EFFECTIVE DIVISORS ON MODULI OF CURVES AND HURWITZ SPACES

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ABSTRACT. We discuss the transition from uniruledness to being of general type for moduli spaces of curves and Hurwitz spaces of covers. First, we construct and calculate the class of a virtual divisor on the moduli space $\overline{\mathcal{M}}_{23}$ of stable curves of genus 23, having slope less than that of the canonical divisor $K_{\overline{\mathcal{M}}_{23}}$. Furthermore, we calculate the class of a series of virtual divisors on the moduli space $\overline{\mathcal{M}}_{2s^2+s+1}$, with $s \geq 3$, having slope less that that of the Brill-Noether divisors. Combined with the recent proofs of the *Strong Maximal Rank Conjecture* obtained independently by Liu, Osserman, Teixidor, Zhang and by Jansen and Payne respectively, this completes the proof that both moduli spaces $\overline{\mathcal{M}}_{22}$ and $\overline{\mathcal{M}}_{23}$ are varieties of general type. Finally, we prove the effectiveness of the canonical bundle of several Hurwitz spaces $\overline{\mathcal{H}}_{g,k}$ of degree k admissible covers of $\mathbf{P}^1$ from curves of genus $14 \leq g \leq 19$.

Following a principle due to Mumford, most moduli spaces that appear in algebraic geometry (classifying curves, abelian varieties, $K3$ surfaces) are of general type, with a finite number of exceptions, which are unirational, or at least uniruled. Understanding the transition from negative Kodaira dimension to being of general type is usually quite difficult. With one exception (the moduli space of spin curves [FV]), for all these moduli spaces there are notorious open cases, when the Kodaira dimension is not known. The aim of this paper is to shed some light on this change of the birational nature of two highly related moduli spaces of curves.

In a series of landmark papers [HM], [H], [EH2] published in the 1980s, Harris, Mumford and Eisenbud proved that $\overline{\mathcal{M}}_g$ is a variety of general type for $g > 23$. This contrasts with the classical result of Severi [Sev] that $\overline{\mathcal{M}}_g$ is unirational for $g \leq 10$ (see [AC] for a beautiful modern treatment) and with the more recent results of Chang-Ran [CR1], [CR2], [CR3], Sernesi [Ser], Verra [Ve] and Schreyer [Sch], which summarized, amount to the statement that $\overline{\mathcal{M}}_g$ is uniruled for $g \leq 16$. The Slope Conjecture of Harris and Morrison [HMo] predicted that the Brill-Noether divisors are the effective divisors on $\overline{\mathcal{M}}_g$ having minimal slope $6 + \frac{12}{g+1}$. This led people to expect that $\overline{\mathcal{M}}_g$ changes from uniruledness to being of general type precisely at genus $g = 23$. However the Slope Conjecture turned out to be false and there are instances of effective divisors on $\overline{\mathcal{M}}_g$ for infinitely many genera $g \geq 10$ having slope less than $6 + \frac{12}{g+1}$, see [FP], [F2], [F3], [Kh], [FR]. In view of these examples it is to be expected that there should be an effective divisor of slope less than $\frac{13}{2} = 6 + \frac{12}{24}$ on $\overline{\mathcal{M}}_{23}$ as well, which would imply that $\overline{\mathcal{M}}_{23}$ is of general type. The best known result on $\overline{\mathcal{M}}_{23}$ is the statement $\kappa(\overline{\mathcal{M}}_{23}) \geq 2$, proven in [F1] via a study of the relative position of the three Brill-Noether divisors on $\overline{\mathcal{M}}_{23}$.

Effective divisors on $\overline{\mathcal{M}}_g$

One central aim of this paper is to address this transition case and reduce the calculation of the Kodaira dimension of $\overline{\mathcal{M}}_{23}$ to a transversality statement for generic curves of genus 23. In two very recent breakthrough papers, this transversality statement, also known as the *Strong Maximal Rank Conjecture* has been established independently by Liu, Osserman, Teixidor, Zhang [LOTZ1] and Jensen, Payne [JP2] respectively.

We begin by describing our construction of an effective divisor on $\mathcal{M}_{23}$. By Brill-Noether theory, a general curve C of genus 23 carries a two dimensional family of linear series $L \in W^6_{26}(C)$, all satisfying $h^1(C,L) = 3$. Each of these linear series is complete and very ample. Consider the multiplication map

$$\phi_L : \mathrm{Sym}^2 H^0(C,L) \to H^0(C, L^{\otimes 2}).$$

By Riemann-Roch $h^0(C, L^{\otimes 2}) = 30$, whereas $\dim \mathrm{Sym}^2 H^0(C,L) = 28$. Imposing the condition that ϕ_L be non-injective, one expects a codimension 3 locus inside the parameter space of pairs $[C,L]$. Since this parameter space has 2-dimensional fibres over $\mathcal{M}_{23}$, by projection, one expects a divisor inside the moduli space $\mathcal{M}_{23}$.

Theorem 0.1. *The following locus consisting of curves of genus* 23

$$\mathfrak{D} := \Big\{[C] \in \mathcal{M}_{23} : \exists L \in W^6_{26}(C) \ \textit{with} \ \mathrm{Sym}^2 H^0(C,L) \stackrel{\phi_L}{\longrightarrow} H^0(C, L^{\otimes 2}) \ \textit{not injective}\Big\}$$

is a virtual divisor on $\mathcal{M}_{23}$. *The virtual class of its compactification inside* $\overline{\mathcal{M}}_{23}$ *equals*

$$[\widetilde{\mathfrak{D}}]^{\mathrm{virt}} = \frac{4}{9}\binom{19}{8}\Big(470749\lambda - 72725\,\delta_0 - 401951\,\delta_1 - \sum_{j=2}^{11} b_j\,\delta_j\Big) \in CH^1(\overline{\mathcal{M}}_{23}),$$

where $b_j \geq b_1$ *for* $j \geq 2$. *In particular,* $s\big([\widetilde{\mathfrak{D}}]^{\mathrm{virt}}\big) = \frac{470749}{72725} = 6.473\ldots < \frac{13}{2}$.

For the precise definition of the virtual class $[\widetilde{\mathfrak{D}}]^{\mathrm{virt}}$ and which coincides with the class of the closure of $\mathfrak{D}$ inside a partial compactification of $\mathcal{M}_{23}$, we refer to Definition 1.3. Recall that the slope of an effective divisor D on the moduli space $\overline{\mathcal{M}}_g$ not containing any boundary divisor Δ_i in its support is defined as the quantity $s(D) := \frac{a}{\inf_{i\geq 0} b_i}$, where $[D] = a\lambda - \sum_{i=0}^{\lfloor \frac{g}{2}\rfloor} b_i\delta_i \in CH^1(\overline{\mathcal{M}}_g)$, with $a, b_i \geq 0$. The canonical class of the moduli space $\overline{\mathcal{M}}_g$ can be expressed as $K_{\overline{\mathcal{M}}_g} = 13\lambda - 2\sum_{i=0}^{\lfloor \frac{g}{2}\rfloor}\delta_i - \delta_1$, see [HM]. Since $s\big([\widetilde{\mathfrak{D}}]^{\mathrm{virt}}\big) < s(K_{\overline{\mathcal{M}}_g}) = \frac{13}{2}$, Theorem 0.1 implies the following (conditional) result on the Kodaira dimension of $\mathcal{M}_{23}$:

Corollary 0.2. *Assuming* $\mathfrak{D}$ *is a genuine divisor on* $\mathcal{M}_{23}$, *that is, the multiplication map*

$$\phi_L : \mathrm{Sym}^2 H^0(C,L) \to H^0(C, L^{\otimes 2})$$

is injective for every linear system $L \in W^6_{26}(C)$ *on a general curve* C *of genus* 23, *then* $\overline{\mathcal{M}}_{23}$ *is a variety of general type.*

The construction of $\widetilde{\mathfrak{D}}$ is inspired by results in [FP], where it is established that any divisor on $\overline{\mathcal{M}}_g$ having slope less that $6 + \frac{12}{g+1}$ must necessarily contain the locus $\mathcal{K}_g \subseteq \mathcal{M}_g$ of curves lying on a $K3$ surface. Finding geometric divisors on $\mathcal{M}_g$ which verify this condition has proven to be difficult, for curves on $K3$ are notorious for behaving generically with respect to most interesting geometric properties, like those of Brill-Noether nature. A construction similar to that in Theorem 0.1 can be carried out for infinitely many genera using linear series having Brill-Noether number equal to 1. We fix $s \geq 2$ and set

$$g := 2s^2 + s + 1, \ d := 2s^2 + 2s + 1 \ \text{ and } \ r := 2s.$$

A general curve $[C] \in \mathcal{M}_{2s^2+s+1}$ carries a one dimensional family of linear systems $L \in W^{2s}_{2s^2+2s+1}(C)$. For each such L we consider the multiplication map

$$\phi_L : \mathrm{Sym}^2 H^0(C, L) \to H^0(C, L^{\otimes 2}).$$

Observe that $h^0(C, L^{\otimes 2}) - \mathrm{Sym}^2 H^0(C, L) = 2d + 1 - g - (s + 1)(2s + 1) = 1$, therefore, the condition that the map ϕ_L be non-injective is expected to lead to a codimension *two* locus in the parameter space of pairs $[C, L]$, where $L \in W^{2s}_{2s^2+2s+1}(C)$.

Theorem 0.3. *Fix $s \geq 2$. The following subvariety inside $\mathcal{M}_{2s^2+s+1}$*

$$\mathfrak{D}_s := \Big\{[C] \in \mathcal{M}_{2s^2+s+1} : \exists L \in W^{2s}_{2s^2+2s+1}(C) \ \textit{with } \phi_L \ \textit{not injective}\Big\}$$

is a virtual divisor on $\mathcal{M}_{2s^2+s+1}$. The slope of the virtual class of its compactification $\widetilde{\mathfrak{D}}_s$ inside $\overline{\mathcal{M}}_{2s^2+s+1}$ equals

$$s\big([\widetilde{\mathfrak{D}}_s]^{\mathrm{virt}}\big) = \frac{3(48s^8 - 56s^7 + 92s^6 - 90s^5 + 86s^4 + 324s^3 + 317s^2 + 182s + 48)}{24s^8 - 28s^7 + 22s^6 - 5s^5 + 43s^4 + 112s^3 + 100s^2 + 50s + 12} =$$

$$6 + \frac{12}{g+1} - \frac{3(120s^6 - 140s^5 - 162s^4 + 67s^3 + 153s^2 + 94s + 24)}{(2s^2 + s + 1)(24s^8 - 28s^7 + 22s^6 - 5s^5 + 43s^4 + 112s^3 + 100s^2 + 50s + 12)}.$$

In particular, $s\big([\widetilde{\mathfrak{D}}_s]^{\mathrm{virt}}\big) < 6 + \frac{12}{g+1}$, for all $s \geq 3$.

The Brill-Noether divisors $\mathcal{M}^d_{g,r}$ of curves $[C] \in \mathcal{M}_g$ such that $W^r_d(C) \neq \emptyset$, where $\rho(g, r, d) := g - (r + 1)(g - d + r) = -1$, have slope $6 + \frac{12}{g+1}$, see [HM] and [EH2]. Theorem 0.3 shows that if , as expected, $\mathfrak{D}_s$ is a genuine divisor on $\mathcal{M}_{2s^2+s+1}$, its slope is smaller than that of $6 + \frac{12}{g+1}$ for all $s \geq 3$. For $s = 2$, applying Theorem 0.3 we obtain the divisor $\widetilde{\mathfrak{D}}_2$ on $\overline{\mathcal{M}}_{11}$, which has already been interpreted in [BF] and earlier in [FO], as the failure locus of the *Mercat Conjecture* for curves of genus 11. Precisely, $\mathfrak{D}_2$ consists of curves $[C] \in \mathcal{M}_{11}$ carrying a stable rank two vector bundle E with $h^0(C, E) \geq 4$ and $\det(E) \in W^4_{13}(C)$. In this case $\mathfrak{D}_2$ is known to be a genuine divisor on $\mathcal{M}_{11}$ and Theorem 0.3 yields $s\big([\widetilde{\mathfrak{D}}_2]\big) = 7$, which agrees with the conclusions of [BF] Theorem 2.

The case $s = 3$ of Theorem 0.3 leads to the virtual divisor $\widetilde{\mathfrak{D}}_3$ on $\overline{\mathcal{M}}_{22}$. In this case $s\big([\widetilde{\mathfrak{D}}_3]\big) = \frac{17121}{2636} = 6.495\ldots < 6.5$, thus if $\mathfrak{D}_3$ is a genuine divisor, then $\overline{\mathcal{M}}_{22}$ is of general type. The calculation of the class $[\widetilde{\mathfrak{D}}_3]^{\mathrm{virt}}$, that is, the case $s = 3$ of Theorem 0.3 has already been carried out in [F4] Theorem 7.1. In *loc.cit.* it was also announced that $\mathfrak{D}_3$ is an effective divisor on $\mathcal{M}_{22}$, but unfortunately the intended proof has not materialized.

One can generalize the construction of the (virtual) divisors $\mathfrak{D}_s$ (and that of $\mathfrak{D}$) and consider for every $a \geq 1$ and $s \geq 2$, the locus $\mathfrak{D}_{s,a}$ of curves $[C] \in \mathcal{M}_{2s^2+s+a}$ such that there exists a linear system $L \in W^{2s}_{2s^2+2s+a}(C)$ with the multiplication map $\phi_L : \mathrm{Sym}^2 H^0(C, L) \to H^0(C, L^{\otimes 2})$ being non-injective. A parameter count shows that $\mathfrak{D}_{s,a}$ is indeed a virtual divisor. In this way, $\mathfrak{D}_{s,1} = \mathfrak{D}_s$, whereas setting $s = 3$ and $a = 2$, we recover the divisor $\mathfrak{D} = \mathfrak{D}_{3,2}$ on $\mathcal{M}_{23}$. However, finding a closed formula for the virtual class $[\widetilde{\mathfrak{D}}_{s,a}]^{\mathrm{virt}}$ of the closure of $\mathfrak{D}_{s,a}$ in $\overline{\mathcal{M}}_{2s^2+s+a}$ is a daunting task. One stumbling block lies in generalizing Proposition 1.8 and 1.9 to this setting, coupled with the major difficulty that the virtual class in question is the push-forward to $\overline{\mathcal{M}}_{2s^2+s+a}$ of a class having codimension $a+1$ (that is, arbitrarily high) in the moduli space of pairs $[C, L]$, when the higher Chow rings of these moduli spaces are poorly understood.

The Strong Maximal Rank Conjecture

The Maximal Rank Conjecture, originally due to Eisenbud and Harris, predicts that for a pair $[C, L]$, where C is a general curve of genus g and $L \in W^r_d(C)$ is a general linear system, the multiplication of global sections $\phi_L : \mathrm{Sym}^2 H^0(C, L) \to H^0(C, L^{\otimes 2})$ is of maximal rank. The conjecture has been the focus of much attention, both a couple of decades ago [BE] using embedded degenerations in projective space, as well as recently in [JP1] using tropical geometry, or in [LOTZ1] using limit linear series. A full solution of the Maximal Rank Conjecture, considering the multiplication maps $\phi^k_L : \mathrm{Sym}^k H^0(C, L) \to H^0(C, L^{\otimes k})$ for *all* $k \geq 2$, has been recently found by Larson [La].

A refined version of the Maximal Rank Conjecture, taking into account *every* linear series $L \in W^r_d(C)$ on a general curve (rather than the general one), has been put forward in [AF] Conjecture 5.4. The *Strong Maximal Rank Conjecture*, motivated by applications to the birational geometry of $\overline{\mathcal{M}}_g$, predicts that for a general curve C of genus g and for positive integers r, d such that $0 \leq \rho(g, r, d) \leq r-2$, the determinantal variety

$$\Big\{L \in W^r_d(C) : \phi_L : \mathrm{Sym}^2 H^0(C, L) \to H^0(C, L^{\otimes 2}) \text{ is not of maximal rank}\Big\}$$

has the expected dimension. The Strong Maximal Rank Conjecture in the cases $g = 23$, $d = 26$ and $r = 6$ and respectively $g = 22$, $d = 25$, $r = 6$ amounts to the statement that the virtual divisors $\mathfrak{D}$ on $\mathcal{M}_{23}$ (discussed in Theorem 0.1), respectively $\mathfrak{D}_3$ on $\mathcal{M}_{22}$ are genuine divisors. These two important cases of the Strong Maximal Conjecture have been recently proved by Liu, Osserman, Teixidor, Zhang [LOTZ2] and independently by Payne and Jensen [JP2]. Putting together Theorem 0.1, Theorem 0.3 (in the case $s = 3$) and the new results [LOTZ2] and [JP2], one can conclude:

Theorem 0.4. *The moduli spaces $\overline{\mathcal{M}}_{22}$ and $\overline{\mathcal{M}}_{23}$ are both of general type.*

Other interesting cases of the Strong Maximal Rank Conjecture for line bundles L with $h^1(C, L) = 2$ have been proved to be equivalent to *Mercat's Conjecture* for rank 2 vector bundles on a generic curve and shown to hold in [BF] using $K3$ surfaces. One can formulate a Strong Maximal Rank Conjecture for the maps ϕ^k_L as well, though I am unaware of applications to moduli when $k \geq 3$.

The Kodaira dimension of Hurwitz spaces

The second goal of this paper is to discuss the transition from unirationality to being of general type in the case of the Hurwitz space $\mathcal{H}_{g,k}$ classifying degree k covers $C \to \mathbf{P}^1$ with only simple ramification, from a smooth curve C of genus g. Hurwitz spaces provide an interesting bridge between the more accessible moduli spaces of pointed rational curves and the moduli space of curves. We denote by $\overline{\mathcal{H}}_{g,k}$ the moduli space of admissible covers constructed by Harris and Mumford [HM] and studied further in [ACV]. It comes equipped with a finite branch map

$$\mathfrak{b} : \overline{\mathcal{H}}_{g,k} \to \overline{\mathcal{M}}_{0,2g+2k-2}/\mathfrak{S}_{2g+2k-2},$$

where the target is the moduli space of *unordered* $(2g+2k-2)$-pointed stable rational curves, as well as with a map

$$\sigma : \overline{\mathcal{H}}_{g,k} \to \overline{\mathcal{M}}_g,$$

obtained by assigning to an admissible cover the stable model of its source.

It is a fundamental question to describe the birational nature of $\overline{\mathcal{H}}_{g,k}$ and one knows much less than in the case of other moduli spaces like $\overline{\mathcal{M}}_g$ or $\overline{cA}_g$. From Brill-Noether theory it follows that when $k \geq \frac{g+2}{2}$, every curve of genus g can be represented as a k-sheeted cover of $\mathbf{P}^1$, that is, $\sigma : \overline{\mathcal{H}}_{g,k} \to \overline{\mathcal{M}}_g$ is dominant, and thus $\overline{\mathcal{H}}_{g,k}$ is of general type whenever $\overline{\mathcal{M}}_g$ is. Classically it has been known that $\mathcal{H}_{g,k}$ is unirational for $k \leq 5$, see again [AC] and references therein. Geiss [G] using liaison techniques showed that most Hurwitz spaces $\mathcal{H}_{g,6}$ with $g \leq 45$ are unirational. Schreyer and Tanturri [ST] put forward the hypothesis that there exist only finitely many pairs (g,k), with $k \geq 6$, such that $\overline{\mathcal{H}}_{g,k}$ is *not* of general type.

We are particularly interested in the range $k \geq \frac{g+2}{2}$ (when $\overline{\mathcal{H}}_{g,k}$ dominates $\overline{\mathcal{M}}_g$) and $14 \leq g \leq 19$, so that $\overline{\mathcal{M}}_g$ is either unirational/uniruled for $g = 14, 15, 16$, or its Kodaira dimension is unknown, when $g = 17, 18, 19$. We summarize our results showing the positivity of the canonical bundle of $\overline{\mathcal{H}}_{g,k}$ and we begin with the cases when $\overline{\mathcal{M}}_g$ is known to be uniruled.

Theorem 0.5. *Suppose $\overline{\mathcal{H}}$ is one of the spaces $\overline{\mathcal{H}}_{14,9}$ or $\overline{\mathcal{H}}_{16,9}$. Then there exists an effective $\mathbb{Q}$-divisor class E on $\overline{\mathcal{H}}$ contracted under the map $\sigma : \overline{\mathcal{H}} \to \overline{\mathcal{M}}_g$, such that the twisted canonical class $K_{\overline{\mathcal{H}}} + E$ is effective.*

A few comments are in order. Firstly, the boundary divisor E (that may be empty) is supported on the locus $\sum_{i\geq 1} \sigma^*(\Delta_i)$ of admissible covers with source curve being of compact type. Secondly, Theorem 0.5 is optimal. In genus 14, Verra [Ve] established the unirationality of the Hurwitz space $\mathcal{H}_{14,8}$, which is a finite cover of $\mathcal{M}_{14}$, and concluded in this way that $\mathcal{M}_{14}$ itself is unirational. In genus 16, the map $\sigma : \overline{\mathcal{H}}_{16,9} \to \overline{\mathcal{M}}_{16}$ is generically finite. Chang and Ran [CR3] showed that $\overline{\mathcal{M}}_{16}$ is uniruled (more precisely, they proved that $K_{\overline{\mathcal{M}}_{16}}$ is not pseudo-effective, which by now, one knows that it implies uniruledness). This is the highest genus for which $\overline{\mathcal{M}}_g$ is known to be uniruled.

Next, we consider the range when the Kodaira dimension of $\overline{\mathcal{M}}_g$ is unknown.

Theorem 0.6. *Suppose $\overline{\mathcal{H}}$ is one of the spaces $\overline{\mathcal{H}}_{17,11}$ or $\overline{\mathcal{H}}_{19,13}$. Then there exists an effective $\mathbb{Q}$-divisor class E on $\overline{\mathcal{H}}$ contracted under the map $\sigma : \overline{\mathcal{H}} \to \overline{\mathcal{M}}_g$, such that the twisted canonical class $K_{\overline{\mathcal{H}}} + E$ is big.*

We observe that the maps $\overline{\mathcal{H}}_{17,11} \to \overline{\mathcal{M}}_{17}$ and $\overline{\mathcal{H}}_{19,13} \to \overline{\mathcal{M}}_{19}$ have generically 3 and respectively 5-dimensional fibres. As in Theorem 0.5, the divisor E is supported on the divisor $\sum_{i\geq 1} \sigma^*(\Delta_i)$.

Both Theorems 0.5 and 0.6 are proven at the level of a partial compactification $\widetilde{\mathcal{G}}^1_{g,k}$ (described in detail in Section 3) and which incorporates only admissible covers with a source curve whose stable model is irreducible. In each relevant genus we produce an explicit effective divisor $\widetilde{\mathcal{D}}_g$ on $\widetilde{\mathcal{G}}^1_{g,k}$ such that the canonical class $K_{\widetilde{\mathcal{G}}^1_{g,k}}$ can be expressed as a positive combination of a multiple of $[\widetilde{\mathcal{D}}_g]$ and the pull-back under the map σ of an effective divisor on $\overline{\mathcal{M}}_g$.

We describe the construction of these divisors in two instances and refer to Section 3 for the remaining cases and further details. First we consider the space $\overline{\mathcal{H}}_{16,9}$ which is a generically finite cover of $\overline{\mathcal{M}}_{16}$.

Let us choose a general pair $[C,A] \in \mathcal{H}_{16,9}$. We set $L := K_C \otimes A^\vee \in W^7_{21}(C)$. Denoting by $I_{C,L}(k) := \mathrm{Ker}\{\mathrm{Sym}^k H^0(C,L) \to H^0(C, L^{\otimes k})\}$, by Riemann-Roch one computes that $\dim I_{C,L}(2) = 9$ and $\dim I_{C,L}(3) = 72$. Therefore the locus where there exists a non-trivial syzygy, that is, the map $\mu_{C,L} : I_{C,L}(2) \otimes H^0(C,L) \to I_{C,L}(3)$ is not an isomorphism is expected to be a divisor on $\mathcal{H}_{16,9}$. This indeed is the case and we define the *syzygy divisor*

$$\mathcal{D}_{16} := \Big\{[C,A] \in \mathcal{H}_{16,9} : I_{C,K_C\otimes A^\vee}(2) \otimes H^0(C, K_C \otimes A^\vee) \stackrel{\ncong}{\to} I_{C,K_C\otimes A^\vee}(3)\Big\}.$$

The ramification divisor of the map $\sigma : \mathcal{H}_{16,9} \to \mathcal{M}_{16}$, viewed as the *Gieseker-Petri divisor*

$$\mathcal{GP} := \Big\{[C,A] \in \mathcal{H}_{16,9} : H^0(C,A) \otimes H^0(C, K_C \otimes A^\vee) \stackrel{\ncong}{\to} H^0(C, K_C)\Big\}$$

also plays an important role. After computing the class of the closure $\widetilde{\mathcal{D}}_{16}$ of the Koszul divisor $\mathcal{D}_{16}$ inside the partial compactification $\widetilde{\mathcal{G}}^1_{16,9}$, we find the following explicit representative for the canonical class

$$K_{\widetilde{\mathcal{G}}^1_{16,9}} = \frac{2}{5}[\widetilde{\mathcal{D}}_{16}] + \frac{3}{5}[\widetilde{\mathcal{GP}}] \in CH^1(\widetilde{\mathcal{G}}^1_{16,9}). \tag{1}$$

This proves Theorem 0.5 in the case $g = 16$. We may wonder whether (1) is the only effective representative of $K_{\widetilde{\mathcal{G}}^1_{16,9}}$, which would imply that the Kodaira dimension of $\overline{\mathcal{H}}_{16,9}$ is equal to zero. To summarize, since the smallest Hurwitz cover of $\overline{\mathcal{M}}_{16}$ has an effective canonical class, it seems unlikely that the method of establishing the uniruledness/unirationality of $\mathcal{M}_{14}$ and $\mathcal{M}_{15}$ by studying a Hurwitz space covering it can be extended to higher genera $g \geq 17$.

The last case we discuss in this introduction is $g = 17$ and $k = 11$. We choose a general pair $[C,A] \in \mathcal{H}_{17,11}$. The residual linear system $L := K_C \otimes A^\vee \in W^6_{21}(C)$ induces an embedding $C \subseteq \mathbf{P}^6$ of degree 21. The multiplication map

$$\phi_L : \mathrm{Sym}^2 H^0(C,L) \to H^0(C, L^{\otimes 2})$$

has a 2-dimensional kernel. We impose the condition that the pencil of quadrics containing the image curve $C \stackrel{|L|}{\hookrightarrow} \mathbf{P}^6$ be *degenerate*, that is, it intersects the discriminant divisor in $\mathbf{P}\big(\mathrm{Sym}^2 H^0(C,L)\big)$ non-transversally. We thus define the locus

$$\mathcal{D}_{17} := \Big\{[C,A] \in \mathcal{H}_{17,11} : \mathbf{P}\big(\mathrm{Ker}(\phi_{K_C\otimes A^\vee})\big) \text{ is a degerate pencil}\Big\}.$$

Using [FR], we can compute the class $[\widetilde{\mathcal{D}}_{17}]$ of the closure of $\mathcal{D}_{17}$ inside $\widetilde{\mathcal{G}}_{17,11}$. Comparing this class to that of the canonical divisor, we obtain the relation

$$K_{\widetilde{\mathcal{G}}^1_{17,11}} = \frac{1}{5}[\widetilde{\mathcal{D}}_{17}] + \frac{3}{5}\sigma^*(7\lambda - \delta_0) \in CH^1(\widetilde{\mathcal{G}}^1_{17,11}). \tag{2}$$

Since the class $7\lambda - \delta_0$ can be shown to be big on $\overline{\mathcal{M}}_{17}$, the conclusion of Theorem 0.6 in the case $g = 17$ now follows.

1. THE CONSTRUCTION OF VIRTUAL DIVISORS ON $\overline{\mathcal{M}}_g$

The (virtual) divisor $\mathfrak{D}$ is constructed as the push-forward of a codimension 3 cycle inside the universal parameter space of linear series of type $\mathfrak{g}^6_{26}$ on curves of genus 23. We describe the construction of this cycle, then extend this determinantal structure over a partial compactification of $\mathcal{M}_{23}$. This will suffice to compute the virtual slope (to be defined below) of $\widetilde{\mathfrak{D}}$ and eventually prove Theorem 0.1. We shall follow the same procedure in the case of the divisors $\mathfrak{D}_s$ on $\mathcal{M}_{2s^2+s+1}$, where $s \geq 2$. As long as the two constructions run parallel, we will treat both Theorems 0.1 and 0.3 unitarily.

In what follows we shall be in one of the following numerical situations

(3) $$g = 23,\ r = 6,\ d = 26, \ \text{ or}$$

(4) $$g = 2s^2 + s + 1,\ r = 2s,\ d = 2s^2 + 2s + 1, \ \text{ where } \ s \geq 2.$$

Note that $\rho(g, r, d) = 2$ in case (3) and $\rho(g, r, d) = 1$ in case (4). We denote by $\mathcal{M}_g^{\mathrm{part}}$ the open substack of $\mathcal{M}_g$ of curves C satisfying both properties $W^r_{d-1}(C) = \emptyset$ and $W^{r+1}_d(C) = \emptyset$. Basic results in Brill-Noether theory imply $\mathrm{codim}(\mathcal{M}_g - \mathcal{M}_g^{\mathrm{part}}, \mathcal{M}_g) \geq 2$. In particular, for every $[C] \in \mathcal{M}_g^{\mathrm{part}}$, the variety $W^r_d(C)$ is pure 2-dimensional in case (3) (respectively 1-dimensional in case (4)), and each linear system $L \in W^r_d(C)$ is complete and base point free.

Let $\Delta_1^p \subseteq \Delta_1 \subseteq \overline{\mathcal{M}}_g$ be the locus of curves $[C \cup_y E]$, where E is an arbitrary elliptic curve and C is a smooth curve of genus $g-1$ that verifies the Brill-Noether Theorem. The point of attachment $y \in C$ is chosen arbitrarily. Furthermore, let $\Delta_0^p \subseteq \Delta_0 \subseteq \overline{\mathcal{M}}_g$ be the locus consisting of curves $[C_{yq} := C/y \sim q] \in \Delta_0$, where $[C, q]$ is a Brill-Noether general curve of genus $g-1$ (in the pointed sense of [EH2]) and $y \in C$ is an arbitrary point, together with their degenerations $[C \cup_q E_\infty]$, where E_∞ is a rational nodal curve (that is, E_∞ is a nodal elliptic curve and $j(E_\infty) = \infty$). Observe that points of this form describe the intersection of Δ_0^p and Δ_1^p. We introduce the partial compactification

$$\overline{\mathcal{M}}_g^{\mathrm{part}} := \mathcal{M}_g^{\mathrm{part}} \cup \Delta_0^p \cup \Delta_1^p \subseteq \overline{\mathcal{M}}_g.$$

Note that $\overline{\mathcal{M}}_g^{\mathrm{part}}$ and $\mathcal{M}_g \cup \Delta_0 \cup \Delta_1$ are isomorphic in codimension one, in particular, we identify their Picard groups. We have that $\mathrm{Pic}_{\mathbb{Q}}(\overline{\mathcal{M}}_g^{\mathrm{part}}) \cong CH^1_{\mathbb{Q}}(\overline{\mathcal{M}}_g^{\mathrm{part}}) = \mathbb{Q}\langle \lambda, \delta_0, \delta_1 \rangle$, where λ is the Hodge class, $\delta_0 := [\Delta_0^p]$ and $\delta_1 := [\Delta_1^p]$.

We consider the following desingularized universal Picard variety of degree d

$$h : \widetilde{\mathcal{P}ic}^d_g \to \overline{\mathcal{M}}_g^{\mathrm{part}},$$

whose properties we describe.

(i) Over a smooth curve $[C] \in \mathcal{M}_g^{\mathrm{part}}$, we have $h^{-1}([C]) \cong \mathrm{Pic}^d(C)$.
(ii) Over a curve $[C \cup E] \in \Delta_1^p$, we have $h^{-1}([C \cup E]) \cong \mathrm{Pic}^d(C) \times \mathrm{Pic}^0(E)$.
(iii) If $[C_{yq}] \in \Delta_0^p$, with $\nu : C \to C_{yq}$ being the normalization map, then

$$h^{-1}\big([C_{yq}]\big) = \widetilde{\mathrm{Pic}}^d(C_{yq}) := \mathbf{P}\big(\mathcal{P}_y \oplus \mathcal{P}_q\big)$$

is the desingularization of the compactified Jacobian $\overline{\mathrm{Pic}}^d(C_{yq})$ of rank one torsion-free sheaves on C_{yq}. Here, $\mathcal{P}$ denotes a Poincaré bundle on $C \times \mathrm{Pic}^d(C)$ and $\mathcal{P}_y$ respectively $\mathcal{P}_q$ denote the restrictions of $\mathcal{P}$ to $\{y\} \times \mathrm{Pic}^d(C)$, respectively $\{q\} \times \mathrm{Pic}^d(C)$. Thus a point in $\widetilde{\mathrm{Pic}}^d(C_{yq})$ can be though of a pair (L, Q), where L is a line bundle of degree d on C and $L_y \oplus L_q \twoheadrightarrow Q$ is a 1-dimensional quotient. The map $\widetilde{\mathrm{Pic}}^d(C_{yq}) \to \overline{\mathrm{Pic}}^d(C_{yq})$ assigns to a pair (L, Q) the sheaf L' on C_{yq}, defined by the exact sequence

$$0 \longrightarrow L' \longrightarrow \nu_* L \longrightarrow Q \longrightarrow 0.$$

Definition 1.1. We denote by $\widetilde{\mathcal{G}}^r_d$ the variety classifying pairs $[C, \ell]$, where $[C] \in \overline{\mathcal{M}}_g^{\mathrm{part}}$ and ℓ is a limit linear series on (the tree-like curve) C in the sense of [EH1]. We set

$$\sigma : \widetilde{\mathcal{G}}^r_d \to \overline{\mathcal{M}}_g^{\mathrm{part}}$$

to be the proper projection map. One has a morphism $\widetilde{\mathcal{G}}^r_d \to \widetilde{Pic}^d_g$ commuting with h and σ, which over curves from Δ_1^p is obtained by retaining the underlying line bundle of the genus $(g-1)$-aspect of the corresponding limit linear series.

Remark 1.2. Suppose $[C_{yq}] \in \Delta_0^p$ and let L' be a rank 1 torsion free sheaf on C_{yq} with $h^0(L') = r+1$. Then there exists a unique line bundle $L \in W^r_d(C)$ such that $\nu_* L = L'$. If L' is non-locally free, then to this point will correspond *two* points in $\widetilde{\mathcal{G}}^r_d$. If $L \in W^r_{d-1}(C)$ is such that $\nu_* L = L'$, then these points are $(L(q), L(q)_q)$ and $(L(y), L(y)_y)$ respectively.

We need further notation. Let $\pi : \overline{\mathcal{C}}_g^{\mathrm{part}} \to \overline{\mathcal{M}}_g^{\mathrm{part}}$ be the universal curve and we introduce the projection map $p_2 : \overline{\mathcal{C}}_g^{\mathrm{part}} \times_{\overline{\mathcal{M}}_g^{\mathrm{part}}} \widetilde{\mathcal{G}}^r_d \to \widetilde{\mathcal{G}}^r_d$. We choose a Poincaré line bundle $\mathcal{L}$ over $\overline{\mathcal{C}}_g^{\mathrm{part}} \times_{\overline{\mathcal{M}}_g^{\mathrm{part}}} \widetilde{\mathcal{G}}^r_d$, such that for a curve $[C \cup E] \in \Delta_1^p$ and a limit linear series $\ell = (\ell_C, \ell_E) \in W^r_d(C) \times W^r_d(E)$, we have that $\mathcal{L}_{[C\cup E, \ell]} \in \mathrm{Pic}^d(C) \times \mathrm{Pic}^0(E)$. Both sheaves

$$\mathcal{E} := (p_2)_*(\mathcal{L}) \ \text{ and } \mathcal{F} := (p_2)_*(\mathcal{L}^{\otimes 2})$$

are locally free and $\mathrm{rank}(\mathcal{E}) = r+1$ and $\mathrm{rank}(\mathcal{F}) = 2d+1-g$ respectively. There is a natural vector bundle morphism over $\widetilde{\mathcal{G}}^r_d$ given by multiplication of sections,

$$\phi : \mathrm{Sym}^2(\mathcal{E}) \to \mathcal{F},$$

and we denote by $\mathcal{U} \subseteq \widetilde{\mathcal{G}}^r_d$ its first degeneracy locus. We set

$$\widetilde{\mathfrak{D}} := \sigma_*(\mathcal{U})$$

in case (3), respectively $\widetilde{\mathfrak{D}}_s := \sigma_*(\mathcal{U})$ in case (4). Since the degeneracy locus $\mathcal{U}$ has expected codimension 3 inside $\widetilde{\mathcal{G}}^6_{26}$ in the case (3), respectively codimension 2 inside $\widetilde{\mathcal{G}}^{2s}_{2s^2+2s+1}$ in the case (4), the loci $\widetilde{\mathfrak{D}}$ and $\widetilde{\mathfrak{D}}_s$ are virtual divisors on $\overline{\mathcal{M}}_g^{\mathrm{part}}$, that is, either they are divisors, or the entire space $\overline{\mathcal{M}}_g^{\mathrm{part}}$.

Definition 1.3. We define the class of the virtual divisor $\mathfrak{D}$ to be the quantity

$$[\widetilde{\mathfrak{D}}]^{\mathrm{virt}} := \sigma_*\Big(c_3(\mathcal{F} - \mathrm{Sym}^2(\mathcal{E}))\Big) \in CH^1(\overline{\mathcal{M}}_{23}^{\mathrm{part}}).$$

Similarly, for $s \geq 2$ the virtual class of the divisor $\mathfrak{D}_s$ is defined as

$$[\widetilde{\mathfrak{D}}_s]^{\mathrm{virt}} := \sigma_*\Big(c_2(\mathcal{F} - \mathrm{Sym}^2(\mathcal{E}))\Big) \in CH^1(\overline{\mathcal{M}}_{2s^2+s+1}^{\mathrm{part}}).$$

If, as the Strong Maximal Conjecture predicts, $\mathfrak{D}$ and $\mathfrak{D}_s$ are divisors on $\mathcal{M}_g$, then their virtual classes defined above coincide with the actual class of the closure of $\mathfrak{D}$ and $\mathfrak{D}_s$ inside $\overline{\mathcal{M}}_g^{\mathrm{part}}$. We now describe the vector bundle morphism ϕ in more detail.

Proposition 1.4. *The vector bundle map $\phi : \mathrm{Sym}^2(\mathcal{E}) \to \mathcal{F}$ has the following local description:*
(i) For $[C, L] \in \mathcal{G}^r_d$, with $[C] \in \mathcal{M}_g^{\mathrm{part}}$ being a smooth curve, one has

$$\mathcal{E}(C, L) = H^0(C, L) \ \textit{and} \ \mathcal{F}(C, L) = H^0(C, L^{\otimes 2})$$

and $\phi(C, L)$ is the usual multiplication of sections.
(ii) For a point $t = (C \cup_y E, \ell_C, \ell_E) \in \sigma^{-1}(\Delta_1^p)$, where C is a curve of genus $g-1$, E is an elliptic curve and $\ell_C = |L_C|$ is the C-aspect of the corresponding limit linear series, such that $L_C \in W^r_d(C)$ verifies $h^0(C, L_C(-2y)) \geq r$, one has $\mathcal{E}(t) = H^0(C, L_C)$ and

$$\mathcal{F}(t) = H^0(C, L_C^{\otimes 2}(-2y)) \oplus \mathbb{C} \cdot u^2,$$

where $u \in H^0(C, L_C)$ is any section such that $\mathrm{ord}_y(u) = 0$.

Assume now L_C has a base point at y. Then $\mathcal{E}(t) = H^0(C, L_C) \cong H^0(C, L_C(-y))$ and the image of the map $\mathcal{F}(t) \to H^0(C, L_C^{\otimes 2})$ is the subspace $H^0\big(C, L_C^{\otimes 2}(-2y)\big) \subseteq H^0(C, L_C^{\otimes 2})$.

(iii) Let $t = [C_{yq}, L] \in \sigma^{-1}(\Delta_0^p)$ be a point with $q, y \in C$ and $L \in \overline{W}^r_d(C_{yq})$ be a torsion-free sheaf of rank 1, such that $h^0(C, \nu^* L(-y-q)) \geq r$, where $\nu : C \to C_{yq}$ is the normalization map. If L is locally free, we have that

$$\mathcal{E}(t) = H^0(C, \nu^* L) \ \textit{and} \ \mathcal{F}(t) = H^0\big(C, \nu^* L^{\otimes 2}(-y-q)\big) \oplus \mathbb{C} \cdot u^2,$$

where $u \in H^0(C, \nu^* L)$ is any section not vanishing at both points y and q.

Assume $L = \nu_*(A)$ is a non locally free torsion-free sheaf of rank 1 on the curve C_{yq}, where $A \in W^r_{d-1}(C)$, the image of the map $\mathcal{F}(t) \to H^0(C, \nu^* L^{\otimes 2})$ equals $H^0(C, A^{\otimes 2})$.

Proof. Both sheaves $(p_2)_*(\mathcal{L})$ and $(p_2)_*(\mathcal{L}^{\otimes 2})$ are reflexive, hence in this case, also locally free of ranks $r+1$ and respectively $2d+1-g$. The local description of the fibres of both $\mathcal{E}$ and $\mathcal{F}$ is straightforward. For $\mathcal{E}$ it follows from Grauert's Theorem. In the case of $\mathcal{F}$, part (i) is immediate, whereas for part (ii), keeping the notation above, one always has the inclusion $H^0(C, L_C^{\otimes 2}(-2y)) \oplus \mathbb{C} \cdot u^2 \subseteq \mathcal{F}(t)$. The conclusion follows comparing dimensions, once we observe that $u \notin H^0(C, L_C(-y))$, because $W^r_{d-2}(C) = \emptyset$. Case (iii) is analogous. $\square$

Our main result concerns the explicit calculation of $[\widetilde{\mathfrak{D}}]^{\mathrm{virt}}$ and $[\widetilde{\mathfrak{D}}_s]^{\mathrm{virt}}$. In order to prove Theorems 0.1 and 0.3, we describe the restriction of the morphism ϕ along the pull-backs of two standard test curves in the boundary of $\overline{\mathcal{M}}_g^{\mathrm{part}}$ which have been used in many other instances: Choose a general pointed curve $[C, q]$ of genus $g-1$ and a pointed elliptic curve $[E, y]$. Then construct the families of stable curves of genus g

$$C_0 := \Big\{ C_{yq} := C/y \sim q : y \in C \Big\} \subseteq \Delta_0^p \subseteq \overline{\mathcal{M}}_g, \quad \text{and}$$
$$C_1 := \Big\{ C \cup_y E : y \in C \Big\} \subseteq \Delta_1^p \subseteq \overline{\mathcal{M}}_g.$$

The intersection of the test curves with the generators of $CH^1(\overline{\mathcal{M}}_g)$ is well-known, see for instance [HM]:

$$C_0 \cdot \lambda = 0, \ C_0 \cdot \delta_0 = \deg(\omega_{C_{yq}}) = 2 - 2g, \ C_0 \cdot \delta_1 = 1 \text{ and } C_0 \cdot \delta_j = 0 \ \text{ for } \ j = 2, \ldots, \lfloor \frac{g}{2} \rfloor,$$

respectively

$$C_1\cdot\lambda=0,\ C_1\cdot\delta_0=0,\ C_1\cdot\delta_1=-\deg(K_C)=4-2g\ \text{ and }\ C_1\cdot\delta_j=0 \text{ for } j=2,\ldots,\lfloor\frac{g}{2}\rfloor.$$

Having fixed a general pointed curve $[C,q]\in\mathcal{M}_{g-1,1}$, we first turn our attention to the pull-back $\sigma^*(C_0)\subseteq\widetilde{\mathcal{G}}^r_d$. We consider the determinantal variety

$$Y:=\Bigl\{(y,L)\in C\times W^r_d(C): h^0(C,L(-y-q))\geq r\Bigr\},$$

together with the projection $\pi_1: Y\to C$. Inside Y we introduce the following loci

$$\Gamma_1:=\Bigl\{(y,A(y)): y\in C,\ A\in W^r_{d-1}(C)\Bigr\}\ \text{ and}$$

$$\Gamma_2:=\Bigl\{(y,A(q)): y\in C,\ A\in W^r_{d-1}(C)\Bigr\}.$$

Let $\vartheta: Y':=\mathrm{Bl}_\Gamma(Y)\to Y$ be the blow-up of Y along the intersection Γ of Γ_1 and Γ_2.

Proposition 1.5. *With notation as above, there is a birational morphism*

$$f:\sigma^*(C_0)\to Y',$$

which is an isomorphism outside $\vartheta^{-1}(\pi_1^{-1}(q))$. The restriction of f to $(\pi_1\circ\vartheta\circ f)^{-1}(q)$ forgets the aspect of each limit linear series on the elliptic curve E_∞. Furthermore, $\mathcal{E}_{|\sigma^*(C_0)}$ and $\mathcal{F}_{|\sigma^*(C_0)}$ are pull-backs under f of vector bundles on Y.

Proof. We pick $y\in C\setminus\{q\}$ and, as usual, $\nu: C\to C_{yq}$ be the normalization. Recall that $\overline{W}^r_d(C_{yq})\subseteq\overline{\mathrm{Pic}}^d(C_{yq})$ is the variety of rank 1 torsion-free sheaves on C_{yq} with $h^0(C_{yq},L)\geq r+1$. A locally free sheaf $L\in\overline{W}^r_d(C_{yq})$ is uniquely determined by its pull-back $\nu^*(L)\in W^r_d(C)$, which has the property $h^0(C,\nu^*L(-y-q))=r$ (note that we have assumed that $W^r_{d-2}(C)=\emptyset$, thus there must exist a section of L that does not vanish simultaneously at both y and q. In other words, the 1-dimensional quotient Q of $L_y\oplus L_q$ is uniquely determined as $\nu_*(\nu^*L)/L$).

Assume now, $L\in\overline{W}^d_r(C_{yq})$ is not locally free, thus $L=\nu_*(A)$, for a linear series $A\in W^r_{d-1}(C)$. Using Remark 1.2, to this point correspond two points in Y, namely $(y,A(y))$ and $(y,A(q))$. Summarizing, for $y\in C\setminus\{q\}$, there is a birational morphism $\pi_1^{-1}(y)\to\overline{W}^r_d(C_{yq})$ which is an isomorphism over the locus of locally free sheaves. More precisely, $\overline{W}^r_d(C_{yq})$ is obtained from $\pi_1^{-1}(y)$ by identifying the disjoint divisors $\Gamma_1\cap\pi_1^{-1}(y)$ and $\Gamma_2\cap\pi_1^{-1}(y)$.

Finally, when $y=q$, then C_{yq} degenerates to $C\cup_q E_\infty$, where E_∞ is a rational nodal cubic. If $(\ell_C,\ell_{E_\infty})\in\sigma^{-1}([C\cup_q E_\infty])$, then the corresponding Brill-Noether numbers with respect to q satisfy $\rho(\ell_C,q)\geq 0$ and $\rho(\ell_{E_\infty},q)\leq 2$ in case (3), respectively $\rho(\ell_C,q)\leq 1$ in case (4). The statement about the restrictions $\mathcal{E}_{|\sigma^*(C_0)}$ and $\mathcal{F}_{|\sigma^*(C_0)}$ follows, because both restrictions are defined by dropping the information coming from the elliptic tail. $\square$

We now describe the pull-back $\sigma^*(C_1)\subseteq\widetilde{\mathcal{G}}^r_d$. To that end, we define the locus

$$X:=\Bigl\{(y,L)\in C\times W^r_d(C): h^0(L(-2y))\geq r\Bigr\},$$

and by slight abuse of notation, we denote again by $\pi_1: X\to C$ the first projection.

In what follows we use notation from [EH1] to denote vanishing sequences of limit linear series on curves from the divisor Δ_1^p. In particular, if Y is a smooth curve of genus g and $y \in Y$, then for a linear series $\ell \in G^r_d(C)$ we denote by

$$\rho(\ell, y) := \rho(g, r, d) - w^{\ell}(y)$$

the *adjusted Brill-Noether number*. Here $w^{\ell} := \sum_{i=0}^r \alpha_i^{\ell}(y)$ is the *weight* of the point y with respect to ℓ and $\alpha^{\ell}(y) : \alpha_0^{\ell}(y) \leq \ldots \leq \alpha_r^{\ell}(y)$ is the *ramification sequence* of ℓ at y.

Proposition 1.6. *The variety X is an irreducible component of $\sigma^*(C_1)$. In case (3)*

$$c_3\big(\mathcal{F} - \mathrm{Sym}^2(\mathcal{E})\big)_{|\sigma^*(C_1)} = c_3\big(\mathcal{F} - \mathrm{Sym}^2(\mathcal{E})\big)_{|X}, \ \textit{and}$$

$$c_2\big(\mathcal{F} - \mathrm{Sym}^2(\mathcal{E})\big)_{|\sigma^*(C_1)} = c_2\big(\mathcal{F} - \mathrm{Sym}^2(\mathcal{E})\big)_{|X}$$

in case (4).

Proof. We deal with the case $(g, r, d) = (23, 6, 26)$, the case (4) being analogous. By the additivity of the Brill-Noether number, if $(\ell_C, \ell_E) \in \sigma^{-1}([C \cup_y E])$ is a limit linear series of type $\mathfrak{g}^6_{26}$, we have that $2 = \rho(23, 6, 26) \geq \rho(\ell_C, y) + \rho(\ell_E, y)$. Since $\rho(\ell_E, y) \geq 0$, we obtain that $\rho(\ell_C, y) \leq 2$. If $\rho(\ell_E, y) = 0$, then $\ell_E = 19y + |\mathcal{O}_E(7y)|$. This shows that ℓ_E is uniquely determined, while the aspect $\ell_C \in G^6_{26}(C)$ is a complete linear series $\mathfrak{g}^6_{26}$ with a cusp at $y \in C$. This gives rise to an element from X and shows that $X \times \{\ell_E\} \cong X$ is a component of $\sigma^*(C_1)$.

The other components of $\sigma^*(C_1)$ are indexed by Schubert indices

$$\alpha := (0 \leq \alpha_0 \leq \ldots \leq \alpha_6 \leq 20 = 26 - 6),$$

such that lexicographically $\alpha > (0, 1, 1, 1, 1, 1, 1)$, and $7 \leq \sum_{j=0}^6 \alpha_j \leq 9$, for we must have $-1 \leq \rho(\ell_C, y) \leq 1$. For such an index α, we set $\bar{\alpha}^c := (20 - \alpha_6, \ldots, 20 - \alpha_0)$ to be the complementary Schubert index, then define

$$X_\alpha := \big\{(y, \ell_C) \in C \times G^6_{26}(C) : \alpha^{\ell_C}(y) \geq \alpha\big\} \text{ and } Z_\alpha := \big\{\ell_E \in G^6_{26}(E) : \alpha^{\ell_E}(y) \geq \alpha^c\big\}.$$

Then the following relation holds

$$\sigma^*(C_1) = X + \sum_{\alpha > (0,1,1,1,1,1,1)} m_\alpha \, X_\alpha \times Z_\alpha,$$

where the multiplicities m_α can be determined via Schubert calculus but play no role in our calculation. Our claim now follows for dimension reasons. Applying the pointed Brill-Noether Theorem [EH2], Theorem 1.1, using that C is general, we obtain the estimate $\dim X_{\bar{\alpha}} = 1 + \rho(22, 6, 26) - \sum_{j=0}^6 \alpha_j < 3$, for every index $\alpha > (0, 1, 1, 1, 1, 1, 1)$. In the definition of the test curve C_1, the point of attachment $y \in E$ is fixed, therefore the restrictions of both $\mathcal{E}$ and $\mathcal{F}$ are pulled-back from X_α and one obtains that $c_3(\mathcal{F} - \mathrm{Sym}^2(\mathcal{E}))_{|X_{\bar{\alpha}} \times Z_{\bar{\alpha}}} = 0$, for dimension reasons. □

1.1. **Chern numbers of tautological classes on Jacobians.** Often in our calculations, we use facts about intersection theory on Jacobians. We refer to [ACGH] Chapter VIII for background on this topic and to [HM], [H] and [F3] for further applications to divisor class calculations on $\overline{\mathcal{M}}_g$. We start with a Brill-Noether general curve C of genus g and denote by $\mathcal{P}$ a Poincaré line bundle on $C \times \mathrm{Pic}^d(C)$ and by

$$\pi_1 : C \times \mathrm{Pic}^d(C) \to C \text{ and } \pi_2 : C \times \mathrm{Pic}^d(C) \to \mathrm{Pic}^d(C)$$

the two projections. We introduce the class $\eta = \pi_1^*([x_0]) \in H^2(C \times \mathrm{Pic}^d(C), \mathbb{Z})$, where $x_0 \in C$ is an arbitrary point. After picking a symplectic basis $\delta_1, \ldots, \delta_{2g} \in H^1(C, \mathbb{Z}) \cong H^1(\mathrm{Pic}^d(C), \mathbb{Z})$, we consider the class

$$\gamma := -\sum_{\alpha=1}^{g} \Big(\pi_1^*(\delta_\alpha)\pi_2^*(\delta_{g+\alpha}) - \pi_1^*(\delta_{g+\alpha})\pi_2^*(\delta_\alpha)\Big) \in H^2(C \times \mathrm{Pic}^d(C), \mathbb{Z}).$$

One has the formula $c_1(\mathcal{P}) = d\eta + \gamma$, which describes the Hodge decomposition of $c_1(\mathcal{P})$, as well as the relations $\gamma^3 = 0,\ \gamma\eta = 0,\ \eta^2 = 0$ and $\gamma^2 = -2\eta\pi_2^*(\theta)$, see [ACGH] p. 335. Assuming $W_d^{r+1}(C) = \emptyset$, that is, when the Brill-Noether number $\rho(g, r+1, d)$ is negative, the smooth variety $W_d^r(C)$ admits a rank $r+1$ vector bundle

$$\mathcal{M} := (\pi_2)_*\Big(\mathcal{P}_{|C \times W_d^r(C)}\Big).$$

In order to compute the Chern numbers of $\mathcal{M}$ we employ repeatedly the Harris-Tu formula [HT]. We write

$$\sum_{i=0}^{r} c_i(\mathcal{M}^\vee) = (1 + x_1) \cdots (1 + x_{r+1}),$$

and then for every class $\zeta \in H^*(\mathrm{Pic}^d(C), \mathbb{Z})$ one has the following formula:

$$x_1^{i_1} \cdots x_{r+1}^{i_{r+1}}\, \zeta = \det\Big(\frac{\theta^{g+r-d+i_j-j+\ell}}{(g+r-d+i_j-j+\ell)!}\Big)_{1 \le j, \ell \le r+1}\, \zeta. \tag{5}$$

One must interpret this formula the following way. Any top Chern number on $W_d^r(C)$ can be expressed as a *symmetric* polynomial in the variables $x_1, \ldots, x_{r+1}$. This Chern number can be calculated by applying formula (5) and symmetrizing.

1.2. **Top intersection products in the Jacobian of a curve of genus** 22**.** We now specialize to the case of a general curve C of genus 22. By Riemann-Roch $W_{26}^6(C) \cong W_{16}^1(C)$ and note that $W_{26}^7(C) = \emptyset$, so we can consider the rank 8 vector bundle $\mathcal{M}$ on $W_{26}^6(C)$. The vector bundle $\mathcal{N} := (R^1\pi_2)_*\Big(\mathcal{P}_{|C \times W_{26}^6(C)}\Big)$ has rank 2 and we explain how its two Chern classes determine all the Chern classes of $\mathcal{M}$.

Proposition 1.7. *For a general curve C of genus 22 we set $c_i := c_i(\mathcal{M}^\vee)$, for $i = 1, \ldots, 8$, and $y_i := c_i(\mathcal{N})$, for $i = 1, 2$. Then the following relations hold in $H^*(W_{26}^6(C), \mathbb{Z})$:*

$$c_1 = \theta - y_1,\ \ c_2 = \frac{1}{2}\theta^2 - \theta y_1 + y_2,\ \ c_3 = \frac{1}{3!}\theta^3 - \frac{1}{2}\theta^2 y_1 + \theta_2,$$

$$c_4 = \frac{1}{4!}\theta^4 - \frac{\theta^3}{3!}\theta^3 y_1 + \frac{1}{2}\theta^2 y_2,\quad c_5 = \frac{1}{5!}\theta^5 - \frac{1}{4!}\theta^4 y_1 + \frac{1}{3!}\theta^3 y_2,$$

$$c_6 = \frac{1}{6!}\theta^6 - \frac{1}{5!}\theta^5 y_1 + \frac{1}{4!}\theta^4 y_2,\ \textit{and}\ \ c_7 = \frac{1}{7!}\theta^7 - \frac{1}{6!}\theta^6 y_1 + \frac{1}{5!}\theta^5 y_2.$$

Proof. Let us fix a divisor $D \in C_e$ of large degree e. We write down the exact sequence

$$0 \to \mathcal{M} \to (\pi_2)_*\Big(\mathcal{P} \otimes \mathcal{O}(\pi^* D)\Big) \to (\pi_2)_*\Big(\mathcal{P} \otimes \mathcal{O}(\pi_1^* D)_{|\pi_1^* D}\Big) \to R^1\pi_{2*}\Big(\mathcal{P}_{|C \times W_{26}^6(C)}\Big) \to 0.$$

Then we use that $c_{\mathrm{tot}}\Bigl((\pi_2)_*(\mathcal{P}\otimes\mathcal{O}(\pi_1^*D))\Bigr)=e^{-\theta}$, whereas the total Chern class of the vector bundle $(\pi_2)_*\Bigl(\mathcal{P}\otimes\mathcal{O}(\pi_1^*D)_{|\pi_1^*D}\Bigr)$ is trivial, for the divisor D is fixed. One obtains

$$c_{\mathrm{tot}}(R^1\pi)_{2*}\Bigl(\mathcal{P}_{|C\times W^6_{26}(C)}\Bigr)\cdot e^{-\theta}=\sum_{i=0}^{8}(-1)^i c_i,$$

Hence $c_{i+2}=\frac{1}{i!}y_2\theta^i-\frac{1}{(i+1)!}y_i\theta^{i+1}+\frac{1}{(i+2)!}\theta^{i+2}$ for all $i\geq 0$, which finishes the proof. □

Using Proposition 1.7, any Chern number on the smooth 8-fold $W^6_{26}(C)$ can be expressed purely in terms of monomials in the classes u_1, u_2 and θ, where u_1 and u_2 are the Chern roots of $\mathcal{N}$, that is,

$$y_1=c_1(\mathcal{N})=u_1+u_2 \ \text{ and } y_2=c_2(\mathcal{N})=u_1\cdot u_2.$$

We record for further use the following formal identities on $H^{\mathrm{top}}(W^6_{26}(C),\mathbb{Z})$, which are obtained by applying formula (5) in the case $g=22$, $r=1$ and $d=16$, using the canonical isomorphism $H^1(C,L)\cong H^0\bigl(C,K_C\otimes L^\vee\bigr)^\vee$ provided by Serre duality.

$$u_1^3\theta^5=\frac{4\cdot 22!}{11!\cdot 7!},\ u_2^3\theta^5=-\frac{2\cdot 22!}{8!\cdot 10!},\ u_1^2\theta^6=\frac{3\cdot 22!}{10!\cdot 7!},\ u_2^2\theta^6=-\frac{22!}{8!\cdot 9!},\ u_1\theta^7=\frac{2\cdot 22!}{7!\cdot 9!},$$

$$u_2\theta^7=0,\ u_1u_2^4\theta^3=-\frac{2\cdot 22!}{9!\cdot 11!},\ u_1^4u_2\theta^3=\frac{4\cdot 22!}{8!\cdot 12!},\ u_1^2u_2\theta^5=\frac{2\cdot 22!}{8!\cdot 10!},\ u_1u_2^2\theta^5=0,$$

$$u_1^2u_2^3\theta^3=0,\ u_1^3u_2^2\theta^3=\frac{2\cdot 22!}{9!\cdot 11!},\ u_1^2u_2^2\theta^4=\frac{22!}{9!\cdot 10!},\ u_1^4\theta^4=\frac{5\cdot 22!}{7!\cdot 12!},\ u_2^4\theta^4=-\frac{3\cdot 22!}{8!\cdot 11!},$$

$$u_1^3u_2\theta^4=\frac{3\cdot 22!}{8!\cdot 11!},\ u_1u_2^3\theta^4=-\frac{22!}{9!\cdot 10!},\ u_1u_2\theta^6=\frac{22!}{8!\cdot 9!},\ \theta^8=\frac{22!}{7!\cdot 8!}.$$

To compute the corresponding Chern numbers on $W^6_{26}(C)$, one uses Proposition 1.7 and the previous formulas. Each Chern number corresponds to a degree 8 polynomial in u_1, u_2 and θ, which is symmetric in u_1 and u_2.

1.3. **Top intersection products in the Jacobian of curve of genus $2s^2+s$.** We next turn our attention to the top intersection products on $W^{2s}_{2s^2+2s+1}(C)$ in case (4), that is, when C is a general curve of genus $2s^2+s$, for $s\geq 2$. We apply systematically (5). Observe that $\rho(2s^2+s,2s,2s^2+2s)=0$ and we denote by

$$C_{2s+1}:=\frac{(2s^2+s)!\,(2s)!\,(2s-1)!\cdots 2!\,1!}{(3s)!\,(3s-1)!\cdots(s+1)!\,s!}=\#\Bigl(W^{2s}_{2s^2+2s}(C)\Bigr),$$

where the last scheme is reduced and 0-dimensional. We collect the following formulas:

Proposition 1.8. *Let C be as above and set $c_i:=c_i(\mathcal{M}^\vee)\in H^{2i}\bigl(W^{2s}_{2s^2+2s+1}(C),\mathbb{Z}\bigr)$ to be the Chern classes of the dual of the tautological bundle on $W^{2s}_{2s^2+2s+1}(C)$. The following hold:*

$$c_{2s+1}=x_1x_2\cdots x_{2s+1}=C_{2s+1},\ c_{2s}\cdot c_1=x_1x_2\cdots x_{2s+1}+x_1^2x_2\cdots x_{2s},$$

$$c_{2s-1}\cdot c_2=x_1x_2\cdots x_{2s+1}+x_1^2x_2\cdots x_{2s}+x_1^2x_2^2x_3\cdots x_{2s-1},$$

$$c_{2s-1}\cdot c_1^2=x_1x_2\cdots x_{2s+1}+2x_1^2x_2\cdots x_{2s}+x_1^2x_2^2x_3\cdots x_{2s-1}+x_1^3x_2x_3\cdots x_{2s-1},$$

$$c_{2s}\cdot\theta = x_1x_2\cdots x_{2s}\cdot\theta = (2s+1)s\, C_{2s+1},\quad c_{2s-1}\cdot c_1\cdot\theta = x_1x_2\cdots x_{2s}\cdot\theta + x_1^2x_2\cdots x_{2s-1}\cdot\theta$$
$$c_{2s-2}\cdot c_2\cdot\theta = x_1x_2\cdots x_{2s}\cdot\theta + x_1^2x_2\cdots x_{2s-1}\cdot\theta + x_1^2x_2^2x_3\cdots x_{2s-2}\cdot\theta$$
$$c_{2s-2}\cdot c_1^2\cdot\theta = x_1x_2\cdots x_{2s}\cdot\theta + 2x_1^2x_2\ldots x_{2s-1}\cdot\theta + x_1^2x_2^2x_3\cdots x_{2s-2}\cdot\theta + x_1^3x_2x_3\cdots x_{2s-2}\cdot\theta$$
$$c_{2s-1}\cdot\theta^2 = x_1x_2\cdots x_{2s-1}\cdot\theta^2,\quad c_{2s-2}\cdot c_1\cdot\theta^2 = x_1x_2\cdots x_{2s-1}\cdot\theta^2 + x_1^2x_2\cdots x_{2s-2}\cdot\theta^2$$

Using Proposition 1.8, any top intersection product on the smooth $(2s+1)$-dimensional variety $W^{2s}_{2s^2+2s+1}(C)$ has been reduced to a sum of monomials in the variables x_i and θ. Next we record the values of these monomials:

Proposition 1.9. *Keep the notation from above. The following hold in* $H^{4s+2}\big(W^{2s}_{2s^2+2s+1}(C),\mathbb{Z}\big)$:

$$x_1x_2\cdots x_{2s+1} = C_{2s+1},\quad x_1^2x_2^2x_3\cdots x_{2s-1} = \frac{s(s-1)(s+1)^2(2s+1)^2}{3s(3s+1)}C_{2s+1}$$
$$x_1^2x_2\cdots x_{2s} = \frac{4s(s+1)}{3s+1}C_{2s+1},\quad x_1^3x_2x_3\cdots x_{2s-1} = \frac{s^2(s+1)^2(2s-1)(2s+3)}{(3s+1)(3s+2)}C_{2s+1}$$
$$x_1x_2\cdots x_{2s}\cdot\theta = (2s+1)sC_{2s+1},\quad x_1^2x_2\cdots x_{2r-1}\cdot\theta = \frac{(s+1)^2(2s-1)}{3s+1}x_1x_2\cdots x_{2s}\cdot\theta,$$
$$x_1^2x_2^2x_3\cdots x_{2s-2} = \frac{(2s-3)(2s+1)(s+1)^2(s+2)}{9(3s+1)}x_1x_2\cdots x_{2s}\cdot\theta$$
$$x_1^3x_2x_3\cdots x_{2s-2}\cdot\theta = \frac{(s-1)(s+1)^2(s+2)(2s-1)(2s+3)}{3(3s+1)(3s+2)}x_1x_2\cdots x_{2s}\cdot\theta$$
$$x_1x_2\cdots x_{2s-1}\cdot\theta^2 = 2s(s+1)(2s+1)C_{2s+1}$$
$$x_1^2x_2\cdots x_{2s-2}\cdot\theta^2 = \frac{4(s+1)(s-1)(s+2)}{3(3s+1)}x_1x_2\cdots x_{2s-1}\cdot\theta^2$$
$$x_1x_2\cdots x_{2s-2}\cdot\theta^3 = \frac{(2s+1)(2s-1)(s+2)(s+1)s^2}{3}C_{2s+1}.$$

We compute the classes of the loci X and Y appearing in Propositions 1.5 and 1.6:

Proposition 1.10. *Let* $[C,q]$ *be general* 1*-pointed curve of genus* 22 *and let* $\mathcal{M}$ *denote the tautological rank* 7 *vector bundle over* $W^6_{26}(C)$ *and* $c_i := c_i(\mathcal{M}^\vee)\in H^{2i}(W^6_{26}(C),\mathbb{Z})$ *as before. Then the following hold:*

(i) $[X] = \pi_2^*(c_6) - 6\eta\theta\pi_2^*(c_4) + (94\eta+2\gamma)\pi_2^*(c_5)\in H^{12}(C\times W^6_{26}(C),\mathbb{Z})$.
(ii) $[Y] = \pi_2^*(c_6) - 2\eta\theta\pi_2^*(c_4) + (25\eta+\gamma)\pi_2^*(c_3)\in H^{12}(C\times W^6_{26}(C),\mathbb{Z})$.

Proof. Recall that $W^6_{26}(C)$ is smooth of dimension 8. We realize X as the degeneracy locus of a vector bundle morphism over $C\times W^6_{26}(C)$. For each pair $(y,L)\in C\times W^6_{26}(C)$, there is a natural map

$$H^0(C,L\otimes\mathcal{O}_{2y})^\vee\to H^0(C,L)^\vee$$

which globalizes to a bundle morphism $\zeta: J_1(\mathcal{P})^\vee\to\pi_2^*(\mathcal{M})^\vee$ over $C\times W^6_{26}(C)$. Here $J_1(\mathcal{P})$ is the jet bundle of $\mathcal{P}$. Then we have the identification $X = Z_1(\zeta)$, that is, X is the first degeneracy locus of ζ. The Porteous formula yields $[X] = c_6\Big(\pi_2^*(\mathcal{M})^\vee - J_1(\mathcal{P})^\vee\Big)$. To evaluate this class, we write down the exact sequence over $C\times \mathrm{Pic}^{26}(C)$ involving the jet bundle

$$0\longrightarrow\pi_1^*(K_C)\otimes\mathcal{P}\longrightarrow J_1(\mathcal{P})\longrightarrow\mathcal{P}\longrightarrow 0.$$

We can compute the total Chern class of the (formal inverse of the) jet bundle

$$c_{\mathrm{tot}}(J_1(\mathcal{P})^\vee)^{-1} = \Big(\sum_{j\geq 0}(\deg(L)\eta+\gamma)^j\Big)\cdot\Big(\sum_{j\geq 0}\big((2g(C)-2+\deg(L))\eta+\gamma\big)^j\Big) =$$

$$\big(1+26\eta+\gamma+\gamma^2+\cdots\big)\cdot\big(1+68\eta+\gamma+\gamma^2+\cdots\big)=1+194\eta+2\gamma-6\eta\theta,$$

leading to the desired formula for $[X]$.

In order to compute $[Y]$ we proceed in a similar way. We denote by

$$\mu,\nu : C\times C\times \mathrm{Pic}^{26}(C)\to C\times \mathrm{Pic}^{26}(C)$$

the two projections, by $\Delta\subseteq C\times C\times \mathrm{Pic}^{26}(C)$ the diagonal and set $\Gamma_q := \{q\}\times \mathrm{Pic}^{26}(C)$. We introduce the rank 2 vector bundle $\mathcal{B} := \mu_*\big(\nu^*(\mathcal{P})\otimes \mathcal{O}_{\Delta+\nu^*(\Gamma_q)}\big)$ over $C\times W^6_{26}(C)$. Note that there is a bundle morphism $\chi : \mathcal{B}^\vee\to(\pi_2)^*(\mathcal{M})^\vee$, such that $Y=Z_1(\chi)$. Since we also have that

$$c_{\mathrm{tot}}(\mathcal{B}^\vee)^{-1} = \Big(1+(\deg(L)\eta+\gamma)+(\deg(L)\eta+\gamma)^2+\cdots\Big)\cdot\big(1-\eta\big)=1+25\eta+\gamma-2\eta\theta,$$

we immediately obtain the stated expression for $[Y]$. □

In case (4), recall that C is a general curve of genus $2s^2+s$ and $W^{2s}_{2s^2+2s+1}(C)$ is smooth of dimension $2s+1$. We record the formulas for the classes of X and Y, the proofs being analogous to those of Proposition 1.10.

Proposition 1.11. *Let $[C,q]\in\mathcal{M}_{2s^2+s,1}$ be a general pointed curve. If $c_i := c_i(\mathcal{M}^\vee)$ are the Chern classes of the tautological vector bundle over $W^{2s}_{2s^2+2s+1}(C)$, then one has:*

(i) $[X]=\pi_2^*(c_{2s})-6\pi_2^*(c_{2s-2})\eta\theta+2\big((4s^2+3s)\eta+\gamma\big)\pi_2^*(c_{2s-1})\in H^{4s}(C\times W^r_d(C))$.

(ii) $[Y]=\pi_2^*(c_{2s})-2\pi_2^*(c_{2s-2})\eta\theta+\big((2s^2+2s)\eta+\gamma\big)\pi_2^*(c_{2s-1})\in H^{4s}(C\times W^r_d(C))$.

Remark 1.12. For future reference we also record the following formulas:

(6) $$c_{2s+1}\big(\pi_2^*(\mathcal{M})^\vee-J_1(\mathcal{P})^\vee\big)=\pi_2^*(c_{2s+1})-6\pi_2^*(c_{2s-1})\eta\theta+\big(2s(4s+3)\eta+2\gamma\big)\pi_2^*(c_{2s}),$$

(7) $$c_{2s+1}\big(\pi_2^*(\mathcal{M})^\vee-\mathcal{B}^\vee\big)=\pi_2^*(c_{2s+1})-2\pi_2^*(c_{2s-1})\eta\theta+\big(2s(s+1)\eta+\gamma\big)\pi_2^*(c_{2s}).$$

The following formulas are applications of Grothendieck-Riemann-Roch:

Proposition 1.13. *Let C be a general curve of genus $g-1$ and consider the vector bundles $\mathcal{A}_2$ and $\mathcal{B}_2$ on $C\times\mathrm{Pic}^d(C)$ having fibres*

$$\mathcal{A}_2(y,L)=H^0\big(C,L^{\otimes 2}(-2y)\big)\ \text{ and }\ \mathcal{B}_2(y,L)=H^0\big(C,L^{\otimes 2}(-y-q)\big),$$

respectively.

(i) *One has the following formulas in case (3):*

$$c_1(\mathcal{A}_2)=-4\theta-4\gamma-146\eta,\quad c_1(\mathcal{B}_2)=-4\theta-2\gamma-51\eta,$$

$$c_2(\mathcal{A}_2)=8\theta^2+560\eta\theta+16\gamma\theta,\quad c_2(\mathcal{B}_2)=8\theta^2+196\eta\theta+8\theta\gamma,$$

$$c_3(\mathcal{A}_2)=-\frac{32}{3}\theta^3-1072\eta\theta^2-32\theta^2\gamma\ \text{ and }\ c_3(\mathcal{B}_2)=-\frac{32}{3}\theta^3-376\eta\theta^2-16\theta^2\gamma.$$

(ii) *In the case (4), that is, when $g-1=2s^2+s$, the following formulas hold:*

$$c_1(\mathcal{A}_2)=-4\theta-4\gamma-2(3s+1)(2s+1)\eta,\quad c_1(\mathcal{B}_2)=-4\theta-2\gamma-(2s+1)^2\eta,$$

$$c_2(\mathcal{A}_2)=8\theta^2+8(6s^2+5s-2)\eta\theta+16\gamma\theta,\quad c_2(\mathcal{B}_2)=8\theta^2+4(4s^2+4s-1)\eta\theta+8\theta\gamma.$$

Proof. Immediate application of Grothendieck-Riemann-Roch with respect to the map $\nu : C \times C \times \mathrm{Pic}^{26}(C) \to C \times \mathrm{Pic}^{26}(C)$ defined above. The vector bundle $\mathcal{A}_2$ is realized as a push-forward under the map ν:

$$\mathcal{A}_2 = \nu_!\Big(\mu^*\big(\mathcal{P}^{\otimes 2} \otimes \mathcal{O}_{C\times C\times \mathrm{Pic}^{26}(C)}(-2\Delta)\big)\Big) = \nu_*\Big(\mu^*\big(\mathcal{P}^{\otimes 2} \otimes \mathcal{O}_{C\times C\times \mathrm{Pic}^{26}(C)}(-2\Delta)\big)\Big),$$

and we apply Grothendieck-Riemann-Roch to ν. One finds that $\mathrm{ch}_2(\mathcal{A}_2) = 8\eta\theta$ and $\mathrm{ch}_n(\mathcal{A}_2) = 0$, for $n \geq 3$. Furthermore, $\nu_*(c_1(\mathcal{P})^2) = -2\theta$. The calculation of $\mathcal{B}_2$ is similar and we skip the details. The case (4) follows along similar lines. □

2. The Kodaira dimension of $\overline{\mathcal{M}}_{23}$

In this section we complete the calculation of the virtual class $[\widetilde{\mathfrak{D}}]^{\mathrm{virt}}$. We shall use repeatedly that if $\mathcal{V}$ is a vector bundle of rank $r+1$ on a stack X, the Chern classes of its second symmetric product can be computed as follows:

(i) $c_1(\mathrm{Sym}^2(\mathcal{V})) = (r+2)c_1(\mathcal{V})$,

(ii) $c_2(\mathrm{Sym}^2(\mathcal{V})) = \frac{r(r+3)}{2}c_1^2(\mathcal{V}) + (r+3)c_2(\mathcal{V})$,

(iii) $c_3(\mathrm{Sym}^2(\mathcal{V})) = \frac{r(r+4)(r-1)}{6}c_1^3(\mathcal{V}) + (r+5)c_3(\mathcal{V}) + (r^2+4r-1)c_1(\mathcal{V})c_2(\mathcal{V})$.

We expand the virtual class

$$[\widetilde{\mathfrak{D}}]^{\mathrm{virt}} = \sigma_*\Big(c_3(\mathcal{F} - \mathrm{Sym}^2(\mathcal{E}))\Big) = a\lambda - b_0\delta_0 - b_1\delta_1 \in CH^1(\overline{\mathcal{M}}_{23}^{\mathrm{part}}).$$

Our task is to determine the coefficients a, b_0 and b_1. As we shall explain at the end of this section, this suffices in order to estimate the other coefficients in the boundary expansion of $[\widetilde{\mathfrak{D}}]^{\mathrm{virt}}$. We begin with the coefficient of δ_1:

Theorem 2.1. *Let C be a general curve of genus 22 and denote by $C_1 \subseteq \Delta_1^p \subseteq \overline{\mathcal{M}}_{23}$ the associated test curve. Then the coefficient of δ_1 in the expansion of $[\widetilde{\mathfrak{D}}]^{\mathrm{virt}}$ is equal to*

$$b_1 = \frac{1}{2g(C)-2}\sigma^*(C_1)\cdot c_3\big(\mathcal{F} - \mathrm{Sym}^2(\mathcal{E})\big) = 13502337992 = \frac{4}{9}\binom{19}{8}401951.$$

Proof. We intersect the degeneracy locus of the map $\phi : \mathrm{Sym}^2(\mathcal{E}) \to \mathcal{F}$ with the 3-fold $\sigma^*(C_1) = X + \sum_{\bar{\alpha}} m_\alpha X_{\bar{\alpha}} \times Z_{\bar{\alpha}}$, where recall that $\dim X_\alpha < 3$, for all Schubert indices $\alpha > (0,1,1,1,1,1,1)$. Therefore, as already explained in Proposition 1.6, it is enough to estimate the contribution coming from X and we can write

$$\sigma^*(C_1)\cdot c_3\big(\mathcal{F} - \mathrm{Sym}^2(\mathcal{E})\big) = c_3(\mathcal{F}_{|X}) - c_3(\mathrm{Sym}^2\mathcal{E}_{|X}) - c_1(\mathcal{F}_{|X})c_2(\mathrm{Sym}^2\mathcal{E}_{|X}) +$$

$$+2c_1(\mathrm{Sym}^2\mathcal{E}_{|X})c_2(\mathrm{Sym}^2\mathcal{E}_{|X}) - c_1(\mathrm{Sym}^2\mathcal{E}_{|X})c_2(\mathcal{F}_{|X}) + c_1^2(\mathrm{Sym}^2\mathcal{E}_{|X})c_1(\mathcal{F}_{|X}) - c_1^3(\mathrm{Sym}^2\mathcal{E}_{|X}).$$

We now evaluate the terms that appear in the right-hand-side of this expression.

In the course of proving Proposition 1.10, we set-up a vector bundle morphism $\zeta : J_1(\mathcal{P})^\vee \to \pi_2^*(\mathcal{M})^\vee$. The kernel $\mathrm{Ker}(\zeta)$ is locally free of rank 1. If U is the line bundle on X with fibre

$$U(y, L) = \frac{H^0(C, L)}{H^0(C, L(-2y))} \hookrightarrow H^0(C, L \otimes \mathcal{O}_{2y})$$

over a point $(y, L) \in X$, then one has the following exact sequence over X

$$0 \longrightarrow U \longrightarrow J_1(\mathcal{P}) \longrightarrow \big(\mathrm{Ker}(\zeta)\big)^\vee \longrightarrow 0.$$

In particular, $c_1(U) = 2\gamma + 94\eta + c_1(\mathrm{Ker}(\zeta))$. The products of the Chern class of $\mathrm{Ker}(\zeta)$ with other classes coming from $C \times W^6_{26}(C)$ can be computed from the formula in [HT]:

$$c_1\big(\mathrm{Ker}(\zeta)\big) \cdot \xi_{|X} = -c_7\Big(\pi_2^*(\mathcal{M})^\vee - J_1(\mathcal{P})^\vee\Big) \cdot \xi_{|X} \tag{8}$$

$$= -\Big(\pi_2^*(c_7) - 6\eta\theta\pi_2^*(c_5) + (94\eta + 2\gamma)\pi_2^*(c_6)\Big) \cdot \xi_{|X},$$

where $\xi \in H^2(C \times W^6_{26}(C), \mathbb{Z})$.

If $\mathcal{A}_3$ denotes the rank 31 vector bundle on X having fibres

$$\mathcal{A}_3(y, L) = H^0(C, L^{\otimes 2})$$

(and constructed in an obvious way as push-forward of a line bundle on the triple product $C \times C \times \mathrm{Pic}^{26}(C)$), then $U^{\otimes 2}$ can be embedded in $\mathcal{A}_3/\mathcal{A}_2$. We consider the quotient

$$\mathcal{G} := \frac{\mathcal{A}_3/\mathcal{A}_2}{U^{\otimes 2}}.$$

The morphism $U^{\otimes 2} \to \mathcal{A}_3/\mathcal{A}_2$ vanishes along the locus of pairs (y, L) where L has a base point, which implies that $\mathcal{G}$ has torsion along the $\Gamma \subseteq X$ consisting of pairs $(q, A(q))$, where $A \in W^6_{25}(C)$.Furthermore, $\mathcal{F}_{|X}$ is identified as a subsheaf of $\mathcal{A}_3$ with the kernel of the map $\mathcal{A}_3 \to \mathcal{G}$. Summarizing, there is an exact sequence of vector bundles on X

$$0 \longrightarrow \mathcal{A}_{2|X} \longrightarrow \mathcal{F}_{|X} \longrightarrow U^{\otimes 2} \longrightarrow 0.$$

Over a general point of $(y, L) \in X$ this sequence reflects the decomposition

$$\mathcal{F}(y, L) = H^0(C, L^{\otimes 2}(-2y)) \oplus \mathbb{C} \cdot u^2,$$

where $u \in H^0(C, L)$ is such that $\mathrm{ord}_y(u) = 1$.

Hence

$$c_1(\mathcal{F}_{|X}) = c_1(\mathcal{A}_{2|X}) + 2c_1(U),\ c_2(\mathcal{F}_{|X}) = c_2(\mathcal{A}_{2|X}) + 2c_1(\mathcal{A}_{2|X})c_1(U) \quad \text{and}$$

$$c_3(\mathcal{F}_{|X}) = c_3(\mathcal{A}_2) + 2c_2(\mathcal{A}_{2|X})c_1(U).$$

Recalling that $\mathcal{E}_{|X} = \pi_2^*(\mathcal{M})_{|X}$, we obtain that:

$$\sigma^*(C_1) \cdot c_3\big(\mathcal{F} - \mathrm{Sym}^2\mathcal{E}\big) = c_3(\mathcal{A}_{2|X}) + c_2(\mathcal{A}_{2|X})c_1(U^{\otimes 2}) - c_3(\mathrm{Sym}^2\pi_2^*\mathcal{M}_{|X}) -$$

$$-\Big(\frac{r(r+3)}{2}c_1(\pi_2^*\mathcal{M}_{|X}) + (r+3)c_2(\pi_2^*\mathcal{M}_{|X})\Big) \cdot \Big(c_1(\mathcal{A}_{2|X}) + c_1(U^{\otimes 2}) - 2(r+2)c_1(\pi_2^*\mathcal{M}_{|X})\Big) -$$

$$-(r+2)c_1(\pi_2^*\mathcal{M}_{|X})c_2(\mathcal{A}_{2|X}) - (r+2)c_1(\pi_2^*\mathcal{M}_{|X})c_1(\mathcal{A}_{2|X})c_1(U^{\otimes 2}) +$$

$$+(r+2)^2c_1^2(\pi_2^*\mathcal{M}_{|X})c_1(\mathcal{A}_{2|X}) + (r+2)^2c_1^2(\pi_2^*\mathcal{M}_{|X})c_1(U^{\otimes 2}) - (r+2)^3c_1^3(\pi_2^*\mathcal{M}_{|X}).$$

Here, $c_i(\pi_2^*\mathcal{M}^\vee_{|X}) = \pi_2^*(c_i) \in H^{2i}(X, \mathbb{Z})$ and $r = \mathrm{rk}(\mathcal{M}) - 1 = 6$. The coefficient of $c_1\big(\mathrm{Ker}(\zeta)\big)$ in the product $\sigma^*(C_1) \cdot c_3\big(\mathcal{F} - \mathrm{Sym}^2(\mathcal{E})\big)$ is evaluated via (8). First we consider the part of this product which *does not* contain $c_1\big(\mathrm{Ker}(\zeta)\big)$ and we obtain

$$36\pi_2^*(c_2)\theta - 148\pi_2^*(c_1^2)\theta + 1554\eta\pi_2^*(c_1^2) - 85\pi_2^*(c_1c_2) - \frac{32}{3}\theta^3 + 304\eta\theta^2 - 1280\eta\theta\pi_2^*(c_1)$$

$$+130\pi_2^*(c_1^3) - 378\eta\pi_2^*(c_2) + 64\theta^2\pi_2^*(c_1) + 11\pi_2^*(c_3) \in H^6\big(C \times W^6_{26}(C), \mathbb{Z}\big).$$

This polynomial of degree 3 gets multiplied by the degree 6 polynomial in θ, η and $\pi_2^*(c_i)$ that gives the class $[X]$ obtained in Proposition 1.10. We add to ot the contribution coming from $c_1\big(\mathrm{Ker}(\zeta)\big)$. One obtains a homogeneous polynomial of degree 9 in η, θ

and $\pi_2^*(c_i)$ for $i=1,\ldots,7$. The only non-zero monomials are those containing η. After retaining only these monomials and dividing by η, the resulting degree 8 polynomial in $\theta, c_i\in H^*(W^6_{26}(C),\mathbb{Z})$ can be brought to a manageable form, by using Proposition 1.7. After lengthy but straightforward manipulations carried out using *Maple*, one finds

$$\sigma^*(C_1)\cdot c_3\big(\mathrm{Sym}^2(\mathcal{E})-\mathcal{F}\big)=\eta\pi_2^*\Big(-780c_1^3c_4\theta+12220c_1^3c_5+888c_1^2c_4\theta^2-13468c_1^2c_5\theta$$
$$-5402c_1^2c_6-384\theta^3c_1c_4+5632\theta^2c_1c_5+510\theta c_1c_2c_4+4480c_1c_6\theta-7990c_1c_2c_5$$
$$+2336c_1c_7-216c_2c_4\theta^2+3276c_2c_5\theta-66c_3c_4\theta+1034c_3c_5+1314c_2c_6+640c_4\theta^4$$
$$-\frac{2720}{3}c_5\theta^5-1072c_6\theta^2-1120c_7\theta\Big).$$

We suppress η and the remaining polynomial lives inside $H^{16}(W^6_{26}(C),\mathbb{Z})$. We have indicated how to calculate explicitly using (5) all top Chern numbers on $W^6_{26}(C)$ and we eventually find that

$$b_1=\frac{1}{42}\sigma^*(C_1)\cdot c_3\big(\mathcal{F}-\mathrm{Sym}^2(\mathcal{E})\big)=13502337992.$$

□

Theorem 2.2. *Let $[C,q]$ be a general pointed curve of genus 22 and let $C_0\subseteq\Delta_0^p\subseteq\overline{\mathcal{M}}_{23}^{\mathrm{part}}$ be the associated test curve. Then*

$$\sigma^*(C_0)\cdot c_3\big(\mathcal{F}-\mathrm{Sym}^2(\mathcal{E})\big)=44b_0-b_1=93988702808.$$

It follows that $b_0=\frac{4}{9}\binom{19}{8}72725$.

Proof. As already noted in Proposition 1.5, the vector bundles $\mathcal{E}_{|\sigma^*(C_0)}$ and $\mathcal{F}_{|\sigma^*(C_0)}$ are both pull-backs of vector bundles on Y and we denote these vector bundles $\mathcal{E}$ and $\mathcal{F}$ as well, that is, $\mathcal{E}_{|\sigma^*(C^0)}=f^*(\mathcal{E}_{|Y})$ and $\mathcal{F}_{|\sigma^*(C_0)}=f^*(\mathcal{F}_{|Y})$. Following broadly the lines of the proof of Theorem 2.1, we evaluate the terms appearing in $\sigma^*(C_0)\cdot c_3(\mathcal{F}-\mathrm{Sym}^2(\mathcal{E}))$.

Let V be the line bundle on Y with fibre

$$V(y,L)=\frac{H^0(C,L)}{H^0(C,L(-y-q))}\hookrightarrow H^0(C,L\otimes\mathcal{O}_{y+q})$$

over a point $(y,L)\in Y$. There is an exact sequence of vector bundles over Y

$$0\longrightarrow V\longrightarrow\mathcal{B}\longrightarrow\big(\mathrm{Ker}(\chi)\big)^\vee\longrightarrow 0,$$

where $\chi:\mathcal{B}^\vee\to\pi_2^*(\mathcal{M})^\vee$ is the bundle morphism defined in the second part of Proposition 1.10. In particular, $c_1(V)=25\eta+\gamma+c_1\big(\mathrm{Ker}(\chi)\big)$, for the Chern class of $\mathcal{B}$ has been computed in the proof of Proposition 1.10. By using again [HT], we find the following formulas for the Chern numbers of $\mathrm{Ker}(\chi)$:

$$c_1(\mathrm{Ker}(\chi))\cdot\xi_{|Y}=-c_7\big(\pi_2^*(\mathcal{M})^\vee-\mathcal{B}^\vee\big)\cdot\xi_{|Y}=-\Big(\pi_2^*(c_7)+\pi_2^*(c_6)(13\eta+\gamma)-2\pi_2^*(c_4)\eta\theta\Big)\cdot\xi_{|Y},$$

for any class $\xi\in H^2(C\times W^6_{26}(C),\mathbb{Z})$. We have previously defined the vector bundle $\mathcal{B}_2$ over $C\times W^6_{26}(C)$ with fibre $\mathcal{B}_2(y,L)=H^0(C,L^{\otimes 2}(-y-q))$. One has the following exact sequence of bundles over Y

$$0\longrightarrow\mathcal{B}_{2|Y}\longrightarrow\mathcal{F}_{|Y}\longrightarrow V^{\otimes 2}\longrightarrow 0.\tag{9}$$

If $\mathcal{B}_3$ is the vector bundle on Y with fibres $\mathcal{B}_3(y,L)=H^0(C,L^{\otimes 2})$, we have an injective morphism of sheaves $V^{\otimes 2}\hookrightarrow \mathcal{B}_3/\mathcal{B}_2$ locally given by

$$v^{\otimes 2}\mapsto v^2 \text{ mod } H^0(C,L^{\otimes 2}(-y-q)),$$

where $v\in H^0(C,L)$ is any section not vanishing at q and y. Then $\mathcal{F}_{|Y}$ is canonically identified with the kernel of the projection morphism

$$\mathcal{B}_3\to \frac{\mathcal{B}_3/\mathcal{B}_2}{V^{\otimes 2}}$$

and the exact sequence (9) now becomes clear. Therefore $c_1(\mathcal{F}_{|Y})=c_1(\mathcal{B}_{2|Y})+2c_1(V)$, $c_2(\mathcal{F}_{|Y})=c_2(\mathcal{B}_{2|Y})+2c_1(\mathcal{B}_{2|Y})c_1(V)$ and $c_3(\mathcal{F}_{|Y})=c_3(\mathcal{B}_{2|Y})+2c_2(\mathcal{B}_{2|Y})c_1(V)$. The part of the intersection number $\sigma^*(C_0)\cdot c_3(\mathcal{F}-\mathrm{Sym}^2(\mathcal{E}))$ that *does not* contain $c_1\big(\mathrm{Ker}(\chi)\big)$ equals

$$36\pi_2^*(c_2)\theta-148\pi_2^*(c_1^2)\theta-37\eta\pi_2^*(c_1^2)-85\pi_2^*(c_1c_2)-\frac{32}{3}\theta^3+$$
$$-8\eta\theta^2+32\eta\theta\pi_2^*(c_1)+130\pi_2^*(c_1^3)+9\eta\pi_2^*(c_2)+64\theta^2\pi_2^*(c_1)+11\pi_2^*(c_3)\in H^6(C\times W^6_{26}(C),\mathbb{Z}).$$

We multiply this expression with the class $[Y]$ computed in Proposition 1.10. The coefficient of $c_1(\mathrm{Ker}(\zeta)^\vee)$ in $\sigma^*(C_0)\cdot c_3\big(\mathcal{F}-\mathrm{Sym}^2\mathcal{E}\big)$ equals

$$-2c_2(\mathcal{B}_{2|Y})-2(r+2)^2\pi_2^*(c_1^2)-2(r+2)c_1(\mathcal{B}_{2|Y})\pi_2^*(c_1)+r(r+3)\pi_2^*(c_1^2)+2(r+3)\pi_2^*(c_2),$$

where recall that $r=\mathrm{rk}(\mathcal{M})-1=6$. All in all, $44b_0-b_1=\sigma^*(C_0)\cdot c_3(\mathcal{F}-\mathrm{Sym}^2\mathcal{E})$ and we evaluate this as explained using (5). □

The following result follows from the definition of the vector bundles $\mathcal{E}$ and $\mathcal{F}$ given in Proposition 1.4. It will provide the third relation between the coefficients of $[\mathfrak{D}]^{\mathrm{virt}}$, and thus complete the calculation of its slope.

Theorem 2.3. *Let $[C,q]$ be a general 1-pointed curve of genus 22 and $R\subseteq\overline{\mathcal{M}}_{23}$ be the pencil obtained by attaching at the fixed point $q\in C$ a pencil of plane cubics at one of the base points of the pencils. Then one has the relation*

$$a-12b_0+b_1=\sigma_*c_3\big(\mathcal{F}-\mathrm{Sym}^2(\mathcal{E})\big)\cdot R=0.$$

Proof of Corollary 0.2 and of Theorem 0.1. We assume that the Strong Maximal Conjecture holds on $\overline{\mathcal{M}}_{23}$, that is, $\mathfrak{D}$ is a genuine divisor on $\overline{\mathcal{M}}_{23}^{\mathrm{part}}$ and let $\overline{\mathfrak{D}}$ denote its closure in $\overline{\mathcal{M}}_{23}$. To determine its slope, we express its class in terms of the standard generators of $CH^1(\overline{\mathcal{M}}_{23})$ and write

$$[\overline{\mathfrak{D}}]=a\lambda-\sum_{j=0}^{11}b_j\delta_j\in CH^1(\overline{\mathcal{M}}_{23}),$$

where the coefficients a,b_0 and b_1 have already been determined. Since

$$\frac{a}{b_0}=6.473...\leq\frac{83}{11},$$

we are in a position to apply Corollary 1.2 from [FP], which gives the inequalities $b_j\geq b_0$ for $1\leq j\leq 11$. (Interestingly, $\overline{\mathcal{M}}_{23}$ is the last moduli space where the argument from [FP], based on pencils of curves on $K3$ surface and which reduces the calculation of the slope of an effective divisor on $\overline{\mathcal{M}}_g$ to a bound on its λ and δ_0-coefficients, can be applied). Therefore $s(\overline{\mathfrak{D}}) = \frac{a}{b_0} < 6 + \frac{12}{13}$. Thus assuming the Strong Maximal Conjecture on $\overline{\mathcal{M}}_{23}$, we can express the canonical class $K_{\overline{\mathcal{M}}_{23}} = 13\lambda - 2\delta_0 - 3\delta_1 - 2\sum_{j=1}^{11}\delta_j$ as a positive combination of $[\overline{\mathfrak{D}}]$, of the big class λ and further boundary classes $\delta_2, \ldots, \delta_{11}$. One concludes that $\overline{\mathcal{M}}_{23}$ is a variety of general type. □

3. Effective divisors of small slope on $\overline{\mathcal{M}}_{2s^2+s+1}$

We now complete the proof of Theorem 0.3. Recall that $g = 2s^2 + s + 1$ and we have expressed the virtual class

$$[\widetilde{\mathfrak{D}}_s]^{\mathrm{virt}} = \sigma_*\big(c_2(\mathcal{F} - \mathrm{Sym}^2(\mathcal{E}))\big) = a\lambda - b_0\delta_0 - b_1\delta_1 \in CH^1(\overline{\mathcal{M}}_g^{\mathrm{part}}).$$

The determination of the coefficients a, b_0 and b_1 follows largely the proof of Theorem 0.1 and we shall highlight the differences. Recall that C_{2s+1} denotes the number of linear systems of type $\mathfrak{g}^{2s}_{2s^2+2s}$ on a general curve of genus $2sr + s$.

Theorem 3.1. *Let C be a general curve of genus $2s^2 + s$ and denote by $C_1 \subseteq \Delta_1^p \subseteq \overline{\mathcal{M}}_{2s^2+s+1}$ the associated test curve. Then the coefficient of δ_1 in the expansion of $[\widetilde{\mathfrak{D}}_s]^{\mathrm{virt}}$ is equal to*

$$b_1 = \frac{1}{2g(C)-2}\sigma^*(C_1)\cdot c_2\big(\mathcal{F} - \mathrm{Sym}^2(\mathcal{E})\big) =$$

$$C_{2s+1}\frac{2s(s-1)(2s+1)}{(2s-1)(3s+1)(3s+2)}\Big(24s^6 - 40s^5 + 18s^4 + 26s^3 + 30s^2 + 47s + 18\Big).$$

Proof. We intersect the degeneracy locus of the map $\phi : \mathrm{Sym}^2(\mathcal{E}) \to \mathcal{F}$ with the surface $\sigma^*(C_1)$, containing X as an irreducible component. It follows from Proposition 1.6 that X is the only component contributing to this intersection product, that is,

(10)
$$\sigma^*(C_1)\cdot c_2\big(\mathcal{F} - \mathrm{Sym}^2(\mathcal{E})\big) = c_2(\mathcal{F}_{|X}) - c_2(\mathrm{Sym}^2\mathcal{E}_{|X}) - c_1(\mathcal{F}_{|X})c_1(\mathrm{Sym}^2\mathcal{E}_{|X}) + c_1^2(\mathrm{Sym}^2\mathcal{E}_{|X}).$$

The kernel $\mathrm{Ker}(\zeta)$ of the vector bundle morphism $\zeta : J_1(\mathcal{P})^\vee \to \pi_2^*(\mathcal{M})^\vee$ defined in the proof of Proposition 1.10 is a line bundle on X. If U is the line bundle on X with fibre

$$U(y, L) = \frac{H^0(C, L)}{H^0(C, L(-2y))} \hookrightarrow H^0(C, L \otimes \mathcal{O}_{2y})$$

over a point $(y, L) \in X$, then one has the following exact sequence over X

$$0 \longrightarrow U \longrightarrow J_1(\mathcal{P}) \longrightarrow \big(\mathrm{Ker}(\zeta)\big)^\vee \longrightarrow 0.$$

From this sequence it follows, $c_1(U) = 2\gamma + 2(4s^2+3s)\eta + c_1(\mathrm{Ker}(\zeta))$, where the products of $c_1\big(\mathrm{Ker}(\zeta)\big)$ with arbitrary classes coming from $\xi \in C \times W^{2s}_{2s^2+2s+1}(C)$ can be computed using once more the Harris-Tu formula [HT]:

$$c_1\big(\mathrm{Ker}(\zeta)\big)\cdot \xi_{|X} = -c_{2s+1}\big(\pi_2^*(\mathcal{M})^\vee - J_1(\mathcal{P})^\vee\big)\cdot \xi_{|X}. \tag{11}$$

The Chern classes on the right hand side of (11) has been evaluated in the formula (6). Using a local analysis identical to the one in Proposition 2.1 in the case $g = 23$ we conclude that the restriction $\mathcal{F}_{|X}$ sits in the following exact sequence of vector bundles

$$0 \longrightarrow \mathcal{A}_{2|X} \longrightarrow \mathcal{F}_{|X} \longrightarrow U^{\otimes 2} \longrightarrow 0.$$

We obtain the following intersection product on the surface X:

$$c_2\big(\mathcal{F}_{|X} - \mathrm{Sym}^2(\mathcal{E})_{|X}\big) = c_2\big(\mathcal{A}_{2|X}\big) + 2c_1(\mathcal{A}_{2|X})\cdot c_1\big(J_1(\mathcal{P})\big) - (2s+3)c_2(\pi_2^*\mathcal{M}^\vee) +$$

$$\Big((2s+2)^2-s(2s+3)\Big)c_1^2\big(\pi_2^*\mathcal{M}^\vee\big)+(2s+2)c_1\big(\mathcal{A}_{2|X}\big)\cdot c_1\big(\pi_2^*\mathcal{M}^\vee\big)$$
$$+c_1\big(\mathrm{Ker}(\zeta)\big)\cdot\Big(2c_1(\mathcal{A}_{2|X})-2(r+2)c_1(\pi_2^*\mathcal{M}^\vee)\Big).$$

This expression gets multiplied with the class $[X]$ computed in Proposition 1.11. The Chern classes of $\mathcal{A}_{2|X}$ have been computed in part (ii) of Proposition 1.13. Finally, the contribution of the terms containing $c_1\big(\mathrm{Ker}(\zeta)\big)$ is evaluated using (6). The resulting intersection product is then evaluated with *Maple*, retaining only monomials of degree $2s+2$ in tautological classes on $C\times W^{2s}_{2s^2+2s+1}(C)$ that contain η. Dropping this class, one is led to a sum of top Chern numbers on $W^{2s}_{2s^2+2s+1}(C)$, which can be evaluated individually using Propositions 1.8 and 1.9 respectively. □

Theorem 3.2. *Let $[C,q]$ be a general pointed curve of genus $2s^2+s$ and $C_0\subseteq \Delta_0^p\subseteq \overline{\mathcal{M}}^{\mathrm{part}}_{2s^2+s+1}$ be the associated test curve. Then the δ_0-coefficient b_0 in the expression of $[\widetilde{\mathfrak{D}}_s]^{\mathrm{virt}}$ is equal to*

$$b_0=\frac{\sigma^*(C_0)\cdot c_2\Big(\mathcal{F}-\mathrm{Sym}^2(\mathcal{E})\Big)+b_1}{2s(2s+1)}=$$

$$C_{2s+1}\frac{2(s-1)(24s^8-28s^7+22s^6-5s^5+43s^4+112s^3+100s^2+50s+12)}{9(2s-1)(3s+1)(3s+2)}.$$

Proof. Using Proposition 1.5, we observe that

$$c_2\big(\mathcal{F}-\mathrm{Sym}^2(\mathcal{E})\big)_{|\sigma^*(C_0)}=c_2\big(\mathcal{F}-\mathrm{Sym}^2(\mathcal{E})\big)_{|Y}. \tag{12}$$

To determine the Chern classes of $\mathcal{F}_{|Y}$, we introduce the line bundle V on Y with fibre

$$V(y,L)=\frac{H^0(C,L)}{H^0(C,L(-y-q))}\hookrightarrow H^0(C,L\otimes\mathcal{O}_{y+q})$$

over a point $(y,L)\in Y$. There is an exact sequence of vector bundles over Y

$$0\longrightarrow V\longrightarrow\mathcal{B}\longrightarrow\big(\mathrm{Ker}(\chi)\big)^\vee\longrightarrow 0,$$

where the morphism $\chi:\mathcal{B}^\vee\to\pi_2^*(\mathcal{M})^\vee$ was defined in the second part of Proposition 1.11. Recalling the vector bundle $\mathcal{B}_2$ defined in Proposition 1.13, a local analysis similar to that in Theorem 2.2, shows that one has an exact sequence on Y:

$$0\longrightarrow\mathcal{B}_{2|Y}\longrightarrow\mathcal{F}_{|Y}\longrightarrow V^{\otimes 2}\longrightarrow 0.$$

This determines $c_i(F_{|Y})$, by also using that $c_1\big(\mathrm{Ker}(\chi)\big)=-c_{2s+1}\big(\pi_2^*(\mathcal{M}^\vee)-\mathcal{B}^\vee\big)$, where the right hand side is estimated via (7). We can now substitute in (12), and as in the proof of Theorem 3.1, after manipulations we obtain a polynomial of degree $2s+1$ on $W^{2s}_{2s^2+2s+1}(C)$, that we estimate by applying (1.8) and (1.9). □

We can now complete the calculation of the slope of $[\widetilde{\mathfrak{D}}_s]^{\mathrm{virt}}$.

Proof of Theorem 0.3. We denote once more by $R\subseteq\overline{\mathcal{M}}_{2s^2+s+1}$ the pencil obtained by attaching at the fixed point of a general curve C of genus $2s^2+s$ a pencil of plane cubics at one of the base points of the pencils. Then one has the relation

$$a-12b_0+b_1=\sigma_*c_2\big(\mathcal{F}-\mathrm{Sym}^2(\mathcal{E})\big)\cdot R=0.$$

We find the following expression for the λ-coefficient

$$a = \frac{2(s-1)(48s^8 - 56s^7 + 92s^6 - 90s^5 + 86s^4 + 324s^3 + 317s^2 + 182s + 48)}{3(3s+2)(2s-1)(3s+1)}.$$

□

4. THE BIRATIONAL GEOMETRY OF HURWITZ SPACES

We denote by $\mathcal{H}_{g,k}^{\mathfrak{o}}$ the Hurwitz space classifying degree k covers $f : C \to \mathbf{P}^1$ with source being a smooth curve C of genus g and having simple ramifications. Note that we choose an *ordering* $(p_1, \ldots, p_{2g+2k-2})$ of the set of the branch points of f. An excellent reference for the algebro-geometric study of Hurwitz spaces is [Ful]. Let $\overline{\mathcal{H}}_{g,k}^{\mathfrak{o}}$ be the (projective) moduli space of admissible covers (with an ordering of the set of branch points). The geometry of $\overline{\mathcal{H}}_{g,k}^{\mathfrak{o}}$ has been described in detail by Harris and Mumford [HM] and further clarified in [ACV]. Summarizing their results, the stack $\overline{H}_{g,k}^{\mathfrak{o}}$ of admissible covers (whose coarse moduli space is precisely $\overline{\mathcal{H}}_{g,k}^{\mathfrak{o}}$) is isomorphic to the stack of *twisted stable* maps into the classifying stack $\mathcal{B}\mathfrak{S}_k$ of the symmetric group $\mathfrak{S}_k$, that is, there is a canonical identification

$$\overline{H}_{g,k}^{\mathfrak{o}} := \overline{M}_{0,2g+2k-2}\Big(\mathcal{B}\mathfrak{S}_k\Big).$$

Points of $\overline{\mathcal{H}}_{g,k}^{\mathfrak{o}}$ can be thought of as admissible covers $[f : C \to R, p_1, \ldots, p_{2g+2k-2}]$, where the source C is a nodal curve of arithmetic genus g, the target R is a tree of smooth rational curves, f is a finite map of degree k satisfying $f^{-1}(R_{\mathrm{sing}}) = C_{\mathrm{sing}}$, and $p_1, \ldots, p_{2g+2k-2} \in R_{\mathrm{reg}}$ denote the branch points of f. Furthermore, the ramification indices on the two branches of C at each ramification point of f at a node of C must coincide. One has a finite *branch* morphism

$$\mathfrak{b} : \overline{\mathcal{H}}_{g,k}^{\mathfrak{o}} \to \overline{\mathcal{M}}_{0,2g+2k-2},$$

associating to a cover its (ordered) branch locus. The symmetric group $\mathfrak{S}_{2g+2k-2}$ operates on $\overline{\mathcal{H}}_{g,k}^{\mathfrak{o}}$ by permuting the branch points of each admissible cover. Denoting by

$$\overline{\mathcal{H}}_{g,k} := \overline{\mathcal{H}}_{g,k}^{\mathfrak{o}}/\mathfrak{S}_{2g+2k-2}$$

the quotient parametrizing admissible covers *without* an ordering of the branch points, we introduce the projection $q : \overline{\mathcal{H}}_{g,k}^{\mathfrak{o}} \to \overline{\mathcal{H}}_{g,k}$. Finally, let

$$\sigma : \overline{\mathcal{H}}_{g,k} \to \overline{\mathcal{M}}_g$$

be the map assigning to an admissible degree k cover the stable model of its source curve, obtained by contracting unnecessary rational components.

We discuss the structure of the boundary divisors on the compactified Hurwitz space. For $i = 0, \ldots, g+k-1$, let B_i be the boundary divisor of $\overline{\mathcal{M}}_{0,2g+2k-2}$, defined as the closure of the locus of unions of two smooth rational curves meeting at one point, such that precisely i of the marked points lie on one component and $2g+2k-2-i$ on the remaining one. To specify a boundary divisor of $\overline{\mathcal{H}}_{g,k}^{\mathfrak{o}}$, one needs the following combinatorial information:

(i) A partition $I \sqcup J = \{1, \ldots, 2g+2k-2\}$, such that $|I| \geq 2$ and $|J| \geq 2$.

(ii) Transpositions $\{w_i\}_{i\in I}$ and $\{w_j\}_{j\in J}$ in $\mathfrak{S}_k$, satisfying

$$\prod_{i\in I} w_i = u, \ \ \prod_{j\in J} w_j = u^{-1},$$

for some permutation $u \in \mathfrak{S}_k$.

To this data, we associate the locus of admissible covers of degree k with labeled branch points

$$\big[f : C \to R, \ p_1, \ldots, p_{2g+2k-2}\big] \in \overline{\mathcal{H}}^{\mathfrak{o}}_{g,k},$$

where $[R = R_1 \cup_p R_2, p_1, \ldots, p_{2g+2k-2}] \in B_{|I|} \subseteq \overline{\mathcal{M}}_{0,2g+2k-2}$ is a pointed union of two smooth rational curves R_1 and R_2 meeting at the point p. The marked points indexed by I lie on R_1, those indexed by J lie on R_2. Let $\mu := (\mu_1, \ldots, \mu_\ell) \vdash k$ be the partition corresponding to the conjugacy class of $u \in \mathfrak{S}_k$. We denote by $E_{i:\mu}$ the boundary divisor on $\overline{\mathcal{H}}^{\mathfrak{o}}_k$ classifying twisted stable maps with underlying admissible cover as above, with $f^{-1}(p)$ having partition type μ, and exactly i of the points $p_1, \ldots, p_{2g+2k-2}$ lying on R_1. Passing to the unordered Hurwitz space, we denote by $D_{i:\mu}$ the image $E_{i:\mu}$ under the map q, with its reduced structure.

The effect of the map $\mathfrak{b}$ on boundary divisors is summarized in the following relation that holds for $i = 2, \ldots, g+k-1$, see [HM] p. 62, or [GK1] Lemma 3.1:

$$\mathfrak{b}^*(B_i) = \sum_{\mu \vdash k} \mathrm{lcm}(\mu) E_{i:\mu}. \tag{13}$$

The Hodge class on the Hurwitz space is by definition pulled back from $\overline{\mathcal{M}}_g$. Its class $\lambda := (\sigma \circ q)^*(\lambda)$ on $\overline{\mathcal{H}}^{\mathfrak{o}}_{g,k}$ has been determined in [KKZ], or see [GK1] Theorem 1.1 for an algebraic proof. Remarkably, unlike on $\overline{\mathcal{M}}_g$, the Hodge class is always a boundary class:

$$\lambda = \sum_{i=2}^{g+k-1} \sum_{\mu \vdash k} \mathrm{lcm}(\mu) \Big(\frac{i(2g+2k-2-i)}{8(2g+2k-3)} - \frac{1}{12}\Big(k - \sum_{j=1}^{\ell(\mu)} \frac{1}{\mu_j}\Big)\Big) [E_{i:\mu}] \in CH^1(\overline{\mathcal{H}}^{\mathfrak{o}}_k). \tag{14}$$

The sum (14) runs over partitions μ of k corresponding to conjugacy classes of permutations that can be written as products of i transpositions. In the formula (14), $\ell(\mu)$ denotes the length of the partition μ. . The only negative coefficient in the expression of $K_{\overline{\mathcal{M}}_{0,2g+2k-2}}$ is that of the boundary divisor B_2. For this reason, the compoents of $\mathfrak{b}^*(B_2)$ play a special role, which we now discuss. We pick an admissible cover

$$[f : C = C_1 \cup C_2 \to R = R_1 \cup_p R_2, \ p_1, \ldots, p_{2g+2k-2}] \in \mathfrak{b}^*(B_2),$$

and set $C_1 := f^{-1}(R_1)$ and $C_2 := f^{-1}(R_2)$ respectively. Note that C_1 and C_2 may well be disconnected. Without loss of generality, we assume $I = \{1, \ldots, 2g+2k-4\}$, thus $p_1, \ldots, p_{2g+2k-4} \in R_1$ and $p_{2g+2k-3}, p_{2g+2k-2} \in R_2$.

Let $E_{2:(1^k)}$ be the closure in $\overline{\mathcal{H}}^{\mathfrak{o}}_{g,k}$ of the locus of admissible covers such that the transpositions $w_{2g+2k-3}$ and $w_{2g+2k-2}$ describing the local monodromy in a neighborhood of the branch points $p_{2g+2k-3}$ and $p_{2g+2k-2}$ respectively, are equal. Let E_0 further denote the component of $E_{2:(1^k)}$ consisting of those admissible cover for which the subcurve C_1 is connected. This is the case precisely when $\langle w_1, \ldots, w_{2g+2k-4}\rangle = \mathfrak{S}_k$. To show that E_0 is irreducible, one uses the classical topological argument due to Clebsch invoked when establishing the irreducibility of $\overline{\mathcal{H}}_{g,k}$. Note that $E_{2:(1^k)}$ has other irreducible components, for instance when C_1 splits as the disjoint union of a smooth rational curve mapping isomorphically onto R_1 and a second component mapping with degree $k-1$ onto R_1.

When the permutations $w_{2g+2k-3}$ and $w_{2g+2k-2}$ are distinct but share one element in their orbit, then $\mu=(3,1^{k-3})\vdash k$ and the corresponding boundary divisor is denoted by $E_{2:(3,1^{k-3})}$. Let E_3 be the subdivisor of $E_{2:(3,1^{k-3})}$ corresponding to admissible covers with $\langle w_1,\ldots,w_{2g+2k-4}\rangle=\mathfrak{S}_k$, that is, C_1 is a connected curve. Finally, in the case when $w_{2g+2k-3}$ and $w_{2g+2k-2}$ are disjoint transpositions, we obtain the boundary divisor $E_{2:(2,2,1^{k-4})}$. Similarly to the previous case, we denote by E_2 the irreducible component of $E_{2:(2,2,1^{k-4})}$ consisting of admissible covers for which $\langle w_1,\ldots,w_{2g+2k-4}\rangle=\mathfrak{S}_k$.

The boundary divisors E_0, E_2 and E_3, when pulled-back under the quotient map $q:\overline{\mathcal{H}}_{g,k}^{\mathfrak{o}}\rightarrow\overline{\mathcal{H}}_{g,k}$, verify the following formulas

$$q^*(D_0)=2E_0,\ q^*(D_2)=E_2\ \text{ and } q^*(D_3)=2E_3,$$

which we now explain. The general point of both E_0 and E_3 has no automorphism that fixes all branch points, but admits an automorphism of order two that fixes C_1 and permutes the branch points $p_{2g+2k-3}$ and $p_{2g+2k-2}$. The general admissible cover in E_2 has an automorphism group $\mathbb{Z}_2\times\mathbb{Z}_2$ (each of the two components of C_2 mapping $2:1$ onto R_2 has an automorphism of order 2). In the stack $\overline{H}_{g,k}^{\mathfrak{o}}$ we have two points lying over this admissible cover and each of them has an automorphism group of order 2. In particular the map $\overline{H}_{g,k}^{\mathfrak{o}}\rightarrow\overline{\mathcal{H}}_{g,k}^{\mathfrak{o}}$ from the stack to the coarse moduli space is ramified with ramification index 1 along the divisor E_2.

One applies now the Riemann-Hurwitz formula to the map $\mathfrak{b}:\overline{\mathcal{H}}_{g,k}^{\mathfrak{o}}\rightarrow\overline{\mathcal{M}}_{0,2g+2k-2}$. Recall also that the canonical bundle of the moduli space of pointed rational curves is given by the formula $K_{\overline{\mathcal{M}}_{0,2g+2k-2}}=\sum_{i=2}^{g+k-1}\bigl(\frac{i(2g+2k-2-i)}{2g+2k-3}-2\bigr)[B_i]$. All in all, we obtain the following formula for the canonical class of the Hurwitz stack:

$$K_{\overline{H}_{g,k}^{\mathfrak{o}}}=\mathfrak{b}^*K_{\overline{\mathcal{M}}_{0,2g+2k-2}}+\mathrm{Ram}(\mathfrak{b}),\tag{15}$$

where $\mathrm{Ram}(\mathfrak{b})=\sum_{i,\mu\vdash k}(\mathrm{lcm}(\mu)-1)[E_{i:\mu}]$.

4.1. **A partial compactification of $\mathcal{H}_{g,k}$.** Like in the paper [FR], we work on a partial compactification of $\mathcal{H}_{g,k}$, for the Koszul-theoretic calculations are difficult to extend over all the boundary divisors $D_{i:\mu}$. The partial compactification of the Hurwitz space we consider is defined in the same spirit as in the part of the paper devoted to divisors on $\overline{\mathcal{M}}_g$. We fix an integer $k\leq\frac{2g+4}{3}$. It follows then that the locus of curves $[C]\in\mathcal{M}_g$ such that $W^2_k(C)\neq\emptyset$ has codimension at least 2 in $\mathcal{M}_g$, hence it will play no role in any divisor class calculation. We denote by $\widetilde{\mathcal{G}}^1_{g,k}$ the space of pairs $[C,A]$, where C is an irreducible nodal curve of genus g satisfying $W^2_k(C)=\emptyset$ and A is a base point free locally free sheaf of degree k on C with $h^0(C,A)=2$. The rational map $\overline{\mathcal{H}}_{g,k}\dashrightarrow\widetilde{\mathcal{G}}^1_{g,k}$ is of course regular outside a subvariety of $\overline{\mathcal{H}}_{g,k}$ of codimension at least 2, but can be made explicit over the boundary divisors D_0, D_2 and D_3, which we now explain.

Retaining the previous notation, to the general point $[f:C_1\cup C_2\rightarrow R_1\cup_p R_2]$ of D_3 (respectively D_2), we assign the pair $[C_1, A_1:=f^*\mathcal{O}_{R_1}(1)]\in\widetilde{\mathcal{G}}^1_k$. Note that C_1 is a smooth curve of genus g and A_1 is a pencil on C_1 having a triple point (respectively two ramification points in the fibre over p). The spaces $\mathcal{H}_{g,k} \cup D_0 \cup D_2 \cup D_3$ and $\widetilde{\mathcal{G}}^1_{g,k}$ differ outside a set of codimension at least 2 and for divisor class calculations they will be identified. Using this, we copy the formula (14) at the level of the parameter space $\widetilde{\mathcal{G}}^1_{g,k}$ and obtain:

$$(16)\quad \lambda = \frac{g+k-2}{4(2g+2k-3)}[D_0] - \frac{1}{4(2g+2k-3)}[D_2] + \frac{g+k-6}{12(2g+2k-3)}[D_3] \in CH^1(\widetilde{\mathcal{G}}^1_{g,k}).$$

We now observe that the canonical class of $\widetilde{\mathcal{G}}^1_k$ has a simple expression in terms of the Hodge class λ and the boundary divisors D_0 and D_3. Quite remarkably, this formula is independent of both g and k!

Theorem 4.1. *The canonical class of the partial compactification $\widetilde{\mathcal{G}}^1_{g,k}$ is given by*

$$K_{\widetilde{\mathcal{G}}^1_{g,k}} = 8\lambda + \frac{1}{6}[D_3] - \frac{3}{2}[D_0].$$

Proof. We combine the equation (15) with the Riemann-Hurwitz formula applied to the quotient $q : \overline{\mathcal{H}}^{\circ}_{g,k} \dashrightarrow \widetilde{\mathcal{G}}^1_{g,k}$ and write:

$$q^*(K_{\widetilde{\mathcal{G}}^1_{g,k}}) = K_{\overline{H}^{\circ}_{g,k}} - [E_0] - [E_2] - [E_3] =$$

$$-\frac{4}{2g+2k-3}[D_2] - \frac{2g+2k-1}{2g+2k-3}[D_0] + \frac{2g+2k-9}{2g+2k-3}[D_3].$$

To justify this formula, observe that the divisors E_0 and E_3 lie in the ramification locus of q, hence they have to subtracted from $K_{\overline{H}^{\circ}_{g,k}}$. The morphism $\overline{H}^{\circ}_{g,k} \to \overline{\mathcal{H}}^{\circ}_{g,k}$ from the stack to the coarse moduli space is furthermore simply ramified along E_2, so this divisor has to be subtracted as well. We now use (16) to express $[D_2]$ in terms of λ, $[D_0]$ and $[D_3]$ and obtain that $q^*(K_{\widetilde{\mathcal{G}}^1_{g,k}}) = 8\lambda + \frac{1}{3}[E_3] - 3[E_0]$, which yields the claimed formula. □

Let $f : \mathcal{C} \to \widetilde{\mathcal{G}}^1_{g,k}$ be the universal curve and we choose a degree k Poincaré line bundle $\mathcal{L}$ on $\mathcal{C}$ (or on an étale cover if necessary). Along the lines of [FR] Section 2 (where only the case $g = 2k-1$ has been treated, though the general situation is analogous), we introduce two tautological codimension one classes:

$$\mathfrak{a} := f_*\big(c_1^2(\mathcal{L})\big) \text{ and } \mathfrak{b} := f_*\big(c_1(\mathcal{L}) \cdot c_1(\omega_f)\big) \in CH^1(\widetilde{\mathcal{G}}^1_{g,k}).$$

The push-forward sheaf $\mathcal{V} := f_*\mathcal{L}$ is locally free of rank 2 on $\widetilde{\mathcal{G}}^1_{g,k}$. Its fibre at a point $[C, A]$ is canonically identified with $H^0(C, A)$. Although $\mathcal{L}$ is not unique, an easy exercise involving first Chern classes, convinces us that the class

$$(17)\qquad \gamma := \mathfrak{b} - \frac{g-1}{k}\mathfrak{a} \in CH^1(\widetilde{\mathcal{G}}^1_{g,k})$$

does not depend of the choice of a Poincaré bundle.

Proposition 4.2. *We have that $\mathfrak{a} = kc_1(\mathcal{V}) \in CH^1(\widetilde{\mathcal{G}}^1_{g,k})$.*

Proof. Simple application of the Porteous formula in the spirit of Proposition 11.2 in [FR]. □

The following locally free sheaves on $\widetilde{\mathcal{G}}^1_{g,k}$ will play an important role in several Koszul-theoretic calculations:

$$\mathcal{E} := f_*(\omega_f \otimes \mathcal{L}^{-1}) \ \text{ and } \ \mathcal{F}_\ell := f_*(\omega_f^\ell \otimes \mathcal{L}^{-\ell}),$$

where $\ell \geq 2$.

Proposition 4.3. *The following formulas hold*

$$c_1(\mathcal{E}) = \lambda - \frac{1}{2}\mathfrak{b} + \frac{k-2}{2k}\mathfrak{a} \ \textit{and} \ c_1(\mathcal{F}_\ell) = \lambda + \frac{\ell^2}{2}\mathfrak{a} - \frac{\ell(2\ell-1)}{2}\mathfrak{b} + \binom{\ell}{2}\big(12\lambda - [D_0]\big).$$

Proof. Use Grothendieck-Riemann-Roch applied to the universal curve f, coupled with Proposition 4.2 in order to evaluate the terms. Use that $R^1 f^*(\omega_f^\ell \otimes \mathcal{L}^{-\ell}) = 0$ for $\ell \geq 2$. Similar to Proposition 11.3 in [FR], so we skip the details. □

We summarize the relation between the class γ and the classes $[D_0], [D_2]$ and $[D_3]$ as follows. Again, we find it remarkable that this formula is independent of g and k.

Proposition 4.4. *One has the formula* $[D_3] = 6\gamma + 24\lambda - 3[D_0]$.

Proof. We form the fibre product of the universal curve $f : \mathcal{C} \to \widetilde{\mathcal{G}}^1_{g,k}$ together with its projections:

$$\mathcal{C} \xleftarrow{\pi_1} \mathcal{C} \times_{\widetilde{\mathcal{G}}^1_{g,k}} \mathcal{C} \xrightarrow{\pi_2} \mathcal{C}.$$

For $\ell \geq 1$, we consider the jet bundle $J_f^\ell(\mathcal{L})$, which sits in an exact sequence:

$$0 \longrightarrow \omega_f^{\otimes \ell} \otimes \mathcal{L} \longrightarrow J_f^\ell(\mathcal{L}) \longrightarrow J_f^{\ell-1}(\mathcal{L}) \longrightarrow 0. \tag{18}$$

One has a sheaf morphism $\nu_2 : f^*(\mathcal{V}) \to J_f^2(\mathcal{L})$, which we think of as the *second Taylor map* associating to a section its first two derivatives. For points $[C, A, p] \in \mathcal{C}$ such that $p \in C$ is a smooth point, this map is simply the evaluation $H^0(C, A) \to H^0(A \otimes \mathcal{O}_{3p})$. Let $Z \subseteq \mathcal{C}$ be the locus where ν_2 is not injective. Over the locus of smooth curves, D_3 is the set-theoretic image of Z. A local analysis that we shall present shows that ν_2 is degenerate with multiplicity 1 at a point $[C, A, p]$, where $p \in C_{\text{sing}}$. Thus, D_0 is to be found with multiplicity 1 in the degeneracy locus of ν_2. The Porteous formula leads to:

$$[D_3] = f_* c_2\Big(\frac{J_f^2(\mathcal{L})}{f^*(\mathcal{V})}\Big) - [D_0] \in CH^1(\widetilde{\mathcal{G}}^1_{g,k}).$$

As anticipated, we now show that D_0 appears with multiplicity 1 in the degeneracy locus of ν_2. To that end, we choose a family $F : X \to B$ of genus g curves of genus over a smooth 1-dimensional base B, such that X is smooth, and there is a point $b_0 \in B$ with $X_b := F^{-1}(b)$ is smooth for $b \in B \setminus \{b_0\}$, whereas X_{b_0} has a unique node $u \in X$. Assume furthermore that $A \in \text{Pic}(X)$ is a line bundle such that $A_b := L_{|X_b} \in W^1_k(X_b)$, for each $b \in B$. We further choose a local parameter $t \in \mathcal{O}_{B,b_0}$ and $x, y \in \mathcal{O}_{X,u}$, such that $xy = t$ represents the local equation of X around the point u. Then ω_F is locally generated by the meromorphic differential τ that is given by $\frac{dx}{x}$ outside the divisor $x = 0$ and by $-\frac{dy}{y}$ outside the divisor $y = 0$. Let us pick sections $s_1, s_2 \in H^0(X, A)$, where $s_1(u) \neq 0$, whereas s_2 vanishes with order 1 at the node u of X_{b_0}, along both its branches.

Passing to germs of functions at u, we have the relation $s_{2,u} = (x+y)s_{1,u}$. Then by direct calculation in local coordinates, the map $H^0\big(X_{b_0}, A_{b_0}\big) \to H^0\big(X_{b_0}, A_{b_0|3u}\big)$ is given by the 2×2 minors of the following matrix:

$$\begin{pmatrix} 1 & 0 & 0 \\ x+y & x-y & x+y \end{pmatrix}.$$

We conclude that D_0 appears with multiplicity 1 in the degeneracy locus of ν_2.

From the exact sequence (18) one computes $c_1(J_f^2(\mathcal{L})) = 3c_1(\mathcal{L}) + 3c_1(\omega_f)$ and $c_2(J_f^2(\mathcal{L})) = c_2(J_f^1(\mathcal{L})) + c_1(J_f^1(\mathcal{L})) \cdot c_1(\omega_f^{\otimes 2} \otimes \mathcal{L}) = 3c_1^2(\mathcal{L}) + 6c_1(\mathcal{L}) \cdot c_1(\omega_f) + 2c_1^2(\omega_f)$. Substituting, we find after routine calculations that

$$f_* c_2\left(\frac{J_f^2(\mathcal{L})}{f^*(\mathcal{V})}\right) = 6\gamma + 2\kappa_1,$$

where $\kappa_1 = f_*(c_1(\omega_f)^{\otimes 2})$. Using Mumford's formula $\kappa_1 = 12\lambda - [D_0] \in CH^1(\widetilde{\mathcal{G}}^1_{g,k})$, see e.g. [HM] top of page 50, we finish the proof. □

5. Effective divisors on Hurwitz spaces

We now describe the construction of four effective divisors on particular Hurwitz spaces interesting in moduli theory. The divisors in question are of syzygetic nature and resemble somehow the (virtual) divisors on $\overline{\mathcal{M}}_g$ discussed in the first part of the paper. Note that these divisors on Hurwitz spaces, unlike say the divisor $\mathfrak{D}$ on $\mathcal{M}_{23}$, are defined directly in terms of a general element $[C, A] \in \mathcal{H}_{g,k}$, without making reference to other attributes of the curve C. This simplifies both the task of computing their classes and showing that the respective codimension 1 conditions in moduli lead to genuine divisor on $\mathcal{H}_{g,k}$. Using the irreducibility of $\mathcal{H}_{g,k}$, this amounts to exhibiting *one* example of a point $[C, A]$ outside the divisor, which can be easily achieved with the use of *Macaulay*. We mention also the papers [GK2], [DP]. where one studies syzygetic loci in Hurwitz spaces using the splitting type of the scroll one canonically associates to a cover $C \to \mathbf{P}^1$.

5.1. **The Hurwitz space $\mathcal{H}_{14,9}$.** We consider the morphism $\sigma : \overline{\mathcal{H}}_{14,9} \to \overline{\mathcal{M}}_{14}$, whose general fibre is 2-dimensional. We choose a general element $[C, A] \in \mathcal{H}_{14,9}$ and set $L := K_C \otimes A^\vee \in W^5_{17}(C)$ to be the residual linear system. Furthermore, L is very ample, else there exist points $x, y \in C$ such that $A(x+y) \in W^2_{13}(C)$, which contradicts the Brill-Noether Theorem. Note that

$$h^0(C, L^{\otimes 2}) = \dim \mathrm{Sym}^2 H^0(C, L) = 21$$

and we set up the (a priori virtual) divisor

$$\widetilde{\mathcal{D}}_{14} := \Big\{[C, A] \in \widetilde{\mathcal{G}}^1_{14,9} : \phi_L : \mathrm{Sym}^2 H^0(C, L) \to H^0(C, L^{\otimes 2}) \text{ is not an isomorphism}\Big\}.$$

Our next results shows that, remarkably, this locus is indeed a divisor and it gives rise to an effective representative of the canonical divisor of $\widetilde{\mathcal{G}}^1_{14,9}$.

Proposition 5.1. *The locus $\mathcal{D}_{14}$ is a divisor on $\mathcal{H}_{14,9}$ and one has the following formula*

$$[\widetilde{\mathcal{D}}_{14}] = 4\lambda + \frac{1}{12}[D_3] - \frac{3}{4}[D_0] = \frac{1}{2}K_{\widetilde{\mathcal{G}}^1_{14,9}} \in CH^1(\widetilde{\mathcal{G}}^1_{14,9}).$$

Proof. The divisor $\widetilde{\mathcal{D}}_{14}$ is the degeneracy locus of the vector bundle morphism

$$\phi : \mathrm{Sym}^2(\mathcal{E}) \to \mathcal{F}_2.$$

The Chern class of both $\mathcal{E}$ and $\mathcal{F}_2$ are computed in Proposition 4.3 and we have the formulas $c_1(\mathcal{E}) = \lambda - \frac{1}{2}\mathfrak{b} + \frac{7}{18}\mathfrak{a}$ and $c_1(\mathcal{F}_2) = 13\lambda + 2\mathfrak{a} - 3\mathfrak{b} - [D_0]$. Taking into account that $\mathrm{rk}(\mathcal{E}) = 6$, we can write:

$$[\widetilde{\mathcal{D}}_{14}] = c_1\big(\mathcal{F}_2 - \mathrm{Sym}^2(\mathcal{E})\big) = c_1(\mathcal{F}_2) - 7c_1(\mathcal{E}) = 6\lambda - [D_0] + \frac{1}{2}\gamma.$$

We now substitute γ in the formula given by Proposition 4.4 involving also the divisor D_3 on $\widetilde{\mathcal{G}}^1_{14,9}$ of pairs $[C, A]$, such that $A \in W^1_9(C)$ has a triple point. We obtain that $[\widetilde{\mathcal{D}}_{14}] = 4\lambda + \frac{1}{12}[D_3] - \frac{3}{4}[D_0]$. Comparing with Theorem 4.1, the fact that $[\widetilde{\mathcal{D}}_{14}]$ is a (half-) canonical representative follows.

It remains to show that $\widetilde{\mathcal{D}}_{14}$ is indeed a divisor. Since $\mathcal{H}_{14,9}$ is irreducible, it suffices to construct one example of a smooth curve $C \subseteq \mathbf{P}^5$ of genus 14 and degree 17 which does not lie on any quadrics. To that end, we consider the *White surface* $X \subseteq \mathbf{P}^5$, obtained by blowing-up $\mathbf{P}^2$ at 15 points $p_1, \ldots, p_{15}$ in general position and embedded into $\mathbf{P}^5$ by the linear system $H := |5h - E_{p_1} - \cdots - E_{p_{15}}|$, where E_{p_i} is the exceptional divisor at the point $p_i \in \mathbf{P}^2$ and $h \in |\mathcal{O}_{\mathbf{P}^2}(1)|$. The White surface is known to projectively Cohen-Macaulay, its ideal being generated by the 3×3-minors of a certain 3×5-matrix of linear forms, see [Gi] Proposition 1.1. In particular, the map

$$\mathrm{Sym}^2 H^0(X, \mathcal{O}_X(1)) \to H^0(X, \mathcal{O}_X(2))$$

is an isomorphism and $X \subseteq \mathbf{P}^5$ lies on no quadrics. We now let $C \subseteq X$ be a general element of the linear system

$$\Big|12h - 3(E_{p_1} + \cdots + E_{p_{13}}) - 2(E_{p_{14}} + E_{p_{15}})\Big|.$$

Note that $\dim |\mathcal{O}_X(C)| = 6$ and a general element is a smooth curve $C \subseteq \mathbf{P}^5$ of degree 17 and genus 14. This finishes the proof, for C lies on no quadrics. □

5.2. **The Hurwitz space $\mathcal{H}_{19,13}$.** The case $g = 19$ is analogous to the situation in genus 14. We have a morphism $\sigma : \overline{\mathcal{H}}_{19,13} \to \overline{\mathcal{M}}_{19}$ with generically 5-dimensional fibres. The Kodaira dimension of both $\overline{\mathcal{M}}_{19}$ and of the Hurwitz spaces $\overline{\mathcal{H}}_{19,11}$ and $\overline{\mathcal{H}}_{19,12}$ is unknown. For a general element $[C, A] \in \mathcal{H}_{19,13}$, we set $L := K_C \otimes A^\vee \in W^6_{23}(C)$. Observe that $W^7_{23}(C) = \emptyset$, that is, L must be a complete linear series. The multiplication

$$\phi_L : \mathrm{Sym}^2 H^0(C, L) \to H^0(C, L^{\otimes 2})$$

is a map between between two vector spaces of the same dimension 28. It is easy to produce an example of a pair $[C, L]$, such that ϕ_L is an isomorphism. We introduce the divisor

$$\widetilde{\mathcal{D}}_{19} := \Big\{[C, A] \in \widetilde{\mathcal{G}}^1_{19,13} : \phi_L : \mathrm{Sym}^2 H^0(C, L) \to H^0(C, L^{\otimes 2}) \text{ is not an isomorphism}\Big\}.$$

Proposition 5.2. *One has the following formula*

$$[\widetilde{\mathcal{D}}_{19}] = \lambda + \frac{1}{6}[D_3] - \frac{1}{2}[D_0] \in CH^1(\widetilde{\mathcal{G}}^1_{19,13}).$$

Proof. Very similar to the proof of Proposition 5.1. □

We can now prove the case $g = 19$ from Theorem 0.6. Indeed, combining Proposition 5.2 and Theorem 4.1, we find

$$K_{\widetilde{\mathcal{G}}^1_{19,13}} = [\widetilde{\mathcal{D}}_{19}] + \sigma^*(7\lambda - \delta_0).$$

Since the class $7\lambda - \delta_0$ is big an $\overline{\mathcal{M}}_{19}$, it follows that $K_{\widetilde{\mathcal{G}}^1_{19,13}}$ is big.

5.3. **The Hurwitz space $\mathcal{H}_{17,11}$.** The minimal Hurwitz cover of $\overline{\mathcal{M}}_{17}$ is $\overline{\mathcal{H}}_{17,10}$, but its Kodaira dimension is unknown. We consider the next case $\sigma : \overline{\mathcal{H}}_{17,11} \to \overline{\mathcal{M}}_{17}$. As described in the Introduction, a general curve $C \subseteq \mathbf{P}^6$ of genus 17 and degree 21 (whose residual linear system is a pencil $A = K_C(-1) \in W^1_{11}(C)$) lies on a pencil of quadrics. The general element of this pencil has full rank 7 and we consider the intersection of the pencil with the discriminant. We define $\mathcal{D}_{17}$ to be the locus of pairs $[C, A] \in \mathcal{H}_{17,11}$ such that this intersection is not reduced.

Theorem 5.3. *The locus $\mathcal{D}_{17}$ is a divisor and the class of its closure $\widetilde{\mathcal{D}}_{17}$ in $\widetilde{\mathcal{G}}^1_{17,11}$ is given by*

$$[\widetilde{\mathcal{D}}_{17}] = \frac{1}{6}\Big(19\lambda - \frac{9}{2}[D_0] + \frac{5}{6}[D_3]\Big) \in CH^1(\widetilde{\mathcal{G}}^1_{17,11}).$$

Proof. We are in a position to apply [FR] Theorem 1.2, which deals precisely with degeneracy loci of this type. We obtain

$$[\widetilde{\mathcal{D}}_{19}] = 6\big(7c_1(F) - 52c_1(\mathcal{E})\big) = 6\big(39\lambda - 7[D_0] + 5\gamma\big).$$

Using once more Proposition 4.4, we obtained the claimed formula. □

Substituting the expression of $[\widetilde{\mathcal{D}}_{17}]$ in the formula of the canonical class of the Hurwitz space, we find

$$K_{\widetilde{\mathcal{G}}^1_{17,11}} = \frac{1}{5}[\widetilde{\mathcal{D}}_{17}] + \frac{3}{5}\sigma^*(7\lambda - \delta_0).$$

Just like in the previous case, since the class $7\lambda - \delta_0$ is big on $\overline{\mathcal{M}}_{17}$ and λ is ample of $\overline{\mathcal{H}}_{17,11}$, Theorem 0.6 follows for $g = 17$ as well.

5.4. **The Hurwitz space $\mathcal{H}_{16,9}$.** This is the most interesting case, for we consider a *minimal* Hurwitz cover $\sigma : \overline{\mathcal{H}}_{16,9} \to \overline{\mathcal{M}}_{16}$ of the uniruled moduli space of curves of genus 16. We fix a general point $[C, A] \in H_{16,9}$ and, set $L := K_C \otimes A^\vee \in W^6_{21}(C)$. It is proven in [F3] Theorem 2.7 that the locus $\mathcal{D}_{16}$ classifying pairs $[C, A]$ such that the multiplication map

$$\mu : I_2(L) \otimes H^0(C, L) \to I_3(L)$$

is not an isomorphism, is a divisor on $\mathcal{H}_{16,9}$.

First we determine the class of the Gieseker-Petri divisor, already mentioned in the introduction.

Proposition 5.4. *One has $[\widetilde{\mathcal{GP}}] = -\lambda + \gamma \in CH^1(\widetilde{\mathcal{G}}^1_{16,9})$.*

Proof. Recall that we have introduced the sheaves $\mathcal{V}$ and $\mathcal{E}$ on $\widetilde{\mathcal{G}}^1_{16,19}$ with fibres canonically isomorphic to $H^0(C, A)$ and $H^0(C, \omega_C \otimes A^\vee)$ over a point $[C, A] \in \widetilde{\mathcal{G}}^1_{16,9}$. We have a natural morphism $\mathcal{E}\otimes\mathcal{V}\to f_*(\omega_f)$ and $\widetilde{\mathcal{GP}}$ is the degeneracy locus of this map. Accordingly,

$$[\widetilde{\mathcal{GP}}]=\lambda-2c_1(\mathcal{E})-8c_1(\mathcal{V})=-\lambda+\big(\mathfrak{b}-\frac{5}{3}\mathfrak{a}\big)=-\lambda+\gamma.$$

□

We can now compute the class of the divisor $\widetilde{\mathcal{D}}_{16}$.

Theorem 5.5. *The locus $\widetilde{\mathcal{D}}_{16}$ is an effective divisor on $\widetilde{\mathcal{G}}^1_{16,9}$ and its class is given by*

$$[\widetilde{\mathcal{D}}_{16}]=\frac{65}{2}\lambda-5[D_0]+\frac{3}{2}[\widetilde{\mathcal{GP}}]\in CH^1(\widetilde{\mathcal{G}}^1_{16,9}).$$

Proof. Recall the definition of the vector bundles $\mathcal{F}_2$ and $\mathcal{F}_3$ on $\widetilde{\mathcal{G}}^1_{16,9}$, as well as the expression of their first Chern classes provided by Proposition 4.3. We define two further vector bundles $\mathcal{I}_2$ and $\mathcal{I}_3$ on $\widetilde{\mathcal{G}}^1_{16,9}$, via the following exact sequences:

$$0\longrightarrow\mathcal{I}_\ell\longrightarrow\mathrm{Sym}^\ell(\mathcal{E})\longrightarrow\mathcal{F}_\ell\longrightarrow 0,$$

for $\ell=2,3$. Note that $\mathrm{rk}(\mathcal{I}_2)=9$, whereas $\mathrm{rk}(\mathcal{I}_3)=72$. To make sure that these sequences are exact on the left outside a set of codimension at least 2 inside $\widetilde{\mathcal{G}}^1_{16,9}$, we invoke [F2], Propositions 3.9 and 3.10. The divisor $\widetilde{\mathcal{D}}_{16}$ is then the degeneracy locus of the morphism

$$\mu:\mathcal{I}_2\otimes\mathcal{E}\to\mathcal{I}_3$$

which globalizes the maps $\mu_{C,L}: I_{C,L}(2)\otimes H^0(C,L)\to I_{C,L}(3)$, where $L=\omega_C\otimes A^\vee$ and $[C,A]\in\widetilde{\mathcal{G}}^1_{16,9}$.

Noting that $c_1\big(\mathrm{Sym}^3(\mathcal{E})\big)=45c_1(\mathcal{E})$ and $c_1\big(\mathrm{Sym}^2(\mathcal{E})\big)=9c_1(\mathcal{E})$, we compute

$$[\widetilde{\mathcal{D}}_{16}]=c_1(\mathcal{I}_3)-8c_1(\mathcal{I}_2)-9c_1(\mathcal{E})=31\lambda-5[D_0]+\frac{3}{2}\gamma.$$

Substituting $\gamma=\lambda+[\widetilde{\mathcal{GP}}]$, we obtain the claimed formula.

It remains to observe that it has already been proved in [F2] Theorem 2.7 that for a general pair $[C,L]$, where $L\in W^6_{21}(C)$, the multiplication map $\mu_{C,L}$ is an isomorphism. □

The formula (1) mentioned in the introduction follows now by using Theorem 5.5 and the Riemann-Hurwitz formula for the map $\sigma:\widetilde{\mathcal{G}}^1_{16,9}\to\overline{\mathcal{M}}_{16}$. One writes

$$K_{\widetilde{\mathcal{G}}^1_{16,9}}=13\lambda-2[D_0]+\frac{3}{5}[\widetilde{\mathcal{GP}}]+\frac{2}{5}[\widetilde{\mathcal{GP}}]=\frac{2}{5}[\widetilde{\mathcal{D}}_{16}]+\frac{3}{5}[\widetilde{\mathcal{GP}}].$$

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