistable curves stably equivalent to C.

Remark 2.1. Suppose $C = C_1 \cup_p C_2$ is a curve of compact type with C_1 and C_2 being smooth curves and $g(C_1) = i$, $g(C_2) = g - i$. Then it is easy to see that there are no spin structures on C itself. In fact, $\pi^{-1}([C])$ consists of spin structures on the quasistable curve $X = C_1 \cup_q R \cup_r C_2$ obtained from C by "blowing-up" C at the node p. Each such spin structure is given by a line bundle L on X such that $L_{C_1}^2 = K_{C_1}$, $L_{C_2}^2 = K_{C_2}$ and $L_R = \mathcal{O}_R(1)$. More generally, a spin structure on any curve of compact type corresponds to a collection of theta-characteristics on the components.

Assume now that $C = C_1 \cup_p C_2$ is a curve of compact type where C_1 and C_2 are smooth curves of genus i and $g - i$ respectively. We define an *r-dimensional limit theta-characteristic* on C (in short, a limit θ_g^r), as being a pair of line bundles (L_1, L_2) with $L_i \in \mathrm{Pic}^{g-1}(C_i)$, together with $(r+1)$-dimensional subspaces $V_i \subset H^0(L_i)$ such that

(1) $\{l_i = (L_i, V_i)\}_{i=1,2}$ is a limit linear series $\mathfrak{g}_{g-1}^r$ in the sense of [EH1].

(2) $L_1^2 = K_{C_1}\big(2(g-i)p\big)$ and $L_2^2 = K_{C_2}(2ip)$.

Using this terminology we now characterize singular curves in $\overline{\mathscr{M}}_g^r$:

Lemma 2.2. *Suppose* $[C = C_1 \cup_p C_2] \in \overline{\mathscr{M}}_g^r$. *Then* C *possesses a* θ_g^r.

Proof. We may assume that there exists a 1-dimensional family of curves $f : \mathscr{C} \to B$ with smooth general fibre C_b and central fibre $C_0 = f^{-1}(0)$ stably equivalent to C, together with a line bundle $\mathscr{L}$ on $\mathscr{C} - C_0$ and a rank $(r+1)$ subvector bundle $V \subset f_*(\mathscr{L})$ over $B^* := B - \{0\}$ such that $\mathscr{L}_{C_b}^2 \equiv \omega_{C_b}$ for all $b \in B^*$. Then for $i = 1, 2$ there are unique line bundles $\mathscr{L}_i$ on $\mathscr{C}$ for extending $\mathscr{L}$ and such that $\deg_Y(\mathscr{L}_i) = 0$ for every component Y of C_0 different from C_i. If we denote by $L_i := \mathscr{L}_i|_{C_i}$ and $V_i \subset H^0(L_i)$ the $(r+1)$-dimensional subspace of sections that are limits in L_i of sections in V, then by [EH1], Theorem 2.6, we know that $\{(L_i, V_i)\}_{i=1,2}$ is a limit $\mathfrak{g}_{g-1}^r$. Finally, since L_i^2 and K_{C_i} are isomorphic off p they must differ by a divisor supported at p which accounts for condition (2) in the definition of a θ_g^r. $\square$

We describe explicitly the points in $\overline{\mathcal{M}}_g^r \cap \Delta_1$, where Δ_1 is the divisor of curves with an elliptic tail:

Proposition 2.3. *Let $[C = C_1 \cup_p E]$ be a stable curve with C_1 smooth of genus $g-1$ and E an elliptic curve. If $[C] \in \overline{\mathcal{M}}_g^r$ then either* (1) *$[C_1] \in \mathcal{M}_{g-1}^r$, or* (2) *there exists a line bundle L_1 on C_1 such that $(C_1, L_1) \in \mathcal{S}_{g-1}^{r-1}$ and $p \in \mathrm{Bs}|L_1|$. If moreover $p \in C_1$ is a general point, then possibility* (2) *does not occur hence $[C_1] \in \mathcal{M}_{g-1}^r$.*

Proof. We know that C carries a limit θ_g^r, say $l = \{l_{C_1}, l_E\}$. By the compatibility relation between l_{C_1} and l_E, the vanishing sequence $a^{l_{C_1}}(p)$ of l_{C_1} at p is $\geqq (0, 2, \dots, r+1)$. If l_{C_1} has a base point at p then if we set $L := L_{C_1}(-p)$ we see that $(C_1, L) \in \mathcal{S}_{g-1}^r$ and we are in case (1). Otherwise we set $M := L_{C_1}(-2p)$ and then $h^0(C_1, M) = r$, $M^2 = K_{C_1}(-2p)$ and $|M + p|$ is a theta-characteristic on C_1 having p as a base point.

For the last statement, we note that a curve has finitely many positive dimensional theta-characteristics each of them having only a finite number of base points, so possibility (2) occurs for at most finitely many points $p \in C_1$. □

We can now prove Theorem 1.2. More precisely we have the following result:

Proposition 2.4. *Fix $r, g \geqq 1$. If $\mathcal{S}_{g-1}^r$ has a component of codimension $\binom{r+1}{2}$ in $\mathcal{S}_{g-1}$, then $\mathcal{S}_g^r$ has a component of codimension $\binom{r+1}{2}$ in $\mathcal{S}_g$.*

Proof. Suppose $[C_1, L_1] \in \mathcal{S}_{g-1}^r$ is a point for which there exists a component $\mathcal{Z} \ni [C_1, L_1]$ of $\mathcal{S}_{g-1}^r$ with $\mathrm{codim}(\mathcal{Z}, \mathcal{S}_{g-1}) = \binom{r+1}{2}$. We fix a general point $p \in C_1$ and set $C := C_1 \cup_p E$, where (E, p) is a general elliptic curve. We denote by $X := C_1 \cup_q R \cup_s E$ the curve obtained from C by blowing-up p, and we construct a spin structure on X given by a line bundle L on X with $L_{C_1} = L_1$, $L_R = \mathcal{O}_R(1)$ and $L_E = \mathcal{O}_E(t - s)$, where $t - s$ is a non-zero torsion point of order 2. Clearly $h^0(X, L) = h^0(C_1, L_1) \geqq r + 1$. We first claim that (X, L) is a smoothable spin structure which would show that $[X, L] \in \overline{\mathcal{S}}_g^r$.

To see this we denote by $(f : \mathcal{X} \to B, \mathcal{L}, \alpha : \mathcal{L}^2 \to \omega_f)$ the versal deformation space of (X, L), so that if B_1 denotes the versal deformation space of the stable model C of X, there is a commutative diagram:

$$\begin{tikzcd}
B \arrow[r, "\sigma"] \arrow[d, "\phi"] & B/\mathrm{Aut}(X, L) \arrow[r, hook] & \overline{\mathcal{S}}_g \arrow[d, "\pi"] \\
B_1 \arrow[r] & B_1/\mathrm{Aut}(C) \arrow[r, hook] & \overline{\mathcal{M}}_g.
\end{tikzcd}$$

We define $B^r := \{b \in B : h^0(X_b, L_b) \geqq r+1, h^0(X_b, L_b) \equiv r + 1 \bmod 2\}$ and Theorem 1.10 from [H] gives that every component of B^r has dimension $\geqq \dim(B) - r(r+1)/2$. We also consider the divisor $\Delta \subset B$ corresponding to singular spin curves. To conclude that (X, L) is smoothable we show that there exists a component $W \ni 0$ of B^r not contained in Δ (here $0 \in B$ is the point corresponding to (X, L)).

Assume that on the contrary, every component of B^r containing 0 sits inside Δ. It is straightforward to describe $B^r \cap \Delta$: if $(X_b = C_b \cup R_b \cup E_b, L_b)$ where $b \in B$, $g(C_b) = g - 1$, $g(E_b) = 1$, is a spin curve with $h^0(X_b, L_b) \geqq r + 1$, then either (1) $h^0(C_b, L_{b|C_b}) \geqq r + 1$ or (2) $h^0(C_b, L_{b|C_b}) = r$ and $L_{b|E_b} = \mathcal{O}_{E_b}$ (put it differently, $L_{b|E_b}$ is the only odd theta characteristic on E_b). Since even and odd theta characteristics do not mix, it follows that any component $0 \in W \subset B^r$ will consist entirely of elements b for which $h^0(C_b, L_{b|C_b}) \geqq r + 1$. Moreover, there is a $1:1$ correspondence between such components of B^r and components of $\mathcal{S}^r_{g-1}$ through $[C_1, L_1]$. But then the locus

$$\mathcal{Z}_1 := \{b \in \Delta : [C_b, L_{b|C_b}] \in \mathcal{Z}, h^0(E_b, L_{b|E_b}) = 0\}$$

is a component of B^r containing 0 and $\dim(\mathcal{Z}_1) = \dim(\mathcal{Z}) + 2 = 3g - 4 - \binom{r+1}{2}$, which contradicts the estimate on $\dim(B^r)$.

Thus (X, L) is smoothable. We now show that at least one component of $\overline{\mathcal{S}}^r_g$ passing through $[C, L_C]$ has codimension $\binom{r+1}{2}$. Suppose this is not the case. Then each component of $\overline{\mathcal{S}}^r_g \cap \sigma(\Delta)$ through $[C, L_C]$ has codimension $\leqq \binom{r+1}{2} - 1$ in $\sigma(\Delta)$. Recalling that $p \in C_1$ was general, Proposition 2.3 says that any such component corresponds to curves $C_1' \cap E'$ where E' is elliptic and $[C_1'] \in \mathcal{M}^r_{g-1}$. But then $\sigma(\mathcal{Z}_1)$ is such a component and we have already seen that $\operatorname{codim}(\mathcal{Z}_1, \Delta) = \binom{r+1}{2}$, which yields the desired contradiction. □

Remark 2.5. Retaining the notation from the proof of Theorem 2.3, if $[C_1, L_1] \in \mathcal{S}^r_{g-1}$ is such that L_1 is very ample, then a smoothing $[C', L_{C'}] \in \mathcal{S}^r_g$ of $[C = C_1 \cup_p E, L_C]$ corresponds to a very ample $L_{C'}$. Indeed, assuming by contradiction that there exist points $x, y \in C'$ such that $h^0\big(L_{C'}(-x - y)\big) \geqq h^0(L_{C'}) - 1$, we have three possibilities depending on the position of the points $r, s \in C$ to which x and y specialize. The case $x, y \in E$ can be ruled out immediately, while $x, y \in C_1$ would contradict the assumption that L_1 is a very ample line bundle. Finally, if $x \in C_1$ and $y \in E$, one obtains that $\{x, p\}$ fails to impose independent conditions on $|L_1|$, a contradiction. Thus $L_{C'}$ is very ample.

3. Gaussian maps and theta-characteristics

It may be helpful to review a few things about Gaussian maps on curves and to explain the connection between Gaussians and theta-characteristics. This will enable us to construct components of $\mathcal{S}^r_g$ of dimension achieving the Harris bound.

For a smooth projective variety X and a line bundle L, we denote by $R(L)$ the kernel of the multiplication map $H^0(L) \otimes H^0(L) \to H^0(L^2)$. Following J. Wahl (see e.g. [W1]), we consider the *Gaussian map* $\Phi_L = \Phi_{X,L} : R(L) \to H^0(\Omega^1_X \otimes L^2)$, defined locally by

$$s \otimes t \mapsto s\,dt - t\,ds.$$

Since $R(L) = \bigwedge^2 H^0(L) \oplus S_2(L)$, with $S_2(L) = \operatorname{Ker}\{\operatorname{Sym}^2 H^0(L) \xrightarrow{\mu_L} H^0(L^2)\}$, it is clear that Φ_L vanishes on symmetric tensors and it makes sense to look at the restriction

$$\psi_L = \psi_{X,L} := \Phi_{L|\bigwedge^2 H^0(L)} : \bigwedge^2 H^0(L) \to H^0(\Omega_X^1 \otimes L^2).$$

If $X \subset \mathbf{P}^r$ is an embedded variety with $L = \mathcal{O}_X(1)$, one has the following interpretation for Φ_L: we pull back the Euler sequence to X to obtain that $R(L) = H^0(\Omega^1_{\mathbf{P}^r|X} \otimes L^2)$ and then Φ_L can be thought of as the map obtained by passing to global sections in the morphism $\Omega^1_{\mathbf{P}^r|X} \otimes L^2 \to \Omega^1_X \otimes L^2$. Furthermore, if N_X is the normal bundle of X in $\mathbf{P}^r$, tensoring the exact sequence

$$(1) \qquad 0 \to N_X^\vee \to \Omega^1_{\mathbf{P}^r|X} \to \Omega^1_X \to 0$$

by $\mathcal{O}_X(2)$, we obtain that $\mathrm{Ker}(\Phi_L) = \mathrm{Ker}(\psi_L) \oplus S_2(L) = H^0\big(N_X^\vee(2)\big)$. If X is projectively normal, from the exact sequence $0 \to \mathcal{I}_X^2 \to \mathcal{I}_X \to N_X^\vee \to 0$ it is straightforward to check that $\mathrm{Ker}(\psi_L) = H^1\big(\mathbf{P}^r, \mathcal{I}_X^2(2)\big)$.

The map ψ_L has been extensively studied especially when X is a curve, in the context of the deformation theory of the cone over X (cf. e.g. [W1]). The connection between Gaussian maps and spin curves is given by the following tangent space computation due to Nagaraj (cf. [N], Theorem 1): for $(C, L) \in \mathcal{S}_g^r$, if we make the standard identifications $T_{[C,L]}(\mathcal{S}_g) = T_{[C]}(\mathcal{M}_g) = H^1(C, T_C) = H^0(C, K_C^2)^\vee$, then

$$T_{[C,L]}(\mathcal{S}_g^r) = \big(\mathrm{Im}(\psi_L) : \bigwedge^2 H^0(L) \to H^0(K_C^2)\big)^\perp.$$

In other words, to show that a component $\mathcal{Z}$ of $\mathcal{S}_g^r$ has codimension $\binom{r+1}{2}$ in $\mathcal{S}_g$, it suffices to exhibit a spin curve $[C, L] \in \mathcal{Z}$ such that $h^0(L) = r+1$ and ψ_L is injective. We construct such curves as sections of certain homogeneous spaces having injective Gaussians and then we apply Theorem 1.2 to increase the range of (g, r) for which we have a component of $\mathcal{S}_g^r$ of codimension $\binom{r+1}{2}$. We will use repeatedly the following result of Wahl relating the Gaussian map of a variety to that of one of its sections (cf. [W2], Propositions 3.2 and 3.6):

Proposition 3.1. 1. *Suppose $X \subset \mathbf{P}^r$ is a smooth, projectively normal variety such that $\psi_{X,\mathcal{O}_X(1)}$ is injective. If $Y \subset X$ is a subvariety with ideal sheaf $\mathcal{I}$ satisfying the conditions*

$$H^1\big(X, \mathcal{I}(1)\big) = 0, \quad H^1\big(X, \mathcal{I}^2(2)\big) = 0, \quad H^1\big(X, N_X^\vee(2) \otimes \mathcal{I}\big) = 0,$$

then the Gaussian $\psi_{Y,\mathcal{O}_Y(1)}$ is injective too.

2. *Let $X \subset \mathbf{P}^r$ be a smooth, projectively normal, arithmetically Cohen-Macaulay variety and $Y = X \cap \mathbf{P}^{r-n} \subset \mathbf{P}^{r-n}$ a smooth codimension n linear section, where $n < r$. If $H^i\big(X, N_X^\vee(2-i)\big) = 0$ for $1 \leqq i \leqq n$ and $\psi_{X,\mathcal{O}_X(1)}$ is injective, then Y is projectively normal and the Gaussian $\psi_{Y,\mathcal{O}_Y(1)}$ is also injective.*

We will apply Proposition 3.1 in the case of the Grassmannian $X = G(2, n)$ of 2-dimensional quotients of $\mathbb{C}^n$ and for the line bundle $L = \mathcal{O}_{G(2,n)}(1)$ which gives the Plücker embedding. In this case $\psi_{\mathcal{O}_{G(2,n)}(1)}$ is bijective (cf. [W2], Theorem 2.11).

We need to compute the cohomology of several vector bundles on $G(2,n)$ and we do this using Bott's theorem (see [FH] for a standard reference). Recall that $G(2,n) = SL_n(\mathbb{C})/P$, where the reductive part of the parabolic subgroup P consists of matrices of type $\mathrm{diag}(A,B) \in SL_n(\mathbb{C})$ where $A \in GL_2(\mathbb{C})$ and $B \in GL_{n-2}(\mathbb{C})$. We denote by $\mathscr{Q}$ the universal rank 2 quotient bundle defined by the tautological sequence

$$0 \to \mathscr{U} \to \mathcal{O}_{G(2,n)}^{\oplus n} \to \mathscr{Q} \to 0.$$

Every irreducible vector bundle over $G(2,n)$ comes from a representation of the reductive part of P. If $e_1, \dots, e_n$ is an orthonormal basis of $\mathbb{R}^n$, the positive roots of $SL_n(\mathbb{C})$ are $\{e_i - e_j\}_{i<j}$ and we use the notation $E(a_1, \dots, a_n)$ for the vector bundle corresponding to the representation with highest weight $a_1 e_1 + \cdots + a_n e_n$. We then have the identifications $\mathscr{Q} = E(1,0,\dots,0)$, $\mathcal{O}_{G(2,n)}(1) = \det(\mathscr{Q}) = E(1,1,0,\dots,0)$ and $\mathscr{U} = E(0,0,1,0,\dots,0)$. The cotangent bundle $\Omega^1_{G(2,n)} = \mathscr{Q}^\vee \otimes \mathscr{U}$ is irreducible and corresponds to the highest weight $(0,-1,1,0,\dots,0)$. Bott's theorem can be interpreted as saying that the cohomology group $H^i\big(G(2,n), E(a_1,\dots,a_n)\big)$ does not vanish if and only if i is the number of strict inversions in the sequence $(n+a_1, n-1+a_2, \dots, 1+a_n)$ and all the entries of this sequence are distinct.

First we establish the following vanishing result:

Proposition 3.2. *Let* $\mathbb{G} = G(2,n) \subset \mathbf{P}^N$ *with* $N = \binom{n}{2} - 1$, *be the Grassmannian of lines in its Plücker embedding. We have the following vanishing statements*:

(1) $H^i\big(N_{\mathbb{G}}^\vee(2-i)\big) = 0$ *for all* $1 \leqq i \leqq 2n-5$, $i \neq 2$ *and for* $i = 2$ *and* $n \leqq 6$.

(2) $H^i\big(\Omega^1_{\mathbb{G}} \otimes \mathscr{Q}(-i)\big) = 0$ *for* $0 \leqq i \leqq 2n-7$.

(3) $H^{i+1}\big(N_{\mathbb{G}}^\vee \otimes \mathscr{Q}(-i)\big) = 0$ *for* $1 \leqq i \leqq \min(n, 2n-7)$.

(4) $H^*\big(\mathscr{Q}(-i)\big) = 0$ *for* $1 \leqq i \leqq n$.

(5) $H^{i+1}\big(N_{\mathbb{G}}^\vee(-i)\big) = 0$ *for* $0 \leqq i \leqq n-1$.

Proof. (1) We start with the case $i \geqq 3$. From the exact sequence (1) it suffices to show that (a) $H^{i-1}\big(\mathbb{G}, \Omega^1_{\mathbb{G}}(2-i)\big) = 0$ and that (b) $H^i\big(\mathbb{G}, \Omega^1_{\mathbf{P}^N|\mathbb{G}}(2-i)\big) = 0$. From the Euler sequence (b) at its turn is implied by the vanishings $H^{i-1}\big(\mathcal{O}_{\mathbb{G}}(2-i)\big) = H^i\big(\mathcal{O}_{\mathbb{G}}(1-i)\big) = 0$ which are obvious, while (a) is a consequence of Bott's theorem (or of Kodaira-Nakano vanishing). When $i = 1$, one checks that $H^0\big(\mathbb{G}, \Omega^1_{\mathbb{G}}(1)\big) = 0$ (Bott again), and that $H^1\big(\mathbb{G}, \Omega^1_{\mathbf{P}^N|\mathbb{G}}(1)\big) = 0$ (Euler sequence). The remaining case $i = 2$ is handled differently and we employ the Griffiths vanishing theorem: since $\mathbb{G}$ is scheme theoretically cut out by quadrics, the vector bundle $E = N_{\mathbb{G}}^\vee(2)$ is globally generated. From the exact sequence (1) one finds that $\det(E) = \mathcal{O}_{\mathbb{G}}\big((n-3)(n-4)/2\big)$ and we can write $N_{\mathbb{G}}^\vee = K_{\mathbb{G}} \otimes E \otimes \det(E) \otimes L$, with L an ample line bundle, precisely when $n \leqq 6$.

Part (2) is a consequence of Le Potier vanishing (cf. [LP]), while (4) follows from Bott vanishing since $\mathscr{Q}(-i) = E(1-i, -i, 0, \dots, 0)$. To prove (3) we tensor the exact sequence

(1) by $\mathcal{Q}(-i)$ and we have to show that $H^i\big(\Omega^1_{\mathbb{G}} \otimes \mathcal{Q}(-i)\big) = H^{i+1}\big(\Omega^1_{\mathbf{P}^N|\mathbb{G}} \otimes \mathcal{Q}(-i)\big) = 0$ which we already treated in parts (2) and (4). Finally, (5) is handled similarly to (1) and we omit the details. $\square$

For certain r we construct half-canonical curves $C \subset \mathbf{P}^r$ of genus $g(r)$ with injective Gaussian. This combined with Theorem 1.2 proves Theorem 1.1.

Proposition 3.3. *For* $3 \leqq r \leqq 11$, $r \neq 10$, *there exists a smooth half-canonical curve* $C \subset \mathbf{P}^r$ *of genus* $g(r)$ (*to be specified in the proof*), *such that the Gaussian map* $\psi_{\mathcal{O}_C(1)}$ *is injective. It follows that* $\mathcal{S}^r_{g(r)}$ *is smooth of codimension* $r(r+1)/2$ *at the point* $[C, \mathcal{O}_C(1)]$.

Proof. Each case will require a different construction. We treat every situation separately in increasing order of difficulty.

$r = 3$. We let C be a $(3,3)$ complete intersection in $\mathbf{P}^3$, hence $g(C) = g(3) = 10$ and $K_C = \mathcal{O}_C(2)$. Clearly $N_C = \mathcal{O}_C(3) \oplus \mathcal{O}_C(3)$, so trivially $H^1\big(N_C^\vee(2)\big) = 0$ which proves that $\psi_{\mathcal{O}_C(1)}$ is injective.

$r = 4$. Now C is a complete intersection of type $(2,2,3)$ in $\mathbf{P}^4$. Then $g(C) = g(4) = 13$ and $N_C = \mathcal{O}_C(2)^2 \oplus \mathcal{O}_C(3)$. Using that C is projectively normal we get that $H^1\big(\mathbf{P}^4, \mathcal{I}_C^2(2)\big) = 0$, hence $\psi_{\mathcal{O}_C(1)}$ is injective again.

$r = 5$. This is the last case when C can be a complete intersection: C is of type $(2,2,2,2)$ in $\mathbf{P}^5$, thus $g(C) = g(5) = 17$ and like in the $r = 4$ case we check that $H^1\big(\mathbf{P}^5, \mathcal{I}_C^2(2)\big) = 0$.

$r = 8$. We choose the Grassmannian $G(2,6) \subset \mathbf{P}^{14}$. A general codimension 6 linear section of $G(2,6)$ is a K3 surface $S \subset \mathbf{P}^8$ with $\deg(S) = 14$ and we let $C := S \cap Q \subset \mathbf{P}^8$ be a quadric section of S. Then C is half-canonical and $g(C) = g(8) = 29$. We claim that $\psi_{S, \mathcal{O}_S(1)}$ is injective, which follows from Proposition 3.1 since $H^i\big(N^\vee_{G(2,6)}(2-i)\big) = 0$ for $1 \leqq i \leqq 6$. To obtain that $H^1\big(\mathbf{P}^8, \mathcal{I}_C^2(2)\big) = 0$, by Proposition 3.1 we have to check that $H^1\big(S, \mathcal{O}_S(-1)\big) = H^1\big(S, \mathcal{O}_S(-2)\big) = 0$ (Kodaira vanishing), and that $H^1(N_S^\vee) = 0 \Leftrightarrow H^1(N_S) = 0$. Note that S is a general K3 surface of genus 8 having $\rho(S) = 1$ and since by transcendental theory, the Hilbert scheme of such K3 surfaces is irreducible, it will suffice to exhibit a single K3 surface of genus 8 having this property: we let S degenerate to a union $R_1 \cup_B R_2$ of two rational scrolls of degree 7 in $\mathbf{P}^8$ joined along an elliptic curve $B \in |-K_{R_i}|$ for $i = 1, 2$. Then $R_1 \cup_B R_2$ is a limit of smooth K3 surfaces $X \subset \mathbf{P}^8$ of degree 14 and $H^1(R_1 \cup_B R_2, N_{R_1 \cup_B R_2}) = 0$ (see [CLM], Theorem 1.2 for more details on this degeneration). It follows that $H^1(X, N_X) = 0$, for a general prime K3 surface $X \subset \mathbf{P}^8$ of degree 14 and then $H^1(S, N_S) = 0$ as well.

$r = 7$. In this situation we choose the 10-dimensional spinor variety $X \subset \mathbf{P}^{15}$ corresponding to a half-spin representation of $\mathrm{Spin}(10)$ (see [M] for a description of the projective geometry of X). One has that X is a homogeneous space for $SO(10)$, $K_X = \mathcal{O}_X(-8)$ and $\deg(X) = 12$. A general codimension 8 linear section of X is a K3 surface $S \subset \mathbf{P}^7$ of degree 12. Take now C to be a quadric section of S and then $K_C = \mathcal{O}_C(2)$ and $g(C) = g(7) = 25$. Since $N_X^\vee$ is irreducible (cf. e.g. [W2], Theorem 2.14), we obtain that the Gaussian map $\psi_{X, \mathcal{O}_X(1)}$ is injective.

To show that $\psi_{S,\mathcal{O}_S(1)}$ is injective we verify that $H^i\big(N_X^{\vee}(2-i)\big)=0$ for $1\leqq i\leqq 8$. For $3\leqq i\leqq 8$ this follows from Kodaira-Nakano vanishing for the twists of sheaves of holomorphic forms on X in a way similar to the proof of Proposition 3.2, while the $i=1$ it is a consequence of Bott vanishing. For $i=2$ we use Griffiths vanishing: since X is cut out by quadrics (see e.g. [M], Proposition 1.9), the vector bundle $E:=N_X^{\vee}(2)$ is globally generated, $\det(E)=\mathcal{O}_X(2)$ and one can write $N_X^{\vee}=K_X\otimes E\otimes\det(E)\otimes\mathcal{O}_X(4)$. In this way we obtain that $H^2(N_X^{\vee})=0$. Thus $\psi_{S,\mathcal{O}_S(1)}$ is injective, and to have the same conclusion for the Gaussian of C, the only non-trivial thing to check is that $H^1(N_S)=0$, which can be seen by letting S degenerate again to a union of two rational scrolls like in the case $r=8$.

$r=6$. We consider the Grassmannian $\mathbb{G}=G(2,5)\subset\mathbf{P}^9$ and we denote by $X\subset\mathbf{P}^6$ a general codimension 3 linear section of $\mathbb{G}$, by $S:=X\cap Q$ a general quadric section of X and by $C:=S\cap Q'$ a general quadric section of S. Then S is a K3 surface of genus 6, $K_C=\mathcal{O}_C(2)$ and $g(C)=g(6)=21$. Using Propositions 3.1 and 3.2 we see easily that $\psi_{X,\mathcal{O}_X(1)}$ is injective. We claim that $\psi_{S,\mathcal{O}_S(1)}$ is injective as well which would follow from $H^1(X,N_X^{\vee})=0$. Since $N_{X/\mathbb{G}}^{\vee}=\mathcal{O}_X(-1)^{\oplus 3}$, the vanishing of $H^1(X,N_X^{\vee})$ is implied by that of $H^1(N_{\mathbb{G}}^{\vee}\otimes\mathcal{O}_X)$ which in its turn is implied by $H^{i+1}\big(N_{\mathbb{G}}^{\vee}(-i)\big)=0$ for $0\leqq i\leqq 3$ (use the Koszul resolution). These last vanishing statements are contained in Proposition 3.2 and in this way we obtain that $\psi_{S,\mathcal{O}_S(1)}$ is injective. We finally descend to C. To conclude that $\psi_{C,\mathcal{O}_C(1)}$ is injective it is enough to verify that $H^1(N_S)=0$. We could check this again via the Koszul complex, but it is more economical to use that S is a general K3 surface of genus 6 and to invoke once more [CLM], Theorem 1.2, like in the previous cases.

$r=11$. We start with the Grassmannian $X=G(2,7)\subset\mathbf{P}^{20}$ for which $K_X=\mathcal{O}_X(-7)$ and we let C be a general codimension 9 linear section of X. Then $C\subset\mathbf{P}^{11}$ is a smooth half-canonical curve of genus $g(C)=g(11)=43$. To conclude that $\psi_{C,\mathcal{O}_C(1)}$ is injective we apply directly the second part of Proposition 3.1: the vanishing $H^i\big(N_{G(2,7)}^{\vee}(2-i)\big)=0$ is guaranteed by Proposition 3.2 for all $1\leqq i\leqq 9$, $i\neq 2$. For $i=2$ we can no longer employ Griffiths vanishing so we proceed differently: we use (1) together with the vanishing $H^2(X,\Omega^1_{\mathbf{P}^{20}|X})=0$ coming from the Euler sequence, to write down the exact sequence

$$0\to H^1(N_X^{\vee})\to H^1(\Omega^1_{\mathbf{P}^{20}|X})\to H^1(\Omega^1_X)\to H^2(N_X^{\vee})\to 0, \tag{2}$$

where $H^1(\Omega^1_{\mathbf{P}^{20}|X})\cong H^0(\mathcal{O}_X)\cong\mathbb{C}$ and $H^1(\Omega^1_X)\cong\mathbb{C}$. From Bott's theorem at most one of the cohomology groups of the irreducible bundle $N_X^{\vee}$ are $\neq 0$, hence either $H^2(N_X^{\vee})=0$ and then we are done, or else, if $H^2(N_X^{\vee})\neq 0$ then $H^1(N_X^{\vee})=0$, and the map in the middle of the sequence (2) is bijective which yields a contradiction.

$r=9$. This is the most involved case. We look at the ample vector bundle $\mathcal{F}:=\mathcal{Q}(1)$ on $\mathbb{G}=G(2,6)\subset\mathbf{P}^{14}$ and choose a general section $s\in H^0(\mathbb{G},\mathcal{F})$. We denote by Z the zero locus of s, by $\mathcal{I}=\mathcal{I}_{Z/\mathbb{G}}$ the ideal of Z inside $\mathbb{G}$, and by $\mathcal{I}_Z$ and $\mathcal{I}_{\mathbb{G}}$ the ideals of Z and $\mathbb{G}$ in $\mathbf{P}^{14}$ respectively. By adjunction, we have that $\mathcal{I}/\mathcal{I}^2=\mathcal{Q}^{\vee}(-1)\otimes\mathcal{O}_Z$ and the Koszul complex gives a resolution for Z:

$$0\to\mathcal{O}_{\mathbb{G}}(-3)\to\mathcal{Q}^{\vee}(-1)\to\mathcal{I}\to 0.$$

We first claim that $Z\subset\mathbf{P}^{14}$ is nondegenerate and projectively normal. This will follow if we show that $H^0\big(\mathbb{G},\mathcal{I}(1)\big)=0$ and $H^1\big(\mathbb{G},\mathcal{I}(r)\big)=0$ for $r\geqq 1$. Using the Koszul resolution, the first vanishing is implied by $H^0(\mathcal{Q}^{\vee})=H^1\big(\mathcal{O}_{\mathbb{G}}(-2)\big)=0$ which is clear. For the second vanishing we have to check that $H^1\big(\mathcal{Q}^\vee(r-1)\big) = H^2\big(\mathcal{O}_{\mathbb{G}}(r-3)\big) = 0$ for $r \geqq 1$. Since $\mathcal{Q}^\vee(r-1) = E(r-1, r-2, 0, 0, 0, 0)$ and $\mathcal{O}_{\mathbb{G}}(r-3) = E(r-3, r-3, 0, 0, 0, 0)$ this can be checked instantly using Bott's theorem.

Next we claim that the $\psi_{Z, \mathcal{O}_Z(1)}$ is injective. By Proposition 3.1, we have to verify that (1) $H^1\big(\mathbb{G}, \mathcal{I}^2(2)\big) = 0$ and that (2) $H^1\big(\mathbb{G}, N_{\mathbb{G}}^\vee(2) \otimes \mathcal{I}\big) = 0$. We start with (1). From the exact sequence

$$0 \to \mathcal{I}^2(2) \to \mathcal{I}(2) \to \mathcal{Q}^\vee(1) \otimes \mathcal{O}_Z \to 0,$$

using that Z is projectively normal, (1) is implied by the bijectivity of the map $H^0\big(\mathcal{I}(2)\big) \to H^0\big(\mathcal{Q}^\vee(1) \otimes \mathcal{O}_Z\big)$. This is a consequence of the isomorphism $\mathcal{Q}^\vee(1) \cong \mathcal{Q}$ and of the Koszul resolution giving that $H^0\big(Z, \mathcal{Q}^\vee(1) \otimes \mathcal{O}_Z\big) = H^0\big(\mathbb{G}, \mathcal{Q}^\vee(1)\big) = H^0\big(\mathbb{G}, \mathcal{I}(2)\big)$, where for the first isomorphism one uses that $H^0(\mathbb{G}, \mathcal{I} \otimes \mathcal{Q}) = H^1(\mathbb{G}, \mathcal{I} \otimes \mathcal{Q}) = 0$, which is straightforward to check via Bott's theorem.

We turn to (2). The cohomology of $\mathcal{I} \otimes N_{\mathbb{G}}^\vee(2)$ is computed from the Koszul complex of $\mathcal{I}$, which yields an isomorphism $H^1\big(N_{\mathbb{G}}^\vee \otimes \mathcal{I}(2)\big) = H^1\big(N_{\mathbb{G}}^\vee \otimes \mathcal{Q}^\vee(1)\big)$ (because we have $H^i\big(N_{\mathbb{G}}^\vee(-1)\big) = 0$ for $i = 1, 2$—this being checked via the sequence (1)). Next we write the cohomology sequence associated to the exact sequence

$$0 \to N_{\mathbb{G}}^\vee \otimes \mathcal{Q}^\vee(1) \to \Omega^1_{\mathbf{P}^{14}|\mathbb{G}} \otimes \mathcal{Q}^\vee(1) \to \Omega^1_{\mathbb{G}} \otimes \mathcal{Q}^\vee(1) \to 0.$$

The map $H^1\big(\Omega^1_{\mathbf{P}^{14}|\mathbb{G}} \otimes \mathcal{Q}^\vee(1)\big) \to H^1\big(\Omega^1_{\mathbb{G}} \otimes \mathcal{Q}^\vee(1)\big)$ is an isomorphism: from the Euler sequence one obtains that $H^1\big(\Omega^1_{\mathbf{P}^{14}|\mathbb{G}} \otimes \mathcal{Q}^\vee(1)\big) = H^0\big(\mathcal{Q}^\vee(1)\big)$, while tensoring by $\Omega^1_{\mathbb{G}}(1)$ the dual of the tautological sequence, one gets that

$$H^1\big(\Omega^1_{\mathbb{G}}(1) \otimes \mathcal{Q}^\vee\big) = H^0\big(\mathcal{U}^\vee \otimes \Omega^1_{\mathbb{G}}(1)\big) = H^0\big(\mathcal{Q}^\vee(1)\big)$$

(or alternatively, use for this [LP], Corollaire 2). Moreover $H^0\big(\Omega^1_{\mathbb{G}} \otimes \mathcal{Q}^\vee(1)\big)$ injects into $H^0\big(\Omega^1_{\mathbb{G}}(1)\big)^{\oplus 6}$ which is zero by Bott's theorem. Hence $H^1\big(N_{\mathbb{G}}^\vee \otimes \mathcal{Q}^\vee(1)\big) = 0$ and this proves that $\psi_{Z, \mathcal{O}_Z(1)}$ is injective.

We now take a general codimension 5 linear section of Z which is a curve $C \subset \mathbf{P}^9$ with $K_C = \mathcal{O}_C(2)$. A routine calculation gives that $\deg(C) = 3\deg(\mathbb{G}) = 42$, hence $g(C) = g(9) = 43$. We claim that $\psi_{C, \mathcal{O}_C(1)}$ is injective. Since $\psi_{Z, \mathcal{O}_Z(1)}$ is injective, by Proposition 3.1 we are left with checking that Z is ACM (this amounts to $H^i\big(\mathcal{O}_Z(j)\big) = 0$ for $i \neq 0, 6 = \dim(Z)$, which easily follows from the Koszul complex) and that $H^i\big(Z, N_Z^\vee(2-i)\big) = 0$ for $1 \leqq i \leqq 5$ (here $N_Z = (\mathcal{I}_Z/\mathcal{I}_Z)^\vee$ is the normal bundle of Z in $\mathbf{P}^{14}$). We employ the exact sequence

$$0 \to N_{\mathbb{G}}^\vee \otimes \mathcal{O}_Z \to N_Z^\vee \to \mathcal{I}/\mathcal{I}^2 \to 0,$$

from which it will suffice to show that (a) $H^i\big(Z, \mathcal{I}/\mathcal{I}^2(2-i)\big) = H^i\big(\mathcal{Q}^\vee(1-i) \otimes \mathcal{O}_Z\big) = 0$ for $1 \leqq i \leqq 5$ and that (b) $H^i\big(N_{\mathbb{G}}^\vee(2-i) \otimes \mathcal{O}_Z\big) = 0$, which in turn is a consequence of $H^i\big(N_{\mathbb{G}}^\vee(2-i)\big) = H^{i+1}\big(N_{\mathbb{G}}^\vee(-1-i)\big) = 0$ and of the vanishing $H^{i+1}\big(N_{\mathbb{G}}^\vee \otimes \mathcal{Q}^\vee(1-i)\big) = 0$ (for all these use Proposition 3.2).

We are left with (a) which is a consequence of $H^i\big(\mathcal{Q}^\vee(1-i)\big) = 0$ (again, use

Proposition 3.2), of $H^{i+2}\big(\mathcal{Q}^\vee(-2-i)\big)=0$, and of $H^{i+1}\big(\mathcal{Q}\otimes\mathcal{Q}(-2-i)\big)=0$. For this last statement use that $\mathcal{Q}\otimes\mathcal{Q}=S^2\mathcal{Q}\oplus\det(\mathcal{Q})$ and each summand being irreducible the vanishing can be easily verified via Bott's theorem. $\square$

We believe that there should be a uniform way of constructing half-canonical curves $C\subset\mathbf{P}^r$ for any $r\geqq 3$ of high genus $g\gg r$ and having injective Gaussian maps (though no longer as sections of homogeneous varieties). Together with Theorem 1.2 this prompts us to make the following:

Conjecture 3.4. For any $r\geqq 3$ and $g\geqq\binom{r+2}{2}$, there exists a component of $\mathcal{S}_g^r$ of codimension $\binom{r+1}{2}$ inside $\mathcal{S}_g$.

The bound $g\geqq\binom{r+2}{2}$ is obtained by comparing the expected dimension $3g-3-\binom{r+1}{2}$ of $\mathcal{S}_g^r$ with the expected dimension of the Hilbert scheme $\mathrm{Hilb}_{g-1,g,r}$ of curves $C\subset\mathbf{P}^r$ of genus g and degree $g-1$. We believe that there exists a component of $\mathrm{Hilb}_{g-1,g,r}$ consisting entirely of half-canonically embedded curves. To prove the Conjecture it would suffice to construct a smooth half-canonical curve $C\subset\mathbf{P}^r$ of genus $g=\binom{r+2}{2}$ such that $H^1(C,N_{C/\mathbf{P}^r})=0$, that is, $\mathrm{Hilb}_{g-1,g,r}$ is smooth at the point $[C]$ and has expected dimension $h^0(C,N_{C/\mathbf{P}^r})=4(g-1)$. Note that for such C, the map $\Psi_{C,\mathcal{O}_C(1)}$ would be injective, in particular C would not sit on any quadrics. This gives the necessary inequality $g\geqq\binom{r+2}{2}$. The main difficulty in proving Conjecture 3.4 lies in the fact that the degeneration techniques one normally uses to construct "regular" components of Hilbert schemes of curves, seem to be at odds with the requirement that C be half-canonical.

4. Gieseker-Petri loci

In this section we construct divisorial components of the loci $GP_{g,k}^1$. The method we use is inductive and close in spirit to the one employed in Section 2 to construct components of $\mathcal{S}_g^r$ of expected dimension. We begin by describing a setup that enables us to analyze the following situation: if $\{L_b\}_{b\in B^*}$ and $\{M_b\}_{b\in B^*}$ are two families of line bundles over a 1-dimensional family of smooth curves $\{X_b\}_{b\in B^*}$, where $B^*=B-\{b_0\}$ with $b_0\in B$, we want to describe what happens to the multiplication map

$$\mu_b=\mu_b(L_b,M_b):H^0(X_b,L_b)\otimes H^0(X_b,M_b)\to H^0(X_b,L_b\otimes M_b)$$

as X_b degenerates to a singular curve of compact type X_0.

Suppose first that C is a smooth curve and $p\in C$. We recall that if $l=(L,V)$ is a linear series of type $\mathfrak{g}_d^r$ with $L\in\mathrm{Pic}^d(C)$ and $V\subset H^0(L)$, the *vanishing sequence* of l at p

$$a^l(p):0\leqq a_0^l(p)<\cdots<a_r^l(p)\leqq d,$$

is obtained by ordering the set $\{\mathrm{ord}_p(\sigma)\}_{\sigma \in V}$. If L and M are line bundles on C and $\rho \in H^0(L) \otimes H^0(M)$ we write that $\mathrm{ord}_p(\rho) \geqq k$, if ρ lies in the span of elements of the form $\sigma \otimes \tau$, where $\sigma \in H^0(L)$ and $\tau \in H^0(M)$ are such that $\mathrm{ord}_p(\sigma) + \mathrm{ord}_p(\tau) \geqq k$.

Let $\mu_{L,M} : H^0(L) \otimes H^0(M) \to H^0(L \otimes M)$ be the multiplication map. We shall use the following observation: suppose $\{\sigma_i\} \subset H^0(L)$ and $\{\tau_j\} \subset H^0(M)$ are bases of global sections *adapted to the point* $p \in C$ in the sense that $\mathrm{ord}_p(\sigma_i) = a_i^L(p)$ and $\mathrm{ord}_p(\tau_j) = a_j^M(p)$ for all i and j. Then if $\rho \in \mathrm{Ker}(\mu_{L,M})$ then there must exist distinct pairs of integers $(i_1, j_1) \neq (i_2, j_2)$ such that

$$\mathrm{ord}_p(\rho) = \mathrm{ord}_p(\sigma_{i_1}) + \mathrm{ord}_p(\tau_{j_1}) = \mathrm{ord}_p(\sigma_{i_2}) + \mathrm{ord}_p(\tau_{j_2}).$$

Suppose now that $\pi : X \to B$ is a family of genus g curves over $B = \mathrm{Spec}(R)$, with R being a complete DVR with local parameter t, and let $0, \eta$ denote the special and the generic point of B respectively. Assume furthermore that X_η is smooth and that X_0 is singular but of compact type. If L_η is a line bundle on X_η then, as explained in [EH1], there is a canonical way to associate to each component Y of X_0 a line bundle L^Y on X such that $\deg_Z(L^Y_{|_Z}) = 0$ for every component Z of X_0 different from Y. We set $L_Y := L^Y_{|_Y}$ which is a line bundle on the smooth curve Y.

We fix $\sigma \in \pi_* L_\eta$ a section on the generic fibre. We denote by α the smallest integer such that $t^\alpha \sigma \in \pi_* L^Y$, that is, $t^\alpha \sigma \in \pi_* L^Y - t\pi_* L^Y$. Then we set

$$\sigma^Y := t^\alpha \sigma \in \pi_* L^Y \quad \text{and} \quad \sigma_Y := \sigma^Y_{|_Y} \in H^0(Y, L_Y).$$

For a different component Z of the special fibre X_0 meeting Y at a point p, we define similarly L^Z, L_Z, σ^Z and σ_Z. If we write $\sigma^Z = t^\beta \sigma^Y \in \pi_* L^Z$ for a unique integer β, we have the following compatibility relation between σ_Y and σ_Z (cf. [EH1], Proposition 2.2):

$$\deg(L_Y) - \mathrm{ord}_p(\sigma_Y) \leqq \beta \leqq \mathrm{ord}_p(\sigma_Z). \tag{3}$$

An immediate consequence of this is the inequality

$$\mathrm{ord}_p(\sigma_Y) + \mathrm{ord}_p(\sigma_Z) \geqq \deg(L_Y) = \deg(L_Z).$$

Assume from now on that we have two line bundles L_η and M_η on X_η and we choose an element $\rho \in H^0(X_\eta, L_\eta) \otimes_{R_\eta} H^0(X_\eta, M_\eta)$. If Y and Z are components of X_0 meeting at p as above, we define $\rho^Y := t^\gamma \rho \in H^0(X, L^Y) \otimes_R H^0(X, M^Y)$, where γ is the minimal integer with this property. We have a similar definition for $\rho^Z \in H^0(X, L^Z) \otimes_R H^0(X, M^Z)$. Between the sections ρ^Y and ρ^Z there is a relation $\rho^Z = t^\alpha \rho^Y$ for a uniquely determined integer α. To determine α we proceed as follows: we choose bases of sections $\{\sigma_i = \sigma_i^Y\}$ for $H^0(X, L^Y)$ and $\{\tau_j = \tau_j^Y\}$ for $H^0(X, M^Y)$ such that $\mathrm{ord}_p(\sigma_{i,Y}) = a_i^{L_Y}(p)$ and $\mathrm{ord}_p(\tau_{j,Y}) = a_j^{M_Y}(p)$, for all relevant i and j (cf. e.g. [EH1], Lemma 2.3, for the fact that this can be done). Then there are integers α_i and β_j defined by $\sigma_i^Z = t^{\alpha_i} \sigma_i$ and $\tau_j^Z = t^{\beta_j} \tau_j$. To obtain a formula for the integer α we write $\rho^Y = \sum\limits_{i,j} f_{ij} \sigma_i \otimes \tau_j$, where $f_{ij} \in R$. We have the identity

$$\rho^Z = \sum_{i,j} (t^{\alpha - \alpha_i - \beta_j} f_{ij})(t^{\alpha_i} \sigma_i) \otimes (t^{\beta_j} \tau_j),$$

from which we easily deduce that $\alpha = \max_{i,j}\{\alpha_i + \beta_j - v(f_{ij})\}$, where v denotes the valuation on R (see also [EH2], Lemma 3.2).

Lemma 4.1. *With the above notations, if* $\rho_Y := \rho^Y_{|_Y} \in H^0(Y, L_Y) \otimes H^0(Y, M_Y)$ *and* $\rho_Z := \rho^Z_{|_Z} \in H^0(Z, L_Z) \otimes H^0(Z, M_Z)$, *then*

$$\mathrm{ord}_p(\rho_Y) + \mathrm{ord}_p(\rho_Z) \geqq \deg(L_Y) + \deg(M_Y).$$

Proof. By definition, there exists a pair of indices (i_1, j_1) such that $v(f_{i_1 j_1}) = 0$ and

$$\mathrm{ord}_p(\rho_Y) = \mathrm{ord}_p(\sigma_{i_1, Y}) + \mathrm{ord}_p(\sigma_{j_1, Y})$$

and clearly $\alpha \geqq \alpha_{i_1} + \beta_{j_1}$. To get an estimate on $\mathrm{ord}_p(\rho_Z)$ we only have to take into account the pairs of indices (i, j) for which $\alpha_i + \beta_j = \alpha + v(f_{ij}) \geqq \alpha_{i_1} + \beta_{j_1}$. For at least one such pair (i, j) we have that

$$\mathrm{ord}_p(\rho_Z) = \mathrm{ord}_p(t^{\alpha_i}\sigma_{i,Z}) + \mathrm{ord}_p(t^{\beta_j}\tau_{j,Z}) \geqq \alpha_i + \beta_j.$$

On the other hand, by applying (3) we can write

$$\mathrm{ord}_p(\rho_Y) = \mathrm{ord}_p(\sigma_{i_1, Y}) + \mathrm{ord}_p(\tau_{j_1, Y}) \geqq \deg(L_Y) + \deg(M_Y) - \alpha_{i_1} - \beta_{j_1},$$

whence we finally have that $\mathrm{ord}_p(\rho_Z) + \mathrm{ord}_p(\rho_Y) \geqq \deg(L_Y) + \deg(M_Y)$. □

We now fix integers g and k such that $g \geqq 4$ and $(g+2)/2 \leqq k \leqq g-1$ and consider the locus $GP^1_{g,k}$ of curves $[C] \in \mathscr{M}_g$ for which the Gieseker-Petri Theorem fails for a base point free pencil $\mathfrak{g}^1_k$. We denote by $\overline{GP}^1_{g,k}$ the closure of $GP^1_{g,k}$ in $\overline{\mathscr{M}}_g$ and we study $GP^1_{g,k}$ inductively by understanding the intersection $\overline{GP}^1_{g,k} \cap \Delta_1$.

Definition 4.2. For a smooth curve C of genus g, a Gieseker-Petri $(gp)^1_k$*-relation* consists of a linear series $(L, V) \in G^1_k(C)$, $V \subset H^0(L)$, together with an element

$$\rho \in \mathbf{P}\,\mathrm{Ker}\{\mu_0(V) : V \otimes H^0(K_C \otimes L^{-1}) \to H^0(K_C)\}.$$

If $C = C_1 \cup_p C_2$ is of compact type with C_1 and C_2 smooth of genus i and $g - i$ respectively, a $(gp)^1_k$*-relation* on C is a collection (l, m, ρ_1, ρ_2), where $l = \{(L_{C_1}, V_{C_1}), (L_{C_2}, V_{C_2})\}$ is a limit $\mathfrak{g}^1_k$ on C,

$$m = \big\{\big(M_{C_1} = K_{C_1}\big(2(g-i)p\big) \otimes L^{-1}_{C_1}, W_1\big), \big(M_{C_2} = K_{C_2}(2ip) \otimes L^{-1}_{C_2}, W_2\big)\big\}$$

is a limit $\mathfrak{g}^{g-k}_{2g-2-k}$ on C, and elements

$$\rho_1 \in \mathbf{P}\,\mathrm{Ker}\big\{V_{C_1} \otimes W_{C_1} \to H^0\big(K_{C_1}\big(2(g-i)p\big)\big)\big\},$$
$$\rho_2 \in \mathbf{P}\,\mathrm{Ker}\big\{V_{C_2} \otimes W_{C_2} \to H^0\big(K_{C_2}(2ip)\big)\big\}$$

satisfying the relation $\mathrm{ord}_p(\rho_1) + \mathrm{ord}_p(\rho_2) \geqq 2g - 2$.

For a curve C of compact type, we denote by $Q^1_k(C)$ the variety of $(gp)^1_k$-relations on C together with the scheme structure coming from its natural description as a determinantal variety. The discussion above shows that if $[C] \in \overline{GP}^1_{g,k}$ then $Q^1_k(C) \neq \emptyset$. Our strategy is to construct $(gp)^1_k$-relations on certain singular curves and prove that they can be deformed to nearby smooth curves filling up a divisor in $\bar{\mathcal{M}}_g$. The most important technical result of this section is the construction of the moduli space of $(gp)^1_k$-relations over the versal deformation space of a curve of compact type inside the divisor Δ_1:

Theorem 4.3. *Fix integers $g \geqq 4$ and k such that $(g+2)/2 \leqq k \leqq g-1$. Let C be a smooth curve of genus $g-1$, $p \in C$ and $X_0 := C \cup_p E$, where E is an elliptic curve. We denote by $\pi : X \to B$ the versal deformation space of X_0, with $X_0 = \pi^{-1}(0)$ and $0 \in B$. Then there exists a scheme $\mathcal{Q}^1_k \to B$, quasi-projective over B and compatible with base change, such that the fibre over $b \in B$ parametrizes $(gp)^1_k$-relations over X_b. Furthermore each component of $\mathcal{Q}^1_k$ has dimension $\geqq \dim(B) - 1 = 3g-4$.*

Proof. The scheme $\mathcal{Q}^1_k$ is going to be the disjoint union of subschemes where the vanishing sequences of the aspects of the two underlying limit linear series of a $(gp)^1_k$-relation are also specified. We will prove the existence for the component corresponding to vanishing sequences $(1,2)$ and $(k-2,k-1)$ for the limit $\mathfrak{g}^1_k$ and $(1,2,\dots,g-k+1)$ and $(g-3,g-2,\dots,2g-3-k)$ for the limit $\mathfrak{g}^{g-k}_{2g-2-k}$ respectively. The construction is entirely similar for the other compatible vanishing sequences. In our proof we will use Theorem 3.3 in [EH1] where a moduli space of limit linear series over the versal deformation space of a curve of compact type is constructed.

We start by setting some notations. We denote by $\Delta \subset B$ the "boundary" divisor corresponding to curves in which the node p is not smoothed. We denote by $\mathscr{C}_p$ and $\mathscr{E}_p$ the closures in X of the components of $\pi^{-1}(\Delta)$ containing $C - \{p\}$ and $E - \{p\}$ respectively. By shrinking B if necessary we can assume that $\mathcal{O}_X(\mathscr{C}_p + \mathscr{E}_p) = \mathcal{O}_X$. We denote by $\pi_{P^C} : P^C \to B$ the relative Picard variety corresponding to the family $X \to B$ such that for $b \in \Delta$ and $\pi^{-1}(b) = X_b = C_b \cup E_b$ with $C_b \subset \mathscr{C}_p$ and $E_b \subset \mathscr{E}_p$, the fibre of P^C over b consists of line bundles L_b on X_b with $\deg_{C_b}(L_b) = k$ and $\deg_{E_b}(L_b) = 0$. Interchanging the role of C and E we get another Picard variety $P^E \to B$ and tensoring with $\mathcal{O}_X(k\mathscr{C}_p)$ gives an isomorphism $P^C \to P^E$. We denote by P the inverse limit of P^C and P^E under this isomorphism. For $b \in B$ and any line bundle L on X_b, we define two new line bundles L_C and L_E as follows: if $b \in B - \Delta$ then $L_C = L_E = L$. If $b \in \Delta$ and $X_b = C_b \cup_q E_b$, then L_C is the restriction to C of the unique line bundle on X_b obtained from L by tensoring with a divisor based at q and whose restriction to E_b is of degree 0 (and a similar definition for L_E with C and E reversed). Proceeding in a way identical to [EH1], pp. 356–360, we construct a space of compatible frames $\phi : \mathscr{F} \to B$ factoring through $\pi_P : P \to B$, and which parametrizes objects

$$x = \{b, L, (\sigma^C_i)_{i=0,1}, (\sigma^E_i)_{i=0,1}, (\tau^C_j)_{j=0,\dots,g-k}, (\tau^E_j)_{j=0,\dots,g-k}\},$$

where $b \in B$, L is a line bundle of degree k on X_b, (σ^C_i) (resp. (σ^E_i)) is a projective frame inside $H^0(L_C)$ (resp. $H^0(L_E)$), while (τ^C_j) (resp. (τ^E_j)) is a projective frame inside $H^0\big((\omega_{X_b} \otimes L^{-1})_C\big)$ (resp. $H^0\big((\omega_{X_b} \otimes L^{-1})_E\big)$), subject to the following identifications: if $b \in B - \Delta$, so X_b is smooth and $L_E = L_C = L$, then we identify $\sigma^C_i = \sigma^E_{1-i}$ for $i = 0,1$ and $\tau^C_j = \tau^E_{g-k-j}$ for $j = 0,\dots,g-k$ (that is, there are only two frames, one inside $H^0(L)$, the other inside $H^0(K_{X_b} \otimes L^{-1})$). If $b \in \Delta$ and $X_b = C_b \cup_q E_b$ then we require that $\mathrm{ord}_q(\sigma^C_i) \geqq i+1$, $\mathrm{ord}_q(\sigma^E_i) \geqq k-2+i$ for $i = 0,1$, while $\mathrm{ord}_q(\tau^C_j) \geqq j+1$ and $\mathrm{ord}_q(\tau^E_j) \geqq g+j-3$. In this latter case $l = \{(L_C, \langle\sigma^C_i\rangle_i), (L_E, \langle\sigma^E_i\rangle_i)\}$ is a limit $\mathfrak{g}^1_k$ and $m = \big\{\big((\omega_X \otimes L^{-1})_C, \langle\tau^C_j\rangle_j\big), \big((\omega_X \otimes L^{-1})_E, \langle\tau^E_j\rangle_j\big)\big\}$ is a limit $\mathfrak{g}^{g-k}_{2g-2-k}$ on X_b.

The scheme $\mathscr{F}$ is determinantal and each of its components has dimension $\geqq \dim(B) + g + 2 + (g-k+1)(g-k-2)$, which is consistent with the naive dimension count for the fibre over $b \in B - \Delta$. We also have tautological line bundles $\tilde{\sigma}_i^C, \tilde{\sigma}_i^E, \tilde{\sigma}_j^C$ and $\tilde{\sigma}_j^E$ over $\mathscr{F}$, with fibres over each point being the 1-dimensional vector space corresponding to the frame denoted by the same symbol. For $2 \leqq i \leqq g-k+2$, we consider the rang g vector bundle $\Psi_i := \pi_*\big(\omega_{X/B} \otimes \mathcal{O}_X(i\mathscr{C}_p)\big)$; hence $\Psi_i(b) = H^0(X_b, L_b)$ for $b \in B - \Delta$, while for $b \in \Delta$ the fibre $\Psi_i(b)$ consists of those sections in $H^0\big(K_{C_b}(-(i-1)q)\big) \oplus H^0\big(\mathcal{O}_{E_b}((i+1)q)\big)$ that are compatible at the node q.

For $1 \leqq i \leqq g-k$ we define a subscheme $\mathscr{G}_i$ of $\mathscr{F}$ by the equations

$$\tilde{\sigma}_0^C \cdot \tilde{\tau}_i^C = \tilde{\sigma}_1^C \cdot \tilde{\tau}_{i-1}^C \quad \text{and} \quad \tilde{\sigma}_1^E \cdot \tilde{\tau}_{g-k-i}^E = \tilde{\sigma}_0^E \cdot \tilde{\tau}_{g-k-i+1}^E. \tag{4}$$

Here by $(\tilde{\sigma}_\alpha^C \cdot \tilde{\tau}_\beta^C)(x)$ we denote the element in $\mathbf{P}H^0\big((\omega_{X_b})_C\big)$ obtained by multiplying representatives of $\tilde{\sigma}_\alpha^C(x)$ and of $\tilde{\tau}_\beta^C(x)$ for each $x \in \mathscr{F}$, $b = \phi(x)$. To make more sense of (4), for each $x \in \mathscr{F}$ the element $\big((\tilde{\sigma}_0^C \cdot \tilde{\tau}_i^C)(x), (\tilde{\sigma}_1^E \cdot \tilde{\tau}_{g-k-i}^E)(x)\big)$ gives rise canonically to a point in $\mathbf{P}\big((\phi^*\Psi_{i+1})(x)\big)$ and abusing the notation we can consider $(\tilde{\sigma}_0^C \cdot \tilde{\tau}_i^C, \tilde{\sigma}_1^E \cdot \tilde{\tau}_{g-k-i}^E)$ and $(\tilde{\sigma}_1^C \cdot \tilde{\tau}_{i-1}^C, \tilde{\sigma}_0^E \cdot \tilde{\tau}_{g-k-i+1}^E)$ as sections of the $\mathbf{P}^{g-1}$ bundle $\mathbf{P}(\phi^*\Psi_{i+1}) \to \mathscr{F}$. Then $\mathscr{G}_i$ is the locus in $\mathscr{F}$ where these sections coincide and therefore each component of $\mathscr{G}_i$ has dimension $\geqq \dim(\mathscr{F}) - g + 1$.

We define $\mathscr{Q}_k^1$ as the union of the scheme theoretic images of $\mathscr{G}_i$ for $1 \leqq i \leqq g-k$ under the map

$$\mathscr{G}_i \ni x \overset{\chi_i}{\mapsto} \big(b, l, m, \rho_1 = (\sigma_0^C \otimes \tau_i^C - \sigma_1^C \otimes \tau_{i-1}^C), \rho_2 = (\sigma_1^E \otimes \tau_{g-k-i}^E - \sigma_0^E \otimes \tau_{g-k-i+1}^E)\big),$$

where we recall that l and m denote the underlying limit $\mathfrak{g}_k^1$ and $\mathfrak{g}_{2g-2-k}^{g-k}$ respectively. From the base point free pencil trick applied on both C_b and E_b, it is easy to see that $\mathscr{Q}_k^1$ contains all $(gp)_k^1$-relations on the curves $X_b = C_b \cup_q E_b$, the points coming from $\mathscr{G}_i$ corresponding to those $(b, l, m, \rho_1, \rho_2)$ for which $\mathrm{ord}_q(\rho_1) \geqq i+2$ and $\mathrm{ord}_q(\rho_2) \geqq 2g-i-4$.

We are left with estimating $\dim(\mathscr{Q}_k^1)$: having fixed $(b, l, m, \rho_1, \rho_2)$ inside $\chi_i(\mathscr{G}_i)$, there are two cases to consider depending on whether X_b is smooth or not. In each case we obtain the same estimate for the fibre dimension of χ_i but here we only present the case $b \in \Delta$, when $X_b = C_b \cup_q E_b$. We have a one dimensional family of choices for each of (σ_0^C, σ_1^C) and (σ_0^E, σ_1^E), and after choosing these, (τ_i^C, τ_{i-1}^C) and $(\tau_{g-k-i}^E, \tau_{g-k-i+1}^E)$ are uniquely determined (again, use the base point free pencil trick). For choosing the remaining τ_α^C, $\alpha \neq i, i-1$ we have a $(g-k)(g-k+1)/2 - (2g-2k-2i+1)$-dimensional family of possibilities, while for τ_β^E, $\beta \neq g-k-i, g-k-i+1$ we get another $(g-k)(g-k+1)/2 - 2i + 1$ dimensions. Adding these together we get that each component of $\mathscr{Q}_k^1$ has dimension $\geqq 3g-4$. □

We can now prove Theorem 1.4. More precisely we have the following inductive result:

Theorem 4.4. *Fix integers g, k such that $g \geqq 4$ and $(g+2)/2 \leqq k \leqq g-1$. Suppose $GP_{g-1,k-1}^1$ has a divisorial component Z for which a general $[C] \in Z$ is such that there exists a 0-dimensional component of $Q_{k-1}^1(C)$ whose general point corresponds to a base point free $\mathfrak{g}_{k-1}^1$. Then $GP_{g,k}^1$ has a divisorial component Z', for which a general curve $[C'] \in Z'$ is* *such that* $Q^1_k(C')$ *has a* 0*-dimensional component corresponding to a base point free* $\mathfrak{g}^1_k$*. Moreover, if* $\varepsilon : \overline{\mathcal{M}}_{g-1,1} \to \overline{\mathcal{M}}_{g-1}$ *is the forgetful morphism, then using the identification* $\Delta_1 = \overline{\mathcal{M}}_{g-1,1} \times \overline{\mathcal{M}}_{1,1}$*, we have that* $\overline{Z}' \cap \Delta_1 \supset \varepsilon^*(Z) \times \overline{\mathcal{M}}_{1,1}$.

Proof. We choose a general curve $[C] \in Z \subset GP^1_{g-1,k-1}$, a general point $p \in C$ and we set $X_0 := C \cup_p E$, where E is an elliptic curve. By assumption, there exists a base point free $(A, V) \in G^1_{k-1}(C)$ and $\rho \in \mathbf{P}\operatorname{Ker}\big(\mu_0(V)\big)$ such that $\dim_{(A,V,\rho)} Q^1_{k-1}(C) = 0$. In particular $\operatorname{Ker}\big(\mu_0(V)\big)$ is 1-dimensional and $h^0(A) = 2$. Let $\pi : X \to B$ be the versal deformation space of X_0, $\Delta \subset B$ the boundary divisor corresponding to singular curves, and we consider the scheme $\nu : \mathcal{Q}^1_k \to B$ parametrizing $(gp)^1_k$-relations, which was constructed in Theorem 4.3. We construct a $(gp)^1_k$-relation $z = (l, m, \rho_1, \rho_2)$ on X_0 as follows: the C-aspect of the limit $\mathfrak{g}^1_k$ denoted by l is obtained by adding p as a base point to (A, V), while the E-aspect of l is constructed by adding $(k-2)p$ as a base locus to $|\mathcal{O}_E(p+q)|$, where $q \in E - \{p\}$ satisfies $2(p-q) \equiv 0$. Thus the vanishing sequences $a^{l_C}(p)$ and $a^{l_E}(p)$ are $(1,2)$ and $(k-2, k-1)$ respectively. The C-aspect of the limit $\mathfrak{g}^{g-k}_{2g-2-k}$ we denote by m, is the complete linear series $|M_C| = |K_C(p) \otimes A^{-1}|$ which by Riemann-Roch has vanishing sequence $(1, 2, \dots, g-k+1)$ at p. Finally the E-aspect of m is the subseries of $\big|\mathcal{O}_E\big((2g-1-k)p - q\big)\big|$ with vanishing $(g-3, g-2, \dots, 2g-k-3)$ at p. From the base point free pencil trick it follows that we can choose uniquely the relations ρ_C on C and ρ_E on E such that $\operatorname{ord}_p(\rho_C) = 3$ and $\operatorname{ord}_p(\rho_E) = 2g-5$ (we use that $h^0(C, K_C \otimes A^{-2}) = \dim\big(\operatorname{Ker}\big(\mu_0(V)\big)\big) = 1$ by assumption, hence ρ_C is essentially ρ up to subtracting the base locus).

From Theorem 4.3, every component of $\mathcal{Q}^1_k$ passing through z has dimension $\geqq 3g-4$. On the other hand we claim that every component of $\nu^{-1}(\Delta)$ passing through z has dimension $\leqq 3g-5$ and that z is an isolated point in $\nu^{-1}([X_0])$. Assuming this for a moment, we obtain that z is a smoothable $(gp)^1_k$-relation in the sense that there is a component of $\mathcal{Q}^1_k$ through z which meets $\nu^{-1}(B - \Delta)$. From this it follows that $[X_0] \in \overline{GP}^1_{g,k} \cap \Delta_1$. Since by construction the curves $[X_0]$ fill up a divisor inside Δ_1, we conclude that $GP^1_{g,k}$ has a divisorial component Z' such that $\overline{Z}' \cap \Delta_1 \supset \varepsilon^*(Z) \times \overline{\mathcal{M}}_{1,1}$.

Furthermore, because the vanishing sequences of the C and E-aspects of l add up precisely to k, every $\mathfrak{g}^1_k$ on a smooth curve "near" X_0 which specializes to l, is base point free (cf. [EH1], Proposition 2.5). We obtain that a point $z' \in \nu^{-1}(B - \Delta)$ near z will satisfy $\dim_{z'}(\mathcal{Q}^1_k) = 3g-4$ and will correspond to a smooth curve $[C'] \in GP^1_{g,k}$, satisfying all the required conditions.

We return now to the estimate for $\dim_z\big(\nu^{-1}(\Delta)\big)$: we consider a curve $X_b = C_b \cup_q E_b$ with $b \in \Delta$, and let $(l, m, \rho_{C_b}, \rho_{E_b}) \in \nu^{-1}(b)$. Hence the underlying limit linear series l and m have vanishing sequences $a^{l_{C_b}}(q) = (1,2)$, $a^{l_{E_b}}(q) = (k-2, k-1)$ and $a^{m_{C_b}}(q) = (1, 2, \dots, g-k+1)$, $a^{m_{E_b}}(q) = (g-3, g-2, \dots, 2g-3+k)$ respectively.

Clearly $\operatorname{ord}_q(\rho_{C_b}) \geqq 3 (= 1+2 = 2+1)$. We set

$$\big(A_b = L_{C_b}(-q), V_{C_b} := V_{C_b}(-q)\big) \in G^1_{k-1}(C_b)$$

and

$$\big(B_b = L_{E_b}\big(-(k-2)q\big), V_{E_b} := V_{E_b}\big(-(k-2)q\big)\big) \in G^1_2(E_b).$$

We claim that in fact $\mathrm{ord}_q(\rho_{C_b}) = 3$ and therefore

$$\mathrm{ord}_q(\rho_{E_b}) = 2g - 5 (= k - 2 + (2g - 3 - k) = k - 1 + (2g - 4 - k)).$$

Indeed assuming that $\mathrm{ord}_q(\rho_{C_b}) \geqq 4$, from the base point free pencil trick we have that $h^0\big(C_b, K_{C_b} \otimes A_b^{-2}(-q)\big) \geqq 1$. But $h^0\big(C, K_C \otimes A^{-2}(-p)\big) = 0$ (use the assumption on C and the fact that $p \in C$ is a general point), which implies that we can assume that $h^0\big(C_b, K_{C_b} \otimes A_b^{-2}(-q)\big) = 0$ for any point in a component of $\nu^{-1}(\Delta)$ passing through z.

After subtracting base points ρ_{C_b} can be viewed as an element in the projectivization of the kernel of the map $\mu_0(V_{C_b}) : V_{C_b} \otimes H^0(K_{C_b} \otimes A_b^{-1}) \to H^0(K_{C_b})$, while ρ_{E_b} is in the projectivized kernel of the map

$$\mu_0(V_{E_b}) : H^0(E_b, B_b) \otimes H^0\big(E_b, B_b^{-1}(4q)\big) \to H^0\big(E_b, \mathcal{O}_{E_b}(4q)\big).$$

In other words $[C_b] \in GP^1_{g-1,k-1}$ and from the base point free pencil trick we get that $H^0\big(E_b, \mathcal{O}_{E_b}(4q) \otimes B_b^{-2}\big) \neq 0$, which leaves only finitely many choices for B_b and ρ_{E_b}. It follows that $\dim_z \nu^{-1}(\Delta) \leqq \dim_{[C]}(GP^1_{g-1,k-1}) + 1 + 1 = 3g - 5$. □

Proof of Theorem 1.4. We apply Theorem 4.4 starting with the base case $k \geqq 3$, $g = 2k - 2$. In this situation the locus $GP^1_{2k-2,k}$ is a divisor in $\mathcal{M}_g$ which can also be viewed as the branch locus of the map to $\mathcal{M}_g$ from the Hurwitz scheme of coverings $C \xrightarrow{k:1} \mathbf{P}^1$ having a genus g source curve (cf. [EH3], Section 5). The locus of $[C] \in \mathcal{M}_g$ having infinitely many base point free $\mathfrak{g}^1_k$'s is of codimension $\geqq 2$, hence by default the general point of $GP^1_{2k-2,k}$ corresponds to a curve with finitely many $(A, V) \in G^1_k(C)$. The fact that for each of these pencils, $\dim \mathrm{Ker}\big(\mu_0(V)\big) \leqq 1$, also follows from [EH3]. Applying now Theorem 4.4 repeatedly we construct divisorial components of $GP^1_{2k-2+a,k+a}$ for all $k \geqq 3$ and $a \geqq 0$. It is easy to check that in this way we fill all the cases claimed in the statement. □

One could also define the loci $GP^1_{g,k}$ for $k \leqq (g+1)/2$. In this case $GP^1_{g,k}$ coincides with the locus of k-gonal curves, which is irreducible of dimension $2g + 2k - 5$. When g is odd, $GP^1_{g,(g+1)/2}$ is the well-known Brill-Noether divisor on $\mathcal{M}_g$ introduced by Harris and Mumford (see [EH3]). The Gieseker-Petri divisors $GP^1_{g,k}$ with $k \geqq (g+2)/2$ that we introduced, share certain properties with the Brill-Noether divisor. For instance the following holds (compare with [EH3], Proposition 4.1):

Proposition 4.5. *We denote by $j : \bar{\mathcal{M}}_{2,1} \to \bar{\mathcal{M}}_g$ the map obtained by attaching a fixed general pointed curve (C_0, p) of genus $g - 2$. Then for $(g+1)/2 \leqq k \leqq g - 1$ we have the relation $j^*(\overline{GP}^1_{g,k}) = q\mathcal{W}$, where $q \geqq 0$ and $\mathcal{W}$ is the divisor of Weierstrass points on $\mathcal{M}_{2,1}$.*

Sketch of proof. We can degenerate (C_0, p) to a string of elliptic curves $(E_1 \cup \cdots \cup E_{g-2}, p)$, where p lies on the last component E_{g-2}. We assume that for all $2 \leqq i \leqq g - 2$, the points of attachment between E_{i-1} and E_i are general. Fix now $[B, p] \in \mathcal{M}_{2,1}$ and assume that $[X_0 := C_0 \cup_p B] \in \overline{GP}^1_{g,k}$. We denote by (l_B, m_B, ρ_B) the B-aspect of a $(gp)^1_k$-relation on X_0. Then using the setup described at the beginning of Section 4 we obtain that $\mathrm{ord}_p(\rho_B) \geqq 2g - 4$. Since l_B is a $\mathfrak{g}^1_k$ and m_B is a $\mathfrak{g}^{g-k}_{2g-2-k}$, the only way this could happen is if $a^{l_B}(p) = (k - 2, k)$ and $a^{m_B}(p) = (\ldots, 2g - 4 - k, 2g - 2 - k)$, which implies that $h^0\big(\mathcal{O}_B(2p)\big) \geqq 2$, that is, $[B, p] \in \mathcal{W}$. □

Remark 4.6. Using methods developed in this section we can also prove the following result useful for the computation of the class $[\overline{GP}^1_{g,k}] \in \operatorname{Pic}(\bar{\mathcal{M}}_g)$: if $\varepsilon : \bar{\mathcal{M}}_{g-1,1} \to \bar{\mathcal{M}}_{g-1}$ is the forgetful morphism and $\phi : \bar{\mathcal{M}}_{g-1,1} \to \bar{\mathcal{M}}_g$ denotes the map attaching an elliptic tail at the marked point, then $\phi^*(\overline{GP}^1_{g,k})$ is set-theoretically the union of two divisors: $\varepsilon^*(\overline{GP}^1_{g-1,k-1})$, and the closure D in $\bar{\mathcal{M}}_{g-1,1}$ of the locus of curves $[C,p] \in \mathcal{M}_{g-1,1}$ for which there exists a base point free $A \in W^1_k(C)$ such that $h^0\big(C, A(-2p)\big) \geqq 1$ and the multiplication map

$$H^0(C,A) \otimes H^0\big(K_C \otimes A^\vee(2p)\big) \to H^0\big(K_C(2p)\big)$$

is not injective. It is natural to view D as a "pointed" Gieseker-Petri divisor on $\bar{\mathcal{M}}_{g,1}$.

We consider now the moduli space $\mathcal{S}_{g,n}$ of n-pointed spin curves of genus g and its subvariety $\mathcal{S}^r_{g,n}$ consisting of elements $(C, p_1, \dots, p_n, L)$, where $[C, p_1, \dots, p_n] \in \mathcal{M}_{g,n}$ and $L \in \operatorname{Pic}^k(C)$ is a line bundle such that $L^2 \otimes \mathcal{O}_C(p_1 + \cdots + p_n) = K_C$ and $h^0(L) \geqq r+1$. Of course we assume that $2k + n = 2g - 2$. The base point free pencil trick relates these loci to the loci $GP^i_{g,k}$ we introduced before. Precisely, if $f : \mathcal{S}_{g,n} \to \mathcal{M}_g$ is given by $[C, p_1, \dots, p_n, L] \mapsto [C]$, then $f(\mathcal{S}^1_{g,n}) = GP^1_{g,k}$.

We now look at the divisor $Z \subset GP^1_{g,k}$ constructed in Theorem 4.4. The condition that for a general $[C] \in Z$, the scheme $Q^1_k(C)$ has a 0-dimensional component with general point corresponding to a base point free $\mathfrak{g}^1_k$, can be translated into saying that $f^{-1}[C]$ has a zero-dimensional component. We obtain in this way that there exists a component Y of $\mathcal{S}^1_{g,n}$ of dimension $3g-4$ such that $f(Y) = Z$. This proves Corollary 1.5.

5. Injectivity of Gaussian maps

We are going to prove Theorem 1.3 by degeneration. Our proof is inspired by the work of Eisenbud and Harris on the Gieseker-Petri Theorem (cf. [EH2]). Suppose we have a family of genus g curves $\pi : X \to B$ over a base $B = \operatorname{Spec}(R)$ with R being a complete DVR with local parameter t and let 0 and η respectively, denote the special and the generic point of R. Assume furthermore that X_η is smooth and that X_0 is a curve of compact type consisting of a string of components of which g of them, $E_1, \dots, E_g$, are elliptic curves, while the rest are rational curves, glued in such a way that the stable model of X_0 is the curve $E_1 \cup_{p_1} E_2 \cup_{p_2} E_3 \cup \cdots \cup E_{g-1} \cup_{p_{g-1}} E_g$. Slightly abusing the notation, for $2 \leqq i \leqq g-1$ we will consider p_{i-1} and $p_i \in E_i$ to be the points of attachment of E_i to $\overline{X_0 - E_i}$ and we will choose X_0 in such a way that $p_i - p_{i-1}$ is not a torsion class in $\operatorname{Pic}^0(E_i)$.

We proceed by contradiction and assume that there exists a line bundle L_η on X_η of degree d, together with a non-zero element

$$\rho_\eta \in \operatorname{Ker}\{\psi_{L\eta} : \textstyle\bigwedge^2 H^0(X_\eta, L_\eta) \to H^0(X_\eta, \Omega^1_{X_\eta} \otimes L^2_\eta)\}.$$

(Note that because the shape of X_0 does not change if we blow-up the surface X, we can assume that we have a bundle L_η on X_η rather than on the geometric generic fibre $X_{\bar{\eta}}$.) As in Section 4, for each component Y of X_0 we have the line bundle L^Y on X extending L_η and having degree 0 restriction to all components $Z \neq Y$ of X_0 and we set $L_Y := L^Y_{|Y}$. Starting with $\rho_\eta \in \bigwedge^2 \pi_*(L_\eta)$ we obtain elements $\rho^Y = t^\alpha \rho_\eta \in \bigwedge^2 \pi_*(L^Y) - t \bigwedge^2 \pi_*(L^Y)$ for uniquely determined integers α, and we define $\rho_Y := \rho^Y_{|Y} \in \bigwedge^2 H^0(Y, L_Y)$.

Lemma 5.1. *For each component Y of X_0 we have that*

$$\rho_Y \in \mathrm{Ker}\{\psi_{L_Y} : \textstyle\bigwedge^2 H^0(Y, L_Y) \to H^0(Y, \Omega^1_Y \otimes L^2_Y)\}.$$

Proof. We use the commutative diagram

$$\begin{tikzcd}
\bigwedge^2 H^0(X_0, L^Y_{|X_0}) \arrow[r, "\mathrm{res}"] \arrow[d, "\psi_{L^Y_{|X_0}}"] & \bigwedge^2 H^0(Y, L_Y) \arrow[d, "\psi_{L_Y}"] \\
H^0(X_0, \Omega^1_{X_0} \otimes L^{Y^{\otimes 2}}_{|X_0}) \arrow[r, "\mathrm{res}"] & H^0(Y, \Omega^1_Y \otimes L^2_Y)
\end{tikzcd}$$

and keep in mind that the upper restriction map is injective. □

We will use the following observation (similar to the one for ordinary multiplication maps): let C be a smooth curve, $p \in C$ and M a line bundle on C. If $\rho \in \mathrm{Ker}(\psi_M)$ and $\{\sigma_i\}$ is a basis of $H^0(M)$ such that $\mathrm{ord}_p(\sigma_i) = a^M_i(p) = a_i$, then there are distinct pairs of integers $(i_1, j_1) \neq (i_2, j_2)$ with $i_1 \neq j_1$ and $i_2 \neq j_2$, such that $\mathrm{ord}_p(\rho) = \mathrm{ord}_p(\sigma_{i_1}) + \mathrm{ord}_p(\sigma_{j_1}) = \mathrm{ord}_p(\sigma_{i_2}) + \mathrm{ord}_p(\sigma_{j_2})$. This follows from a local calculation: if t is a local parameter for C at p, then

$$\psi_M(\sigma_i \wedge \sigma_j) = \big((a_i - a_j)t^{a_i + a_j - 1} + \text{h.o.t.}\big)\, dt,$$

and since $\psi_M(\rho) = 0$, the number $\mathrm{ord}_p(\rho)$ must be attained for at least two pairs (i, j).

Proposition 5.2. *Suppose Y and Z are two components of X_0 meeting at a point q and let p be a general point on Y. We have the following inequalities*:

(1) $\mathrm{ord}_q(\rho_Z) \geqq \mathrm{ord}_p(\rho_Y)$.

(2) *If Y is one of the elliptic components of X_0, then* $\mathrm{ord}_q(\rho_Z) \geqq \mathrm{ord}_p(\rho_Y) + 2$.

Proof. Although (1) is essentially Proposition 3.1 from [EH2] we will briefly go through the proof and in doing so we will also prove (2). We pick a basis $\{\sigma_i = \sigma^Y_i\}$ of $\pi_*(L^Y)$ such that $\mathrm{ord}_p(\sigma_{i|Y}) = a^L_i(p)$ and for which there are integers α_i with the property that $\{\sigma^Z_i = t^{\alpha_i}\sigma_i\}$ form a basis for $H^0(X, L^Z)$ (see [EH1], Lemma 2.3 for the fact that such a basis can be chosen). We then write $\rho^Y = \sum\limits_{i \neq j} f_{ij}\sigma_i \wedge \sigma_j$, with $f_{ij} \in R$, and we can express $\rho^Z = t^\gamma \rho^Y$, where $\gamma = \max\limits_{i \neq j}\{\alpha_i + \alpha_j - v(f_{ij})\}$. Here v denotes the valuation on the ring R. From the definition of γ it follows that there exists a pair (i, j), $i \neq j$, with $\gamma = \alpha_i + \alpha_j - v(f_{ij})$, such that we have a string of inequalities

$$\mathrm{ord}_q(\rho_Z) = \mathrm{ord}_q(\sigma^Z_{i|Z}) + \mathrm{ord}_q(\sigma^Z_{j|Z}) \geqq \alpha_i + \alpha_j \geqq \gamma, \tag{5}$$

(see also Section 4). On the other hand there exists a pair (i', j'), $i' \neq j'$ such that $v(f_{i'j'}) = 0$, for which we can write the inequalities

$$\begin{aligned} \mathrm{ord}_p(\rho_Y) &= \mathrm{ord}_p(\sigma_{i'|Y}) + \mathrm{ord}_p(\sigma_{j'|Y}) \\ &\leqq \big(d - \mathrm{ord}_q(\sigma_{i'|Y})\big) + \big(d - \mathrm{ord}_q(\sigma_{j'|Y})\big) \leqq \alpha_{i'} + \alpha_{j'} \leqq \gamma. \end{aligned} \tag{6}$$

Combining (5) and (6) we get the first part of the proposition. When moreover the curve Y is elliptic, since $\psi_{L_Y}(\rho_Y) = 0$, there must exist at least two pairs (i_1, j_1) and (i_2, j_2) for which (6) holds. On the other hand $p - q \in \mathrm{Pic}^0(Y)$ can be assumed not to be a torsion class, and we obtain that $\mathrm{ord}_p(\sigma_{i|Y}) + \mathrm{ord}_q(\sigma_{i|Y}) \leqq d - 1$ for all indices i except at most one. This and the fact that the vanishing orders $\mathrm{ord}_p(\sigma_{i|Y})$ are all distinct, quickly lead to the inequality $\mathrm{ord}_q(\rho_Z) \geqq \gamma \geqq \mathrm{ord}_p(\rho_Y) + 2$. $\square$

A repeated application of Proposition 5.2 gives the following result:

Proposition 5.3. *Let X_0 be the curve described in the degeneration above and which has the stable model $\bigcup_{i=1}^{g} E_i$, where E_i are elliptic curves. We denote by p_{i-1} and p_i the points of attachment of E_i to the rest of X_0. If $\psi_{L_\eta}(\rho_\eta) = 0$, then* $\mathrm{ord}_{p_{g-1}}(\rho_{E_g}) \geqq \mathrm{ord}_{p_1}(\rho_{E_2}) + 2g - 4$.

We are now in a position to prove Theorem 1.3. In fact we have a more general result:

Theorem 5.4. *For a general genus g curve C and for any line bundle L on C of degree $d \leqq a + g + 2$, where $a \geqq 0$, we have that* $\dim \mathrm{Ker}(\psi_L) \leqq a(a+1)$. *In particular, if $d \leqq g + 2$ then ψ_L is injective.*

Proof. We apply Proposition 5.3 and degenerate C to $X_0 = E_1 \cup \cdots \cup E_g$. We assume that $\mathrm{Ker}(\psi_{C,L})$ is at least $1 + a(a+1)$-dimensional. Then

$$\dim \mathrm{Ker}(\psi_{X_0, L^{E_2}_{|X_0}}) \geqq 1 + a(a+1)$$

and since the restriction map $\bigwedge^2 H^0(X_0, L^{E_2}_{|X_0}) \to \bigwedge^2 H^0(E_2, L_{E_2})$ is injective we obtain that $\mathrm{Ker}(\psi_{E_2, L_{E_2}})$ is at least $1 + a(a+1)$-dimensional as well. For simplicity let us denote $E_2 = E$, $L_{E_2} = L$ and $p_1 = p \in E_2$ (recall that $p_1 \in E_2 \cap E_1$).

If we choose a basis $\{\sigma_i\}$ of $H^0(L)$ adapted to the point p, then as we noticed before for each $\rho \in \mathrm{Ker}(\psi_L)$ there will be at least two distinct pairs of integers $(i_1, j_1) \neq (i_2, j_2)$ where $i_1 \neq j_1$, $i_2 \neq j_2$ such that

$$\mathrm{ord}_p(\rho) = \mathrm{ord}_p(\sigma_{i_1}) + \mathrm{ord}_p(\sigma_{j_1}) = \mathrm{ord}_p(\sigma_{i_2}) + \mathrm{ord}_p(\sigma_{j_2}).$$

The vanishing sequence $a^{L_{E_1}}(p)$ is $\leqq(\ldots, d-3, d-2, d)$, hence the vanishing sequence of $L = L_{E_2}$ at p is $\geqq(0, 2, 3, 4, 5, \ldots)$, which yields that $\mathrm{ord}_p(\rho) \geqq 5(= 0 + 5 = 2 + 3)$ for every $\rho \in \mathrm{Ker}(\psi_L)$. Since $\dim \mathrm{Ker}(\psi_L) \geqq 1 + a(a+1)$, there is a subspace $W_1 \subset \mathrm{Ker}(\psi_L)$ of dimension $\geqq a(a+1)$ such that $\mathrm{ord}_p(\rho) \geqq 6(= 0 + 6 = 2 + 4)$ for each $\rho \in W_1$.

Repeating this reasoning for W_1 instead of $\mathrm{Ker}(\psi_L)$ we obtain a subspace $W_2 \subset W_1$ with $\dim(W_2) \geqq \dim(W_1) - 1$ such that $\mathrm{ord}_p(\rho) \geqq 7(= 0 + 7 = 2 + 5 = 3 + 4)$ for every $\rho \in W_2$, and then a subspace $W_3 \subset W_2$ with $\dim(W_3) \geqq \dim(W_2) - 2$ with the property that $\mathrm{ord}_p(\rho) \geqq 8(= 0 + 8 = 2 + 6 = 3 + 5)$ for all $\rho \in W_3$. At the end of this argument we find at least one element $\rho = \rho_{E_2} \in \mathrm{Ker}(\psi_L)$ such that $\mathrm{ord}_p(\rho) \geqq 2a + 5$. Since this reasoning works if we replace $\mathrm{Ker}(\psi_L)$ with any of its subspaces having dimension $\geqq 1 + a(a+1)$, we can assume that ρ_{E_2} is the restriction to E_2 of an element ρ_η in the kernel of the corresponding Gaussian map on the general curve X_η, which according to the procedure described before Lemma 5.1 will produce elements $\rho_{E_i} \in \mathrm{Ker}(\psi_{L_{E_i}})$ for $1 \leqq i \leqq g$. Applying Proposition 5.3 we have that $\mathrm{ord}_{p_{g-1}}(\rho_{E_g}) \geqq \mathrm{ord}_p(\rho) + 2g - 4 = 2(a+g) + 1$. The vanishing sequence of L_{E_g} at p_g is $\leqq (\dots, d-3, d-2, d)$ from which we obtain that on the other hand

$$\mathrm{ord}_{p_{g-1}}(\rho_{E_g}) \leqq 2d - 5 (= d + (d-5) = (d-2) + (d-3)),$$

which combined with the previous inequality yields $d \geqq a + g + 3$ which is a contradiction. □

Note that Theorem 5.4 is valid for an *arbitrary* line bundle on a general genus g curve. It is clear that Proposition 5.3 would give better sufficient conditions for the injectivity of ψ_L if we restricted ourselves to line bundles on C having a prescribed ramification sequence at a given point $p \in C$. In this case we degenerate (C, p) to $(X_0 = E_1 \cup \cdots \cup E_g, p)$, where X_0 is as in Theorem 5.4 and $p \in E_1$ is such that $p - p_1 \in \mathrm{Pic}^0(E_1)$ is not a torsion class. We leave it to the interested reader to work out the numerical details. We can also improve on Theorem 5.4 if we look only at a suitably general line bundle L on C:

Proposition 5.5. *Fix integers g, d and $r \geqq 2$ such that $d \leqq g + r$, $\rho = g - (r+1)(g-d+r) \geqq 0$ and moreover $d < g + 3 + \dfrac{\rho}{2(r-1)}$. Then if C is a general curve of genus g and $L \in W^r_d(C)$ is general, the Gaussian map ψ_L is injective.*

Proof. We degenerate C to X_0, fix a general point $p \in E_1$ and set $a := [\rho/(r-1)] + 2$. Our numerical assumptions imply that $\rho - (a-2)(r-1) \geqq 0$. From the general theory of limit linear series in [EH1] reducing the Brill-Noether theory of X_0 to Schubert calculus, we know that there exists a smoothable limit linear series of type $\mathfrak{g}^r_d$ on X_0, say $l = \{L_{E_i} \in W^r_d(E_i)\}_{i=0,\dots,g}$ having vanishing sequence $\geqq (0, 1, a, a+1, a+2, \dots, a+r-2)$ at the point p.

Assume by contradiction that there are elements $\rho_{E_i} \in \mathrm{Ker}(\psi_{L_{E_i}})$ coming from an element $\rho \neq 0$ in the kernel of the corresponding Gaussian on the general curve. Then $\mathrm{ord}_p(\rho_{E_1}) \geqq a + 1 (= 1 + a = 0 + (a+1))$ and from Proposition 5.2 we get that $\mathrm{ord}_{p_{g-1}}(\rho_{E_g}) \geqq \mathrm{ord}_p(\rho_{E_1}) + 2g - 2 = 2g + a - 1$. On the other hand, as we noticed before $\mathrm{ord}_{p_{g-1}}(\rho_{E_g}) \leqq 2d - 5$ which gives a contradiction. □

Remark 5.6. The techniques from this section also allow us to study the kernel $S_2(L)$ of the multiplication map $\mu_L : \mathrm{Sym}^2 H^0(L) \to H^0(L^2)$. In a way similar to the proof of Theorem 5.4 we can show that if L is an arbitrary line bundle of degree $d \leqq g + a + 1$ on a general curve C of genus g then $\dim S_2(L) \leqq a(a+1)$. The $a = 0$ case of this result has been established by Teixidor (cf. [T2]). We also note that this result as well as Theorem 5.4, are meaningful when the bundle L is special. On the other hand the case when L is non-special (when, under suitable assumptions, we expect surjectivity for both ψ_L and μ_L), has been extensively covered in the literature (see e.g. [Pa]).

Theorem 1.3 answers Question 5.8.1 from Wahl's survey [W1], where the problem is raised in terms of self-correspondences on a curve. Suppose that C is a smooth curve and we consider the diagonal $\Delta \subset C \times C$ and the projections $p_i : C \times C \to C$ for $i = 1, 2$. For a line bundle L on C we denote $L_i := p_i^* L$ for $i = 1, 2$. We can rephrase Theorem 1.3 as follows:

Proposition 5.7. *If L is a line bundle of degree $d \leqq g + 1$ on a general curve C of genus g, then $H^0(C \times C, L_1 + L_2 - 2\Delta) = 0$.*

Proof. We use that $H^0(C \times C, L_1 + L_2 - 2\Delta) = \operatorname{Ker}(\Phi_L) = S_2(L) \oplus \operatorname{Ker}(\psi_L)$. We have proved that $\operatorname{Ker}(\psi_L) = 0$ while $S_2(L) = 0$ follows from [T2]. $\square$

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