Brill-Noether Loci and the Gonality Stratification of $\mathcal{M}_g$

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1 Introduction

For an irreducible smooth projective complex curve C of genus g, the gonality defined as $\mathrm{gon}(C) = \min\{d \in \mathbb{Z}_{\geq 1} : \text{there exists a } \mathfrak{g}^1_d \text{ on } C\}$ is perhaps the second most natural invariant: it gives an indication of how far C is from being rational, in a way different from what the genus does. For $g \geq 3$ we consider the stratification of the moduli space $\mathcal{M}_g$ of smooth curves of genus g given by gonality:

$$\mathcal{M}^1_{g,2} \subseteq \mathcal{M}^1_{g,3} \subseteq \ldots \subseteq \mathcal{M}^1_{g,k} \subseteq \ldots \subseteq \mathcal{M}_g,$$

where $\mathcal{M}^1_{g,k} := \{[C] \in \mathcal{M}_g : C \text{ has a } \mathfrak{g}^1_k\}$. It is well-known that the k-gonal locus $\mathcal{M}^1_{g,k}$ is an irreducible variety of dimension $2g + 2k - 5$ when $k \leq (g+2)/2$; when $k \geq [(g+3)/2]$ one has that $\mathcal{M}^1_{g,k} = \mathcal{M}_g$ (see for instance [AC]). The number $[(g+3)/2]$ is thus the *generic gonality* for curves of genus g.

For positive integers g, r and d, we introduce the *Brill-Noether* locus

$$\mathcal{M}^r_{g,d} = \{[C] \in \mathcal{M}_g : C \text{ carries a } \mathfrak{g}^r_d\}.$$

The Brill-Noether Theorem (cf. [ACGH]) asserts that when the *Brill-Noether number* $\rho(g,r,d) = g - (r+1)(g-d+r)$ is negative, the general curve of genus g has no $\mathfrak{g}^r_d$'s, hence in this case the locus $\mathcal{M}^r_{g,d}$ is a proper subvariety of $\mathcal{M}_g$. We study the relative position of the loci $\mathcal{M}^r_{g,d}$ when $r \geq 3$ and $\rho(g,r,d) < 0$ with respect to the gonality stratification of $\mathcal{M}_g$. Typically, we would like to know the gonality of a 'general' point $[C] \in \mathcal{M}^r_{g,d}$, or equivalently the gonality of a 'general' smooth curve $C \subseteq \mathbb{P}^r$ of genus g and degree d. Since the geometry of the loci $\mathcal{M}^r_{g,d}$ is very messy (existence of many components, some nonreduced and/or not of expected dimension), we will content ourselves with computing $\mathrm{gon}(C)$ when $[C]$ is a general point of a 'nice' component of $\mathcal{M}^r_{g,d}$ (i.e. a component which is generically smooth, of the expected dimension and with general point corresponding to a curve with a very ample $\mathfrak{g}^r_d$).

Our main result is the following:

Theorem 1 *Let $g \geq 15$ and $d \geq 14$ be integers with g odd and d even, such that $d^2 > 8g$, $4d < 3g + 12$, $d^2 - 8g + 8$ is not a square and either $d \leq 18$ or $g < 4d - 31$. If*

$$(d', g') \in \{(d,g), (d+1, g+1), (d+1, g+2), (d+2, g+3)\},$$

then there exists a regular component of the Hilbert scheme $\mathrm{Hilb}_{d',g',3}$ *whose general point* $[C']$ *is a smooth curve such that* $\mathrm{gon}(C') = \min(d' - 4, [(g'+3)/2])$.

Here by $\mathrm{Hilb}_{d,g,r}$ we denote the Hilbert scheme of curves $C \subseteq \mathbb{P}^r$ with $p_a(C) = g$ and $\deg(C) = d$. A component of $\mathrm{Hilb}_{d,g,r}$ is said to be *regular* if its general point corresponds to a smooth irreducible curve $C \subseteq \mathbb{P}^r$ such that the normal bundle $N_{C/\mathbb{P}^r}$ satisfies $H^1(C, N_{C/\mathbb{P}^r}) = 0$. By standard deformation theory (cf. [Mod] or [Se]), a regular component of $\mathrm{Hilb}_{d,g,r}$ is generically smooth of the expected dimension $\chi(C, N_{C/\mathbb{P}^r}) = (r+1)d - (r-3)(g-1)$. Note that for $r = 3$ the expected dimension of the Hilbert scheme is just $4d$. We refer to Section 4 for a natural extension of Theorem 1 for curves in higher dimensional projective spaces.

As for the numerical conditions entering Theorem 1, we note that the inequality $d^2 > 8g$ ensures the existence of smooth curves $C \subseteq \mathbb{P}^3$ with $g(C) = g$ and $\deg(C) = d$ (see Section 2), $4d < 3g + 12 \Leftrightarrow \rho(g, 3, d) < 0$ is just the condition that $\mathcal{M}^3_{g,d}$ is a proper subvariety of $\mathcal{M}_g$, while the remaining requirements are mild technical conditions.

A remarkable application of Theorem 1 is a new proof of our result (cf. [Fa]):

Theorem 2 *The Kodaira dimension of the moduli space of curves of genus* 23 *is* ≥ 2.

We recall that for $g \geq 24$ Harris, Mumford and Eisenbud proved (cf. [HM],[EH]) that $\mathcal{M}_g$ is of general type whereas for $g \leq 16, g \neq 14$ we have that $\kappa(\mathcal{M}_g) = -\infty$. The famous Slope Conjecture of Harris and Morrison predicts that $\mathcal{M}_g$ is uniruled for all $g \leq 22$ (see [Mod]). Therefore the moduli space $\mathcal{M}_{23}$ appears as an intriguing transition case between two extremes: uniruledness and being of general type.

To put our Theorem 1 into perspective, let us note that for $r = 2$ we have the following result of M. Coppens (cf. [Co]): let $\nu : C \to \Gamma$ be the normalization of a general, irreducible plane curve of degree d with $\delta = g - \binom{d-1}{2}$ nodes. Assume that $0 < \delta < (d^2 - 7d + 18)/2$. Then $\mathrm{gon}(C) = d - 2$.

This theorem says that there are no $\mathfrak{g}^1_{d-3}$'s on C. On the other hand a $\mathfrak{g}^1_{d-2}$ is given by the lines through a node of Γ. The condition $\delta < (d^2 - 7d + 18)/2$ from the statement is equivalent with $\rho(g, 1, d-3) < 0$. This is the range in which the problem is non-trivial: if $\rho(g, 1, d-3) \geq 0$, the Brill-Noether Theorem provides $\mathfrak{g}^1_{d-3}$'s on C.

For $r \geq 3$ we might hope for a similar result. Let $C \subseteq \mathbb{P}^r$ be a suitably general smooth curve of genus g and degree d, with $\rho(g, r, d) < 0$. We can always assume that $d \leq g - 1$ (by duality $\mathfrak{g}^r_d \mapsto |K_C - \mathfrak{g}^r_d|$ we can always land in this range). One can expect that a $\mathfrak{g}^1_k$ computing $\mathrm{gon}(C)$ is of the form $\mathfrak{g}^r_d(-D) = \{E - D : E \in \mathfrak{g}^r_d, E \geq D\}$ for some effective divisor D on C. Since the expected dimension of the variety of e-secant $(r-2)$-plane divisors

$$V^{r-1}_e(\mathfrak{g}^r_d) := \{D \in C_e : \dim \mathfrak{g}^r_d(-D) \geq 1\}$$

is $2r - 2 - e$ (cf. [ACGH]), we may ask whether C has finitely many $(2r-2)$-secant $(r-2)$-planes (and no $(2r-1)$-secant $(r-2)$-planes at all). This is known to be true for curves with general moduli, that is, when $\rho(g, r, d) \geq 0$ (cf. [Hir]): for instance a smooth curve $C \subseteq \mathbb{P}^3$ with general moduli has only finitely many 4-secant lines and no 5-secant lines. No such principle appears to be known for curves with special moduli.

Definition: We call the number $\min(d - 2r + 2, [(g+3)/2])$ the *expected gonality* of a smooth nondegenerate curve $C \subseteq \mathbb{P}^r$ of degree d and genus g.

One can approach such problems from a different angle: find recipes to compute the

gonality of various classes of curves $C \subseteq \mathbb{P}^r$. Our knowledge in this respect is very scant: we know how to compute the gonality of extremal curves $C \subseteq \mathbb{P}^r$ (that is, curves attaining the Castelnuovo bound, see [ACGH]) and the gonality of complete intersections in $\mathbb{P}^3$ (cf. [Ba]): If $C \subseteq \mathbb{P}^3$ is a smooth complete intersection of type (a, b) then $\mathrm{gon}(C) = ab - l$, where l is the degree of a maximal linear divisor on C. Hence an effective divisor $D \subseteq C$ computing $\mathrm{gon}(C)$ is residual to a linear divisor of degree l in a plane section of C.
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2 Linear systems on $K3$ surfaces in $\mathbb{P}^r$

We will construct smooth curves $C \subseteq \mathbb{P}^r$ having the expected gonality starting with sections of smooth $K3$ surfaces. We recall a few basic facts about linear systems on $K3$ surfaces (cf. [SD]).

Let S be a smooth $K3$ surface. For an effective divisor $D \subseteq S$, we have $h^1(S, D) = h^0(D, \mathcal{O}_D) - 1$. If $C \subseteq S$ is an irreducible curve then $H^1(S, C) = 0$, and by Riemann-Roch we have that $\dim|C| = 1 + C^2/2 = p_a(C)$. In particular $C^2 \geq -2$ for every irreducible curve C. Moreover we have equivalences

$C^2 = -2 \Longleftrightarrow \dim|C| = 0 \Longleftrightarrow C$ is a smooth rational curve and

$C^2 = 0 \Longleftrightarrow \dim|C| = 1 \Longleftrightarrow p_a(C) = 1$.

For a $K3$ surface one also has a 'strong Bertini' Theorem (cf. [SD]):

Proposition 2.1 *Let $\mathcal{L}$ be a line bundle on a $K3$ surface S such that $|\mathcal{L}| \neq \emptyset$. Then $|\mathcal{L}|$ has no base points outside its fixed components. Moreover, if $\mathrm{bs}|\mathcal{L}| = \emptyset$ then either*

- $\mathcal{L}^2 > 0,\ h^1(S, \mathcal{L}) = 0$ *and the general member of $|\mathcal{L}|$ is a smooth, irreducible curve of genus $\mathcal{L}^2/2 + 1$, or*
- $\mathcal{L}^2 = 0$ *and $\mathcal{L} = \mathcal{O}_S(kE)$, where $k \in \mathbb{Z}_{\geq 1}$, $E \subseteq S$ is an irreducible curve with $p_a(E) = 1$. We have that $h^0(S, \mathcal{L}) = k + 1,\ h^1(S, \mathcal{L}) = k - 1$ and all divisors in $|\mathcal{L}|$ are of the form $E_1 + \cdots + E_k$ with $E_i \sim E$.*

We are interested in space curves sitting on $K3$ surfaces and the starting point is Mori's Theorem (cf. [Mo]): if $d > 0$, $g \geq 0$, there is a smooth curve $C \subseteq \mathbb{P}^3$ of degree d and genus g, lying on a smooth quartic surface S, if and only if (1) $g = d^2/8 + 1$, or (2) $g < d^2/8$ and $(d, g) \neq (5, 3)$. Moreover, we can choose S such that $\mathrm{Pic}(S) = \mathbb{Z}H = \mathbb{Z}(4/d)C$ in case (1) and such that $\mathrm{Pic}(S) = \mathbb{Z}H \oplus \mathbb{Z}C$, with $H^2 = 4, C^2 = 2g - 2$ and $H \cdot C = d$, in case (2). In each case H denotes a plane section of S. Note that from the Hodge Index Theorem one has the necessary condition $(C \cdot H)^2 - H^2C^2 = d^2 - 8(g - 1) \geq 0$.

Mori's result has been extended by Rathmann to curves in higher dimensional projective spaces (cf. [Ra], see also [Kn]): For integers $d > 0, g > 0$ and $r \geq 3$ such that $d^2 \geq 4g(r - 1) + (r - 1)^2$, there exists a smooth $K3$ surface $S \subseteq \mathbb{P}^r$ of degree $2r - 2$ and a smooth curve $C \subseteq S$ of genus g and degree d such that $\mathrm{Pic}(S) = \mathbb{Z}H \oplus \mathbb{Z}C$, where H is a hyperplane section of S.

We will repeatedly use the following simple observation:

Proposition 2.2 *Let $S \subseteq \mathbb{P}^r$ be a smooth $K3$ surface of degree $2r-2$ with a smooth curve $C \subseteq S$ such that $\mathrm{Pic}(S) = \mathbb{Z}H \oplus \mathbb{Z}C$ and assume that S has no (-2) curves. A divisor class D on S is effective if and only if $D^2 \geq 0$ and $D \cdot H > 2$.*

Remark: If $S \subseteq \mathbb{P}^r$ is a smooth $K3$ surface of degree $2r-2$ with Picard number 2 as above, S has no (-2) curves when the equation

$$(r-1)m^2 + mnd + (g-1)n^2 = -1 \tag{1}$$

has no solutions $m, n \in \mathbb{Z}$. This is the case for instance when d is even and g and r are odd. Furthermore, a necessary condition for S to have genus 1 curves is that $d^2 - 4(g-1)(r-1)$ is a square.

3 Brill-Noether special linear series on curves on $K3$ surfaces

The first important result in the study of special linear series on curves lying on $K3$ surfaces was Lazarsfeld's proof of the Brill-Noether-Petri Theorem (cf. [Laz]). He noticed that there is no Brill-Noether type obstruction to embed a curve in a $K3$ surface: if $C_0 \subseteq S$ is a smooth curve of genus $g \geq 2$ on a $K3$ surface such that $\mathrm{Pic}(S) = \mathbb{Z}C_0$, then the general curve $C \in |C_0|$ satisfies the Brill-Noether-Petri Theorem, that is, for any line bundle A on C, the Petri map $\mu_0(C, A) : H^0(C, A) \otimes H^0(C, K_C \otimes A^\vee) \to H^0(C, K_C)$ is injective. We mention that Petri's Theorem implies (trivially) the Brill-Noether Theorem.

The general philosophy when studying linear series on a $K3$-section $C \subseteq S$ of genus $g \geq 2$, is that the type of a Brill-Noether special $\mathfrak{g}^r_d$ often does not depend on C but only on its linear equivalence class in S, i.e. a $\mathfrak{g}^r_d$ on C with $\rho(g, r, d) < 0$ is expected to propagate to all smooth curves $C' \in |C|$. This expectation, in such generality, is perhaps a bit too optimistic, but it was proved to be true for the Clifford index of a curve (see [GL]): for $C \subseteq S$ a smooth $K3$-section of genus $g \geq 2$, one has that $\mathrm{Cliff}(C') = \mathrm{Cliff}(C)$ for every smooth curve $C' \in |C|$. Furthermore, if $\mathrm{Cliff}(C) < [(g-1)/2]$ (the generic value of the Clifford index), then there exists a line bundle $\mathcal{L}$ on S such that for all smooth $C' \in |C|$ the restriction $\mathcal{L}_{|C'}$ computes $\mathrm{Cliff}(C')$. Recall that the *Clifford index* of a curve C of genus g is defined as

$$\mathrm{Cliff}(C) := \min\{\mathrm{Cliff}(D) : D \in \mathrm{Div}(C), h^0(D) \geq 2, h^1(D) \geq 2\},$$

where for an effective divisor D on C, we have $\mathrm{Cliff}(D) = \deg(D) - 2(h^0(D) - 1)$. Note that in the definition of $\mathrm{Cliff}(C)$ the condition $h^1(D) \geq 2$ can be replaced with $\deg(D) \leq g-1$. Another invariant of a curve is the *Clifford dimension* of C defined as

$$\text{Cliff-dim}(C) := \min\{r \geq 1 : \exists \mathfrak{g}^r_d \text{ on } C \text{ with } d \leq g-1, \text{ such that } d - 2r = \mathrm{Cliff}(C)\}.$$

Curves with Clifford dimension ≥ 2 are rare: smooth plane curves are precisely the curves of Clifford dimension 2, while curves of Clifford dimension 3 occur only in genus 10 as complete intersections of two cubic surfaces in $\mathbb{P}^3$.

Harris and Mumford during their work in [HM] conjectured that the gonality of a $K3$-section should stay constant in a linear system: if $C \subseteq S$ carries an exceptional $\mathfrak{g}^1_d$ then every smooth $C' \in |C|$ carries an equally exceptional $\mathfrak{g}^1_d$. This conjecture was later disproved by Donagi and Morrison (cf. [DMo]). They came up with the following counterexample: let $\pi : S \to \mathbb{P}^2$ be a $K3$ surface, double cover of $\mathbb{P}^2$ branched along a smooth sextic and let $\mathcal{L} = \pi^*\mathcal{O}_{\mathbb{P}^2}(3)$. The genus of a smooth $C \in |\mathcal{L}|$ is 10. The general $C \in |\mathcal{L}|$ carries a very ample $\mathfrak{g}^2_6$, hence $\mathrm{gon}(C) = 5$. On the other hand, any curve in the codimension 1 linear system $|\pi^* H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(3))|$ is bielliptic, therefore has gonality 4. Under reasonable assumptions this turns out to be the only counterexample to the Harris-Mumford conjecture. Ciliberto and Pareschi proved that if $C \subseteq S$ is such that $|C|$ is base-point-free and ample, then either $\mathrm{gon}(C') = \mathrm{gon}(C)$ for all smooth $C' \in |C|$, or (S, C) are as in the previous counterexample (cf. [CilP]).

Although $\mathrm{gon}(C)$ can drop as C varies in a linear system, base-point-free $\mathfrak{g}^1_d$'s on $K3$-sections do propagate:

Proposition 3.1 (Donagi-Morrison) *Let S be a $K3$ surface, $C \subseteq S$ a smooth, non-hyperelliptic curve and $|Z|$ a complete, base-point-free $\mathfrak{g}^1_d$ on C such that $\rho(g,1,d) < 0$. Then there is an effective divisor $D \subseteq S$ such that:*

- $h^0(S, D) \geq 2,\ h^0(S, C - D) \geq 2, \mathrm{deg_C(D_{|C})} \leq \mathrm{g} - 1.$
- $\mathrm{Cliff}(C', D_{|C'}) \leq \mathrm{Cliff}(C, Z)$, *for any smooth $C' \in |C|$.*
- *There is $Z_0 \in |Z|$, consisting of distinct points such that $Z_0 \subseteq D \cap C$.*

Throughout this paper, for a smooth curve C we denote, as usual, by $W^r_d(C)$ the scheme whose points are line bundles $A \in \mathrm{Pic}^d(C)$ with $h^0(C, A) \geq r + 1$, and by $G^r_d(C)$ the scheme parametrizing $\mathfrak{g}^r_d$'s on C.

4 The gonality of curves in $\mathbb{P}^r$

For a wide range of d, g and r we construct curves $C \subseteq \mathbb{P}^r$ of degree d and genus g having the expected gonality. We start with a case when we can realize our curves as sections of $K3$ surfaces.

Theorem 3 *Let $r \geq 3, d \geq r^2 + r$ and $g \geq 0$ be integers such that $\rho(g,r,d) < 0$ and with $d^2 > 4(r-1)(g+r-2)$ when $r \geq 4$ while $d^2 > 8g$ when $r = 3$. Let us assume moreover that 0 and -1 are not represented by the quadratic form*

$$Q(m,n) = (r-1)m^2 + mnd + (g-1)n^2, \quad m, n \in \mathbb{Z}.$$

Then there exists a smooth curve $C \subseteq \mathbb{P}^r$ of degree d and genus g such that $\mathrm{gon}(C) = \min(d - 2r + 2, [(g+3)/2])$. *If* $\mathrm{gon}(C) = d - 2r + 2 < [(g+3)/2]$ *then* $\dim W^1_{d-2r+2}(C) = 0$ *and every $\mathfrak{g}^1_{d-2r+2}$ is given by the hyperplanes through a $(2r-2)$-secant $(r-2)$-plane.*

Proof: By Rathmann's Theorem there exists a smooth $K3$ surface $S \subseteq \mathbb{P}^r$ with $\deg(S) = 2r - 2$ and $C \subseteq S$ a smooth curve of degree d and genus g such that $\mathrm{Pic}(S) = \mathbb{Z}H \oplus \mathbb{Z}C$, where H is a hyperplane section. The conditions d, g and r are subject to, ensure that S does not contain (-2) curves or genus 1 curves.

We prove first that Cliff-dim$(C) = 1$. It suffices to show that $C \subseteq S$ is an ample divisor, because then by using Prop.3.3 from [CilP] we obtain that either Cliff-dim$(C) = 1$ or C is a smooth plane sextic, $g = 10$ and (S, C) are as in Donagi-Morrison's example (then Cliff-dim$(C) = 2$). The latter case obviously does not happen.

We prove that $C \cdot D > 0$ for any effective divisor $D \subseteq S$. Let $D \sim mH + nC$, with $m, n \in \mathbb{Z}$, such a divisor. Then $D^2 = (2r-2)m^2 + 2mnd + n^2(2g-2) \geq 0$ and $D \cdot H = (2r-2)m + dn > 2$. The case $m \leq 0, n \leq 0$ is impossible, while the case $m \geq 0, n \geq 0$ is trivial. Let us assume $m > 0, n < 0$. Then $D \cdot C = md + n(2g-2) > -n\big(d^2/(2r-2) - 2g + 2\big) + d/(r-1) > 0$, because $d^2/(2r-2) > 2g$. In the remaining case $m < 0, n > 0$ we have that $nD \cdot C \geq -mD \cdot H > 0$, so C is ample by Nakai-Moishezon.

Our assumptions imply that $d \leq g-1$, so $\mathcal{O}_C(1)$ is among the line bundles from which Cliff(C) is computed. We get thus the following estimate on the gonality of C:

$$\text{gon}(C) = \text{Cliff}(C) + 2 \leq \text{Cliff}(C, H_{|C}) + 2 = d - 2r + 2,$$

which yields gon$(C) \leq \min(d - 2r + 2, [(g+3)/2])$.

For the rest of the proof let us assume that gon$(C) < [(g+3)/2]$. We will then show that gon$(C) = d - 2r + 2$. Let $|Z|$ be a complete, base point free pencil computing gon(C). By applying Prop.3.1, there exists an effective divisor $D \subseteq S$ satisfying

$$h^0(S, D) \geq 2, h^0(S, C-D) \geq 2, \deg(D_{|C}) \leq g-1, \ \text{gon}(C) = \text{Cliff}(D_{|C}) + 2 \text{ and } Z \subseteq D \cap C.$$

We consider the exact cohomology sequence:

$$0 \to H^0(S, D-C) \to H^0(S, D) \to H^0(C, D_{|C}) \to H^1(S, D-C).$$

Since $C - D$ is effective and $\nsim 0$, one sees that $D - C$ cannot be effective, so $H^0(S, D-C) = 0$. The surface S does not contain (-2) curves, so $|C - D|$ has no fixed components; the equation $(C-D)^2 = 0$ has no solutions, therefore $(C-D)^2 > 0$ and the general element of $|C - D|$ is smooth and irreducible. Then it follows that $H^1(S, D-C) = H^1(S, C-D)^\vee = 0$. Thus $H^0(S, D) = H^0(C, D_{|C})$ and

$$\text{gon}(C) = 2 + \text{Cliff}(D_{|C}) = 2 + D \cdot C - 2 \dim|D| = D \cdot C - D^2.$$

We consider the following family of effective divisors

$$\mathcal{A} := \{D \in \text{Div}(S) : h^0(S, D) \geq 2, h^0(S, C-D) \geq 2, \ C \cdot D \leq g-1\}.$$

Since we already know that $d - 2r + 2 \geq \text{gon}(C) \geq \alpha$, where $\alpha = \min\{D \cdot C - D^2 : D \in \mathcal{A}\}$, we are done if we prove that $\alpha \geq d - 2r + 2$. Take $D \in \mathcal{A}$ such that $D \sim mH + nC$, $m, n \in \mathbb{Z}$. The conditions $D^2 > 0, D \cdot C \leq g-1$ and $2 < D \cdot H < d-2$ (use Prop.2.2 for the last inequality) can be rewritten as

$$(r-1)m^2 + mnd + n^2(g-1) > 0 \text{ (i)}, \ 2 < (2r-2)m + nd < d-2 \text{ (ii)}, \ md + (2n-1)(g-1) \leq 0 \text{ (iii)}.$$

We have to prove that for any $D \in \mathcal{A}$ the following inequality holds

$$f(m,n) = D \cdot C - D^2 = -(2r-2)m^2 + m(d - 2nd) + (n - n^2)(2g-2) \geq f(1,0) = d - 2r + 2.$$

We solve this standard calculus problem. Denote by

$$a := \frac{d + \sqrt{d^2 - 4(r-1)(g-1)}}{2r-2} \quad \text{and} \quad b := \frac{d - \sqrt{d^2 - 4(r-1)(g-1)}}{2r-2} .$$

We dispose first of the case $n < 0$. Assuming $n < 0$, from (i) we have that either $m < -bn$ or $m > -an$. If $m < -bn$ from (ii) we obtain that $2 < n(d - (2r-2)b) < 0$, because $n < 0$ and $d - (2r-2)b = \sqrt{d^2 - 4(r-1)(g-1)} > 0$, so we have reached a contradiction.

We assume now that $n < 0$ and $m > -an$. From (iii) we get that $m \leq (g-1)(1-2n)/d$. If $-an > (g-1)(1-2n)/d$ we are done because there are no $m, n \in \mathbb{Z}$ satisfying (i), (ii) and (iii), while in the other case for any $D \in \mathcal{A}$ with $D \sim mH + nC$, one has the inequality

$$f(m,n) > f(-an,n) = \frac{(d^2 - 4(r-1)(g-1)) + d\sqrt{d^2 - 4(r-1)(g-1)}}{2r-2}(-n).$$

When $r \geq 4$ since we assume that $\sqrt{d^2 - 4(r-1)(g-1)} \geq 2r-2$, it immediately follows that $f(m,n) \geq d > d - 2r + 2$. In the case $r = 3$ when we only have the weaker assumption $d^2 > 8g$, we still get that $f(-an,n) > d - 4$ unless $n = -1$ and $d^2 - 8g < 8$. In this last situation we obtain $m \geq (d+4)/4$ so $f(m,-1) \geq f((d+4)/4, -1) > d - 4$.

The case $n > 0$ can be treated rather similarly. From (i) we get that either $m < -an$ or $m > -bn$. The first case can be dismissed immediately. When $m > -bn$ we use that for any $D \in \mathcal{A}$ with $D \sim mH + nC$,

$$f(m,n) \geq \min\{f(-(g-1)(2n-1)/d, n), \max\{f(-bn,n), f\big((2-nd)/(2r-2), n\big)\}\}.$$

Elementary manipulations give that

$$f(-(g-1)(2n-1)/d, n) = (g-1)/2\ [(2n-1)^2(d^2 - 4(r-1)(g-1))/d^2 + 1] \geq d - 2r + 2$$

(use only that $d \leq g-1$ and $d^2 > 4(r-1)g$, so we cover both cases $r = 3$ and $r \geq 4$ at once). Note that in the case $n > 0$ we have equality if and only if $n = 1, m = -1$ and $d = g - 1$.

Moreover $f(-bn,n) = n(2g-2-bd) \geq 2g-2-bd$ and $2g-2-bd > d-2r+2 \Leftrightarrow 2r-2 < \sqrt{d^2 - 4(r-1)(g-1)} < d - 2r + 2$. When this does not happen we proceed as follows: if $\sqrt{d^2 - 4(r-1)(g-1)} \geq d - 2r + 2$ then if $n = 1$ we have that $m > -b \geq -1$, that is $m \geq 0$, but this contradicts (ii). When $n \geq 2$, we have $f((2-nd)/(2r-2), n) = [(d^2 - 4(r-1)(g-1))(n^2-n) + (2d-4)]/(2r-2) > d - 2r + 2$. Finally, the remaining possibility $2r - 2 \geq \sqrt{d^2 - 4(r-1)(g-1)}$ does not occur when $r \geq 4$ while in the case $r = 3$ we either have $f(-bn,n) > d - 4$ or else $n = 1$ and then $m > (-d+4)/4$ hence $f(m,1) > f((-d+4)/4, 1) = d - 4$.

All this leaves us with the case $n = 0$, when $f(m,0) = -(2r-2)m^2 + md$. Clearly $f(m,0) \geq f(1,0)$ for all m complying with (i),(ii) and (iii).

Thus we proved that $\text{gon}(C) = d - 2r + 2$. We have equality $D \cdot C - D^2 = d - 2r + 2$ where $D \in \mathcal{A}$, if and only if $D = H$ or in the case $d = g - 1$ also when $D = C - H$. The latter possibility can be ruled out since $d = g - 1$ is not compatible with the assumptions $d \geq r^2 + r$ and $d - 2r + 2 < [(g+3)/2]$. Therefore we can always assume that the divisor on S cutting a $\mathfrak{g}^1_{d-2r+2}$ on C is the hyperplane section of S. Since $Z \subseteq H \cap C$, if we denote by Δ the residual divisor of Z in $H \cap C$, we have that $h^0(C, H_{|C} - \Delta) = 2$, so Δ spans a $\mathbb{P}^{r-2}$ hence $|Z|$ is given by the hyperplanes through the $(2r-2)$-secant $(r-2)$-plane $\langle \Delta \rangle$. This shows that every pencil computing $\mathrm{gon}(C)$ is given by the hyperplanes through a $(2r-2)$-secant $(r-2)$-plane.

There are a few ways to see that C has only finitely many $(2r-2)$-secant $(r-2)$-planes. The shortest is to invoke Theorem 3.1 from [CilP]: since $\mathrm{gon}(C') = d - 2r + 2$ is constant as C' varies in $|C|$, for the general smooth curve $C' \in |C|$ one has $\dim W^1_{d-2r+2}(C') = 0$. □

Remarks: 1. Keeping the assumptions and the notations of Theorem 3 we note that when $d - 2r + 2 < [(g+3)/2]$ the linear system $|C|$ is $(d-2r-1)$-very ample, i.e. for any 0-dimensional subscheme $Z \subseteq S$ of length $\leq d - 2r$ the map $H^0(S, C) \to H^0(S, C \otimes \mathcal{O}_Z)$ is surjective. Indeed, by applying Theorem 2.1 from [BS] if $|C|$ is not $(d-2r-1)$-very ample, there exists an effective divisor D on S such that $C - 2D$ is $\mathbb{Q}$-effective and

$$C \cdot D - (d - 2r) \leq D^2 \leq C \cdot D/2 < d - 2r,$$

hence $C \cdot D - D^2 \leq d - 2r$. On the other hand clearly $D \in \mathcal{A}$, thus $C \cdot D - D^2 \geq d - 2r + 2$, a contradiction.

2. One can find quartic surfaces $S \subseteq \mathbb{P}^3$ containing a smooth curve C of degree d and genus g in the case $g = d^2/8 + 1$ (which is outside the range Theorem 3 deals with). Then $d = 4m, g = 2m^2 + 1$ with $m \geq 1$ and C is a complete intersection of type $(4, m)$. For such a curve, $\mathrm{gon}(C) = d - l$, where l is the degree of a maximal linear divisor on C (cf. [Ba]). If S is sufficiently general so that it contains no lines, by Bezout, C cannot have 5-secant lines so $\mathrm{gon}(C) = d - 4$ in this case too.

When $r = 3$ we want to find out when the curves constructed in Theorem 3 correspond to smooth points of $\mathrm{Hilb}_{d,g,3}$. We have the following:

Proposition 4.1 *Let $C \subseteq S \subseteq \mathbb{P}^3$ be a smooth curve sitting on a quartic surface such that $\mathrm{Pic}(S) = \mathbb{Z}H \oplus \mathbb{Z}C$ with H being a plane section and assume furthermore that S contains no (-2) curves. Then $H^1(C, N_{C/\mathbb{P}^3}) = 0$ if and only if $d \leq 18$ or $g < 4d - 31$.*

Proof: We use the exact sequence

$$0 \longrightarrow N_{C/S} \longrightarrow N_{C/\mathbb{P}^3} \longrightarrow N_{S/\mathbb{P}^3} \otimes \mathcal{O}_C \longrightarrow 0, \tag{2}$$

where $N_{S/\mathbb{P}^3} \otimes \mathcal{O}_C = \mathcal{O}_C(4)$ and $N_{C/S} = K_C$. We claim that there is an isomorphism $H^1(C, N_{C/\mathbb{P}^3}) = H^1(C, \mathcal{O}_C(4))$. Suppose this is not the case. Then the injective map $H^1(C, K_C) \to H^1(C, N_{C/\mathbb{P}^3})$ provides a section $\sigma \in H^0(N^\vee_{C/\mathbb{P}^3} \otimes K_C)$ which yields a splitting of the dual of the exact sequence (2), hence (2) is split as well. Using a result from [GH, p.252] we obtain that C is a complete intersection with S. This is clearly a contradiction. Therefore one has $H^1(C, N_{C/\mathbb{P}^3}) = H^1(C, \mathcal{O}_C(4))$.

We have isomorphisms $H^1(C, 4H_{|C}) = H^2(S, 4H - C) = H^0(S, C - 4H)^\vee$. According to Prop.2.2 the divisor $C - 4H$ is effective if and only if $(C-4H)^2 \geq 0$ and $(C-4H) \cdot H > 2$, from which the conclusion follows. □

We need to determine the gonality of nodal curves not of compact type and which consist of two components meeting at a number of points. We have the following result:

Proposition 4.2 *Let $C = C_1 \cup_\Delta C_2$ be a quasi-transversal union of two smooth curves C_1 and C_2 meeting at a finite set Δ. Denote by $g_1 = g(C_1), g_2 = g(C_2), \delta = \mathrm{card}(\Delta)$. Let us assume that C_1 has only finitely many pencils $\mathfrak{g}^1_d$, where $\delta \le d$ and that the points of Δ do not occur in the same fibre of one of these pencils. Then $\mathrm{gon}(C) \ge d+1$. Moreover if $\mathrm{gon}(C) = d+1$ then either (1) C_2 is rational and there is a degree d map $f_1 : C_1 \to \mathbb{P}^1$ and a degree 1 map $f_2 : C_2 \to \mathbb{P}^1$ such that $f_{1\ |\Delta} = f_{2\ |\Delta}$, or (2) there is a $\mathfrak{g}^1_{d+1}$ on C_1 containing Δ in a fibre.*

Proof: Let us assume that C is k-gonal, that is, a limit of smooth k-gonal curves. If $g = g_1 + g_2 + \delta - 1$, we consider the space $\overline{\mathcal{H}}_{g,k}$ of Harris-Mumford admissible coverings of degree k and we denote by $\pi : \overline{\mathcal{H}}_{g,k} \to \overline{\mathcal{M}}_g$ the proper map sending a covering to the stable model of its domain (see [HM]). Since $[C] \in \overline{\mathcal{M}}^1_{g,k} = \mathrm{Im}(\pi)$, it follows that there exists a semistable curve C' whose stable model is C and a degree k admissible covering $f : C' \to Y$, where Y is a semistable curve of arithmetic genus 0. We thus have that $f^{-1}(Y_{sing}) = C'_{sing}$ and if $p \in C'_1 \cap C'_2$ with C'_1 and C'_2 components of C', then $f(C'_1)$ and $f(C'_2)$ are distinct components of Y and the ramification indices at the point p of the restrictions $f_{|C'_1}$ and $f_{|C'_2}$ are the same.

We have that $C' = C_1 \cup C_2 \cup R_1 \cup \ldots \cup R_\delta$, where for $1 \le i \le \delta$ the curve R_i is a (possibly empty) destabilizing chain of $\mathbb{P}^1$'s inserted at the nodes of C. Let us denote $\{p_i\} = C_1 \cap R_i$ and $\{q_i\} = C_2 \cap R_i$; if $R_i = \emptyset$ then we take $p_i = q_i \in \Delta \subseteq C$.

We first show that $k \ge d+1$. Suppose $k \le d$. Since C_1 has no $\mathfrak{g}^1_{d-1}$'s it follows that $k = d$ and that $f^{-1}f(C_1) = C_1$. If there were distinct points p_i and p_j such that $f(p_i) \ne f(p_j)$, then $f(R_i) \ne f(R_j)$ and the image curve Y would no longer have genus 0. Therefore $f(p_i) = f(p_j)$ for all $i, j \in \{1, \ldots, \delta\}$, that is Δ appears in the fibre of a $\mathfrak{g}^1_d$ on C_1, a contradiction.

Assume now that $k = d+1$. Then either $\deg(f_{|C_1}) = d$ or $\deg(f_{|C_1}) = d+1$. If $\deg(f_{|C_1}) = d+1$, then again $f^{-1}f(C_1) = C_1$ and by the same reasoning f maps all the p_i's to the same point and this yields case (2) from the statement of the Proposition. If $\deg(f_{|C_1}) = d$ then $f^{-1}f(C_1) = C_1 \cup D$, where D is a smooth rational curve mapped isomorphically to its image via f. If $D = C_2$ then the condition that the dual graph of Y is a tree implies that $f(p_i) = f(q_i)$ for all i and this yields case (1) from the statement. Finally, if $D \ne C_2$ then $f(C_1) \ne f(C_2)$. We know that there are $1 \le i < j \le \delta$ such that $f(p_i) \ne f(p_j)$. The image $f(C_2)$ belongs to a chain R of $\mathbb{P}^1$'s such that either $R \cap f(C_1) = \{f(p_i)\}$ or $R \cap f(C_1) = \{f(p_j)\}$. In the former case $f(p) = f(p_i)$ for all $p \in \Delta - \{p_j\}$ while in the latter case $f(p) = f(p_j)$ for all $p \in \Delta - \{p_i\}$. In each case by adding a base point we obtain a $\mathfrak{g}^1_{d+1}$ on C_1 containing Δ in a fibre. □

Theorem 3 provides curves $C \subseteq \mathbb{P}^3$ of expected gonality when d is even and g is odd (equation (1) has no solutions in this case). Naturally, we would like to have such curves when d and g have other parities as well. We will achieve this by attaching to a 'good' curve of expected gonality either a 2 or 3-secant line or a 4-secant conic.

Theorem 1 *Let $g \geq 15$ and $d \geq 14$ be integers with g odd and d even, such that $d^2 > 8g, 4d < 3g+12$, $d^2 - 8g + 8$ is not a square and either $d \leq 18$ or $g < 4d - 31$. If*

$$(d', g') \in \{(d, g), (d+1, g+1), (d+1, g+2), (d+2, g+3)\},$$

then there exists a regular component of $\mathrm{Hilb}_{\mathrm{d}',\mathrm{g}',3}$ *with general point* $[C']$ *a smooth curve such that* $\mathrm{gon}(C') = \min(\mathrm{d}'-4, [(\mathrm{g}'+3)/2])$.

Proof: For d and g as in the statement we know by Theorem 3 and by Prop.4.1 that there exists a smooth nondegenerate curve $C \subseteq \mathbb{P}^3$ of degree d and genus g, with $\mathrm{gon}(C) = \min(d-4, [(g+3)/2])$ and $H^1(C, N_{C/\mathbb{P}^3}) = 0$. We can also assume that C sits on a smooth quartic surface S and $\mathrm{Pic}(S) = \mathbb{Z}H \oplus \mathbb{Z}C$. Moreover, in the case $d - 4 < [(g+3)/2]$ the curve C has only finitely many $\mathfrak{g}^1_{d-4}$'s, all given by planes through a 4-secant line.

i) Let us settle first the case $(d', g') = (d+1, g+1)$. Take $p, q \in C$ general points, $L = \overline{pq} \subseteq \mathbb{P}^3$ and $X := C \cup L$. By applying Lemma 1.2 from [BE], we know that $H^1(X, N_X) = 0$ and the curve X is smoothable in $\mathbb{P}^3$, that is, there exists a flat family of curves $\{X_t\}$ in $\mathbb{P}^3$ over a smooth and irreducible base, with the general fibre X_t smooth while the special fibre X_0 is X. If $d-4 < [(g+3)/2]$, then since C has only finitely many $\mathfrak{g}^1_{d-4}$'s, by applying Prop.4.2 we get that $\mathrm{gon}(X) = d-3$. In the case $d - 4 \geq [(g+3)/2]$ we just notice that $\mathrm{gon}(X) \geq \mathrm{gon}(C) = [(g'+3)/2]$.

ii) Next we tackle the case $(d', g') = (d+1, g+2)$. Assume first that $d - 4 < [(g+3)/2] \Leftrightarrow d' - 4 < [(g'+3)/2]$. We apply Lemma 1.2 from [BE] to a curve $X := C \cup L$, where L is a suitable trisecant line to C. In order to conclude that X is smoothable in $\mathbb{P}^3$ and that $H^1(X, N_X) = 0$, we have to make sure that the trisecant line $L = \overline{pqq'}$ with $p, q, q' \in C$ can be chosen in such a way that

$$L, T_p(C), T_q(C) \text{ and } T_{q'}(C) \text{ do not all lie in the same plane.} \tag{3}$$

We claim that when $C \in |C|$ is general in its linear system, at least one of its trisecants satisfies (3). Suppose not. Then for *every* smooth curve $C \in |C|$ and for *every* trisecant line L to C condition (3) fails.

We consider a 0-dimensional subscheme $Z \subseteq S$ where $Z = p + q + q' + u + u'$, with $p, q, q' \in S$ being collinear points while u and u' are general infinitely near points to q and q' respectively. The linear system $|C|$ is at least 5-very ample (cf. Remark 1), hence a general curve $C \in |C - Z|$ is smooth and possesses a trisecant line for which (3) holds, a contradiction.

Since the scheme of trisecants to a space curve is of pure dimension 1, it follows that for a general curve $C \in |C|$, through a general point $p \in C$ there passes a trisecant line L for which (3) holds. We have that $X := C \cup L$ is smoothable in $\mathbb{P}^3$ and $H^1(X, N_X) = 0$. We conclude that $\mathrm{gon}(X) = d-3$ by proving that there is no $\mathfrak{g}^1_{d-4}$ on C containing $L \cap C$ in a fibre.

If $C \in |C|$ is general, any line in $\mathbb{P}^3$ (hence also a 4-secant line to C) can meet only finitely many trisecants. Indeed, assuming that $m \subseteq \mathbb{P}^3$ is a line meeting infinitely many trisecants, we consider the correspondence

$$T = \{(p, t) \in C \times m : \overline{pt} \text{ is a trisecant to } C\}$$

and the projections $\pi_1 : T \to C$ and $\pi_2 : T \to m$. If π_2 is surjective, then $\mathrm{Nm}_{\pi_1}(\pi_2)$ yields a $\mathfrak{g}^1_3$ on C, a contradiction. If π_2 is not surjective then there exists a point $t \in \mathbb{P}^3$ such that $\overline{pt}$ is a trisecant to C for each $p \in C$. This possibility cannot occur for a general $C \in |C|$: Otherwise we take general points $t \in \mathbb{P}^3$ and $p, p' \in S$ and if we denote

$$\mathcal{B} := \{C \in |C| : p, p' \in C \text{ and } \overline{tx} \text{ is a trisecant to } C \text{ for each } x \in C\},$$

we have that $\dim \mathcal{B} \geq g - 5$. On the other hand since $\overline{tp}$ and $\overline{tp'}$ are trisecants for all curves $C \in \mathcal{B}$, there must be a 0-dimensional subscheme $Z \subseteq (\overline{tp} \cup \overline{tp'}) \cap S$ of length 6 such that $\mathcal{B} \subseteq |C - Z|$, hence $\dim \mathcal{B} \leq \dim|C - Z| = g - 6$ (use again that $|C|$ is 5-very ample), a contradiction. In this way the case $d - 4 < [(g + 3)/2]$ is settled.

When $d - 4 \geq [(g + 3)/2]$ we apply Theorem 3 to obtain a smooth curve $C_1 \subseteq \mathbb{P}^3$ of degree d and genus $g + 2$ such that $\mathrm{gon}(C_1) = (g + 5)/2$ and $H^1(C_1, N_{C_1}) = 0$. We take $X_1 := C_1 \cup L_1$ with L_1 being a general 1-secant line to C_1. Then X_1 is smoothable and $\mathrm{gon}(X_1) = \mathrm{gon}(C_1) = (g + 5)/2$.

iii) Finally, we turn to the case $(d', g') = (d + 2, g + 3)$. Take $H \subseteq \mathbb{P}^3$ a general plane meeting C in d distinct points in general linear position and pick 4 of them: $p_1, p_2, p_3, p_4 \in C \cap H$. Choose $Q \subseteq H$ a general conic such that $Q \cap C = \{p_1, p_2, p_3, p_4\}$. Theorem 5.2 from [Se] ensures that $X := C \cup Q$ is smoothable in $\mathbb{P}^3$ and $H^1(X, N_X) = 0$.

Assume first that $d' - 4 \leq [(g' + 3)/2]$. We claim that $\mathrm{gon}(X) \geq \mathrm{gon}(C) + 2$. According to Prop.4.2 the opposite could happen only in 2 cases: a) There exists a $\mathfrak{g}^1_{d-3}$ on C, say $|Z|$, such that $|Z|(-p_1 - p_2 - p_3 - p_4) \neq \emptyset$. b) There exists a degree $d - 4$ map $f : C \to \mathbb{P}^1$ and a degree 1 map $f' : Q \to \mathbb{P}^1$ such that $f(p_i) = f'(p_i)$, for $i = 1, \ldots, 4$.

Assume that a) does happen. We denote by $U = \{D \in C_4 : |\mathcal{O}_C(1)|(-D) \neq \emptyset\}$ the irreducible 3-fold of divisors of degree 4 spanning a plane and also consider the correspondence

$$\Sigma = \{(L, D) \in W^1_{d-3}(C) \times U : |L|(-D) \neq \emptyset\},$$

with the projections $\pi_1 : \Sigma \to W^1_{d-3}(C)$ and $\pi_2 : \Sigma \to U$. We know that π_2 is dominant, hence $\dim \Sigma \geq 3$ and therefore $\dim W^1_{d-3}(C) \geq 2$.

If $\rho(g, 1, d - 3) < 0$ by Prop.3.1 we get that every base-point-free $\mathfrak{g}^1_{d-3}$ on C is cut out by a divisor D on S such that $D \in \mathcal{A}$ (see the proof of Theorem 3 for this notation) and $C \cdot D - D^2 = \mathrm{Cliff}(C, D_{|C}) + 2 \leq d - 3$, hence $C \cdot D - D^2 \leq d - 4$ for parity reasons. As pointed out at the end of the proof of Theorem 3 this forces $D \sim H$, that is, all base-point-free $\mathfrak{g}^1_{d-3}$'s on C are given by planes through a trisecant line. Thus C has ∞^2 trisecants, a contradiction.

If $\rho(g, 1, d - 3) \geq 0$, then $g = 2d - 9$ and we can assume that there is $L \in \pi_1(\Sigma)$ such that $|\mathcal{O}_C(1) - L| \neq \emptyset$. The map π_1 is either generically finite hence $\dim W^1_{d-4}(C) \geq \dim W^1_{d-3}(C) - 2 \geq 1$ (cf. [FHL]), a contradiction, or otherwise π_1 has fibre dimension 1. This is possible only when there is a component A of $W^1_{d-3}(C)$ with $\dim(A) \geq 2$ and such that the general $L \in A$ satisfies $|\mathcal{O}_C(1) - L| \neq \emptyset$ and *every* $L \in A$ has non-ordinary ramification so that the monodromy of each $\mathfrak{g}^1_{d-3}$ is not the full symmetric group. Applying again [FHL] there is $L \in W^1_{d-4}(C)$ such that $\{L\} + W^0_1(C) \subseteq A$, in particular L has non-ordinary ramification too. It is easy to see that this contradicts the $(d - 7)$-very ampleness of $|C|$ asserted by Remark 1.

We now rule out case b). Suppose that b) does happen and denote by $L \subseteq \mathbb{P}^3$ the 4-secant line corresponding to f. Let $\{p\} = L \cap H$, and pick $l \subseteq H$ a general line. As Q was a general conic through $p_1, \dots, p_4$ we may assume that $p \notin Q$. The map $f' : Q \to l$ is (up to a projective isomorphism of l) the projection from a point $q \in Q$, while $f(p_i) = \overline{p_i p} \cap l$, for $i = 1, \dots, 4$. By Steiner's Theorem from classical projective geometry, the condition $(f(p_1)f(p_2)f(p_3)f(p_4)) = (f'(p_1)f'(p_2)f'(p_3)f'(p_4))$ is equivalent with p_1, p_2, p_3, p_4, p and q being on a conic, a contradiction since $p \notin Q$.

Finally, when $d' - 4 > [(g'+3)/2]$, we have to show that $\mathrm{gon}(X) \geq \mathrm{gon}(C) + 1$. We note that $\dim G^1_{(g+3)/2}(C) = 1$ (for any curve one has the inequality $\dim G^1_{\mathrm{gon}} \leq 1$). By taking $H \in (\mathbb{P}^3)^\vee$ general enough, we obtain that $p_1, \dots, p_4$ do not occur in the same fibre of a $\mathfrak{g}^1_{(g+3)/2}$. □

Remark: Theorem 1 can be viewed as a non-containment relation $\mathcal{M}^3_{g',d'} \nsubseteq \mathcal{M}^1_{g',d'-5}$ between different Brill-Noether loci when d' and g' are as in Theorem 1 and moreover $d' - 4 \leq [(g'+3)/2]$. We can turn this problem on its head and ask the following question: given g and k such that $k < (g+2)/2$, when is it true that the general k-gonal curve of genus g has no other linear series $\mathfrak{g}^r_d$ with $\rho(g,r,d) < 0$, that is, the pencil computing the gonality is the only Brill-Noether exceptional linear series?

In [Fa2] we prove using limit linear series the following result: fix g and k positive integers such that $-3 \leq \rho(g,1,k) < 0$. If $\rho(g,1,k) = -3$ assume furthermore that $k \geq 6$. Then the general k-gonal curve C of genus g has no $\mathfrak{g}^r_d$'s with $\rho(g,r,d) < 0$ except $\mathfrak{g}^1_k$ and $|K_C - \mathfrak{g}^1_k|$. In other words the k-gonal locus $\mathcal{M}^1_{g,k}$ is not contained in any other proper Brill-Noether locus $\mathcal{M}^r_{g,d}$ with $r \geq 2, d \leq g-1$ and $\rho(g,r,d) < 0$.

In seems that other methods are needed to extend this result for more negative values of $\rho(g,1,k)$.

5 The Kodaira dimension of $\mathcal{M}_{23}$

In this section we explain how Theorem 1 gives a new proof of our result $\kappa(\mathcal{M}_{23}) \geq 2$ (cf. [Fa]). We refer to [Fa] for a detailed analysis of the geometry of $\mathcal{M}_{23}$; in that paper we also conjecture that $\kappa(\mathcal{M}_{23}) = 2$ and we present evidence for such a possibility.

Let us denote by $\overline{\mathcal{M}}_g$ the moduli space of Deligne-Mumford stable curves of genus g. We study the multicanonical linear systems on $\overline{\mathcal{M}}_{23}$ by exhibiting three explicit multicanonical divisors on $\overline{\mathcal{M}}_{23}$ which are (modulo a positive combination of boundary classes coming from $\overline{\mathcal{M}}_{23} - \mathcal{M}_{23}$) of Brill-Noether type, that is, loci of curves having a $\mathfrak{g}^r_d$ when $\rho(23,r,d) = -1$.

On $\mathcal{M}_{23}$ there are three Brill-Noether divisors corresponding to the solutions of the equation $\rho(23,r,d) = -1$: the 12-gonal divisor $\mathcal{M}^1_{23,12}$, the divisor $\mathcal{M}^2_{23,17}$ of curves having a $\mathfrak{g}^2_{17}$ and finally the divisor $\mathcal{M}^3_{23,20}$ of curves possessing a $\mathfrak{g}^3_{20}$. If we denote by $\overline{\mathcal{M}}^r_{g,d}$ the closure of $\mathcal{M}^r_{g,d}$ inside $\overline{\mathcal{M}}_g$, the classes $[\overline{\mathcal{M}}^r_{g,d}] \in \mathrm{Pic}_{\mathbb{Q}}(\overline{\mathcal{M}}_g)$ when $\rho(g,r,d) = -1$ have been computed (see [EH],[Fa]). It is quite remarkable that for fixed g all classes $[\overline{\mathcal{M}}^r_{g,d}]$ are proportional. One also knows the canonical divisor class (cf. [HM]):

$$K_{\overline{\mathcal{M}}_g} = 13\lambda - 2\delta_0 - 3\delta_1 - 2\delta_2 - \cdots - 2\delta_{[g/2]},$$

and by comparing for $g = 23$ this formula with the expression of the classes $[\overline{\mathcal{M}}^r_{23,d}]$, we find that there are constants $m, m_1, m_2, m_3 \in \mathbb{Z}_{>0}$ such that

$$mK_{\overline{\mathcal{M}}_{23}} = m_1[\overline{\mathcal{M}}^1_{23,12}] + E = m_2[\overline{\mathcal{M}}^2_{23,17}] + E = m_3[\overline{\mathcal{M}}^3_{23,20}] + E,$$

where E is the same positive combination of the boundary classes $\delta_1, \ldots, \delta_{11}$.

As explained in [Fa], since $\overline{\mathcal{M}}^1_{23,12}$, $\overline{\mathcal{M}}^2_{23,17}$ and $\overline{\mathcal{M}}^3_{23,20}$ are mutually distinct irreducible divisors, we can show that the multicanonical image of $\overline{\mathcal{M}}_{23}$ cannot be a curve once we construct a smooth curve of genus 23 lying in the support of exactly two of the divisors $\mathcal{M}^1_{23,12}$,$\mathcal{M}^2_{23,17}$ and $\mathcal{M}^3_{23,20}$. In this way we rule out the possibility of all three intersections of two Brill-Noether divisors being equal to base-locus$(|mK_{\overline{\mathcal{M}}_{23}}|) \cap \mathcal{M}_{23}$.

In [Fa] we found such genus 23 curves using an intricate construction involving limit linear series (cf. Proposition 5.4 in [Fa]). Here we can construct such curves in a much simpler way by applying Theorem 1 when $(d, g) = (18, 23)$: there exists a smooth curve $C \subseteq \mathbb{P}^3$ of genus 23 and degree 18 such that $\mathrm{gon}(C) = 13$; moreover C sits on a smooth quartic surface $S \subseteq \mathbb{P}^3$ such that $\mathrm{Pic}(S) = \mathbb{Z}C \oplus \mathbb{Z}H$.

Since C has a very ample $\mathfrak{g}^3_{18}$, by adding 2 base points it will also have plenty of $\mathfrak{g}^3_{20}$'s and also $\mathfrak{g}^2_{17}$'s of the form $\mathfrak{g}^3_{18}(-p) = \{D \in \mathfrak{g}^3_{18} : D \geq p\}$, for any $p \in C$. Therefore $[C] \in (\mathcal{M}^3_{23,20} \cap \mathcal{M}^2_{23,17}) - \mathcal{M}^1_{23,12}$, and Theorem 2 now follows.

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