THE UNIVERSAL K3 SURFACE OF GENUS 14 VIA CUBIC FOURFOLDS

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ABSTRACT. Using Hassett’s isomorphism between the Noether-Lefschetz moduli space $C_{26}$ of special cubic fourfolds $X \subset \mathbb{P}^5$ of discriminant 26 and the moduli space $F_{14}$ of polarized $K3$ surfaces of genus 14, we use the family of 3-nodal scrolls of degree seven in $X$ to show that the universal $K3$ surface over $F_{14}$ is rational.

1. Introduction

For a very general cubic fourfold $X \subset \mathbb{P}^5$, the lattice $A(X) := H^{2,2}(X) \cap H^4(X, \mathbb{Z})$ of middle Hodge classes contains only classes of complete intersection surfaces, so $A(X) = \langle h^2 \rangle$, where $h \in \text{Pic}(X)$ is the hyperplane class (see [V]). Hassett, in his influential paper [H1], initiated the study of Noether-Lefschetz special cubic fourfolds. If $C$ is the 20-dimensional coarse moduli space of smooth cubic fourfolds $X \subset \mathbb{P}^5$, let $C_d$ be the locus of special cubic fourfolds $X$ characterized by the existence of an embedding of a saturated rank 2 lattice $L := \langle h^2, [S] \rangle \hookrightarrow A(X)$, of discriminant $\text{disc}(L) = d$, where $S \subset X$ is an algebraic surface not homologous to a complete intersection. Hassett [H1] showed that $C_d \subset C$ is an irreducible divisor, which is nonempty if and only if $d > 6$ and $d \equiv 0, 2, 4 \pmod{6}$. The study of the divisors $C_d$ for small $d$ has received considerable attention. For instance, $C_8$ consists of cubic fourfolds containing a plane, whereas $C_14$ corresponds to cubic fourfolds containing a quintic del Pezzo surface, see [H2]. Relying on Fano’s work [Fa], recently Bolognesi and Russo [BR] have shown that all fourfolds $[X] \in C_{14}$ are rational.

For every $[X] \in C$, we denote by $F(X) := \{ \ell \in G(1,5) : \ell \subset X \}$ the Hilbert scheme of the lines contained in $X$. It is well known [BD] that $F(X)$ is a hyperkähler fourfold deformation equivalent to the Hilbert square of a $K3$ surface. For discriminant $d = 2(n^2 + n + 1)$, where $n \geq 2$, it is shown in [H1] that $F(X)$ is isomorphic to the Hilbert scheme $S^{[2]}$ of a polarized $K3$ surface $(S, H)$ with $H^2 = d$. If $F_g$ denotes the moduli space of polarized $K3$ surfaces of genus $g$, the previous assignment induces a rational map

$$F_{d+1} \dashrightarrow C_d,$$

which is a birational isomorphism for $d \equiv 2(\text{mod } 6)$ and a degree 2 cover for $d \equiv 0(\text{mod } 6)$. This map, though non-explicit for it is defined at the level of moduli spaces of weight-2 Hodge structures, opens the way to the study of $F_{n^2+n+2}$ via the concrete geometry of cubic fourfolds, without making a direct reference to $K3$ surfaces! The main result of this paper concerns the universal $K3$ surface $F_{14,1} \rightarrow F_g$.

Theorem 1.1. The universal $K3$ surface $F_{14,1}$ of genus 14 is rational.

Nuer [Nu] proved that $C_{26}$ (and hence $F_{14}$ as well) is unirational. His proof relies on the fact that a general fourfold $[X] \in C_{26}$ contains certain smooth rational surfaces, whose
parameter space forms a unirational family. One can also show that \( C_{44} \) is unirational, for a general \( [X] \in C_{44} \) contains a Fano embedded Enriques surface and their moduli space is unirational, see [Ve2] and also [Nu]. Recently, Lai [L] showed that \( C_{42} \) is uniruled.

Mukai in a celebrated series of papers [M1], [M2], [M3], [M4], [M5] established structure theorems for polarized \( K3 \) surfaces of genus \( g \leq 12 \), as well as \( g = 13, 16, 18, 20 \). In particular, \( F_g \) is unirational for those value of \( g \). No structure theorem for the general \( K3 \) surface of genus 14 is known. A quick inspection of Mukai’s methods shows that the universal \( K3 \) surface \( F_{g,1} \) is unirational for \( g \leq 11 \) as well. On the other hand, Gritsenko, Hulek and Sankaran [GHS] have proved that \( F_g \) is a variety of general type for \( g > 62 \), as well as for \( g = 47, 51, 53, 55, 58, 59, 61 \). In a similar vein, recently it has been established in [TVA] that \( C_d \) is of general type for all \( d \) sufficiently large. As pointed out in Remark 5.4, whenever \( F_g \) is of general type, the Kodaira dimension of \( F_{g,1} \) is equal to 19.

The proof of Theorem 1.1 relies on the connection between singular scrolls and special cubic fourfolds. We fix a general point \( [X] \in C_{26} \) and denote by \( S \) the associated \( K3 \) surface, such that \( S^{[2]} \cong F(X) \hookrightarrow G(1, 5) \). For each \( p \in S \), we introduce the rational curve

\[
\Delta_p := \{ \xi \in S^{[2]} : \{p\} = \text{supp}(\xi) \}.
\]

Under the Plücker embedding \( G(1, 5) \subseteq \mathbb{P}^{14} \), the degree of \( \Delta_p \subseteq F(X) \) is equal to 7, which suggests that each point of \( p \in S \) parametrizes a septic scroll \( R = R_p \subseteq X \). Imposing the condition \( \text{disc}(h^2, [R]) = 26 \), one obtains \( R^2 = 25 \). Assuming \( R \) has isolated non-normal nodal singularities, the double point formula implies that \( R \) has precisely 3 non-normal nodes. We shall prove that indeed, a general fourfold \( [X] \in C_{26} \) carries a 2-dimensional family of 3-nodal scrolls \( R \subseteq X \) with \( \text{deg}(R) = 7 \). Furthermore, this family of scrolls is parametrized by the \( K3 \) surface \( S \) associated to \( X \).

We now describe the moduli space of 3-nodal septic scrolls. We start with the Hirzebruch surface \( F_1 := \text{Bl}_\sigma(\mathbb{P}^2) \), where \( \sigma \in \mathbb{P}^2 \), and denote by \( \ell \) the class of a line and by \( E \) the exceptional divisor. The smooth septic scroll \( R' = S_{3,4} \subseteq \mathbb{P}^8 \) is the image of the linear system

\[
\phi_{|4\ell-3E|} : F_1 \hookrightarrow \mathbb{P}^8.
\]

We shall show in Section 3 that the secant variety \( \text{Sec}(R') \subseteq \mathbb{P}^8 \) is as expected 5-dimensional. Choose general points \( a_1, a_2, a_3 \in \text{Sec}(R') \) and denote by \( \Lambda := \langle a_1, a_2, a_3 \rangle \in G(2, 8) \) their linear span. The image \( R \subseteq \mathbb{P}^5 \) of the projection with center \( \Lambda \)

\[
\pi_\Lambda : R' \to \mathbb{P}^5
\]

is a 3-nodal septic scroll. Conversely, up to the action of \( PGL(6) \) on the ambient projective space \( \mathbb{P}^5 \), each such scroll appears in this way. We denote by \( \mathfrak{H}_{\text{scr}} \) the moduli space of unparametrized 3-nodal septic scrolls in \( \mathbb{P}^5 \), that is, the quotient of the corresponding Hilbert scheme under the action of \( PGL(6) \). Then as showed in Proposition 3.6, the space \( \mathfrak{H}_{\text{scr}} \) turns out to be birationally isomorphic to the 9-dimensional unirational variety

\[
\mathfrak{H}_{\text{scr}} \cong \text{Sym}^3(\text{Sec}(R')) / \text{Aut}(R').
\]

Fix a general 3-nodal septic scroll \( R \subseteq \mathbb{P}^5 \). A general \( X \in \mathbb{P}(H^0(\mathcal{I}_R/\mathcal{I}_R(3))) = \mathbb{P}^{12} \) is a smooth cubic fourfold. Since \( R \) has no further singularities apart from the three non-normal nodes, the double point formula implies that \( [X] \in C_{26} \). One sets up the following incidence
correspondence between scrolls and cubic fourfolds of discriminant 26:

$$\mathcal{X} := \left\{ (X, R) : R \subseteq X \right\} / \text{PGL}(6)$$

Thus $\mathcal{X}$ is birational to a $\mathbb{P}^{12}$-bundle over the unirational variety $\mathcal{H}_{\text{scr}}$. We then show that the fibre over a general cubic fourfold $[X] \in C_{26}$ of the projection $\pi_1$ is 2-dimensional and isomorphic to the $K3$ surface $S$ appearing in the identification $F(X) \cong S^{[2]}$. We summarize the discussion above.

**Theorem 1.2.** The universal $K3$ surface $\mathcal{F}_{14,1}$ is birational to the $\mathbb{P}^{12}$-bundle $\mathcal{X}$ over the moduli space $\mathcal{H}_{\text{scr}}$ of 3-nodal septic scrolls $R \subseteq \mathbb{P}^5$. A general fourfold $[X] \in C_{26}$ contains a two-dimensional family of such scrolls $R \subseteq X \subseteq \mathbb{P}^5$. The space of such scrolls is isomorphic to the $K3$ surface associated to $X$.

Theorem 1.2 allows us to elucidate the structure of $\mathcal{F}_{14,1}$ even further and prove its rationality. We fix a 3-nodal septic scroll $R \subseteq \mathbb{P}^5$ as above and denote its nodes by $p_1, p_2, p_3$. The curve $\Gamma_R \subseteq G(1,5)$ induced by the rulings of $R$ is a smooth rational septic curve admitting bisecant lines $L_1, L_2$ and $L_3$ in the Plücker embedding of $G(1,5)$. Precisely, $L_i$ parametrizes the lines passing through $p_i$ and contained in the 2-plane $P_i$ spanned by the two rulings of $R$ that intersect at the node $p_i$, for $i = 1, 2, 3$. Since $\Gamma_R$ spans a 7-dimensional linear space in projective space $\mathbb{P}^{14}$ containing $G(1,5)$, using Mukai’s work [M6] on realizing canonical genus 8 curves as linear sections of the Grassmannian $G(1,5)$, it follows that the intersection $G(1,5) \cdot \langle \Gamma_R \rangle$ is a semi-stable curve of genus 8. We denote by $Q \subseteq \langle \Gamma_R \rangle = \mathbb{P}^7$ the residual curve defined by the following equality:

$$1.\quad G(1,5) \cdot \langle \Gamma_R \rangle = \Gamma_R + L_1 + L_2 + L_3 + Q.$$

We shall establish in Lemmas 4.1 and 4.2 that $Q$ is a smooth rational quartic curve and $Q \cdot L_i = 1$ for $i = 1, 2, 3$, as well as $Q \cdot \Gamma_R = 3$. Therefore $Q$ is the curve of rulings of a quartic scroll $R_Q \subseteq \mathbb{P}^5$, which contains three rulings $\ell_1, \ell_2, \ell_3$, such that that $p_i \in \ell_i$ and $\ell_i \in P_i$ for $i = 1, 2, 3$. In particular, $R_Q$ contains the three nodes of the septic scroll $R$. We can show furthermore that $R_Q$ is smooth and isomorphic to $F_0$, see Theorem 4.10.

The construction above can be reversed. Using the automorphism group of the scroll $R_Q \subseteq \mathbb{P}^5$, we fix three of its rulings $\ell_1, \ell_2, \ell_3 \in G(1,5)$, as well as points $p_i \in \ell_i$. We set

$$P_i^3 := \{ P_i \in G(2,5) : \ell_i \subseteq P_i \},$$

for $i = 1, 2, 3$, then define a map

$$\varpi : P_1^3 \times P_2^3 \times P_3^3 / \mathfrak{S}_3 \rightarrow \mathcal{H}_{\text{scr}},$$

by reversing the above construction and using the decomposition $1$. Along with the fixed point $p_i$, each 2-plane $P_i \in P_i^3$ defines a line $L_i \subseteq G(1,5)$ meeting the curve $Q$ at the point $\ell_i$. Precisely, $L_i$ is the line of lines in $P_i$ passing through the point $p_i$. To the triple $(P_1, P_2, P_3)$ we associate the scroll $R \subseteq \mathbb{P}^5$ whose associated curve of rulings $\Gamma_R$ is defined by the formula $1$. The above discussion indicates that $\varpi$ is dominant. In fact more can be proved:

**Theorem 1.3.** The moduli space of scrolls $\mathcal{H}_{\text{scr}}$ is birational to $P_1^3 \times P_2^3 \times P_3^3 / \mathfrak{S}_3$ and is thus rational.
Indeed, using the theorem on symmetric functions, see [Ma] or [GKZ] Theorem 2.8 for a recent reference, all symmetric products of projective spaces are known to be rational. It is now clear that Theorem 1.3 coupled with Theorem 1.2 implies that $\mathcal{F}_{14,1}$ is a rational variety.

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2. $K^3$ SURFACES AND CUBIC FOURFOLDS

We begin by setting some notation. Let $U \subseteq |O_{P^3}(3)|$ be the locus of smooth cubic fourfolds and set

$$C := U/PGL(6)$$

to be the 20-dimensional moduli space of cubic fourfolds. For an integer $d \equiv 0,2 \, (\text{mod} \, 6)$, as pointed out in the Introduction, $C_d$ denotes the irreducible divisor of $C$ consisting of special cubic fourfolds of discriminant $d$. As usual, $\mathcal{F}_g$ is the irreducible 19-dimensional moduli space of smooth polarized $K^3$ surfaces $(S,H)$ of genus $g$, that is, with $H^2 = 2g - 2$. We denote by $u : \mathcal{F}_{g,1} \to \mathcal{F}_g$ the universal $K^3$ surface of genus $g$ in the sense of stacks. Each fibre $u^{-1}([S,H])$ is identified with the $K^3$ surface $S$.

Using the Hodge-theoretic similarity between $K^3$ surfaces of genus $g = n^2 + n + 1$ and special cubic fourfolds of degree $2g - 2$, Hassett [HT] constructed a morphism of moduli spaces

$$\varphi : \mathcal{F}_{n^2+n+2} \to C_{2(n^2+n+1)},$$

which is birational for $n \equiv 0,2 \, (\text{mod} \, 3)$, and of degree 2 for $n \equiv 1 \, (\text{mod} \, 3)$ respectively. In particular, for $n = 3$ there is a birational isomorphism of spaces of weight 2 Hodge structures

$$\varphi : \mathcal{F}_{14} \isom \to C_{20},$$

that will be of use throughout the paper. At the moment, there is no geometric construction of the polarized $K^3$ surface $\varphi^{-1}([X])$ associated to a general fourfold $X \in C_{20}$.

We recall basic facts on Hilbert squares of $K^3$ surfaces and refer to [HT] for a general reference on these matters. Let $(S,H)$ be a polarized $K^3$ surface with Pic$(S) = \mathbb{Z} \cdot H$ and $H^2 = 2g - 2$. We denote by $S[2]$ the Hilbert scheme of length two 0-dimensional subschemes on $S$. Then $H^2(S[2],\mathbb{Z})$ is endowed with the Beauville-Bogomolov quadratic form $q$. We denote by $\Delta \subseteq S[2]$ the diagonal divisor consisting of zero-dimensional subschemes supported only at a single point and by $\delta := |\Delta| \subseteq H^2(S[2],\mathbb{Z})$ the reduced diagonal class. One has $q(\delta,\delta) = -2$. Note the canonical identification

$$\Delta = \text{P}(T_S) = \cup \{ \Delta_p : p \in S \},$$

where $\Delta_p$ is the rational curve consisting of those 0-dimensional subschemes $\xi \in \Delta$ such that supp$(\xi) = \{ p \}$. We set $\delta_p := |\Delta_p| \in H_2(S[2],\mathbb{Z})$.

For a curve $C \in |H|$ in the polarization class, we introduce the divisor

$$f_C := \{ \xi \in S[2] : \text{supp}(\xi) \cap C \neq \emptyset \}$$

and set $f := [f_C] \in H^2(S[2],\mathbb{Z})$. If $p \in S$ is a general point, we also define the curve

$$F_p := \{ \xi = p + x \in S[2] : x \in C \}$$

and set $f_p := [F_p] \in H_2(S[2],\mathbb{Z})$. The Beauville-Bogomolov form can be extended to a quadratic form on $H_2(S[2],\mathbb{Z})$, by setting $q(\alpha,\alpha) := q(w_\alpha,w_\alpha)$, with $w_\alpha \in H^2(S[2],\mathbb{Z})$ being the
class characterized by the property $\alpha \cdot u = q(w_\alpha, u)$, for every $u \in H^2(S^{[2]}, \mathbb{Z})$. Here $\alpha \cdot u$ denotes the usual intersection product.

One has the following decompositions, orthogonal with respect to $q$, both for the Picard group and for the group $N_1(S^{[2]}, \mathbb{Z})$ of 1-cycles modulo numerical equivalence:

$$\text{Pic}(S^{[2]}) \cong \mathbb{Z} \cdot f \oplus \mathbb{Z} \cdot \delta \quad \text{and} \quad N_1(S^{[2]}, \mathbb{Z}) \cong \mathbb{Z} \cdot f_p \oplus \mathbb{Z} \cdot \delta_p.$$ 

We record, the more or less obvious relations:

$$f \cdot f_p = 2g - 2, \quad \delta \cdot \delta_p = -1, \quad f \cdot \delta_p = 0 \quad \text{and} \quad \delta \cdot f_p = 0. \quad (2)$$

Assume now that $X \subseteq \mathbb{P}^5$ is a general special cubic fourfold of discriminant 26 and let

$$[S, H] = \varphi^{-1}([X]) \in \mathcal{F}_{14}$$

be the associated polarized $K3$ surface such that

$$S^{[2]} \cong F(X) \subseteq G(1, 5) \hookrightarrow \mathbb{P}^{14}. \quad (3)$$

Following [BD], let $\gamma_S := |O_{S^{[2]}}(1)|$ be the hyperplane class of $G(1, 5)$ restricted to the Hilbert square under the identification $S^{[2]}$. Since $q(\gamma_S, \gamma_S) = 6$, using (2), it quickly follows that $\gamma_S = 2f - 7\delta \in H^2(S^{[2]}, \mathbb{Z})$.

**Proposition 2.1.** Suppose $[S, H] \in \mathcal{F}_{26}$ is a general element and let $R \subseteq S^{[2]}$ be an effective 1-cycle such that $R \cdot \gamma_S = 7$. Then $R$ is one of the rational irreducible curves $\Delta_p$, for $p \in S$. In particular, $R$ is smooth.

**Proof.** Assume that $R$ is an effective 1-cycle and write $[R] = af_p - b\delta_p \in N_1(S^{[2]}, \mathbb{Z})$. Since $7 = R \cdot \gamma_S = 52a - 7b$, hence we can write $a = 7a_1$, with $a_1 \in \mathbb{Z}$, and then $b = 52a_1 - 1$. Using [BM] Proposition 12.6, we have $q(R, R) \geq -\frac{s}{2}$. We obtain $39a_1^2 - 26a_1 - 1 \leq 0$, and the only integer solution of this inequality is $a_1 = 0$, therefore $[R] = \delta_p$.

Since $[R] \cdot \delta = -1$, it follows that $R \subseteq \Delta$. We claim that $R$ lies in one of the fibres of the $\mathbb{P}^1$-bundle $\pi : \Delta = \mathbb{P}(T_S) \to S$, which implies that $R = \Delta_p$, for some $p \in S$. Indeed, otherwise $\pi(R) \equiv mH$, for some $m > 0$. Accordingly, we write

$$mH^2 = R \cdot \pi^{-1}(H) = R \cdot f = \delta_p \cdot f = 0,$$

which is a contradiction. \hfill \square

**Remark 2.2.** Unlike degree 26, for other values of $d$, a general $[X] \in C_d$ may contain several types of scrolls. For instance when $d = 14$ and $\gamma_S = 2f - 5\delta$, the curves $\Delta_p$ with $p \in S$ correspond to quintic scroll, but $X$ also contains quartic scrolls corresponding to rational curves $R \subseteq F(X)$ with $[R] = 3f_p - 16\delta_p$. Note that $q(R, R) = -2$.

We now recall the correspondence between scrolls and rational curves in Grassmannians. Following for instance [Do] 10.4, we define a **rational scroll** to be the image $R \subseteq \mathbb{P}^n$ of a $\mathbb{P}^1$-bundle $\pi : R' = \mathbb{P}(\mathcal{E}) \to \mathbb{P}^1$ under a map $\phi : R' \to \mathbb{P}^n$ given by a linear subsystem of $|O_{R}(\mathcal{E})(1)|$, thus sending the fibres of $\pi$ to lines in $\mathbb{P}^n$. Let $f_R : \mathbb{P}^1 \to G(1, n)$ be the map

$$f_R(t) := [\phi(\pi^{-1}(t))]$$

and denote by $\Gamma_R$ its image. Conversely, start with a non-degenerate map $f : \mathbb{P}^1 \to G(1, n)$, then consider the pull-back under $f$ of the projectivization of tautological rank 2 vector over $G(1, n)$, that is,

$$\Xi := \{(t, x) : t \in \mathbb{P}^1, x \in L_f(t)\} \subseteq \mathbb{P}^1 \times \mathbb{P}^n. \quad (4)$$
Here \( L_{f(t)} \subseteq \mathbb{P}^n \) denotes the line whose moduli point in \( G(1, n) \) is precisely \( f(t) \).

The projection \( \pi_2 : \Xi \to \mathbb{P}^n \) is a finite map and its image is a scroll \( R \subseteq \mathbb{P}^n \) of degree

\[
\deg(\Gamma_R) = \deg f^* \left( \mathcal{O}_{G(1, n)}(1) \right).
\]

Throughout the paper, we interpret scrolls in terms of their associated curves of rulings. It will be useful to determine, using this language, when a scroll is smooth.

**Proposition 2.3.** Let \( R \subseteq \mathbb{P}^n \) be a scroll which is not a cone and such that \( \Gamma_R \) is a smooth rational curve in \( G(1, n) \) which is not contained in a plane. Then there is a bijective correspondence between singularities of \( R \) and bisection lines to \( \Gamma_R \) lying on \( G(1, n) \). In particular, if \( \Gamma_R \) admits no bisection lines contained in \( G(1, n) \), then \( R \) is smooth.

**Proof.** We consider the projection \( \pi_2 : \Xi \to \mathbb{P}^n \) defined by (4). Then \( \Xi \) is a smooth variety and the assumptions made on \( R \) imply that \( \pi_2 \) is a finite map. If a point \( x \in R \) corresponds to a singularity, then one of the two following possibilities occur: (i) the fibre \( \pi_2^{-1}(x) \) consists of more than point, or (ii) the differential of \( \pi_2 \) at a point of \( (t, x) \in \pi_2^{-1}(x) \) is not an isomorphism.

In case (i), we choose distinct points \( t_1, t_2 \in \pi_1(\pi_2^{-1}(x)) \). Denoting by \( \ell_1 := f_R(t_1) \) and \( \ell_2 := f_R(t_2) \) the rulings of \( \Xi \) corresponding to these points, we observe that \( x \in \ell_1 \cap \ell_2 \). The set \( L \) of lines in the 2-plane \( \langle \ell_1, \ell_2 \rangle \) passing through \( x \) is a line in \( G(1, n) \) such that \( \Gamma_R \cap L \supseteq \langle \ell_1, \ell_2 \rangle \), that is, \( \Gamma_R \) possesses a secant line lying inside \( G(1, n) \) in its Plücker embedding. Note that \( L \) is a genuine secant line in the sense that it meets the curve \( \Gamma_R \) in two distinct points \( \ell_1 \) and \( \ell_2 \). All lines lying inside \( G(1, n) \) in its Plücker embedding correspond to pencils of lines in a 2-plane passing through a point in \( \mathbb{P}^n \). Thus conversely, when such a line meets \( \Gamma_R \) in two distinct points, these will correspond to two incident rulings of \( R \). In particular \( R \) is singular at their point of intersection.

To deal with case (ii), we carry out a local calculation. Assume \((t_0, x) \in \Xi \) is a point at which the differential of \( \pi_2 \) is not an isomorphism. We set \( \ell_0 := f_R(t_0) \) and denote by

\[
p_{ij}(t) = a_i(t)b_j(t) - a_j(t)b_i(t), \quad \text{where } 0 \leq i < j \leq n
\]

the Plücker coordinates of the curve \( \Gamma_R \) in a neighborhood of \( \ell_0 \), where \( a(t) = (a_0(t), \ldots, a_n(t)) \) and \( b(t) = (b_0(t), \ldots, b_n(t)) \).

In local coordinates, the map \( \pi_2 \) is given by \( \mathbb{P}^1 \times \mathbb{C} \ni ([\lambda, \mu], t) \mapsto \left( \lambda a_i(t) + \mu b_i(t) \right) =: x \). By direct calculation, the condition that \((d\pi_2)_{(t_0, x)}\) is not an isomorphism is equivalent to

\[
b'(t_0) \wedge a(t_0) = 0 \in \bigwedge^2 \mathbb{C}^{n+1}.
\]

Setting \( a_i := a_i(t_0), b_i := b_i(t_0), a_i' := a_i'(t_0) \) and \( b_i' := b_i(t_0) \), we then observe that the Plücker coordinates of a point on the tangent line \( T_{\ell_0}(\Gamma_R) \subseteq \mathbb{P}^{(n+1)} \) are given by

\[
a_i b_j - a_j b_i + \mu(a_i b_j + a_j b_i - a_j b_i - a_i b_j) = b_j(a_i + \mu a_i') - b_i(a_j + \mu a_j'),
\]

for some scalar \( \mu \). It follows that the tangent plane to \( \Gamma_R \) at \( \ell_0 \) is contained in \( G(1, n) \). The argument being reversible, we finish the proof. \( \square \)

The scrolls \( R \subseteq \mathbb{P}^n \) we consider most of the time have at worst non-normal nodal singularities \( x \in R \), corresponding to the case \(|\phi^{-1}(x)| = 2 \). The tangent cone of \( R \) at \( x \) is isomorphic to the union of two 2-planes in \( \mathbb{P}^4 \) meeting in one point. According to Proposition 2.3, to each
such singularity corresponds a line in the Plücker embedding of \( G(1,n) \) meeting \( \Gamma_R \) in two distinct points.

Suppose now that \( R \subseteq X \subseteq \mathbb{P}^5 \) is a rational scroll with isolated nodal singularities contained in a cubic fourfold. Using the double point formula \[ \text{9.3} \] applied to the map \( \phi : R' \to X \), we find the number of singularities of \( R = \phi(R') \):

\[
D(\phi) = R^2 - 6h^2 - K_R^2 - 3h \cdot K_R + 2\chi_{\text{top}}(R).
\]

When \([X] \in C_{26}\), assuming that \( A(X) = \langle h^2, [R] \rangle \), where \( h^2 \cdot [R] = \deg(R) = 7 \), necessarily \( R^2 = 25 \). From formula \[ \text{5} \], we compute \( D(\phi) = 3 \), that is, if \( R \) has only (isolated) improper nodes, then it is 3-nodal.

Before stating our next result, we recall that \( \mathcal{M}_0(F(X), 7) \) denotes the space of stable maps \( f : C \to F(X) \), from a nodal curve \( C \) of genus zero such that \( \deg(f^*(\mathcal{O}_{F(X)}(1))) = 7 \). We denote by \( \mathcal{M}_0(F(X), 7) \) the open sublocus consisting of maps with source \( \mathbb{P}^1 \) and denote by \( \mathcal{M}_7(X) \) the closure of \( \mathcal{M}_0(F(X), 7) \) inside \( \mathcal{M}_0(F(X), 7) \).

**Corollary 2.4.** Let \([X] \in C_{26}\) a general special fourfold of discriminant 26 and \([S, H] \in F_{26}\) its associated K3 surface. Then there is an isomorphism \( S \cong \mathcal{M}_7(X) \).

**Proof.** Using the identification \( S^{[2]} \cong F(X) \), we define the map \( j : S \to \mathcal{M}_7(X) \), by setting \( j(p) := \Delta_p \subseteq F(X) \). All points in the image of \( j \) consist of embedded smooth rational curves \( \mathbb{P}^1 \cong \Delta_p \) and we identify \( \Delta_p \) with the corresponding map \( \mathbb{P}^1 \to F(X) \). In a neighborhood of this map, the moduli space \( \mathcal{M}_0(F(X), 7) \) is locally isomorphic to the Hilbert scheme of septic rational curves on \( F(X) \).

The tangent space of \( \mathcal{M}_7(X) \) at the point \([\Delta_p]\) is canonically isomorphic to \( H^0(N_{\Delta_p/F(X)}) \). Using the following exact sequence on \( \Delta_p \cong \mathbb{P}^1 \)

\[
0 \to N_{\Delta_p/\Delta} \to N_{\Delta_p/F(X)} \to \mathcal{O}_{\Delta_p}(\Delta) \to 0,
\]

since \( N_{\Delta_p/\Delta} = \mathcal{O}_{\Delta_p}^{\oplus 2} \) and \( \mathcal{O}_{\Delta_p}(\Delta) = \mathcal{O}_{\Delta_p}(-1) \), we compute \( N_{\Delta_p/F(X)} = \mathcal{O}_{\Delta_p}^{\oplus 2} \oplus \mathcal{O}_{\Delta_p}(-1) \). It follows that \( H^1(\Delta_p, N_{\Delta_p/F(X)}) = 0 \), hence the obstruction space for deformations vanishes and

\[
\dim T_{[\Delta_p]}(\mathcal{M}_0(F(X), 7)) = h^0(\Delta_p, N_{\Delta_p/F(X)}) = 2.
\]

We conclude that \([\Delta_p]\) is a smooth point of expected dimension of \( \mathcal{M}_7(X) \), for every \( p \in S \).

Furthermore, \( j \) is injective, because for distinct points \( p, q \in S \), since \( \Delta_p \cap \Delta_q = \emptyset \), the associated scrolls \( R_p \) and \( R_q \) share no rulings. We finally observe that \( j \) is an immersion. Indeed, for each \( p \in S \), we have the identification \( \Delta_p = \mathbb{P}\left(T_p(S) \oplus T_p(S)/T_p(S)\right) \), the quotient being given by the diagonal embedding. Thus the differential \( dj(p) \) is essentially the identity map, via the identification \( \mathbb{P}(T_S) \cong \bigcup_{p \in S} \mathbb{P}(N_{\Delta_p/\Delta}) \). Since according to Proposition 2.1 we have that \( \mathcal{M}_0(F(X), 7) \subseteq \text{Im}(j) \), we can conclude the proof.

\[ \square \]

### 3. Nodal Septic Scrolls and Cubic Fourfolds

In this section we study in more detail the moduli space \( \mathfrak{H}_{\text{scr}} \) of 3-nodal septic scrolls that will be used to parametrize the universal K3 surface of genus 14 via cubic fourfolds. We fix once and for all the smooth septic scroll

\[
R' := S_{3,4} \hookrightarrow \mathbb{P}^8,
\]
Every smooth septic scroll in $\mathbb{P}^8$ is obtained from $R'$ by applying a linear transformation of $\mathbb{P}^8$. In particular, the Hilbert scheme of septic scrolls in $\mathbb{P}^8$ has dimension equal to
\[
\dim \text{Aut}(R') = \dim \text{Aut}(F_1) = 6.
\]

Using coordinates in $\mathbb{P}^8$, if $P_{x_0\ldots,x_3}^3 \subseteq \mathbb{P}^8$ is the linear span of the twisted cubic $E$ corresponding to the exceptional divisor on $F_1$ and $P_{y_0\ldots,y_4}^3 \subseteq \mathbb{P}^8$ is the linear span of a rational quartic curve linearly equivalent to $\ell$, then the ideal of $R'$ in $\mathbb{P}^8$ is given by the following determinantal condition, see for instance [Ha] Lecture 9:
\[
\text{rk}\left(\begin{array}{cccc}
 x_0 & x_1 & x_2 & x_3 \\
 x_1 & x_2 & x_3 & x_4 \\
 x_2 & x_3 & x_4 & y_3
\end{array}\right) \leq 1.
\]

The secant variety $\text{Sec}(R') \subseteq \mathbb{P}^5$ is also determinantal, with equations given by the $3 \times 3$ minors of the following $1$-generic matrix:
\[
\text{rk}\left(\begin{array}{cccc}
 x_0 & x_1 & x_2 & y_0 \\
 x_1 & x_2 & x_3 & y_1 \\
 x_2 & x_3 & x_4 & y_3
\end{array}\right) \leq 2.
\]

It follows from [CC] Lemma 3.1 that, as expected, $\text{Sec}(R')$ is 5-dimensional. Furthermore, applying e.g. [Ei] Corollary 3.3, it follows that the singular locus of $\text{Sec}(R')$ coincides with the scroll $R'$.

**Lemma 3.1.** Let $a_1, a_2, a_3 \in \text{Sec}(R')$ be general points and set $\Lambda := \langle a_1, a_2, a_3 \rangle \in G(2, 8)$. The image $R$ of the projection $\pi : R' \to P^5$ with center $\Lambda$ has three non-normal nodes corresponding to the three bisecant lines passing through $a_1, a_2$ and $a_3$ and no further singularities.

**Proof.** The chosen points $a_1, a_2, a_3$ can be assumed to lie in $\text{Sec}(R') - (R' \cup \tan(R'))$. Since $\dim \text{Sec}(R') = 5$, by using the Trisecant lemma, see for instance [CC] Proposition 2.6, it follows that the scheme-theoretic intersection of $\text{Sec}(R')$ with $\Lambda$ consists only of the points $a_1, a_2, a_3$. In particular, $\Lambda \cap R' = \emptyset$, hence the projection $\pi = \pi_\Lambda : R' \to R$ is a regular morphism. Furthermore, each point $a_i$ lies on a unique bisecant line $\langle x_i, y_i \rangle$, where $x_i$ and $y_i$ are distinct points of $R'$, for $i = 1, 2, 3$.

Suppose now that for $x, y \in R'$, one has $\pi(x) = \pi(y)$. This happens if and only if $\langle x, y \rangle \nsubseteq \Lambda$, hence $\emptyset \neq \langle x, y \rangle \cap \Lambda \subseteq \{a_1, a_2, a_3\}$ and then necessarily $\{x, y\} = \{x_i, y_i\}$, for $i \in \{1, 2, 3\}$. Since $\Lambda \cap \tan(R') = \emptyset$, it follows that the differential of $\pi$ is everywhere injective. To summarize, the only singularities of $R$ are the three non-normal nodes $\pi(x_i) = \pi(y_i)$, for $i = 1, 2, 3$. \qed
We now fix a general projection \( \pi = \pi_\Lambda : R' \to \mathbb{P}^5 \) as in Lemma 3.1. We denote by \( p_i \) the three singularities of the image scroll \( R \). The map \( \pi_\Lambda \) is defined by the 6-dimensional subspace \( V := H^0(\mathbb{P}^5, \mathcal{I}_\Lambda/\mathcal{I}_R(1)) \) of \( H^0(\mathbb{P}^1, G) \). To give \( \Lambda \) amounts to specifying \( V \subset H^0(\mathbb{P}^1, G) \). Since \( \Lambda \cap R' = \emptyset \), it follows that the evaluation map \( \text{ev}_V : V \otimes \mathcal{O}_{\mathbb{P}^1} \to G \) is surjective. Hence \( \text{ev}_V \) defines a morphism
\[
f : \mathbb{P}^1 \to G(1,5).
\]
This map is induced by the ruling of the image scroll \( R \), that is, \( f_R = f \) is the map given by \( f_R(t) := [\pi(h^{-1}(t))] \), for \( t \in \mathbb{P}^1 \). Set \( \Gamma_R := \text{Im}(f_R) \).

**Proposition 3.2.** For a general choice of the 3-secant plane \( \Lambda \) to \( \text{Sec}(R') \), the following hold:
\[
(i) \quad \dim \langle p_1, p_2, p_3 \rangle = 2.
(ii) \quad \langle p_1, p_2, p_3 \rangle \cap R = \{ p_1, p_2, p_3 \}.
\]

**Proof.** It suffices to consider a codimension 2 general linear section \( Z \subset R' \subset \mathbb{P}^5 \). Then \( Z \) is a smooth 0-dimensional scheme supported at seven distinct points \( x_1, y_1, x_2, y_2, x_3, y_3 \) and \( z \), spanning a 6-dimensional linear space in \( \mathbb{P}^5 \). In particular, \( z \) does not lie in the 5-plane spanned by the points \( \{ x_i, y_i \}_{i=1}^3 \) and no line intersecting the lines \( \langle x_1, y_1 \rangle, \langle x_2, y_2 \rangle, \langle x_3, y_3 \rangle \) exists. Pick general points \( a_i \in \{ x_i, y_i \} \) for \( i = 1, 2, 3 \). Then the projection \( \pi_\Lambda \) defined by the plane \( \Lambda = \langle a_1, a_2, a_3 \rangle \) satisfies both conditions (i) and (ii).

For a projection \( \pi_\Lambda \) satisfying the assumptions of Lemma 3.1, the map \( f_R : \mathbb{P}^1 \to G(1,5) \) is an embedding, for \( \Lambda \) intersects no ruling of \( R' \). We record the conclusion of Proposition 2.3 for a scroll \( R \) as above:

**Proposition 3.3.** The rational curve \( \Gamma_R \subset G(1,5) \) admits three secant lines that lie in \( G(1,5) \). Conversely, a rational septic curve \( \Gamma \subset G(1,5) \) having three secant lines lying in \( G(1,5) \) is the curve of rulings of a 3-nodal septic scroll in \( \mathbb{P}^5 \).

We establish a couple of properties concerning the linear system of cubic fourfolds containing a 3-nodal septic scroll:

**Proposition 3.4.** The following statements hold for a general 3-nodal septic scroll \( R \subset \mathbb{P}^5 \):
\[
(i) \quad \dim \mathcal{I}_R/\mathcal{I}_R(3) = 12 \quad \text{and} \quad (ii) \quad H^1(\mathbb{P}^5, \mathcal{I}_R/\mathcal{I}_R(3)) = 0.
\]

**Proof.** Recall that \( R \) is the image of a projection \( \pi = \pi_\Lambda : R' \to R \) with center \( \Lambda \), and denote by \( p_1, p_2, p_3 \in R \) the three (non-normal) singularities of \( R \) and by \( \{ x_i, y_i \} = \pi^{-1}(p_i) \), for \( i = 1, 2, 3 \). By Proposition 5.2, the points \( p_1, p_2 \) and \( p_3 \) are in general linear position in \( \mathbb{P}^5 \) and thus impose independent conditions on cubic hypersurfaces, that is, \( H^1(\mathbb{P}^5, \mathcal{I}_{\text{Sing}(R)}/\mathcal{I}_R(3)) = 0 \).

By passing to cohomology in the short exact sequence
\[
0 \longrightarrow \mathcal{I}_R/\mathcal{I}_R(3) \longrightarrow \mathcal{I}_{\text{Sing}(R)}/\mathcal{I}_R(3) \longrightarrow \mathcal{I}_{\text{Sing}(R)/R}(3) \longrightarrow 0,
\]
we write the following exact sequence:
\[
0 \longrightarrow H^0(\mathcal{I}_R/\mathcal{I}_R(3)) \longrightarrow H^0(\mathcal{I}_{\text{Sing}(R)}/\mathcal{I}_R(3)) \longrightarrow H^0(\mathcal{I}_{\text{Sing}(R)/R}(3)) \longrightarrow H^1(\mathcal{I}_R/\mathcal{I}_R(3)) \longrightarrow 0.
\]
Clearly \( h^0(\mathbb{P}^5, \mathcal{I}_{\text{Sing}(R)}/\mathcal{I}_R(3)) = \binom{8}{3} - 3 = 53 \). Furthermore, we have the following identification of linear systems:
\[
(6) \quad \pi^* \left( \mathcal{I}_{\text{Sing}(R)/R}(3) \right) = \mathcal{I}_{\{ x_1, y_1, x_2, y_2, x_3, y_3 \}/R'}(12\ell - 9E).
\]
The scroll $[R] \in \mathcal{H}_{\text{scr}}$ is obtained as a general projection like in Lemma 3.1. In particular, the points $\{x_i, y_i\}_{i=1}^3 \subseteq R'$ are general as well, hence impose independent conditions on the linear system $|12\ell - 9E|$ on $R'$. Using the identification (6), we compute:

$$h^0(R, \mathcal{I}_{\text{Sing}(R)/R}(3)) = h^0(R', \mathcal{O}_{R'}(12\ell - 9E)) - 6 = h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(12)) - \binom{10}{2} - 6 = 40.$$  

Therefore $h^0(\mathbb{P}^5, \mathcal{I}_{R/\mathbb{P}^5}(3)) = 13$ if and only if $h^1(\mathbb{P}^5, \mathcal{I}_{R/\mathbb{P}^5}(3)) = 0$. This last statement can be proved via a simple Macaulay calculation by choosing the points $a_1, a_2, a_3$ randomly in the variety $\text{Sec}(R')$ whose equations have been given explicitly. \hfill \Box

**Remark 3.5.** It is possible to realize the rational curve $\Gamma_R$ inside the linear system $|\mathcal{O}_R(1)|$ as follows. Recall that we have denoted by $\phi : F_1 \hookrightarrow \mathbb{P}^5$ the embedding whose image is the smooth scroll $R'$. In $|4\ell - 3E| \cong \mathbb{P}^5$, we consider the space of reducible hyperplane sections:

$$\left\{ A' + L' : A' \in [3\ell - 2E], L' \in [\ell - E] \right\}.$$  

Note that $L'$ is a ruling of $R'$, whereas $A' \subseteq \mathbb{P}^5$ is a sextic with $\langle A' \rangle = \mathbb{P}^6$ and with $L' \cdot A' = 1$. In the linear system $|3\ell - 2E|$ there exists a unique sextic $A_0'$ such that $\Lambda \subseteq \langle A_0' \rangle \subseteq \mathbb{P}^5$. The curve $A_0'$ corresponds to the unique curve in the linear system

$$\mathcal{I}_{(x_1, y_1, x_2, y_2, x_3, y_3)/R'(3\ell - 2E)}$$  

on $R'$. Indeed, $x_i, y_i \in A_0'$, therefore $a_i \in \langle x_i, y_i \rangle \subseteq \langle A_0' \rangle$, for $i = 1, 2, 3$. It then follows that $\Lambda = \langle a_1, a_2, a_3 \rangle \subseteq \langle A_0' \rangle$. The projection $A_0 := \pi(A_0') \subseteq \mathbb{P}^5$ is a sextic curve on $R$ passing through the nodes $p_1, p_2, p_3$. One identifies $\Gamma_R$ with $A_0$ via the map $L \mapsto L \cdot A_0$.

We denote by $\mathcal{H}_{\text{scr}}$ the Hilbert scheme of 3-nodal septic scrolls in $R \subseteq \mathbb{P}^5$ and set

$$\mathcal{H}_{\text{scr}} := \mathcal{H}_{\text{scr}}/\text{PGL}(6).$$

We regard $\mathcal{H}_{\text{scr}}$ as the coarse moduli space of 3-nodal septic scrolls.

**Proposition 3.6.** The parameter space $\mathcal{H}_{\text{scr}}$ is birationally isomorphic to $\text{Sym}^3(\text{Sec}(R'))/\text{Aut}(R')$. In particular, $\mathcal{H}_{\text{scr}}$ is a unirational 9-dimensional variety.

**Proof.** We identify $\text{Aut}(R')$ with the group consisting of linear automorphisms $\sigma \in \text{PGL}(9)$ such that $\sigma(R') = R'$. Every $\sigma \in \text{Aut}(R')$ clearly invariates $\text{Sec}(R')$. Since $\text{Sing}(\text{Sec}(R')) = R'$, conversely, every automorphism $\sigma \in \text{PGL}(9)$ inverting $\text{Sec}(R')$ belongs actually to $\text{Aut}(R')$. One has a birational action of $\text{Aut}(R')$ on $\text{Sym}^3(\text{Sec}(R'))$ given by

$$\sigma(a_1, a_2, a_3) := (\sigma(a_1), \sigma(a_2), \sigma(a_3)),$$

for $\sigma \in \text{Aut}(R')$ and $a_1, a_2, a_3 \in \text{Sec}(R')$. We can now define a birational morphism

$$\vartheta : \text{Sym}^3(\text{Sec}(R'))/\text{Aut}(R') \dasharrow \mathcal{H}_{\text{scr}},$$  

by setting

$$\Lambda := \langle a_1, a_2, a_3 \rangle \mapsto \pi_{\Lambda}(R') \mod \text{PGL}(6),$$

where $\pi_{\Lambda} : \mathbb{P}^5 \dasharrow \mathbb{P}^5$ is a projection of center $\Lambda$. The assignment is clearly $\text{Aut}(R')$-invariant, hence $\vartheta$ is well-defined. Applying Lemma 3.1, we obtain that $\vartheta$ is a birational isomorphism.

The secant variety $\text{Sec}(R')$ being a $\mathbb{P}^1$-bundle over the rational variety $\text{Sym}^2(R')$ is unirational. This implies that $\text{Sym}^3(\text{Sec}(R'))$ and thus $\mathcal{H}_{\text{scr}}$ are unirational as well. \hfill \Box
Over the Hilbert scheme $\mathcal{H}_{\text{scr}}$ we consider the universal family of scrolls:
\[
\mathcal{H}_{\text{scr}} \xleftarrow{p_1} \mathcal{Y}_{\text{scr}} \xrightarrow{p_2} \mathbb{P}^5.
\]

We introduce the incidence correspondence between cubic fourfolds of discriminant 26 and nodal septic scrolls in $\mathbb{P}^5$:
\[
|O_{\mathbb{P}^5}(3)| \xleftarrow{} X := \mathbb{P}\left((p_1)_*\left(\mathcal{I}_{\mathcal{Y}_{\text{scr}}/\mathcal{H}_{\text{scr}} \times \mathbb{P}^5} \otimes p_2^*O_{\mathbb{P}^5}(3)\right)\right) \longrightarrow \mathcal{H}_{\text{scr}}
\]

The group $PGL(6)$ acts on the entire diagram. By quotienting out this action, if we set $X := X/PGL(6)$, we obtain two projections:
\[
C_{26} \xleftarrow{\pi_1} X \xrightarrow{\pi_2} \mathcal{H}_{\text{scr}}
\]

The 21-dimensional variety $X$ being a $\mathbb{P}^{12}$-bundle over the unirational variety $\mathcal{H}_{\text{scr}}$ is unirational as well. A general scroll $[R] \in \mathcal{H}_{\text{scr}}$ has precisely 3 non-normal nodes. Checking that a general cubic fourfold $X \supseteq R$ is smooth, reduces to a standard Macaulay calculation. Applying (5), we obtain that the lattice $A(X)$ contains a 2-dimensional lattice $\langle h^2, [R] \rangle$ of discriminant 26, therefore the map $\pi_1$ is well-defined. Proposition 2.1 implies $\dim \pi_1^{-1}([X]) \leq 2$, for all $[X] \in C_{26}$, hence $X$ dominates $C_{26}$. In fact one can prove something more precise and establish in the process Theorem 1.2.

**Theorem 3.7.** The incidence correspondence $X$ is birational to the universal K3 surface $F_{14,1}$.

**Proof.** We define a map $\theta : X \to F_{14,1}$ as follows. We start with a pair $[X, R] \in X$ and denote by $f_R : \mathbb{P}^1 \to F(X)$ the rational curve of rulings described in Proposition 3.3. Denoting by $[S, H] := \phi^{-1}([X]) \in F_{14}$ the polarized K3 surface provided by the identification (3), applying Proposition 2.1 there exists a uniquely determined point $p \in S$ such that $\Delta_p = \Gamma_R$.

The map $\theta$ is clearly generically injective. Since both $X$ and $F_{14,1}$ are irreducible varieties of the same dimension 21, it follows that $\theta$ is birational. In particular, in the isomorphism $S \approx \mathcal{M}_7(X)$ constructed in Corollary 2.4 the general point on both sides corresponds to a septic scroll $R \subseteq X$ which is 3-nodal and has no further singularities.

4. The rationality of $F_{14,1}$

In this section, using in an essential way the characterization given in Proposition 3.3 of the rational curves $\Gamma_R$ of rulings of 3-nodal scrolls $R \subseteq \mathbb{P}^5$, we show that the universal K3 surface of genus 14 is rational.

We begin by recalling the structure of the moduli space of curves of genus 8. Consider the Grassmannian $G(1, 5) \subseteq \mathbb{P}^{14}$ in its Plücker embedding. Denote by
\[
\mathcal{M}_8 := G\left(7, \mathbb{P}\left(\bigwedge^2 \mathcal{O}\right)\right)/PGL(6)
\]

the space of codimension 7 linear sections of $G(1, 5)$. Mukai [M6] has shown that the map
\[
\mathcal{M}_8 \longrightarrow \mathcal{M}_8,
\]

sending a general 7-plane $[\mathbb{P}(V) \subset \mathbb{P}^{14}] \in \mathcal{M}_8$ to the intersection $[G(1, 5) \cdot \mathbb{P}(V)] \in \mathcal{M}_8$ viewed as a canonical curve of genus 8, is a birational isomorphism. For more details on how to extend Mukai’s isomorphism over parts of the boundary of $\mathcal{M}_8$, see also [PV2].

Recall that we introduced in Section 3 the smooth septic scroll $R' \cong F_1 \subseteq \mathbb{P}^5$, then considered a singular scroll $R \subseteq \mathbb{P}^5$, defined as the image of a linear projection $\pi_A : R' \to \mathbb{P}^5$
whose center is a general plane \( \Lambda \subset \mathbb{P}^8 \), which is 3-secant to \( \text{Sec}(R') \). We denote by \( p_1, p_2, p_3 \) the three nodes of \( R \) and \( \{ x_i, y_i \} = \pi^{-1}(p_i) \). As explained in the Introduction, \( P_i \subset \mathbb{P}^9 \) denotes the 2-plane spanned by the rulings of \( R \) passing through \( p_i \), for \( i = 1, 2, 3 \). The line \( L_i \subset \mathbb{G}(1, 5) \subset \mathbb{P}^{14} \) parametrizes the lines in the plane \( P_i \) passing through the point \( p_i \). If \( \Gamma = \Gamma_R \subset \mathbb{G}(1, 5) \) is the curve of rulings associated to \( R \) introduced in Proposition 3.3, then \( L_i \) meets \( \Gamma \) in two distinct points. We keep this notation throughout this section.

Due to the results of the previous section, our strategy is now to describe the family

\[ \mathcal{U} \subset \text{Hom}(\mathbb{P}^1, \mathbb{G}(1, 5)) \]

of smooth rational septic curves \( \Gamma_R \subset \mathbb{G}(1, 5) \) carrying three bisecant lines contained in \( \mathbb{G}(1, 5) \). From Proposition 3.3 it follows that \( \mathcal{U} \) is birational to the Hilbert scheme \( \mathcal{H}_{\text{scr}} \) of 3-nodal septic scrolls in \( \mathbb{P}^5 \). Then we show that the quotient \( \mathcal{U}/\text{PGL}(6) \) is rational. Since \( \mathcal{U}/\text{PGL}(6) \) is birational to \( \mathcal{H}_{\text{scr}} \) and, as proven in Theorem 1.2, the universal \( K3 \) surface of genus 14 is a \( \mathbb{P}^{12} \)-bundle over \( \mathcal{H}_{\text{scr}} \), its rationality will follow.

The nodal curve \( \Gamma + L_1 + L_2 + L_3 \subset \langle \Gamma \rangle \cdot \mathbb{G}(1, 5) \) has arithmetic genus 3. It follows from Mukai’s work [MI] that the intersection \( \langle \Gamma \rangle \cdot \mathbb{G}(1, 5) \) is a canonical curve of genus 8, provided (i) it is proper and reduced and (ii) \( \dim \langle \Gamma \rangle = 7 \). Using the surjectivity of the period map for polarized \( K3 \) surfaces of genus 8, we shall show that both assumptions (i) and (ii) are satisfied. Granting both (i) and (ii) for the moment, we consider the canonically embedded curve in \( \langle \Gamma \rangle = \mathbb{P}^7 \), pictured also below:

\[ C := \langle \Gamma \rangle \cdot \mathbb{G}(1, 5) = Q + \Gamma + L_1 + L_2 + L_3. \]

\[ \text{Figure 1. The canonical curve } C = \Gamma + Q + L_1 + L_2 + L_3. \]

Bertini’s Theorem implies that a general 8-dimensional space \( \langle \Gamma \rangle \subset \mathbb{P}^8 \subset \mathbb{P}^{14} \) cuts out on \( \mathbb{G}(1, 5) \) a smooth 2-dimensional linear section \( T \), see also [Ve1], Propositions 3.2 and 3.3. By the adjunction formula, \( T \hookrightarrow \mathbb{P}^8 \) is a smooth \( K3 \) surface (of genus 8) polarized by \( \mathcal{O}_T(C) \). We now describe the Picard lattice of \( T \):

**Lemma 4.1.** One has the following intersection products on \( T \):

\[ Q^2 = -2, \quad Q \cdot \Gamma = 3, \quad Q \cdot L_i = 1, \quad \Gamma \cdot L_i = 2, \quad L_i \cdot L_j = -2 \delta_{ij}, \quad \text{for } i, j = 1, 2, 3. \]
The generality assumptions ensure that $L_i$ and $L_j$ are disjoint lines, for $i \neq j$. Else, if $L_i \cap L_j \neq \emptyset$, then $\langle p_i, p_j \rangle \subseteq P_i \cap P_j \subseteq \mathbb{P}^5$. It follows that the four rulings of $R'$ passing through the points $x_i, y_i, x_j, y_j$ respectively, span a 6-dimensional space in $\mathbb{P}^8$, which is impossible for

$$h^0\left( R', \mathcal{O}_{R'}(1)(-4(\ell - E)) \right) = h^0(R', \mathcal{O}_{R'}(E)) = 1,$$

where recall that $\ell, E \in \text{Pic}(R')$ denote the line class and the exceptional divisor respectively. This implies that there exists a unique hyperplane in $\mathbb{P}^8$ containing the four rulings, therefore they must span a 7-dimensional linear space.

Since $L_i^2 = -2$, by intersecting (7) with $L_i$, we obtain $Q \cdot L_i = 1$. Furthermore $7 = \Gamma \cdot C$ and since $\Gamma^2 = -2$, we obtain $\Gamma \cdot Q = 3$. Finally, $C \cdot Q = \deg(Q) = 4$, therefore $Q^2 + \Gamma \cdot Q + 3 = 4$, implying $Q^2 = -2$ and thus finishing the proof.

In particular $Q \subseteq \langle T \rangle = \mathbb{P}^8$ is a reduced, connected quartic curve of arithmetic genus zero. Since $C - Q \equiv \Gamma + L_1 + L_2 + L_3$, we obtain $h^0(T, \mathcal{O}_T(C - Q)) = 4$. The next lemma summarizes the situation.

**Lemma 4.2.** The span $\langle Q \rangle$ is 4-dimensional and $Q$ is a connected nodal quartic curve with $p_a(Q) = 0$.

In fact, we shall construct a $K3$ surface $T$, such that the curve $Q$ described in Lemma 4.2 is actually smooth.

To establish the validity of the assumptions (i) and (ii) and thus the existence of the special $K3$ surface $T$, we use Hodge theory. We consider the following sublattice

$$\mathbb{L} := Z \cdot [Q] \oplus Z \cdot [\Gamma] \oplus Z \cdot [L_1] \oplus Z \cdot [L_2] \oplus Z \cdot [L_3]$$

generated by the $(-2)$ classes corresponding to $Q, \Gamma, L_1, L_2$ and $L_3$ respectively, and with intersection pairing as given in Lemma 4.1. We invoke the surjectivity of the period map for $K3$ surfaces. The rank 5 lattice $\mathbb{L}$ is even and has signature $(1, 4)$. Applying [Mo] Corollary 2.9, there exists a smooth $K3$ surface $T$, such that $\text{Pic}(T) \cong \mathbb{L}$. We define the following class on $T$

$$C := \Gamma + Q + L_1 + L_2 + L_3.$$

The genus zero curves $\Gamma, Q, L_1, L_2, L_3 \subseteq T$ cannot have multiple components, for that would make $\text{Pic}(T)$ larger than $\mathbb{L}$, therefore they are all smooth, rational curves on $T$.

**Lemma 4.3.** The linear system $|\mathcal{O}_T(C)|$ is very ample.

**Proof.** We use Reider’s Theorem [R], which, in the case of $K3$ surfaces, had been proven before in [SD]. It suffices to show that there exists no curve $E$ on $T$ with $E^2 = 0$ and $E \cdot C \in \{1, 2\}$, nor a curve $F$ on $T$ with $F^2 = -2$ and $F \cdot C = 0$. We prove the first statement, the second follows similarly. Assuming there is such a curve $E$, we express it as an integral combination $E \equiv x\Gamma + yQ + z_1L_1 + z_2L_2 + z_3L_3$ of the generators of $\text{Pic}(T)$. If $C \cdot E = 1$, we obtain

$$-15x^2 - 12xy - 5y^2 + 2x + y = z_1^2 + z_2^2 + z_3^2.$$

By comparing the signs of the two sides, one concludes that this equation has no integral solutions. The case $C \cdot E = 2$ is similar. Finally, if $F \equiv x\Gamma + yQ + z_1L_1 + z_2L_2 + z_3L_3$ is a $(-2)$-curve with $C \cdot F = 0$, we obtain

$$-15x^2 - 12xy - 5y^2 + 1 = z_1^2 + z_2^2 + z_3^2,$$

which implies $x = y = 0$ and, say $z_2 = z_3 = 0$ and then $z_1 = 1$. Thus $F = L_1$, but $C \cdot L_1 = 1$, hence this case does not appear. We conclude that $C$ is very ample. □
We show that the $K3$ surface $T$ constructed in Lemma 4.3 is a linear section of $G(1,5)$. In particular, Mukai's results [M6] will apply for its hyperplane section $C$.

**Proposition 4.4.** The $K3$ surface $T$ carries a globally generated rank two vector bundle $T$ with \( \det(T) = O_T(C) \), providing an embedding $T \hookrightarrow G(1,5)$ such that

\[
\langle T \rangle \cdot G(1,5) = S.
\]

**Proof.** We use [M7] and need to show that the polarized $K3$ surface $(T, O_T(C))$ is Brill-Noether general, that is, for all pairs of line bundles $M, N$ on $T$ such that $M \otimes N = O_T(C)$, one has $h^0(T, M) \cdot h^0(T, N) < h^0(T, C)$. Under these circumstances, it is shown in loc.cit. that $T$ carries a rigid, globally generated, stable rank 2 vector bundle $E$ with $h^0(T, E) = 6$ and $\det(E) = O_T(C)$, inducing a map $\varphi_E : T \to G(1,5)$. Reasoning along the lines of [M7] Theorem 3.10, the $K3$ surface $T$ is then a linear section of $G(1,5)$ in its Plücker embedding, that is, $T = G(1,5) \cdot \langle T \rangle$.

To establish the Brill-Noether generality of $(T, O_T(C))$, we use for instance [GLT] Lemma 2.8. It suffices to show that in the lattice $L$ there exists no vector $D$ such that $D^2 = 2$ and $D \cdot C \in \{7, 6\}$, nor is there a vector $D$ with $D^2 = 0$ and $D \cdot C \leq 4$.

We treat in detail only the first case, the remaining ones being similar. We write

\[
D \equiv x\Gamma + yQ + z_1L_1 + z_2L_2 + z_3L_3.
\]

The conditions $D^2 = 2$ and $D \cdot C = 7$ translate into the equalities $z_1 + z_2 + z_3 + 7x + 4y = 7$ and $-15x^2 - 5y^2 - 12xy + 14x + 7y + 1 = z_1^2 + z_2^2 + z_3^2 \geq 0$. It is elementary to see that there are no integral solutions.

Using Proposition 4.4, we conclude that the intersection (\ref{7}) corresponding to a general curve $\Gamma_R \in \mathcal{U}$ corresponds to a semistable canonical curve of genus 8.

It will be useful to have a criterion for determining when the curve $\Gamma$ spans a space of maximal possible dimension in the Plücker space $P^{14} \subseteq G(1,5)$. To that end, recall that the Plücker embedding of the dual Grassmannian $G(1,5)^\vee = G(3,5) \hookrightarrow (P^{14})^\vee$ assigns to a point $p \in G(1,5)^\vee$ corresponding to a 3-plane $P^3_p \subseteq P^9$ the Schubert cycle

\[
\sigma_p := \{ \ell \in G(1,5) : \ell \cap P^3_p \neq \emptyset \}.
\]

Note that $\dim(\langle \Gamma \rangle) + 1 = \text{codim}(\Gamma)^\perp$. Setting

\[
W^1(\Gamma) := G(3,5) \cap \langle \Gamma \rangle^\perp = \{ p \in G(3,5) : \Gamma \subseteq \sigma_p \},
\]

for dimension reasons, the next lemma follows immediately:

**Lemma 4.5.** Assume $W^1(\Gamma)$ is finite. Then $\dim(\langle \Gamma \rangle) = 7$.

Keeping the previous notation, let $f_R : P^1 \to G(1,5)$ be a sufficiently general element of $\mathcal{U}$ and set again $\Gamma = \Gamma_R$. Then under the assumption $R^I = S_{3,4}$, we can prove that:

**Theorem 4.6.** The set $W^1(\Gamma)$ is finite. In particular $\dim(\langle \Gamma \rangle) = 7$ and $\Gamma$ is a rational normal septic curve.

**Proof.** If $p \in W^1(\Gamma)$, then $P^3_p$ contains an integral curve intersecting each line of $R$. Its strict transform by $\pi_\Lambda : R' \to R$ is an integral section $A$ of the ruled surface $R'$. Set $d := \deg(A)$, hence $A \equiv (d - 3)\ell - (d - 4)E \in \text{Pic}(F_1)$. Clearly $\langle A \rangle \subseteq \pi_{\Lambda}^{-1}(P^3_p)$, implying $\dim(\langle A \rangle) \leq 6$. 


Let $I_A := |H - A|$ be the linear system of hyperplanes in $P^5$ containing the curve $A \subseteq R'$. By direct calculation, we find $\dim(I_A) = \dim |H - A| = 7 - d \geq 1$ and $\dim |A| = 2d - 6$. It follows that $3 \leq d \leq 6$. Recalling that $V = H^0(P^5, I_A/P^8(1))$, the condition

$$\dim(\mathbb{P}V \cap I_A) \geq 1$$

is equivalent to the condition that the curve $\pi(A)$ be contained in a $3$-space $P^3$. For $3 \leq d \leq 6$ let $G(7 - d, |H|)$ denote the Grassmannian of $(7 - d)$-subspaces of $|H| \cong P^8$ and introduce the $(2d - 6)$-dimensional variety

$$S_d := \left\{ I_{A'} \in G(7 - d, |H|) : A' \in |(d - 3)\ell - (d - 4)E| \right\}.$$  

For an integer $k \geq 1$, we consider the Schubert cycle

$$\sigma^k \cdot S_d := \{ I \in G(7 - d, |H|) : \dim(\mathbb{P}V \cap I) \geq k \}.$$  

The cycle $\sigma^k \cdot S_d$ is finite for $k = 1$ and empty for $k \geq 2$, provided the intersection is proper. By Kleiman’s transversality of a general translate this is true for a general translate of $\sigma^k \cdot S_d$ in $G(d - 7, |H|)$, that is, for a general choice of $\Lambda$ (or equivalently, of $V$). Hence $W^1(\Gamma)$ is finite. □

**Remark 4.7.** The theorem above fails for rational septic scrolls in $P^8$ containing sections of degree $d \leq 2$, that is, for the scrolls $S_a, 7 - a$, where $a \neq 3$.

We turn to the smooth residual rational curve $Q \subseteq G(1, 5)$ defined by (7). Let

$$R_Q \subseteq P^5$$

be the quartic scroll whose rulings are parametrized by the curve $Q$.

**Lemma 4.8.** $R_Q$ is a non-degenerate smooth rational normal scroll in $P^5$.

**Proof.** First, observe that $R_Q$ cannot be a cone. Let us assume $R_Q$ is a cone of vertex $v \in P^5$. Then $\langle Q \rangle \cong P^4 \subseteq G(1, 5)$ parametrizes the lines passing through $v$. This is a contradiction because $\langle Q \rangle \subseteq \langle \Gamma \rangle$:$G(1, 5) = C$. Now assume that $R_Q$ is contained in a hyperplane $H \subseteq P^5$. Then $Q$ is contained in the Grassmannian $G_H := G(1, H) \subseteq G(1, 5)$ of lines of $H$. Since $K_{G_H} = O_{G_H}(-5)$, we observe that, by adjunction, the curvilinear sections of $G_H$ are curves of arithmetic genus $1$. Because of this fact and since $\deg(G_H) = 5$, it follows that

$$\langle Q \rangle \cdot G_H = Q + L \subseteq C,$$

where $L$ is a bisecant line to $Q$. But the only line components in $C$ are $L_1, L_2, L_3$ and none of them is bisecant to $Q$. Via Proposition 2.3, the same argument shows that the scroll $R_Q$ has no incident rulings, therefore $R_Q$ is smooth. □

**Lemma 4.9.** The scroll $R_Q$ contains no other lines except the ruling parametrized by $Q$.

**Proof.** Assume $R_Q$ contains a line $\ell_0$ not parametrized by a point of $Q$. We prove that this implies that $W^1(\Gamma)$ is not finite, thus contradicting Theorem 4.6. Consider the family $G$ of codimension $1$ Schubert cycles $\sigma_p$ defined by $3$-space $P^3 \cong \ell_0$. Note that $G \cong G(1, 3)$. We have $G \subseteq \langle Q \rangle$. Since $\langle Q \rangle \subseteq \langle \Gamma \rangle$, we also have $\langle \Gamma \rangle^\perp \subseteq \langle Q \rangle^\perp$. Counting dimensions it follows

$$\dim(G \cap \langle \Gamma \rangle^\perp) \geq 1,$$

which implies that $W^1(\Gamma)$ is not finite. □

There are two types of smooth quartic scrolls in $P^5$, namely $S_{1,3} = P(O_{P^1}(1) \oplus O_{P^1}(3))$ and $S_{2,2} = P(O_{P^1}(2) \oplus O_{P^1}(2))$. The latter case is characterized by the property that every line contained in the scroll is a ruling. Lemma 4.9 implies the following:
Theorem 4.10. Let $\Gamma \subseteq G(1,5)$ be a smooth septic rational curve corresponding to a general element of $\mathcal{U}$ and $Q \subseteq G(1,5)$ the residual quartic curve. Then $R_Q$ is isomorphic to $S_{2,2}$.

To summarize, to a general rational curve $\Gamma = \Gamma_R \in \mathcal{U}$, we associated the quartic scroll $R_Q$, equipped with three rulings $\ell_1, \ell_2, \ell_3$ corresponding to the points $L_i : Q \in G(1,5)$, for $i = 1, 2, 3$. Each ruling $\ell_i$ passes through the node $p_i$ of the scroll $R$ and is contained in the 2-plane $P_i$ whose existence is established in Proposition 3.

To prove the rationality of $S_{\text{scr}}$ and thus that $F_{14,1}$, we reverse this construction. We denote by $\mathcal{V}$ the variety classifying elements $(R_Q, p_1, p_2, p_3)$, where $R_Q \subseteq P^5$ is a smooth quartic scroll isomorphic to $S_{2,2}$ and $p_i \in R_Q$ for $i = 1, 2, 3$.

Lemma 4.11. The $PGL(6)$-stabilizer of a general point $(R_Q, p_1, p_2, p_3) \in \mathcal{V}$ is trivial. In particular, $PGL(6)$ acts transitively on $\mathcal{V}$.

Proof. The automorphism group of $S_{2,2} \cong F_0$ is the semidirect product of $PGL(2) \times PGL(2)$ with $\mathbb{Z}/2\mathbb{Z}$. The last factor corresponds to the automorphism $u \in \text{Aut}(F_0)$ permuting the two factors. In particular, $\text{Aut}(S_{2,2})$ is 6-dimensional. This implies that the space $\mathcal{V}$ has dimension

$$\dim PGL(6) - \dim \text{Aut}(S_{2,2}) + 3\dim(R_Q) = 35 = \dim PGL(6).$$

Choose general points $p_i = (a_i, b_i) \in F_0 \cong S_{2,2}$, with $a_i \neq b_i$, for $i = 1, 2, 3$. Up to the action of $u \in \text{Aut}(F_0)$, the stabilizer $\text{Stab}_{PGL(6)}(R_Q, p_1, p_2, p_3)$ corresponds to pairs of automorphism $(\sigma_1, \sigma_2) \in PGL(2) \times PGL(2)$, such that $\sigma_1(a_i) = a_i$ and $\sigma_2(b_i) = b_i$. Thus $\sigma_1 = \sigma_2 = 1$. The points $p_i$ not lying on the diagonal of $F_0$, the automorphism $u$ does not fix any of them, thus the stabilizer in question is trivial. Since $\mathcal{V}$ and $PGL(6)$ have the same dimension, this also implies the transitivity of the $PGL(6)$-action on $\mathcal{V}$, as claimed.

We can thus start by fixing once and for all the quartic scroll $R_Q$. Precisely, we embed the surface $F_0 := P^1 \times P^1$ in $P^5$ via the linear system $|O_{F_0}(1,2)|$ and denote by

$$R_Q \subseteq P^5$$

the image quartic scroll. The rulings on $R_Q$ are the elements of the linear system $|O_{F_0}(0,1)|$. Let $Q_0 \subseteq G(1,5)$ be the curve of rulings of $R_Q$. We then fix three points in $F_0$, for instance

$$a_1 := ([1:0],[0:1]), \quad a_2 := ([0:1],[1:0]) \quad \text{and} \quad a_3 := ([1:1],[-1:-1]),$$

which we identify with their images in $R_Q$. As explained in Lemma 4.11, the stabilizer subgroup $G$ of $PGL(6)$ fixing both $R_Q$ as well as the set $\{a_1, a_2, a_3\}$ is isomorphic to the subgroup of $PGL(2) \times PGL(2)$ fixing the set $\{a_1, a_2, a_3\}$. Therefore $G = \mathcal{S}_3$.

For $i = 1, 2, 3$, we denote by $\ell_i$ the ruling of $R_Q$ passing through the point $a_i$. Then, let $P^3_i$ be the projective space consisting of 2-planes $\Pi_i \subseteq P^5$ containing the line $\ell_i$. Giving a plane $\Pi_i$ is equivalent to specifying a line $L_i \subseteq G(1,5)$ in the Plücker embedding of the Grassmannian. Note that $L_i$ meets $Q_0$ transversally at precisely one point, namely $\ell_i \in G(1,5)$.

We introduce a rational map

$$\varphi : P^3_1 \times P^3_2 \times P^3_3 / \mathcal{S}_3 \longrightarrow S_{\text{scr}}$$

defined as follows. To a triple of planes $((\Pi_1, \Pi_2, \Pi_3))$, we attach the lines $L_1, L_2, L_3 \subseteq G(1,5)$. Since $Q_0 \subseteq G(1,5)$ is a smooth rational quartic curve, in the Plücker embedding we have that $\langle Q_0 \rangle \cong P^4$. Attaching one general 1-secant line to $Q_0$ increases the dimension of the linear span of the union by one, therefore by attaching three general 1-secant lines, we have

$$\langle Q_0 + L_1 + L_2 + L_3 \rangle \cong P^7 \subseteq P^{14}.$$
We write
\[(Q_0 + L_1 + L_2 + L_3) \cdot G(1, 5) = Q_0 + L_1 + L_2 + L_3 + \Gamma,
\]
where \(\Gamma\) is a degree 7 curve. Applying Lemma 4.11 it follows that \(\Gamma\) is a rational curve and
\[\Gamma \cdot L_i = 2,\text{ for } i = 1, 2, 3.\]
We denote by \(L'_i\) and \(L''_i\) the intersection points \(L_i \cdot \Gamma\). From Proposition 3.3 it follows that the scroll \(R := R_\Gamma\) induced by \(\Gamma\) is 3-nodal, with nodes given by the
intersection \(L'_i \cap L''_i\) taken in the 2-plane \(\Pi_i\). We set
\[\varkappa(\Pi_1 + \Pi_2 + \Pi_3) := [R].\]

We conclude the proof of the rationality of the Hilbert scheme of 3-nodal scrolls in \(P^5\):

**Proof of Theorem 1.3.** We first observe that \(\varkappa\) is well-defined. To that end, we choose the polarized \(K3\) surface \((T, O_T(C))\) constructed in Propositions 4.3 and 4.4 and we keep the notation used there. Applying Theorem 4.10, the residual quartic rational curve \(Q \subseteq G(1, 5)\) parametrizes the rulings of a quartic scroll \(R_Q \subseteq P^5\), which is isomorphic to \(S_{2,2}\). Applying Lemma 4.11, there exists a unique automorphisms \(\sigma \in PGL(6)\) such that \(\sigma(R_Q) = R_\Gamma\) and \(\sigma(p_i) = a_i\), for \(i = 1, 2, 3\). Set \(\sigma(P_i) =: \Pi_i \in P^3\) and then \(\varkappa(\Pi_1 + \Pi_2 + \Pi_3) = [R_\Gamma]\).

To finish the proof it suffices to observe that \(\varkappa\) is generically injective. A general septic curve \(\Gamma \in U\) corresponding to a 3-nodal septic scroll \([R_\Gamma]\) in \(P^5\) has precisely 3 bisecant lines lying in \(G(1, 5)\). Giving \(\Gamma\) determines its linear span \(\langle \Gamma \rangle\), hence the set \(\{L_1, L_2, L_3\}\) as well. \(\square\)

5. The unirationality of the universal \(K3\) surface of genus at most 12

We denote by \(F_{g,n}\) the universal \(n\)-pointed \(K3\) surface of genus \(g\). Thus \(F_{g,n}\) is an irreducible variety of dimension \(19 + 2n\). Similarly, one can consider the universal Hilbert scheme of \(0\)-dimensional cycles of length \(n\), that is, \(u^{[n]} : F_{g}^{[n]} \rightarrow F_{g}\). We also introduce the notation \(\mathcal{C}_{g,n} := \mathcal{M}_{g,n}/\mathfrak{S}_n\) for the degree \(n\) universal symmetric product over \(\mathcal{M}_g\), where the symmetric group \(\mathfrak{S}_n\) acts by permuting the marked points.

The aim of this short last section is to point out how Mukai’s results determine the birational type of \(F_{g,n}\) and that of \(F_{g}^{[n]}\) for small \(g\), and thus put our Theorem 1.1 better into context:

**Theorem 5.1.** The following results on the Kodaira dimension of \(F_{g,n}\) hold:

(i) \(F_{g, g+1}\) is unirational for \(g \leq 10\).

(ii) \(F_{11,1}\) is unirational. The Kodaira dimension of both \(F_{11,11}\) and \(F_{11}^{[11]}\) equals 19.

**Proof.** For \(g \leq 5\), the general \(K3\) surface of genus \(g\) is a complete intersection in a projective space and the result follows easily. For details, see the table after Theorem 1.10 in [M7].

For \(6 \leq g \leq 10\), Mukai [M1] has constructed a rational homogeneous variety \(V_g \subseteq P^{N_g}\), where \(N_g = g + \dim(V_g) - 2\), such that the general \(K3\) surface of genus \(g\) is obtained as a general linear section \(S = V_g \cap \mathcal{A}_g\), where \(\mathcal{A}_g \subseteq P^{N_g}\) is a \(g\)-dimensional plane, with the polarization being the one induced by \(O_{P^{N_g}}(1)\). Moreover, one has the following birational isomorphism, see [M1] Corollary 0.3:

\[F_g \cong G(g, N_g)/Aut(V_g)\]

These results imply the existence of a dominant map \(\chi_g : V_g^{g+1} \rightarrow F_{g,g+1}\) given by
\[\chi(x_1, \ldots, x_{g+1}) := [V_g \cap \langle x_1, \ldots, x_{g+1}\rangle, x_1, \ldots, x_{g+1}]\].

This proves that \(F_{g,g+1}\) (and hence \(F_{g,n}\) for \(n \leq g + 1\)) is unirational in this range.
For $g = 11$, we use [M8], where it is shown that a general curve $[C] \in \mathcal{M}_{11}$ lies on a unique $K3$ surface $C \subseteq S$ as a hyperplane section, with $\text{Pic}(S) = \mathbb{Z} \cdot C$. This implies the existence of a rational map $\chi_n : \mathcal{M}_{11,n} \dashrightarrow \mathcal{F}_{11,n}$ defined by

$$\chi_n([C, x_1, \ldots, x_n]) := [S, x_1, \ldots, x_n].$$

The map $\chi_n$ is dominant for $n \leq 11$ and a birational isomorphism for $n = 11$. Indeed, in this last case, given an embedded $K3$ surface $S \overset{[H]}{\rightarrow} \mathbb{P}^{11}$ and general points $x_1, \ldots, x_{11} \in S$, the hyperplane $\langle x_1, \ldots, x_{11} \rangle \cong \mathbb{P}^{10}$ cuts out a canonical genus 11 curve $C$ on $S$, which comes equipped with the marked points $x_1, \ldots, x_{11}$. By quotienting the action of the symmetric group $S_{11}$, the map $\chi_{11}$ induces a birational isomorphism between the universal symmetric product $\mathcal{C}_{11,11}$ and $\mathcal{F}_{11}$. Now we use [FV1] Theorem 0.5. Both varieties $\mathcal{M}_{11,11}$ and $\mathcal{C}_{11,11}$ have Kodaira dimension 19, hence we conclude.

We now pass on to the universal $K3$ surface $\mathcal{F}_{11,1}$. To that end we define a rational map

$$\vartheta : \mathcal{M}_{10,2} \dashrightarrow \mathcal{F}_{11,1},$$

associating to a 2-pointed curve $[C, p_1, p_2] \in \mathcal{M}_{10,2}$, the unique $K3$ surface $S$ of genus 11 containing the curve $[X := C/p_1 \sim p_2]$ obtained from $C$ by identifying $p_1$ and $p_2$. To show that $\vartheta$ is well-defined, that is, Mukai's construction [M8] can be also carried out for the 1-nodal curve $[X] \in \mathcal{M}_{11}$, we use [CLM] Proposition 4.4. Observe that the $K3$ surface $S$ has a distinguished point corresponding to the image of the singularity of $X$. The map $\vartheta$ is clearly dominant, for in each linear system on a $K3$ surface, the 1-nodal curves fill-up a divisor. The unirationality of $\mathcal{F}_{11,1}$ now follows from that of $\mathcal{M}_{10,2}$, which can be established in a variety of ways, see for instance [BCF] Theorem B.

\[ \square \]

Remark 5.2. It is claimed incorrectly in [L] Table 3, that $\mathcal{M}_{11,n}$ is unirational for $n \leq 10$. The argument sketched in loc.cit. only establishes the uniruledness of $\mathcal{M}_{11,n}$ when $n \leq 10$, precisely using the map $\chi_n : \mathcal{M}_{11,n} \rightarrow \mathcal{F}_{11,n}$, which is birationally a $\mathbb{P}^{11-n}$-bundle. But this argument alone offers no indications concerning the birational nature of the base variety $\mathcal{F}_{11,n}$. One can establish partial results on the birational nature of $\mathcal{F}_{11,n}$, for $n \leq 10$. For instance, it is shown in [Ve1] that the universal product $\mathcal{C}_{11,6}$ is unirational, which implies that $\mathcal{F}_{11}$ is unirational as well.

Remark 5.3. Mukai [M4] gives an explicit orbit space realization over a projective space for the universal $K3$ surface $\mathcal{F}_{13,1}$. The unirationality of $\mathcal{F}_{13,1}$ thus follows. Presumably, a similar argument works for genus 12, when $\mathcal{F}_{12}$ is known to be birational to a $\mathbb{P}^{13}$-bundle over the rational moduli space $\mathcal{M}F_{22}$ of Fano 3-folds $V_{22} \subseteq \mathbb{P}^{13}$, see again [Mi].

Remark 5.4. Since $u : \mathcal{F}_{g,1} \rightarrow \mathcal{F}_{g}$ is a morphism fibred in Calabi-Yau varieties, by Iitaka’s easy addition formula $\kappa(\mathcal{F}_{g,1}) \leq \dim(\mathcal{F}_{g}) = 19$, in particular, $\mathcal{F}_{g,1}$ is never of general type. Furthermore, by [K], we also write $\kappa(\mathcal{F}_{g,1}) \geq \kappa(\mathcal{F}_{g})$. In particular, when $\mathcal{F}_{g}$ is of general type, then $\kappa(\mathcal{F}_{g,1}) = 19$.

References


