EFFECTIVE DIVISORS ON HURWITZ SPACES

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ABSTRACT. We prove the effectiveness of the canonical bundle of several Hurwitz spaces $\overline{\mathcal{H}}_{g,k}$ of degree k admissible covers of $\mathbf{P}^1$ from curves of genus $14 \leq g \leq 19$.

Following a principle due to Mumford, most moduli spaces that appear in algebraic geometry (classifying curves, abelian varieties, $K3$ surfaces) are of general type, with a finite number of exceptions, which are unirational, or at least uniruled. Understanding the transition from negative Kodaira dimension to being of general type is usually quite difficult. The aim of this paper is to investigate to some extent this principle in the case of another prominent parameter space of curves, namely the Hurwitz space $\mathcal{H}_{g,k}$ classifying degree k covers $C \to \mathbf{P}^1$ with only simple ramification, from a smooth curve C of genus g. Hurwitz spaces provide an interesting bridge between the more accessible moduli spaces of pointed rational curves and the moduli space of curves. They have been classically used by Clebsch and Hurwitz to establish the irreducibility of the moduli space of curves (see [Ful] for a modern treatment in arbitrary characteristic), as well as by Harris and Morrison [HMo] to bound from below the slope of all effective divisors on $\overline{\mathcal{M}}_g$. We denote by $\overline{\mathcal{H}}_{g,k}$ the moduli space of admissible covers constructed by Harris and Mumford [HM] and studied further in [ACV]. It comes equipped with a finite branch map

$$\mathfrak{b}\colon \overline{\mathcal{H}}_{g,k} \to \overline{\mathcal{M}}_{0,2g+2k-2}/\mathfrak{S}_{2g+2k-2},$$

where the target is the moduli space of *unordered* $(2g+2k-2)$-pointed stable rational curves, as well as with a map

$$\sigma\colon \overline{\mathcal{H}}_{g,k} \to \overline{\mathcal{M}}_g,$$

obtained by assigning to an admissible cover the stable model of its source.

It is a fundamental question to describe the birational nature of $\overline{\mathcal{H}}_{g,k}$ and one knows much less than in the case of other moduli spaces like $\overline{\mathcal{M}}_g$ or $\overline{\mathcal{A}}_g$. Recall that in a series of landmark papers [HM], [H], [EH2] published in the 1980s, Harris, Mumford and Eisenbud proved that $\overline{\mathcal{M}}_g$ is a variety of general type for $g > 23$. This contrasts with the classical result of Severi [Sev] that $\overline{\mathcal{M}}_g$ is unirational for $g \leq 10$. Later it has been shown that the Kodaira dimension of $\overline{\mathcal{M}}_g$ is negative for $g \leq 15$. Precisely, $\overline{\mathcal{M}}_g$ is unirational for $g \leq 14$, see [CR1], [Ser], [Ve], whereas $\overline{\mathcal{M}}_{15}$ is rationally connected [BV]. It has been recently established in [FJP] that both $\overline{\mathcal{M}}_{22}$ and $\overline{\mathcal{M}}_{23}$ are of general type.

From Brill-Noether theory it follows that when $k \geq \frac{g+2}{2}$, every curve of genus g can be represented as a k-sheeted cover of $\mathbf{P}^1$, that is, $\sigma\colon \overline{\mathcal{H}}_{g,k} \to \overline{\mathcal{M}}_g$ is dominant, and thus $\overline{\mathcal{H}}_{g,k}$ is of general type whenever $\overline{\mathcal{M}}_g$ is. Classically it has been known that $\mathcal{H}_{g,k}$ is unirational for $k \leq 5$, see [AC] and references therein for a modern treatment. Geiss [G] using liaison techniques showed that most Hurwitz spaces $\mathcal{H}_{g,6}$ with $g \leq 45$ are unirational. Schreyer and Tanturri [ST] put forward the hypothesis that there exist only finitely many pairs (g,k), with $k \geq 6$, such that $\overline{\mathcal{H}}_{g,k}$ is *not* of general type. Further results on the unirationality of some particular Hurwitz spaces are obtained in [KT].

We are particularly interested in the range $k \geq \frac{g+2}{2}$ (when $\overline{\mathcal{H}}_{g,k}$ dominates $\overline{\mathcal{M}}_g$) and $14 \leq g \leq 19$, so that $\overline{\mathcal{M}}_g$ is either unirational/uniruled for $g = 14, 15$, or its Kodaira dimension is unknown, when $g = 16, 17, 18, 19$. We summarize our results showing the positivity of the canonical bundle of $\overline{\mathcal{H}}_{g,k}$ and we begin with the cases when $\overline{\mathcal{M}}_g$ is known to be (or thought to be) uniruled.

Theorem 0.1. *Suppose $\overline{\mathcal{H}}$ is one of the spaces $\overline{\mathcal{H}}_{14,9}$ or $\overline{\mathcal{H}}_{16,9}$. Then there exists an effective $\mathbb{Q}$-divisor class E on $\overline{\mathcal{H}}$ contracted under the map $\sigma\colon \overline{\mathcal{H}} \to \overline{\mathcal{M}}_g$, such that the twisted canonical class $K_{\overline{\mathcal{H}}} + E$ is effective.*

A few comments are in order. Firstly, the boundary divisor E (that may be empty) is supported on the locus $\sum_{i\geq 1} \sigma^*(\Delta_i)$ of admissible covers with source curve being of compact type. Secondly, Theorem 0.1 is optimal. In genus 14, Verra [Ve] established the unirationality of the Hurwitz space $\mathcal{H}_{14,8}$, which is a finite cover of $\mathcal{M}_{14}$, and concluded in this way that $\mathcal{M}_{14}$ itself is unirational. In genus 16, the map

$$\sigma\colon \overline{\mathcal{H}}_{16,9} \to \overline{\mathcal{M}}_{16}$$

is generically finite. Chang and Ran [CR3] claimed that $\overline{\mathcal{M}}_{16}$ is uniruled (more precisely, that $K_{\overline{\mathcal{M}}_{16}}$ is not pseudo-effective, which by now, one knows that it implies uniruledness). However, it has become recently clear [Ts] that their proof contains a fatal error, so the Kodaira dimension of $\overline{\mathcal{M}}_{16}$ is currently unknown.

Theorem 0.2. *Suppose $\overline{\mathcal{H}}$ is one of the spaces $\overline{\mathcal{H}}_{17,11}$ or $\overline{\mathcal{H}}_{19,13}$. Then there exists an effective $\mathbb{Q}$-divisor class E on $\overline{\mathcal{H}}$ contracted under the map $\sigma\colon \overline{\mathcal{H}} \to \overline{\mathcal{M}}_g$, such that the twisted canonical class $K_{\overline{\mathcal{H}}} + E$ is big.*

We observe that the maps $\overline{\mathcal{H}}_{17,11} \to \overline{\mathcal{M}}_{17}$ and $\overline{\mathcal{H}}_{19,13} \to \overline{\mathcal{M}}_{19}$ have generically 3 and respectively 5-dimensional fibres. As in Theorem 0.1, the divisor E is supported on the divisor $\sum_{i\geq 1} \sigma^*(\Delta_i)$.

Both Theorems 0.1 and 0.2 are proven at the level of a partial compactification $\widetilde{\mathcal{G}}^1_{g,k}$ (described in detail in Section 2) and which incorporates only admissible covers with a source curve whose stable model is irreducible. In each relevant genus we produce an explicit effective divisor $\widetilde{\mathcal{D}}_g$ on $\widetilde{\mathcal{G}}^1_{g,k}$ such that the canonical class $K_{\widetilde{\mathcal{G}}^1_{g,k}}$ can be expressed as a positive combination of a multiple of $[\widetilde{\mathcal{D}}_g]$ and the pull-back under the map σ of an effective divisor on $\overline{\mathcal{M}}_g$.

We describe the construction of these divisors in two instances and refer to Section 2 for the remaining cases and further details. First we consider the space $\overline{\mathcal{H}}_{16,9}$ which is a generically finite cover of $\overline{\mathcal{M}}_{16}$.

Let us choose a general pair $[C,A] \in \mathcal{H}_{16,9}$. We set $L := K_C \otimes A^{\vee} \in W^7_{21}(C)$. Denoting by $I_{C,L}(k) := \mathrm{Ker}\big\{\mathrm{Sym}^k H^0(C,L) \to H^0(C, L^{\otimes k})\big\}$, by Riemann-Roch one computes that $\dim I_{C,L}(2) = 9$ and $\dim I_{C,L}(3) = 72$. Therefore the locus where there exists a non-trivial syzygy, that is, the map

$$\mu_{C,L} : I_{C,L}(2) \otimes H^0(C,L) \to I_{C,L}(3)$$

is not an isomorphism is expected to be a divisor on $\mathcal{H}_{16,9}$. This indeed is the case and we define the *syzygy divisor*

$$\mathcal{D}_{16} := \Big\{[C,A] \in \mathcal{H}_{16,9} : I_{C,K_C\otimes A^\vee}(2) \otimes H^0(C, K_C \otimes A^\vee) \stackrel{\ncong}{\to} I_{C,K_C\otimes A^\vee}(3)\Big\}.$$

The ramification divisor of the map $\sigma : \mathcal{H}_{16,9} \to \mathcal{M}_{16}$, viewed as the *Gieseker-Petri divisor*

$$\mathcal{GP} := \Big\{[C,A] \in \mathcal{H}_{16,9} \colon H^0(C,A) \otimes H^0(C, K_C \otimes A^\vee) \stackrel{\ncong}{\to} H^0(C, K_C)\Big\}$$

also plays an important role. After computing the class of the closure $\widetilde{\mathcal{D}}_{16}$ of the Koszul divisor $\mathcal{D}_{16}$ inside the partial compactification $\widetilde{\mathcal{G}}^1_{16,9}$, we find the following explicit representative for the canonical class

$$K_{\widetilde{\mathcal{G}}^1_{16,9}} = \frac{2}{5}[\widetilde{\mathcal{D}}_{16}] + \frac{3}{5}[\widetilde{\mathcal{GP}}] \in CH^1(\widetilde{\mathcal{G}}^1_{16,9}). \tag{1}$$

This proves Theorem 0.1 in the case $g = 16$. We may wonder whether (1) is the only effective representative of $K_{\widetilde{\mathcal{G}}^1_{16,9}}$, which would imply that the Kodaira dimension of $\overline{\mathcal{H}}_{16,9}$ is equal to zero. To summarize, since the smallest Hurwitz cover of $\overline{\mathcal{M}}_{16}$ has an effective canonical class, it seems unlikely that the method of establishing the uniruledness/unirationality of $\mathcal{M}_{14}$ and $\mathcal{M}_{15}$ by studying a Hurwitz space covering it can be extended to higher genera $g \geq 17$.

The last case we discuss in this introduction is $g = 17$ and $k = 11$. We choose a general pair $[C,A] \in \mathcal{H}_{17,11}$. The residual linear system $L := K_C \otimes A^\vee \in W^6_{21}(C)$ induces an embedding $C \subseteq \mathbf{P}^6$ of degree 21. The multiplication map

$$\phi_L : \mathrm{Sym}^2 H^0(C,L) \to H^0(C, L^{\otimes 2})$$

has a 2-dimensional kernel. We impose the condition that the pencil of quadrics containing the image curve $C \stackrel{|L|}{\hookrightarrow} \mathbf{P}^6$ be *degenerate*, that is, it intersects the discriminant divisor in $\mathbf{P}\big(\mathrm{Sym}^2 H^0(C,L)\big)$ non-transversally. We thus define the locus

$$\mathcal{D}_{17} := \Big\{[C,A] \in \mathcal{H}_{17,11} : \mathbf{P}\big(\mathrm{Ker}(\phi_{K_C\otimes A^\vee})\big) \text{ is a degerate pencil}\Big\}.$$

Using [FR], we can compute the class $[\widetilde{\mathcal{D}}_{17}]$ of the closure of $\mathcal{D}_{17}$ inside $\widetilde{\mathcal{G}}_{17,11}$. Comparing this class to that of the canonical divisor, we obtain the relation

$$K_{\widetilde{\mathcal{G}}^1_{17,11}} = \frac{1}{5}[\widetilde{\mathcal{D}}_{17}] + \frac{3}{5}\sigma^*(7\lambda - \delta_0) \in CH^1(\widetilde{\mathcal{G}}^1_{17,11}). \tag{2}$$

Since the class $7\lambda - \delta_0$ can be shown to be big on $\overline{\mathcal{M}}_{17}$, the conclusion of Theorem 0.2 in the case $g = 17$ now follows.

1. The birational geometry of Hurwitz spaces

We denote by $\mathcal{H}^{\mathfrak{o}}_{g,k}$ the Hurwitz space classifying degree k covers $f \colon C \to \mathbf{P}^1$ with source being a smooth curve C of genus g and having simple ramifications. Note that we choose an *ordering* $(p_1, \ldots, p_{2g+2k-2})$ of the set of the branch points of f. An excellent reference for the algebro-geometric study of Hurwitz spaces is [Ful]. Let $\overline{\mathcal{H}}^{\mathfrak{o}}_{g,k}$ be the (projective) moduli space of admissible covers (with an ordering of the set of branch points). The geometry of $\overline{\mathcal{H}}^{\mathfrak{o}}_{g,k}$ has been described in detail by Harris and Mumford [HM] and further clarified in [ACV]. Summarizing their results, the stack $\overline{H}^{\mathfrak{o}}_{g,k}$ of admissible covers (whose coarse moduli space is precisely $\overline{\mathcal{H}}^{\mathfrak{o}}_{g,k}$) is isomorphic to the stack of *twisted stable* maps into the classifying stack $\mathcal{B}\mathfrak{S}_k$ of the symmetric group $\mathfrak{S}_k$, that is, there is a canonical identification

$$\overline{H}^{\mathfrak{o}}_{g,k} := \overline{M}_{0,2g+2k-2}\Big(\mathcal{B}\mathfrak{S}_k\Big).$$

Points of $\overline{\mathcal{H}}^{\mathfrak{o}}_{g,k}$ can be thought of as admissible covers $[f\colon C \to R, p_1,\ldots,p_{2g+2k-2}]$, where the source C is a nodal curve of arithmetic genus g, the target R is a tree of smooth rational curves, f is a finite map of degree k satisfying $f^{-1}(R_{\mathrm{sing}}) = C_{\mathrm{sing}}$, and $p_1,\ldots,p_{2g+2k-2} \in R_{\mathrm{reg}}$ denote the branch points of f. Furthermore, the ramification indices on the two branches of C at each ramification point of f at a node of C must coincide. One has a finite *branch* morphism

$$\mathfrak{b}\colon \overline{\mathcal{H}}^{\mathfrak{o}}_{g,k} \to \overline{\mathcal{M}}_{0,2g+2k-2},$$

associating to a cover its (ordered) branch locus. The symmetric group $\mathfrak{S}_{2g+2k-2}$ operates on $\overline{\mathcal{H}}^{\mathfrak{o}}_{g,k}$ by permuting the branch points of each admissible cover. Denoting by

$$\overline{\mathcal{H}}_{g,k} := \overline{\mathcal{H}}^{\mathfrak{o}}_{g,k}/\mathfrak{S}_{2g+2k-2}$$

the quotient parametrizing admissible covers *without* an ordering of the branch points, we introduce the projection $q\colon \overline{\mathcal{H}}^{\mathfrak{o}}_{g,k} \to \overline{\mathcal{H}}_{g,k}$. Finally, let

$$\sigma\colon \overline{\mathcal{H}}_{g,k} \to \overline{\mathcal{M}}_g$$

be the map assigning to an admissible degree k cover the stable model of its source curve, obtained by contracting unnecessary rational components.

We discuss the structure of the boundary divisors on the compactified Hurwitz space. For $i = 0,\ldots,g+k-1$, let B_i be the boundary divisor of $\overline{\mathcal{M}}_{0,2g+2k-2}$, defined as the closure of the locus of unions of two smooth rational curves meeting at one point, such that precisely i of the marked points lie on one component and $2g+2k-2-i$ on the remaining one. To specify a boundary divisor of $\overline{\mathcal{H}}^{\mathfrak{o}}_{g,k}$, one needs the following combinatorial information:

(i) A partition $I \sqcup J = \{1,\ldots,2g+2k-2\}$, such that $|I| \geq 2$ and $|J| \geq 2$.
(ii) Transpositions $\{w_i\}_{i\in I}$ and $\{w_j\}_{j\in J}$ in $\mathfrak{S}_k$, satisfying

$$\prod_{i\in I} w_i = u,\ \ \prod_{j\in J} w_j = u^{-1},$$

for some permutation $u \in \mathfrak{S}_k$.

To this data, we associate the locus of admissible covers of degree k with labeled branch points

$$\big[f\colon C \to R,\ p_1,\ldots,p_{2g+2k-2}\big] \in \overline{\mathcal{H}}^{\mathfrak{o}}_{g,k},$$

where $[R = R_1 \cup_p R_2, p_1,\ldots,p_{2g+2k-2}] \in B_{|I|} \subseteq \overline{\mathcal{M}}_{0,2g+2k-2}$ is a pointed union of two smooth rational curves R_1 and R_2 meeting at the point p. The marked points indexed by I lie on R_1, those indexed by J lie on R_2. Let $\mu := (\mu_1,\ldots,\mu_\ell) \vdash k$ be the partition corresponding to the conjugacy class of $u \in \mathfrak{S}_k$. We denote by $E_{i:\mu}$ the boundary divisor on $\overline{\mathcal{H}}^{\mathfrak{o}}_k$ classifying twisted stable maps with underlying admissible cover as above, with $f^{-1}(p)$ having partition type μ, and exactly i of the points $p_1,\ldots,p_{2g+2k-2}$ lying on R_1. Passing to the unordered Hurwitz space, we denote by $D_{i:\mu}$ the image $E_{i:\mu}$ under the map q, with its reduced structure.

The effect of the map $\mathfrak{b}$ on boundary divisors is summarized in the following relation that holds for $i = 2, \ldots, g+k-1$, see [HM] p. 62, or [GK1] Lemma 3.1:

$$\mathfrak{b}^*(B_i) = \sum_{\mu \vdash k} \mathrm{lcm}(\mu) E_{i:\mu}. \tag{3}$$

The Hodge class on the Hurwitz space is by definition pulled back from $\overline{\mathcal{M}}_g$. Its class $\lambda := (\sigma \circ q)^*(\lambda)$ on $\overline{\mathcal{H}}^{\mathfrak{o}}_{g,k}$ has been determined in [KKZ], or see [GK1] Theorem 1.1 for an algebraic proof. Remarkably, unlike on $\overline{\mathcal{M}}_g$, the Hodge class is always a boundary class:

$$\lambda = \sum_{i=2}^{g+k-1} \sum_{\mu \vdash k} \mathrm{lcm}(\mu) \Big(\frac{i(2g+2k-2-i)}{8(2g+2k-3)} - \frac{1}{12}\Big(k - \sum_{j=1}^{\ell(\mu)} \frac{1}{\mu_j}\Big)\Big) [E_{i:\mu}] \in CH^1(\overline{\mathcal{H}}^{\mathfrak{o}}_k). \tag{4}$$

The sum (4) runs over partitions μ of k corresponding to conjugacy classes of permutations that can be written as products of i transpositions. In the formula (4), $\ell(\mu)$ denotes the length of the partition μ. . The only negative coefficient in the expression of $K_{\overline{\mathcal{M}}_{0,2g+2k-2}}$ is that of the boundary divisor B_2. For this reason, the compoents of $\mathfrak{b}^*(B_2)$ play a special role, which we now discuss. We pick an admissible cover

$$[f \colon C = C_1 \cup C_2 \to R = R_1 \cup_p R_2,\ p_1, \ldots, p_{2g+2k-2}] \in \mathfrak{b}^*(B_2),$$

and set $C_1 := f^{-1}(R_1)$ and $C_2 := f^{-1}(R_2)$ respectively. Note that C_1 and C_2 may well be disconnected. Without loss of generality, we assume $I = \{1, \ldots, 2g+2k-4\}$, thus $p_1, \ldots, p_{2g+2k-4} \in R_1$ and $p_{2g+2k-3}, p_{2g+2k-2} \in R_2$.

Let $E_{2:(1^k)}$ be the closure in $\overline{\mathcal{H}}^{\mathfrak{o}}_{g,k}$ of the locus of admissible covers such that the transpositions $w_{2g+2k-3}$ and $w_{2g+2k-2}$ describing the local monodromy in a neighborhood of the branch points $p_{2g+2k-3}$ and $p_{2g+2k-2}$ respectively, are equal. Let E_0 further denote the component of $E_{2:(1^k)}$ consisting of those admissible cover for which the subcurve C_1 is connected. This is the case precisely when $\langle w_1, \ldots, w_{2g+2k-4}\rangle = \mathfrak{S}_k$. To show that E_0 is irreducible, one uses the classical topological argument due to Clebsch invoked when establishing the irreducibility of $\overline{\mathcal{H}}_{g,k}$. Note that $E_{2:(1^k)}$ has other irreducible components, for instance when C_1 splits as the disjoint union of a smooth rational curve mapping isomorphically onto R_1 and a second component mapping with degree $k-1$ onto R_1.

When the permutations $w_{2g+2k-3}$ and $w_{2g+2k-2}$ are distinct but share one element in their orbit, then $\mu = (3, 1^{k-3}) \vdash k$ and the corresponding boundary divisor is denoted by $E_{2:(3,1^{k-3})}$. Let E_3 be the subdivisor of $E_{2:(3,1^{k-3})}$ corresponding to admissible covers with $\langle w_1, \ldots, w_{2g+2k-4}\rangle = \mathfrak{S}_k$, that is, C_1 is a connected curve. Finally, in the case when $w_{2g+2k-3}$ and $w_{2g+2k-2}$ are disjoint transpositions, we obtain the boundary divisor $E_{2:(2,2,1^{k-4})}$. Similarly to the previous case, we denote by E_2 the irreducible component of $E_{2:(2,2,1^{k-4})}$ consisting of admissible covers for which $\langle w_1, \ldots, w_{2g+2k-4}\rangle = \mathfrak{S}_k$.

The boundary divisors E_0, E_2 and E_3, when pulled-back under the quotient map $q \colon \overline{\mathcal{H}}^{\mathfrak{o}}_{g,k} \to \overline{\mathcal{H}}_{g,k}$, verify the following formulas

$$q^*(D_0) = 2E_0,\ q^*(D_2) = E_2 \ \text{ and } q^*(D_3) = 2E_3,$$

which we now explain. The general point of both E_0 and E_3 has no automorphism that fixes all branch points, but admits an automorphism of order two that fixes C_1 and permutes the branch points $p_{2g+2k-3}$ and $p_{2g+2k-2}$. The general admissible cover in E_2 has an automorphism group $\mathbb{Z}_2 \times \mathbb{Z}_2$ (each of the two components of C_2 mapping $2:1$ onto R_2 has an automorphism of order 2). In the stack $\overline{H}^{\mathfrak{o}}_{g,k}$ we have two points lying over this admissible cover and each of them has an automorphism group of order 2. In particular the map $\overline{H}^{\mathfrak{o}}_{g,k} \to \overline{\mathcal{H}}^{\mathfrak{o}}_{g,k}$ from the stack to the coarse moduli space is ramified with ramification index 1 along the divisor E_2.

One applies now the Riemann-Hurwitz formula to the map $\mathfrak{b}\colon \overline{\mathcal{H}}^{\mathfrak{o}}_{g,k} \to \overline{\mathcal{M}}_{0,2g+2k-2}$. Recall also that the canonical bundle of the moduli space of pointed rational curves is given by the formula $K_{\overline{\mathcal{M}}_{0,2g+2k-2}} = \sum_{i=2}^{g+k-1}\bigl(\frac{i(2g+2k-2-i)}{2g+2k-3}-2\bigr)[B_i]$. All in all, we obtain the following formula for the canonical class of the Hurwitz stack:

$$K_{\overline{H}^{\mathfrak{o}}_{g,k}} = \mathfrak{b}^* K_{\overline{\mathcal{M}}_{0,2g+2k-2}} + \mathrm{Ram}(\mathfrak{b}), \tag{5}$$

where $\mathrm{Ram}(\mathfrak{b}) = \sum_{i,\mu\vdash k}(\mathrm{lcm}(\mu)-1)[E_{i:\mu}]$.

1.1. **A partial compactification of $\mathcal{H}_{g,k}$.** Like in the paper [FR], we work on a partial compactification of $\mathcal{H}_{g,k}$, for the Koszul-theoretic calculations are difficult to extend over all the boundary divisors $D_{i:\mu}$. The partial compactification of the Hurwitz space we consider is defined in the same spirit as in the part of the paper devoted to divisors on $\overline{\mathcal{M}}_g$. We fix an integer $k \leq \frac{2g+4}{3}$. It follows then that the locus of curves $[C] \in \mathcal{M}_g$ such that $W^2_k(C) \neq \emptyset$ has codimension at least 2 in $\mathcal{M}_g$, hence it will play no role in any divisor class calculation. We denote by $\widetilde{\mathcal{G}}^1_{g,k}$ the space of pairs $[C, A]$, where C is an irreducible nodal curve of genus g satisfying $W^2_k(C) = \emptyset$ and A is a base point free locally free sheaf of degree k on C with $h^0(C, A) = 2$. The rational map $\overline{\mathcal{H}}_{g,k} \dashrightarrow \widetilde{\mathcal{G}}^1_{g,k}$ is of course regular outside a subvariety of $\overline{\mathcal{H}}_{g,k}$ of codimension at least 2, but can be made explicit over the boundary divisors D_0, D_2 and D_3, which we now explain.

Retaining the previous notation, to the general point $[f\colon C_1 \cup C_2 \to R_1 \cup_p R_2]$ of D_3 (respectively D_2), we assign the pair $[C_1, A_1 := f^*\mathcal{O}_{R_1}(1)] \in \widetilde{\mathcal{G}}^1_k$. Note that C_1 is a smooth curve of genus g and A_1 is a pencil on C_1 having a triple point (respectively two ramification points in the fibre over p). The spaces $\mathcal{H}_{g,k} \cup D_0 \cup D_2 \cup D_3$ and $\widetilde{\mathcal{G}}^1_{g,k}$ differ outside a set of codimension at least 2 and for divisor class calculations they will be identified. Using this, we copy the formula (4) at the level of the parameter space $\widetilde{\mathcal{G}}^1_{g,k}$ and obtain:

$$\lambda = \frac{g+k-2}{4(2g+2k-3)}[D_0] - \frac{1}{4(2g+2k-3)}[D_2] + \frac{g+k-6}{12(2g+2k-3)}[D_3] \in CH^1(\widetilde{\mathcal{G}}^1_{g,k}). \tag{6}$$

We now observe that the canonical class of $\widetilde{\mathcal{G}}^1_k$ has a simple expression in terms of the Hodge class λ and the boundary divisors D_0 and D_3. Quite remarkably, this formula is independent of both g and k!

Theorem 1.1. *The canonical class of the partial compactification $\widetilde{\mathcal{G}}^1_{g,k}$ is given by*

$$K_{\widetilde{\mathcal{G}}^1_{g,k}} = 8\lambda + \frac{1}{6}[D_3] - \frac{3}{2}[D_0].$$

Proof. We combine the equation (5) with the Riemann-Hurwitz formula applied to the quotient $q : \overline{\mathcal{H}}^{\circ}_{g,k} \dashrightarrow \widetilde{\mathcal{G}}^1_{g,k}$ and write:

$$q^*(K_{\widetilde{\mathcal{G}}^1_{g,k}}) = K_{\overline{H}^{\circ}_{g,k}} - [E_0] - [E_2] - [E_3] =$$

$$-\frac{4}{2g+2k-3}[D_2] - \frac{2g+2k-1}{2g+2k-3}[D_0] + \frac{2g+2k-9}{2g+2k-3}[D_3].$$

To justify this formula, observe that the divisors E_0 and E_3 lie in the ramification locus of q, hence they have to subtracted from $K_{\overline{H}^{\circ}_{g,k}}$. The morphism $\overline{H}^{\circ}_{g,k} \to \overline{\mathcal{H}}^{\circ}_{g,k}$ from the stack to the coarse moduli space is furthermore simply ramified along E_2, so this divisor has to be subtracted as well. We now use (6) to express $[D_2]$ in terms of λ, $[D_0]$ and $[D_3]$ and obtain that $q^*(K_{\widetilde{\mathcal{G}}^1_{g,k}}) = 8\lambda + \frac{1}{3}[E_3] - 3[E_0]$, which yields the claimed formula. $\square$

Let $f : \mathcal{C} \to \widetilde{\mathcal{G}}^1_{g,k}$ be the universal curve and we choose a degree k Poincaré line bundle $\mathcal{L}$ on $\mathcal{C}$ (or on an étale cover if necessary). Along the lines of [FR] Section 2 (where only the case $g = 2k-1$ has been treated, though the general situation is analogous), we introduce two tautological codimension one classes:

$$\mathfrak{a} := f_*\big(c_1^2(\mathcal{L})\big) \text{ and } \mathfrak{b} := f_*\big(c_1(\mathcal{L}) \cdot c_1(\omega_f)\big) \in CH^1(\widetilde{\mathcal{G}}^1_{g,k}).$$

The push-forward sheaf $\mathcal{V} := f_*\mathcal{L}$ is locally free of rank 2 on $\widetilde{\mathcal{G}}^1_{g,k}$. Its fibre at a point $[C, A]$ is canonically identified with $H^0(C, A)$. Although $\mathcal{L}$ is not unique, an easy exercise involving first Chern classes, convinces us that the class

$$\gamma := \mathfrak{b} - \frac{g-1}{k}\mathfrak{a} \in CH^1(\widetilde{\mathcal{G}}^1_{g,k}) \tag{7}$$

does not depend of the choice of a Poincaré bundle.

Proposition 1.2. *We have that* $\mathfrak{a} = kc_1(\mathcal{V}) \in CH^1(\widetilde{\mathcal{G}}^1_{g,k})$.

Proof. Simple application of the Porteous formula in the spirit of Proposition 11.2 in [FR]. $\square$

The following locally free sheaves on $\widetilde{\mathcal{G}}^1_{g,k}$ will play an important role in several Koszul-theoretic calculations:

$$\mathcal{E} := f_*(\omega_f \otimes \mathcal{L}^{-1}) \text{ and } \mathcal{F}_\ell := f_*(\omega_f^{\ell} \otimes \mathcal{L}^{-\ell}),$$

where $\ell \geq 2$.

Proposition 1.3. *The following formulas hold*

$$c_1(\mathcal{E}) = \lambda - \frac{1}{2}\mathfrak{b} + \frac{k-2}{2k}\mathfrak{a} \text{ and } c_1(\mathcal{F}_\ell) = \lambda + \frac{\ell^2}{2}\mathfrak{a} - \frac{\ell(2\ell-1)}{2}\mathfrak{b} + \binom{\ell}{2}\big(12\lambda - [D_0]\big).$$

Proof. Use Grothendieck-Riemann-Roch applied to the universal curve f, coupled with Proposition 1.2 in order to evaluate the terms. Use that $R^1f^*(\omega_f^{\ell} \otimes \mathcal{L}^{-\ell}) = 0$ for $\ell \geq 2$. Similar to Proposition 11.3 in [FR], so we skip the details. $\square$

We summarize the relation between the class γ and the classes $[D_0]$, $[D_2]$ and $[D_3]$ as follows. Again, we find it remarkable that this formula is independent of g and k.

Proposition 1.4. *One has the formula* $[D_3] = 6\gamma + 24\lambda - 3[D_0]$.

Proof. We form the fibre product of the universal curve $f\colon \mathcal{C} \to \widetilde{\mathcal{G}}^1_{g,k}$ together with its projections:

$$\mathcal{C} \xleftarrow{\ \pi_1\ } \mathcal{C} \times_{\widetilde{\mathcal{G}}^1_{g,k}} \mathcal{C} \xrightarrow{\ \pi_2\ } \mathcal{C}.$$

For $\ell \geq 1$, we consider the jet bundle $J^{\ell}_f(\mathcal{L})$, which sits in an exact sequence:

$$0 \longrightarrow \omega_f^{\otimes \ell} \otimes \mathcal{L} \longrightarrow J^{\ell}_f(\mathcal{L}) \longrightarrow J^{\ell-1}_f(\mathcal{L}) \longrightarrow 0. \tag{8}$$

One has a sheaf morphism $\nu_2\colon f^*(\mathcal{V}) \to J^2_f(\mathcal{L})$, which we think of as the *second Taylor map* associating to a section its first two derivatives. For points $[C, A, p] \in \mathcal{C}$ such that $p \in C$ is a smooth point, this map is simply the evaluation $H^0(C, A) \to H^0(A \otimes \mathcal{O}_{3p})$. Let $Z \subseteq \mathcal{C}$ be the locus where ν_2 is not injective. Over the locus of smooth curves, D_3 is the set-theoretic image of Z. A local analysis that we shall present shows that ν_2 is degenerate with multiplicity 1 at a point $[C, A, p]$, where $p \in C_{\mathrm{sing}}$. Thus, D_0 is to be found with multiplicity 1 in the degeneracy locus of ν_2. The Porteous formula leads to:

$$[D_3] = f_* c_2\Bigl(\frac{J^2_f(\mathcal{L})}{f^*(\mathcal{V})}\Bigr) - [D_0] \in CH^1(\widetilde{\mathcal{G}}^1_{g,k}).$$

As anticipated, we now show that D_0 appears with multiplicity 1 in the degeneracy locus of ν_2. To that end, we choose a family $F : X \to B$ of genus g curves of genus over a smooth 1-dimensional base B, such that X is smooth, and there is a point $b_0 \in B$ with $X_b := F^{-1}(b)$ is smooth for $b \in B \setminus \{b_0\}$, whereas X_{b_0} has a unique node $u \in X$. Assume furthermore that $A \in \mathrm{Pic}(X)$ is a line bundle such that $A_b := L_{|X_b} \in W^1_k(X_b)$, for each $b \in B$. We further choose a local parameter $t \in \mathcal{O}_{B,b_0}$ and $x, y \in \mathcal{O}_{X,u}$, such that $xy = t$ represents the local equation of X around the point u. Then ω_F is locally generated by the meromorphic differential τ that is given by $\frac{dx}{x}$ outside the divisor $x = 0$ and by $-\frac{dy}{y}$ outside the divisor $y = 0$. Let us pick sections $s_1, s_2 \in H^0(X, A)$, where $s_1(u) \neq 0$, whereas s_2 vanishes with order 1 at the node u of X_{b_0}, along both its branches. Passing to germs of functions at u, we have the relation $s_{2,u} = (x + y)s_{1,u}$. Then by direct calculation in local coordinates, the map $H^0\bigl(X_{b_0}, A_{b_0}\bigr) \to H^0\bigl(X_{b_0}, A_{b_0|3u}\bigr)$ is given by the 2×2 minors of the following matrix:

$$\begin{pmatrix} 1 & 0 & 0 \\ x+y & x-y & x+y \end{pmatrix}.$$

We conclude that D_0 appears with multiplicity 1 in the degeneracy locus of ν_2.

From the exact sequence (8) one computes $c_1(J^2_f(\mathcal{L})) = 3c_1(\mathcal{L}) + 3c_1(\omega_f)$ and $c_2(J^2_f(\mathcal{L})) = c_2(J^1_f(\mathcal{L})) + c_1(J^1_f(\mathcal{L})) \cdot c_1(\omega_f^{\otimes 2} \otimes \mathcal{L}) = 3c_1^2(\mathcal{L}) + 6c_1(\mathcal{L}) \cdot c_1(\omega_f) + 2c_1^2(\omega_f)$. Substituting, we find after routine calculations that

$$f_* c_2\left(\frac{J^2_f(\mathcal{L})}{f^*(\mathcal{V})}\right) = 6\gamma + 2\kappa_1,$$

where $\kappa_1 = f_*(c_1(\omega_f)^{\otimes 2})$. Using Mumford's formula $\kappa_1 = 12\lambda - [D_0] \in CH^1(\widetilde{\mathcal{G}}^1_{g,k})$, see e.g. [HM] top of page 50, we finish the proof. $\square$

2. EFFECTIVE DIVISORS ON HURWITZ SPACES

We now describe the construction of four effective divisors on particular Hurwitz spaces interesting in moduli theory. The divisors in question are of syzygetic nature and resemble somehow the (virtual) divisors on $\overline{\mathcal{M}}_g$ discussed in the first part of the paper. Note that these divisors on Hurwitz spaces, unlike say the divisor $\mathfrak{D}$ on $\mathcal{M}_{23}$, are defined directly in terms of a general element $[C, A] \in \mathcal{H}_{g,k}$, without making reference to other attributes of the curve C. This simplifies both the task of computing their classes and showing that the respective codimension 1 conditions in moduli lead to genuine divisor on $\mathcal{H}_{g,k}$. Using the irreducibility of $\mathcal{H}_{g,k}$, this amounts to exhibiting *one* example of a point $[C, A]$ outside the divisor, which can be easily achieved with the use of *Macaulay*. We mention also the papers [GK2], [DP]. where one studies syzygetic loci in Hurwitz spaces using the splitting type of the scroll one canonically associates to a cover $C \to \mathbf{P}^1$.

2.1. **The Hurwitz space $\mathcal{H}_{14,9}$.** We consider the morphism $\sigma : \overline{\mathcal{H}}_{14,9} \to \overline{\mathcal{M}}_{14}$, whose general fibre is 2-dimensional. We choose a general element $[C, A] \in \mathcal{H}_{14,9}$ and set $L := K_C \otimes A^{\vee} \in W^5_{17}(C)$ to be the residual linear system. Furthermore, L is very ample, else there exist points $x, y \in C$ such that $A(x + y) \in W^2_{13}(C)$, which contradicts the Brill-Noether Theorem. Note that

$$h^0(C, L^{\otimes 2}) = \dim \mathrm{Sym}^2 H^0(C, L) = 21$$

and we set up the (a priori virtual) divisor

$$\widetilde{\mathcal{D}}_{14} := \Big\{[C, A] \in \widetilde{\mathcal{G}}^1_{14,9} : \phi_L \colon \mathrm{Sym}^2 H^0(C, L) \to H^0(C, L^{\otimes 2}) \text{ is not an isomorphism}\Big\}.$$

Our next results shows that, remarkably, this locus is indeed a divisor and it gives rise to an effective representative of the canonical divisor of $\widetilde{\mathcal{G}}^1_{14,9}$.

Proposition 2.1. *The locus $\mathcal{D}_{14}$ is a divisor on $\mathcal{H}_{14,9}$ and one has the following formula*

$$[\widetilde{\mathcal{D}}_{14}] = 4\lambda + \frac{1}{12}[D_3] - \frac{3}{4}[D_0] = \frac{1}{2}K_{\widetilde{\mathcal{G}}^1_{14,9}} \in CH^1(\widetilde{\mathcal{G}}^1_{14,9}).$$

Proof. The divisor $\widetilde{\mathcal{D}}_{14}$ is the degeneracy locus of the vector bundle morphism

$$\phi : \mathrm{Sym}^2(\mathcal{E}) \to \mathcal{F}_2.$$

The Chern class of both $\mathcal{E}$ and $\mathcal{F}_2$ are computed in Proposition 1.3 and we have the formulas $c_1(\mathcal{E}) = \lambda - \frac{1}{2}\mathfrak{b} + \frac{7}{18}\mathfrak{a}$ and $c_1(\mathcal{F}_2) = 13\lambda + 2\mathfrak{a} - 3\mathfrak{b} - [D_0]$. Taking into account that $\mathrm{rk}(\mathcal{E}) = 6$, we can write:

$$[\widetilde{\mathcal{D}}_{14}] = c_1\big(\mathcal{F}_2 - \mathrm{Sym}^2(\mathcal{E})\big) = c_1(\mathcal{F}_2) - 7c_1(\mathcal{E}) = 6\lambda - [D_0] + \frac{1}{2}\gamma.$$

We now substitute γ in the formula given by Proposition 1.4 involving also the divisor D_3 on $\widetilde{\mathcal{G}}^1_{14,9}$ of pairs $[C, A]$, such that $A \in W^1_9(C)$ has a triple point. We obtain that $[\widetilde{\mathcal{D}}_{14}] = 4\lambda + \frac{1}{12}[D_3] - \frac{3}{4}[D_0]$. Comparing with Theorem 1.1, the fact that $[\widetilde{\mathcal{D}}_{14}]$ is a (half-) canonical representative follows.

It remains to show that $\widetilde{\mathcal{D}}_{14}$ is indeed a divisor. Since $\mathcal{H}_{14,9}$ is irreducible, it suffices to construct one example of a smooth curve $C \subseteq \mathbf{P}^5$ of genus 14 and degree 17 which does not lie on any quadrics. To that end, we consider the *White surface* $X \subseteq \mathbf{P}^5$, obtained by blowing-up $\mathbf{P}^2$ at 15 points $p_1, \ldots, p_{15}$ in general position and embedded into $\mathbf{P}^5$ by the linear system $H := |5h - E_{p_1} - \cdots - E_{p_{15}}|$, where E_{p_i} is the exceptional divisor at the point $p_i \in \mathbf{P}^2$ and $h \in |\mathcal{O}_{\mathbf{P}^2}(1)|$. The White surface is known to projectively Cohen-Macaulay, its ideal being generated by the 3×3-minors of a certain 3×5-matrix of linear forms, see [Gi] Proposition 1.1. In particular, the map

$$\mathrm{Sym}^2 H^0(X, \mathcal{O}_X(1)) \to H^0(X, \mathcal{O}_X(2))$$

is an isomorphism and $X \subseteq \mathbf{P}^5$ lies on no quadrics. We now let $C \subseteq X$ be a general element of the linear system

$$\Big|12h - 3(E_{p_1} + \cdots + E_{p_{13}}) - 2(E_{p_{14}} + E_{p_{15}})\Big|.$$

Note that $\dim |\mathcal{O}_X(C)| = 6$ and a general element is a smooth curve $C \subseteq \mathbf{P}^5$ of degree 17 and genus 14. This finishes the proof, for C lies on no quadrics. $\square$

2.2. **The Hurwitz space $\mathcal{H}_{19,13}$.** The case $g = 19$ is analogous to the situation in genus 14. We have a morphism $\sigma : \overline{\mathcal{H}}_{19,13} \to \overline{\mathcal{M}}_{19}$ with generically 5-dimensional fibres. The Kodaira dimension of both $\overline{\mathcal{M}}_{19}$ and of the Hurwitz spaces $\overline{\mathcal{H}}_{19,11}$ and $\overline{\mathcal{H}}_{19,12}$ is unknown. For a general element $[C, A] \in \mathcal{H}_{19,13}$, we set $L := K_C \otimes A^\vee \in W^6_{23}(C)$. Observe that $W^7_{23}(C) = \emptyset$, that is, L must be a complete linear series. The multiplication

$$\phi_L \colon \mathrm{Sym}^2 H^0(C, L) \to H^0(C, L^{\otimes 2})$$

is a map between between two vector spaces of the same dimension 28. It is easy to produce an example of a pair $[C, L]$, such that ϕ_L is an isomorphism. We introduce the divisor

$$\widetilde{\mathcal{D}}_{19} := \Big\{[C, A] \in \widetilde{\mathcal{G}}^1_{19,13} : \phi_L \colon \mathrm{Sym}^2 H^0(C, L) \to H^0(C, L^{\otimes 2}) \text{ is not an isomorphism}\Big\}.$$

Proposition 2.2. *One has the following formula*

$$[\widetilde{\mathcal{D}}_{19}] = \lambda + \frac{1}{6}[D_3] - \frac{1}{2}[D_0] \in CH^1(\widetilde{\mathcal{G}}^1_{19,13}).$$

Proof. Very similar to the proof of Proposition 2.1. $\square$

We can now prove the case $g = 19$ from Theorem 0.2. Indeed, combining Proposition 2.2 and Theorem 1.1, we find

$$K_{\widetilde{\mathcal{G}}^1_{19,13}} = [\widetilde{\mathcal{D}}_{19}] + \sigma^*(7\lambda - \delta_0).$$

Since the class $7\lambda - \delta_0$ is big an $\overline{\mathcal{M}}_{19}$, it follows that $K_{\widetilde{\mathcal{G}}^1_{19,13}}$ is big.

2.3. **The Hurwitz space $\mathcal{H}_{17,11}$.** The minimal Hurwitz cover of $\overline{\mathcal{M}}_{17}$ is $\overline{\mathcal{H}}_{17,10}$, but its Kodaira dimension is unknown. We consider the next case $\sigma \colon \overline{\mathcal{H}}_{17,11} \to \overline{\mathcal{M}}_{17}$. As described in the Introduction, a general curve $C \subseteq \mathbf{P}^6$ of genus 17 and degree 21 (whose residual linear system is a pencil $A = K_C(-1) \in W^1_{11}(C)$) lies on a pencil of quadrics. The general element of this pencil has full rank 7 and we consider the intersection of the pencil with the discriminant. We define $\mathcal{D}_{17}$ to be the locus of pairs $[C, A] \in \mathcal{H}_{17,11}$ such that this intersection is not reduced.

Theorem 2.3. *The locus $\mathcal{D}_{17}$ is a divisor and the class of its closure $\widetilde{\mathcal{D}}_{17}$ in $\widetilde{\mathcal{G}}^1_{17,11}$ is given by*

$$[\widetilde{\mathcal{D}}_{17}] = \frac{1}{6}\Big(19\lambda - \frac{9}{2}[D_0] + \frac{5}{6}[D_3]\Big) \in CH^1(\widetilde{\mathcal{G}}^1_{17,11}).$$

Proof. We are in a position to apply [FR] Theorem 1.2, which deals precisely with degeneracy loci of this type. We obtain

$$[\widetilde{\mathcal{D}}_{19}] = 6\big(7c_1(F) - 52c_1(\mathcal{E})\big) = 6\big(39\lambda - 7[D_0] + 5\gamma\big).$$

Using once more Proposition 1.4, we obtained the claimed formula. □

Substituting the expression of $[\widetilde{\mathcal{D}}_{17}]$ in the formula of the canonical class of the Hurwitz space, we find

$$K_{\widetilde{\mathcal{G}}^1_{17,11}} = \frac{1}{5}[\widetilde{\mathcal{D}}_{17}] + \frac{3}{5}\sigma^*(7\lambda - \delta_0).$$

Just like in the previous case, since the class $7\lambda - \delta_0$ is big on $\overline{\mathcal{M}}_{17}$ and λ is ample of $\overline{\mathcal{H}}_{17,11}$, Theorem 0.2 follows for $g = 17$ as well.

2.4. **The Hurwitz space $\mathcal{H}_{16,9}$.** This is the most interesting case, for we consider a *minimal* Hurwitz cover $\sigma : \overline{\mathcal{H}}_{16,9} \to \overline{\mathcal{M}}_{16}$ of the uniruled moduli space of curves of genus 16. We fix a general point $[C, A] \in H_{16,9}$ and, set $L := K_C \otimes A^\vee \in W^6_{21}(C)$. It is proven in [F2] Theorem 2.7 that the locus $\mathcal{D}_{16}$ classifying pairs $[C, A]$ such that the multiplication map

$$\mu : I_2(L) \otimes H^0(C, L) \to I_3(L)$$

is not an isomorphism, is a divisor on $\mathcal{H}_{16,9}$.

First we determine the class of the Gieseker-Petri divisor, already mentioned in the introduction.

Proposition 2.4. *One has* $[\widetilde{\mathcal{GP}}] = -\lambda + \gamma \in CH^1(\widetilde{\mathcal{G}}^1_{16,9})$.

Proof. Recall that we have introduced the sheaves $\mathcal{V}$ and $\mathcal{E}$ on $\widetilde{\mathcal{G}}^1_{16,19}$ with fibres canonically isomorphic to $H^0(C, A)$ and $H^0(C, \omega_C \otimes A^\vee)$ over a point $[C, A] \in \widetilde{\mathcal{G}}^1_{16,9}$. We have a natural morphism $\mathcal{E} \otimes \mathcal{V} \to f_*(\omega_f)$ and $\widetilde{\mathcal{GP}}$ is the degeneracy locus of this map. Accordingly,

$$[\widetilde{\mathcal{GP}}] = \lambda - 2c_1(\mathcal{E}) - 8c_1(\mathcal{V}) = -\lambda + \big(\mathfrak{b} - \frac{5}{3}\mathfrak{a}\big) = -\lambda + \gamma.$$

□

We can now compute the class of the divisor $\widetilde{\mathcal{D}}_{16}$.

Theorem 2.5. *The locus $\widetilde{\mathcal{D}}_{16}$ is an effective divisor on $\widetilde{\mathcal{G}}^1_{16,9}$ and its class is given by*

$$[\widetilde{\mathcal{D}}_{16}] = \frac{65}{2}\lambda - 5[D_0] + \frac{3}{2}[\widetilde{\mathcal{GP}}] \in CH^1(\widetilde{\mathcal{G}}^1_{16,9}).$$

Proof. Recall the definition of the vector bundles $\mathcal{F}_2$ and $\mathcal{F}_3$ on $\widetilde{\mathcal{G}}^1_{16,9}$, as well as the expression of their first Chern classes provided by Proposition 1.3. We define two further vector bundles $\mathcal{I}_2$ and $\mathcal{I}_3$ on $\widetilde{\mathcal{G}}^1_{16,9}$, via the following exact sequences:

$$0 \longrightarrow \mathcal{I}_\ell \longrightarrow \mathsf{Sym}^\ell(\mathcal{E}) \longrightarrow \mathcal{F}_\ell \longrightarrow 0,$$

for $\ell = 2, 3$. Note that $\mathrm{rk}(\mathcal{I}_2) = 9$, whereas $\mathrm{rk}(\mathcal{I}_3) = 72$. To make sure that these sequences are exact on the left outside a set of codimension at least 2 inside $\widetilde{\mathcal{G}}^1_{16,9}$, we invoke [F1], Propositions 3.9 and 3.10. The divisor $\widetilde{\mathcal{D}}_{16}$ is then the degeneracy locus of the morphism $\mu : \mathcal{I}_2 \otimes \mathcal{E} \to \mathcal{I}_3$ which globalizes the multiplication maps $\mu_{C,L}$: $I_{C,L}(2) \otimes H^0(C,L) \to I_{C,L}(3)$, where $L = \omega_C \otimes A^\vee$ and $[C,A] \in \widetilde{\mathcal{G}}^1_{16,9}$.

Noting that $c_1\bigl(\mathrm{Sym}^3(\mathcal{E})\bigr) = 45c_1(\mathcal{E})$ and $c_1\bigl(\mathrm{Sym}^2(\mathcal{E})\bigr) = 9c_1(\mathcal{E})$, we compute

$$[\widetilde{\mathcal{D}}_{16}] = c_1(\mathcal{I}_3) - 8c_1(\mathcal{I}_2) - 9c_1(\mathcal{E}) = 31\lambda - 5[D_0] + \frac{3}{2}\gamma.$$

Substituting $\gamma = \lambda + [\widetilde{\mathcal{GP}}]$, we obtain the claimed formula.

It remains to observe that it has already been proved in [F1] Theorem 2.7 that for a general pair $[C,L]$, where $L \in W^6_{21}(C)$, the multiplication map $\mu_{C,L}$ is an isomorphism. □

The formula (1) mentioned in the introduction follows now by using Theorem 2.5 and the Riemann-Hurwitz formula for the map $\sigma\colon \widetilde{\mathcal{G}}^1_{16,9} \to \overline{\mathcal{M}}_{16}$. One writes

$$K_{\widetilde{\mathcal{G}}^1_{16,9}} = 13\lambda - 2[D_0] + \frac{3}{5}[\widetilde{\mathcal{GP}}] + \frac{2}{5}[\widetilde{\mathcal{GP}}] = \frac{2}{5}[\widetilde{\mathcal{D}}_{16}] + \frac{3}{5}[\widetilde{\mathcal{GP}}].$$

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