

TOPOLOGICAL INVARIANTS OF GROUPS AND KOSZUL MODULES

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ABSTRACT. We provide a uniform vanishing result for the graded components of the finite length Koszul module associated to a subspace $K \subseteq \wedge^2 V$, as well as a sharp upper bound for its Hilbert function. This purely algebraic statement has interesting applications to the study of a number of invariants associated to finitely generated groups, such as the Alexander invariants, the Chen ranks, or the degree of growth and nilpotency class. For instance, we explicitly bound the aforementioned invariants in terms of the first Betti number for the maximal metabelian quotients of (1) the Torelli group associated to the moduli space of curves; (2) nilpotent fundamental groups of compact Kähler manifolds; (3) the Torelli group of a free group.

1. INTRODUCTION

A well-established way of studying the properties of a finitely generated group G is by considering its associated graded Lie algebra

$$\mathrm{gr}(G) := \bigoplus_{q \geq 1} \Gamma_q(G)/\Gamma_{q+1}(G),$$

defined in terms of its *lower central series*

$$G = \Gamma_1(G) \geq \dots \geq \Gamma_q(G) \geq \Gamma_{q+1}(G) \geq \dots \text{ where } \Gamma_{q+1}(G) = [\Gamma_q(G), G].$$

The quotients $\mathrm{gr}_q(G) = \Gamma_q(G)/\Gamma_{q+1}(G)$ are finitely generated abelian groups and their ranks $\phi_q(G) := \mathrm{rk} \mathrm{gr}_q(G)$ encode fundamental numerical information on G and are in general extremely difficult to compute. For example, $\Gamma_2(G) = G'$ is the commutator subgroup of G and $\mathrm{gr}_1(G)$ is the abelianization of G , whose rank $b_1(G)$ is the first Betti number of G .

Following Chen [8], Massey [33], Papadima-Suciu [37] and many others, a more manageable task is to pass instead to the *metabelian quotient* G/G'' , where $G'' = [G', G']$ is the second commutator subgroup of G , and to consider instead its graded Lie algebra $\mathrm{gr}(G/G'')$ and the corresponding *Chen ranks*

$$\theta_q(G) := \phi_q(G/G'') = \mathrm{rk} \mathrm{gr}_q(G/G'').$$

One sample application of the results of our work is the following (see Theorem 4.10).

Theorem 1.1. *Let G be the fundamental group of a compact Kähler manifold, and suppose its first Betti number is $n = b_1(G) \geq 3$. If G/G'' is nilpotent, then $\theta_q(G) = 0$ for $q \geq n - 1$ and*

$$\theta_q(G) \leq \binom{n+q-3}{n-1} \cdot \frac{(n-2)(n-1-q)}{q}, \quad \text{for } q = 2, \dots, n-2.$$

[†]Stefan Papadima passed away on January 10, 2018.

The first Chen rank always satisfies $\theta_1(G) = \phi_1(G) = b_1(G)$, and in general one has the inequality $\phi_q(G) \geq \theta_q(G)$. For basic properties of these invariants initially considered in knot theory we refer to [37], whereas for applications of Chen ranks to hyperplane arrangements one can consult [11], [12] and the references therein.

It turns out that in many cases $\text{gr}(G/G'')$ is an object belonging essentially to commutative algebra, which can be studied to great effect with homological and algebro-geometric methods. The precise context where these methods are most fruitful is when G is a 1-formal group in the sense of Sullivan [44]. We recall that for instance fundamental groups of compact Kähler manifolds, hyperplane arrangement groups, or the Torelli group T_g of the mapping class group are all known to be 1-formal. For such groups we prove the following in Section 4.3.

Theorem 1.2. *Let G be a finitely generated 1-formal group, and let $M = G/G''$ denote its maximal metabelian quotient. If $\Gamma_q(M)$ is finite for $q \gg 0$ (in particular if M is nilpotent) and if $n = b_1(G) \geq 3$, then*

- (1) $\Gamma_{n-1}(M)$ is finite.
- (2) M contains a nilpotent finite index subgroup with nilpotency class at most $n - 2$.
- (3) The group M has polynomial growth of degree $d(M)$ bounded above by

$$d(M) \leq n + (n - 2) \cdot \binom{2n - 3}{n - 4}.$$

We recall that the *nilpotency class* of a nilpotent group H is the largest value of c for which $\Gamma_c(H) \neq \{1\}$, and that a group is *virtually nilpotent* if it contains a nilpotent finite index subgroup. We can then define the *virtual nilpotency class* (*vnc*) of a group to be the minimal nilpotency class of a finite index subgroup, so that the second conclusion of Theorem 1.2 states that the metabelian quotient G/G'' has $\text{vnc}(G/G'') \leq n - 2$. A celebrated theorem of Gromov [22] asserts that virtually nilpotent groups are precisely the ones that exhibit polynomial growth, and our theorem gives a precise bound for the degree of growth of $M = G/G''$ solely in terms of the first Betti number of G . As we show in Section 4.3, this bound is a consequence of the Bass–Guivarc’h formula and the estimate on $\theta_q(G)$ as in Theorem 1.1. As explained in Example 4.9 and Remark 4.13, in the absence of 1-formality (or some appropriate replacement) there can be no such bound for either $\text{vnc}(M)$ or for $d(M)$.

An important special case of Theorem 1.2 concerns the Torelli group T_g which measures from a homotopical point of view the difference between the moduli space of curves of genus g and that of principally polarized abelian varieties of dimension g . If Mod_g denotes the mapping class group, recall that the Torelli group is defined via the exact sequence

$$1 \longrightarrow T_g \longrightarrow \text{Mod}_g \longrightarrow \text{Sp}_{2g}(\mathbb{Z}) \longrightarrow 1.$$

Using Johnson’s fundamental calculation $H^1(T_g, \mathbb{Q}) \cong \bigwedge^3 H_{\mathbb{Q}}/H_{\mathbb{Q}}$, where H is the first integral homology of a genus g Riemann surface (and hence $\text{rk}(H) = 2g$), together with the recent result by Ershov and He [19, Corollary 1.5] asserting that T_g/T_g'' is nilpotent for $g \geq 12$, and the 1-formality of T_g proved in [24], our Theorem 1.2 yields the following consequence (see Section 4.5).

Theorem 1.3. *If $g \geq 12$, the metabelian quotient T_g/T_g'' has virtual nilpotency class at most*

$$\binom{2g}{3} - 2g - 2.$$

We note that some of the results of [19] have been extended to $g \geq 4$ in [9], but the nilpotence of T_g/T_g'' is still open, preventing us from extending the range for g in Theorem 1.3.

We have similar applications to the Torelli group OA_g of the free group F_g on g generators, discussed in Section 4.5. We show in Theorem 4.17 that

$$\text{vnc}(\text{OA}_g / \text{OA}_g'') \leq \frac{g(g+1)(g-2)}{2} - 2 \quad \text{for all } g \geq 4,$$

which is an analogue of Theorem 1.3. We note that in contrast with the Torelli groups T_g , the groups OA_g are not known to be 1-formal [38, Question 10.6].

The proofs of Theorems 1.1 and 1.2 are fundamentally algebraic. In order to make the transition to commutative algebra and algebraic geometry, for every finitely generated group G we consider its *Alexander invariant*

$$B(G) := H_1(G', \mathbb{Z}) = G'/G'',$$

viewed as a module over the group ring $\mathbb{Z}[G/G']$, with the action being given by conjugation. Using the *augmentation ideal* $I \subseteq \mathbb{Z}[G/G']$, one then constructs the graded module $\text{gr} B(G)$ over the ring $\text{gr} \mathbb{Z}[G/G'] = \bigoplus_{q \geq 0} I^q/I^{q+1}$, by setting

$$\text{gr}_q B(G) := I^q \cdot B(G)/I^{q+1} \cdot B(G).$$

Massey proves in [33] the existence of canonical isomorphisms $\Gamma_{q+2}(G/G'') \cong I^q \cdot B(G)$, which in particular yield

$$\theta_{q+2}(G) = \text{rk} \text{gr}_q B(G).$$

Following [37] and especially [39], it turns out that the graded ring $\text{gr} B(G)$ can be studied via Koszul cohomology using a purely algebraic construction which we describe next.

Koszul modules. Suppose that V is an n -dimensional complex vector space and fix an m -dimensional subspace $K \subseteq \Lambda^2 V$. We denote by $K^\perp = (\Lambda^2 V/K)^\vee \subseteq \Lambda^2 V^\vee$ the orthogonal of K . We also denote by $S := \text{Sym}(V)$ the polynomial algebra over V and consider the Koszul complex

$$\cdots \longrightarrow \bigwedge^3 V \otimes S \xrightarrow{\delta_3} \bigwedge^2 V \otimes S \xrightarrow{\delta_2} V \otimes S \xrightarrow{\delta_1} S \longrightarrow \mathbb{C} \longrightarrow 0.$$

According to [39], the *Koszul module* associated to (V, K) is the graded S -module

$$W(V, K) := \text{Coker} \left\{ \bigwedge^3 V \otimes S \longrightarrow \left(\bigwedge^2 V/K \right) \otimes S \right\},$$

where the map in question is the projection $\bigwedge^2 V \otimes S \rightarrow (\bigwedge^2 V/K) \otimes S$ composed with the Koszul differential δ_3 . Our grading conventions are so that $W(V, K)$ is generated in

degree 0. Hence, passing to individual graded pieces one has the identification

$$W_q(V, K) \cong \bigwedge^2 V \otimes \text{Sym}^q(V) / \left(\text{Ker}(\delta_{2,q}) + K \otimes \text{Sym}^q(V) \right).$$

If G is a finitely generated group, we define its *Koszul module*

$$W(G) := W(H_1(G, \mathbb{C}), K),$$

by setting $K^\perp := \text{Ker}\left\{ \cup_G : \bigwedge^2 H^1(G, \mathbb{C}) \rightarrow H^2(G, \mathbb{C}) \right\}$, or equivalently by letting K be the dual of $\text{Im}(\cup_G)$, which embeds naturally into $\bigwedge^2 H_1(G, \mathbb{C})$. It is shown in [16] and [37] that the following inequality holds

$$(1) \quad \theta_{q+2}(G) = \text{rk gr}_q B(G) \leq \dim W_q(G),$$

with equality if the group G is *1-formal*. We prove the following result about $W(G)$.

Theorem 1.4. *Let G be a finitely generated group and assume that the cup product map*

$$\cup_G : \bigwedge^2 H^1(G, \mathbb{C}) \rightarrow H^2(G, \mathbb{C})$$

does not vanish on any non-zero decomposable elements. If $n = b_1(G) \geq 3$, then

$$(2) \quad W_q(G) = 0 \quad \text{for all } q \geq n - 3.$$

If moreover $M = G/G''$ satisfies the condition that $\Gamma_q(M)$ be finite for $q \gg 0$, then its virtual nilpotency class and degree of polynomial growth satisfy the inequalities in Theorem 1.2.

The vanishing (2) follows from Theorem 4.2, while the inequalities for $d(M)$ and $\text{vnc}(M)$ are explained in the proof of Theorem 1.2 (see Remark 4.14). An important application of Theorem 1.4 is when X is a compact Kähler manifold and $G = \pi_1(X)$, in which case $H^1(G, \mathbb{C}) = H^1(X, \mathbb{C})$. In this case, the kernel of the cup-product map \cup_G equals the kernel of the map

$$\cup_X : \bigwedge^2 H^1(X, \mathbb{C}) \rightarrow H^2(X, \mathbb{C}).$$

By a well-known generalization of the Castelnuovo–de Franchis Theorem [6], the condition that \cup_X does vanish on decomposable elements amounts to the existence of a fibration $X \rightarrow C$ over a smooth curve of genus $g \geq 2$. Thus, the first part of Theorem 1.4 applies to all compact Kähler manifolds X with irregularity $q(X) \geq 3$ which are not fibred over curves.

We obtain the vanishing in Theorem 1.4 as a special instance of a general vanishing result for Koszul modules, as explained next. We return to the general situation when V is a vector space, $\iota : K \hookrightarrow \bigwedge^2 V$ is a fixed subspace, and $K^\perp := \text{Ker}(\iota^\vee) \subseteq \bigwedge^2 V^\vee$ is the space of skew-symmetric bilinear forms on V which vanish identically on K . It is shown in [39, Lemma 2.4] that the support of the Koszul module $W(V, K)$ in the affine space V^\vee coincides (if non-empty) with the *resonance variety* $\mathcal{R}(V, K)$ defined as

$$\mathcal{R}(V, K) := \left\{ a \in V^\vee \mid \text{there exists } b \in V^\vee \text{ such that } a \wedge b \in K^\perp \setminus \{0\} \right\} \cup \{0\}.$$

In particular, $W(V, K)$ has finite length, that is, $W_q(V, K) = 0$ for $q \gg 0$ if and only if $\mathcal{R}(V, K) = \{0\}$. This last condition is equivalent to the fact that the projective subspace $\mathbb{P}(K^\perp) \subseteq \mathbb{P}(\wedge^2 V^\vee)$ is disjoint from the Grassmann variety $\text{Gr}_2(V^\vee)$ in its Plücker embedding. Our vanishing result applies uniformly to all subspaces K satisfying this condition, as follows (see Theorem 3.1).

Main Theorem. *Let V be a complex n -dimensional vector space and let $K \subseteq \wedge^2 V$ be a subspace such that $\mathcal{R}(V, K) = \{0\}$. We have that $W_q(V, K) = 0$ for all $q \geq n - 3$.*

In a forthcoming paper, we prove the equivalence between the Main Theorem and Green's Vanishing Conjecture for syzygies of generic canonical curves. This leads to an alternate approach to Green's conjecture from the one of Voisin's [45], [46].

The vanishing result in the Main Theorem is responsible for the estimates of the virtual nilpotency classes that appear in Theorems 1.2, 1.3 and 1.4. In addition to the vanishing result, we also provide a sharp upper bound for the Hilbert function of a finite length Koszul module in Theorem 3.2. In combination with (1) this gives the bound on the Chen ranks in Theorem 1.1, and in conjunction with the Bass–Guivarc'h formula (recalled as formula (29)) it provides the estimate for the degree of polynomial growth in Theorem 1.2.

To emphasize the robust relationship between algebra and topology in the 1-formal setting, we end our Introduction summarizing the connection between some of the invariants discussed in this setting.

Theorem 1.5. *Let G be a finitely generated 1-formal group, and assume that $n = b_1(G) \geq 3$. The following statements are equivalent:*

- (a) $\theta_q(G) = 0$ for some (any) $q \gg 0$.
- (b) $\theta_{n-1}(G) = 0$.
- (c) \cup_G does not vanish on any non-zero decomposable elements.
- (d) $W_q(G) = 0$ for some (any) $q \gg 0$.
- (e) $W_{n-3}(G) = 0$.

The equivalence of (c), (d), (e) comes from the Main Theorem, since all three conditions characterize finite dimensional Koszul modules. The equivalence of these statements with (a) and (b) follows from the fact that (1) is an equality for 1-formal groups.

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2. KOSZUL MODULES

Throughout this article, we denote by V a complex vector space of dimension $n \geq 2$, we write $V^\vee = \text{Hom}_{\mathbb{C}}(V, \mathbb{C})$ for its dual, and $S = \text{Sym } V$ for the symmetric algebra of V . We consider the standard grading on S where the elements in V are of degree 1. We fix a subspace $K \subseteq \bigwedge^2 V$ of dimension m , we denote by $\iota : K \rightarrow \bigwedge^2 V$ the inclusion and let $K^\perp := \text{Ker}(\iota^\vee) = \text{Coker}(\iota)^\vee$, where $\iota^\vee : \bigwedge^2 V^\vee \rightarrow K^\vee$ is the dual of ι .

2.1. Definitions and basic properties. We recall from [39] the definition of Koszul modules. They are defined starting from the classical Koszul differentials

$$\delta_p : \bigwedge^p V \otimes S \rightarrow \bigwedge^{p-1} V \otimes S,$$

$$\delta_p(v_1 \wedge \cdots \wedge v_p \otimes f) = \sum_{j=1}^p (-1)^{j-1} v_1 \wedge \cdots \wedge \widehat{v}_j \wedge \cdots \wedge v_p \otimes v_j f.$$

Note that we have a decomposition $\delta_p = \bigoplus_q \delta_{p,q}$, where

$$(3) \quad \delta_{p,q} : \bigwedge^p V \otimes \text{Sym}^q V \rightarrow \bigwedge^{p-1} V \otimes \text{Sym}^{q+1} V,$$

and that the differentials δ_p fit together into the Koszul complex which gives a minimal resolution by free graded S -modules of the residue field \mathbb{C} .

Definition 2.1 ([39, §2]). The *Koszul module* associated to the pair (V, K) is a graded S -module denoted by $W(V, K)$, and defined as

$$(4) \quad W(V, K) := \text{Coker} \left\{ \bigwedge^3 V \otimes S \rightarrow \left(\bigwedge^2 V/K \right) \otimes S \right\}$$

where the map $\bigwedge^3 V \otimes S \rightarrow \left(\bigwedge^2 V/K \right) \otimes S$ is the composition of the quotient map $\bigwedge^2 V \otimes S \rightarrow \left(\bigwedge^2 V/K \right) \otimes S$ with the Koszul differential δ_3 . The grading is inherited from the symmetric algebra if we make the convention that $\bigwedge^2 V/K$ is placed in degree 0 and $\bigwedge^3 V$ is in degree 1. The degree q component of $W(V, K)$ is then given by

$$(5) \quad W_q(V, K) = \text{Coker} \left\{ \bigwedge^3 V \otimes \text{Sym}^{q-1} V \rightarrow \left(\bigwedge^2 V/K \right) \otimes \text{Sym}^q V \right\}.$$

Note that this grading convention guarantees that $W(V, K)$ is generated in degree 0. In particular, if $W_q(V, K) = 0$ for some $q \geq 0$, then $W_p(V, K) = 0$ for all $p \geq q$.

Since the Koszul complex is exact in homological degree one, we can realize $W(V, K)$ as the middle cohomology of the complex

$$(6) \quad K \otimes S \xrightarrow{\delta_2|_{K \otimes S}} V \otimes S \xrightarrow{\delta_1} S$$

Using the fact that $\text{Ker}(\delta_{1,q+1}) = \text{Im}(\delta_{2,q})$, we obtain as in [39, (2.1)]

$$(7) \quad W_q(V, K) = \text{Im}(\delta_{2,q}) / \text{Im}(\delta_{2,q}|_{K \otimes S}).$$

This construction is natural in the following sense: for $K \subseteq K'$, the identity map of $\bigwedge^2 V \otimes S$ induces a surjective morphism of graded S -modules

$$(8) \quad W(V, K) \twoheadrightarrow W(V, K').$$

Example 2.2. In the extremal cases the Koszul modules are easy to compute:

- If $K = \bigwedge^2 V$ then $W(V, K) = 0$;
- If $K = 0$ then $W_q(V, K) = H^0(\mathbb{P}(V^\vee), \Omega_{\mathbb{P}(V^\vee)}^1(q+2)) \neq 0$ for all $q \geq 0$.

We will therefore assume when necessary that $1 \leq m < \binom{n}{2}$.

Example 2.3 ([39, §6.2]). Any finitely generated group G gives rise to a pair (V, K) by letting $V := H_1(G, \mathbb{C}) \cong H^1(G, \mathbb{C})^\vee$ and K be the dual of the image of the cup-product map $\cup_G : \bigwedge^2 H^1(G, \mathbb{C}) \rightarrow H^2(G, \mathbb{C})$. The corresponding Koszul modules are denoted $W(G)$. Conversely, it was noted in [39, Proposition 6.2] that every Koszul module $W(V, K)$ corresponding to a rational subspace $K \subseteq \bigwedge^2 V$ has the form $W(G)$ for some finitely presented group G . A more thorough analysis of the Koszul modules associated to groups will be discussed in §4.

2.2. Resonance. Inspired by work of Green and Lazarsfeld in [21], Papadima and Suciu defined in [39] resonance varieties in a purely algebraic context as follows.

Definition 2.4. The *resonance variety* associated to the pair (V, K) is the locus

$$(9) \quad \mathcal{R}(V, K) := \left\{ a \in V^\vee \mid \text{there exists } b \in V^\vee \text{ such that } a \wedge b \in K^\perp \setminus \{0\} \right\} \cup \{0\}$$

The resonance variety $\mathcal{R}(V, K)$ is the union of 2-dimensional subspaces of V^\vee parametrized by the intersection $\mathbb{P}(K^\perp) \cap \text{Gr}_2(V^\vee)$. More precisely, via the canonical diagram

$$\begin{array}{ccc} (\mathbb{P}V^\vee \times \mathbb{P}V^\vee) \setminus \text{Diag} & \xrightarrow{\pi} & \text{Gr}_2(V^\vee) \\ \downarrow p_1 & & \\ \mathbb{P}V^\vee & & \end{array}$$

$\mathcal{R}(V, K)$ is the affine cone over $p_1(\pi^{-1}(\mathbb{P}(K^\perp) \cap \text{Gr}_2(V^\vee)))$. It was showed in [39, Lemma 2.4] that resonance and Koszul modules are strongly related: the support of the Koszul module in the affine space V^\vee coincides with the resonance variety away from 0, and hence $\mathcal{R}(V, K) = \{0\}$ if and only if $W(V, K)$ is of finite length. In particular

$$(10) \quad \mathbb{P}(K^\perp) \cap \text{Gr}_2(V^\vee) = \emptyset \iff \mathcal{R}(V, K) = \{0\} \iff \dim_{\mathbb{C}} W(V, K) < \infty.$$

It was also proved in [39, Proposition 2.10] that there exists a uniform bound $q(n, m)$ such that $W_q(V, K) = 0$ for all $q \geq q(n, m)$ and for all (V, K) satisfying $\mathbb{P}(K^\perp) \cap \text{Gr}_2(V^\vee) = \emptyset$. A central goal of our paper is to determine this bound, which we do in Theorem 3.1. For dimension reasons, if $\mathbb{P}(K^\perp) \cap \text{Gr}_2(V^\vee) = \emptyset$ then $m \geq 2n - 3$. The *borderline case* $m = 2n - 3$ is therefore of maximal interest, as illustrated for instance in §3.3 below.

3. FINITE LENGTH KOSZUL MODULES

The goal of this section is to prove the main theorem of our paper, characterizing finite dimensional Koszul modules (those for which the equivalent conditions in (10) hold) in terms of the vanishing of a fixed graded component. More precisely, we prove:

Theorem 3.1. *Let V be a vector space of dimension $n \geq 3$ and let $K \subseteq \bigwedge^2 V$ be an arbitrary subspace. We have that*

$$(11) \quad \mathcal{R}(V, K) = \{0\} \iff W_{n-3}(V, K) = 0.$$

The assumption that $n \geq 3$ in the above theorem is made in order to avoid trivialities: when $n = 1$ we have $\bigwedge^2 V = 0$, while for $n = 2$ we have that either $K = \bigwedge^2 V$ or $K = 0$, both of which have been discussed in Example 2.2. In addition to the vanishing result above, in the borderline case $m = 2n - 3$ we give an exact description of the Hilbert series of the Koszul modules with vanishing resonance, which in turn provides an upper bound for such Hilbert series in the general case $m \geq 2n - 3$, as follows.

Theorem 3.2. *If $\mathcal{R}(V, K) = \{0\}$, then for $q = 0, \dots, n - 4$, we have that*

$$\dim W_q(V, K) \leq \binom{n+q-1}{q} \frac{(n-2)(n-q-3)}{q+2},$$

and equality holds for all $0 \leq q \leq n - 4$ when $\dim(K) = 2n - 3$.

We prove Theorem 3.1 using Bott's theorem and a hypercohomology spectral sequence that was featured in the work of Voisin on the Green conjecture [45]. We recall the special case of Bott's theorem needed for our argument in Section 3.1, and prove the vanishing theorem in Section 3.2. The proof of Theorem 3.2 is presented in Section 3.3, and is summarized as follows. We reinterpret the characterization (11) in terms of a transversality condition on K with respect to the kernel of the Koszul differential δ_2 , which allows us to prove the conclusion of Theorem 3.2 in the borderline case. The general case of Theorem 3.2 reduces to the borderline case via a generic projection argument.

3.1. Bott's Theorem for Grassmannians. Our main reference for this topic is [47, Ch. 4]. Let $\mathbf{G} = \text{Gr}_2(V^\vee)$ denote the Grassmannian of 2-dimensional subspaces of V^\vee , which we think of by duality as parametrizing 2-dimensional quotients of V . On \mathbf{G} we then have the tautological exact sequence

$$(12) \quad 0 \longrightarrow \mathcal{U} \longrightarrow V \otimes \mathcal{O}_{\mathbf{G}} \longrightarrow \mathcal{Q} \longrightarrow 0,$$

where \mathcal{U} (respectively \mathcal{Q}) denotes the universal rank $(n - 2)$ sub-bundle (respectively rank 2 quotient bundle) of the trivial bundle V . We write $\mathcal{O}_{\mathbf{G}}(1)$ for $\bigwedge^2 \mathcal{Q}$, which is the Plücker line bundle giving an embedding of \mathbf{G} into $\mathbb{P}(\bigwedge^2 V^\vee)$ as the projectivization of the set of decomposable 2-forms $a \wedge b$, with $a, b \in V^\vee$.

We write $\mathbb{Z}_{\text{dom}}^r$ for the set of *dominant weights* in \mathbb{Z}^r , that is, tuples $\nu = (\nu_1, \dots, \nu_r) \in \mathbb{Z}^r$ with $\nu_1 \geq \nu_2 \geq \dots \geq \nu_r$. We write \mathbb{S}_ν for the *Schur functor* associated with ν , and recall that if $\nu = (a, 0, \dots, 0)$ then $\mathbb{S}_\nu = \text{Sym}^a$, and if $\nu = (1, 1, \dots, 1)$ then $\mathbb{S}_\nu = \bigwedge^r$. A special case that will be of interest to us is when $\alpha = (\alpha_1, \alpha_2) \in \mathbb{Z}_{\text{dom}}^2$, in which case we have

$$(13) \quad \mathbb{S}_\alpha \mathcal{Q} = \text{Sym}^{\alpha_1 - \alpha_2} \mathcal{Q} \otimes \mathcal{O}_{\mathbf{G}}(\alpha_2).$$

The cohomology groups of sheaves of the form $\mathbb{S}_\alpha \mathcal{Q} \otimes \mathbb{S}_\beta \mathcal{U}$ are completely described by Bott's Theorem [47, Corollary 4.1.9]. We will only need a special case of this theorem, which we record next.

Theorem 3.3 (Bott). *Let $\alpha \in \mathbb{Z}_{\text{dom}}^2$ and $\beta \in \mathbb{Z}_{\text{dom}}^{n-2}$, and let $\gamma = (\alpha|\beta) \in \mathbb{Z}^n$ denote their concatenation. If γ is dominant then*

$$(14) \quad H^0(\mathbf{G}, \mathbb{S}_\alpha \mathcal{Q} \otimes \mathbb{S}_\beta \mathcal{U}) = \mathbb{S}_\gamma V, \text{ and } H^j(\mathbf{G}, \mathbb{S}_\alpha \mathcal{Q} \otimes \mathbb{S}_\beta \mathcal{U}) = 0 \text{ for all } j > 0.$$

If there exist $1 \leq x < y \leq n$ with $\gamma_x - x = \gamma_y - y$, then

$$(15) \quad H^j(\mathbf{G}, \mathbb{S}_\alpha \mathcal{Q} \otimes \mathbb{S}_\beta \mathcal{U}) = 0, \text{ for all } j \geq 0.$$

In particular,

- (1) *If $\alpha_1 \geq \alpha_2 \geq 0$, then $H^0(\mathbf{G}, \mathbb{S}_\alpha \mathcal{Q}) = \mathbb{S}_\alpha V$ and $H^j(\mathbf{G}, \mathbb{S}_\alpha \mathcal{Q}) = 0$ for all $j > 0$.*
- (2) *For all $a, j \geq 0$ we have that $H^j(\mathbf{G}, \text{Sym}^a \mathcal{Q} \otimes \mathcal{U}) = 0$.*
- (3) *If $-2 \geq \alpha_1 \geq 1 - n$, or $-1 \geq \alpha_2 \geq 2 - n$, then $H^j(\mathbf{G}, \mathbb{S}_\alpha \mathcal{Q}) = 0$ for all $j \geq 0$.*

To see how (1) and (3) follow from the general statement of the theorem we specialize to $\beta = (0, 0, \dots, 0)$, so that $\gamma = (\alpha_1, \alpha_2, 0, \dots, 0)$. In case (1) we have that γ is dominant so we can apply (14), while in case (3) we apply (15) with $(x, y) = (1, 1 - \alpha_1)$ when $-2 \geq \alpha_1 \geq 1 - n$, and with $(x, y) = (2, 2 - \alpha_2)$ when $-1 \geq \alpha_2 \geq 2 - n$. To see how (2) follows from (15) we note that $\alpha = (a, 0)$, $\beta = (1, 0, \dots, 0)$, and take $(x, y) = (2, 3)$.

3.2. Vanishing for Koszul modules. We begin by giving a geometric construction of $\text{Ker}(\delta_1)$, where $\delta_1 : V \otimes S \rightarrow S$ is the first Koszul differential. We let $\mathbf{G} = \text{Gr}_2(V^\vee)$ as in the previous section, and consider

$$\mathcal{S} = \text{Sym}_{\mathcal{O}_{\mathbf{G}}}(\mathcal{Q}) = \mathcal{O}_{\mathbf{G}} \oplus \mathcal{Q} \oplus \text{Sym}^2 \mathcal{Q} \oplus \dots$$

as a sheaf of graded $\mathcal{O}_{\mathbf{G}}$ -algebras on \mathbf{G} . Locally on \mathbf{G} , the sheaf \mathcal{S} can be identified with a polynomial ring in two variables, where \mathcal{Q} is the space of linear forms. We can then define the Koszul differentials

$$\delta_1^{\mathcal{Q}} : \mathcal{Q} \otimes_{\mathcal{O}_{\mathbf{G}}} \mathcal{S} \rightarrow \mathcal{S} \quad \text{and} \quad \delta_2^{\mathcal{Q}} : \bigwedge^2 \mathcal{Q} \otimes_{\mathcal{O}_{\mathbf{G}}} \mathcal{S} \rightarrow \mathcal{Q} \otimes_{\mathcal{O}_{\mathbf{G}}} \mathcal{S}$$

as in Section 2.1. Recalling that $\bigwedge^2 \mathcal{Q} = \mathcal{O}_{\mathbf{G}}(1)$, we write $\bigwedge^2 \mathcal{Q} \otimes_{\mathcal{O}_{\mathbf{G}}} \mathcal{S}$ simply as $\mathcal{S}(1)$.

Lemma 3.4. *We have a commutative diagram of graded \mathcal{S} -modules*

$$(16) \quad \begin{array}{ccccc} \bigwedge^2 V \otimes S & \xrightarrow{\delta_2} & V \otimes S & \xrightarrow{\delta_1} & S \\ f_2 \downarrow & & f_1 \downarrow & & f_0 \downarrow \\ 0 \longrightarrow & H^0(\mathbf{G}, \mathcal{S}(1)) & \xrightarrow{H^0(\mathbf{G}, \delta_2^{\mathcal{Q}})} & H^0(\mathbf{G}, \mathcal{Q} \otimes \mathcal{S}) & \xrightarrow{H^0(\mathbf{G}, \delta_1^{\mathcal{Q}})} & H^0(\mathbf{G}, \mathcal{S}) \end{array}$$

where the maps f_1 and f_0 are isomorphisms. Moreover, we have a natural isomorphism between $\text{Ker}(\delta_1)$ and $H^0(\mathbf{G}, \mathcal{S}(1))$.

Proof. If we identify $S \otimes \mathcal{O}_{\mathbf{G}}$ with $\text{Sym}_{\mathcal{O}_{\mathbf{G}}}(V \otimes \mathcal{O}_{\mathbf{G}})$ then the tautological quotient map $V \otimes \mathcal{O}_{\mathbf{G}} \rightarrow \mathcal{Q}$ in (12) yields a natural surjection

$$S \otimes \mathcal{O}_{\mathbf{G}} \rightarrow \mathcal{S}$$

which can be thought of locally on \mathbf{G} as realizing the polynomial ring S in two variables as a quotient of S by the ideal generated by the linear forms in \mathcal{U} . The naturality of the Koszul complex yields a commutative diagram

$$\begin{array}{ccccc} \Lambda^2 V \otimes S \otimes \mathcal{O}_{\mathbf{G}} & \longrightarrow & V \otimes S \otimes \mathcal{O}_{\mathbf{G}} & \longrightarrow & S \otimes \mathcal{O}_{\mathbf{G}} \\ \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \Lambda^2 \mathcal{Q} \otimes S & \longrightarrow & \mathcal{Q} \otimes S & \longrightarrow & S \end{array}$$

where the bottom row is the full Koszul complex since \mathcal{Q} has rank 2. The diagram (16) is then obtained by applying the global sections functor $H^0(\mathbf{G}, \bullet)$. It is sufficient to prove that f_0 and f_1 are isomorphisms, since this implies that f_1 induces an isomorphism between $\text{Ker}(\delta_1)$ and $\text{Ker}(H^0(\mathbf{G}, \delta_1^{\mathcal{Q}}))$, and the latter is in turn isomorphic to $H^0(\mathbf{G}, S(1))$ via the map $H^0(\mathbf{G}, \delta_2^{\mathcal{Q}})$.

To prove that f_0 is an isomorphism it suffices to prove it is injective, since for each q we have that $H^0(\mathbf{G}, \text{Sym}^q \mathcal{Q}) = \text{Sym}^q V$ by conclusion (1) of Theorem 3.3. If we let $Y = \text{Spec}_{\mathbf{G}}(S)$ denote the total space of the bundle \mathcal{Q}^{\vee} (see [26, Exercise II.5.18]), then Y is a subbundle of the trivial bundle V^{\vee} over \mathbf{G} , which can be described explicitly as

$$Y = \left\{ (c, [a \wedge b]) \in V^{\vee} \times \mathbf{G} : c \in \text{Span}(a, b) \right\}$$

If we write $\pi : Y \rightarrow V^{\vee}$ for the natural projection map, then π is surjective and therefore the induced pull-back map $f_0 : S \rightarrow H^0(V^{\vee}, \pi_* \mathcal{O}_Y) = H^0(\mathbf{G}, S)$ is injective, as desired.

To prove that f_1 is an isomorphism, we observe that it factors as the composition of

$$V \otimes S \xrightarrow{\text{id}_V \otimes f_0} V \otimes H^0(\mathbf{G}, S) = H^0(\mathbf{G}, V \otimes S),$$

which is an isomorphism, with the map $H^0(\mathbf{G}, V \otimes S) \rightarrow H^0(\mathbf{G}, \mathcal{Q} \otimes S)$ induced by the tautological surjection $V \otimes \mathcal{O}_{\mathbf{G}} \rightarrow \mathcal{Q}$. To prove that this map is an isomorphism, it is enough to check that

$$H^0(\mathbf{G}, \mathcal{U} \otimes \text{Sym}^q \mathcal{Q}) = H^1(\mathbf{G}, \mathcal{U} \otimes \text{Sym}^q \mathcal{Q}) = 0 \text{ for all } q \geq 0,$$

which follows from conclusion (2) of Theorem 3.3. \square

We next recall that $W(V, K)$ can be thought of as the middle cohomology of (6), or equivalently as

$$\begin{aligned} W(V, K) &= \text{Coker} \left\{ K \otimes S \xrightarrow{\delta_2|_{K \otimes S}} \text{Ker}(\delta_1) \right\} \\ &\cong \text{Coker} \left\{ K \otimes S \xrightarrow{f_2|_{K \otimes S}} H^0(\mathbf{G}, S(1)) \right\} \end{aligned}$$

where the isomorphism above follows from Lemma 3.4. Using again the isomorphism $f_0 : S \rightarrow H^0(\mathbf{G}, S)$ and the fact that $K \otimes H^0(\mathbf{G}, S) = H^0(\mathbf{G}, K \otimes S)$, we can rewrite

$$(17) \quad W(V, K) = \text{Coker} \left\{ H^0(\mathbf{G}, K \otimes S) \xrightarrow{H^0(\mathbf{G}, \eta)} H^0(\mathbf{G}, S(1)) \right\}$$

where

$$(18) \quad \eta : K \otimes \mathcal{S} \longrightarrow \mathcal{S}(1) \text{ is induced by } K \otimes \mathcal{O}_{\mathbf{G}} \hookrightarrow \bigwedge^2 V \otimes \mathcal{O}_{\mathbf{G}} \twoheadrightarrow \bigwedge^2 \mathcal{Q} = \mathcal{O}_{\mathbf{G}}(1).$$

Equipped with the description (17) of the Koszul modules, we can now give the proof of our main theorem.

Proof of Theorem 3.1. If $W_{n-3}(V, K) = 0$, then $\dim W(V, K) < \infty$, since the graded module $W(V, K)$ is generated in degree 0. It follows from (10) that $\mathcal{R}(V, K) = \{0\}$.

Conversely, suppose that $\mathcal{R}(V, K) = \{0\}$, which by (10) is equivalent to the fact that $\mathbb{P}(K^\perp)$ is disjoint from the Grassmannian \mathbf{G} , which in turn is equivalent to the fact that the linear system of divisors $(K, \mathcal{O}_{\mathbf{G}}(1))$ is base point free, or to the fact that the map $K \otimes \mathcal{O}_{\mathbf{G}} \longrightarrow \mathcal{O}_{\mathbf{G}}(1)$ is surjective. The Koszul complex associated to this map is therefore exact, and takes the form

$$\mathcal{K}^\bullet : 0 \longrightarrow \bigwedge^m K \otimes \mathcal{O}_{\mathbf{G}}(1-m) \longrightarrow \cdots \longrightarrow \bigwedge^2 K \otimes \mathcal{O}_{\mathbf{G}}(-1) \longrightarrow K \otimes \mathcal{O}_{\mathbf{G}} \longrightarrow \mathcal{O}_{\mathbf{G}}(1) \longrightarrow 0$$

where we use a cohomological grading with $\mathcal{K}^{-i} = \bigwedge^i K \otimes \mathcal{O}_{\mathbf{G}}(1-i)$. Tensoring with $\text{Sym}^{n-3} \mathcal{Q}$ preserves exactness, so the hypercohomology of $\mathcal{K}^\bullet \otimes \text{Sym}^{n-3} \mathcal{Q}$ vanishes identically. The hypercohomology spectral sequence associated with $\mathcal{K}^\bullet \otimes \text{Sym}^{n-3} \mathcal{Q}$ then yields that

$$E_1^{-i,j} = H^j\left(\mathbf{G}, \bigwedge^i K \otimes \mathcal{O}_{\mathbf{G}}(1-i) \otimes \text{Sym}^{n-3} \mathcal{Q}\right) \implies 0.$$

Since $\text{Sym}^{n-3} \mathcal{Q} \otimes \mathcal{O}_{\mathbf{G}}(1-i) = \mathbb{S}_{(n-2-i, 1-i)} \mathcal{Q}$ by (13), we can rewrite the terms in the spectral sequence as

$$(19) \quad E_1^{-i,j} = H^j\left(\mathbf{G}, \bigwedge^i K \otimes \mathbb{S}_{(n-2-i, 1-i)} \mathcal{Q}\right) = \bigwedge^i K \otimes H^j(\mathbf{G}, \mathbb{S}_{(n-2-i, 1-i)} \mathcal{Q}).$$

We have using (17) the following identification

$$W_{n-3}(V, K) = \text{Coker}\{E_1^{-1,0} \longrightarrow E_1^{0,0}\}$$

and we suppose by contradiction that this map is not surjective. Since $E_\infty^{0,0} = 0$, there must be some *non-zero* differential

$$E_r^{-r, r-1} \longrightarrow E_r^{0,0} \text{ for } r \geq 2, \text{ or } E_r^{0,0} \longrightarrow E_r^{r, 1-r} \text{ for } r \geq 1.$$

Since $E_1^{r, 1-r} = 0$ for all $r \geq 1$ it follows that $E_r^{r, 1-r} = 0$ as well, so the latter case does not occur. To prove that the former case does not occur either and obtain a contradiction, it suffices to check that $E_1^{-r, r-1} = 0$ for all $r \geq 2$, which follows from (19) if we can prove that

$$(20) \quad H^{r-1}(\mathbf{G}, \mathbb{S}_{(n-2-r, 1-r)} \mathcal{Q}) = 0 \text{ for } r \geq 2.$$

If we write $\alpha = (n-2-r, 1-r)$ then for $2 \leq r \leq n-1$ we have that $-1 \geq \alpha_2 \geq 2-n$, and for $n \leq r \leq 2n-3$ we have that $-2 \geq \alpha_1 \geq 1-n$, so (20) follows from conclusion (3) of Theorem 3.3. When $r > 2n-3$ we get that $r-1 > 2n-4 = \dim(\mathbf{G})$, so (20) is a consequence of Grothendieck's vanishing theorem [26, Theorem III.2.7]. \square

3.3. The Hilbert series of Koszul modules. The goal of this section is to prove Theorem 3.2. Recall from equation (7) that $W_q(V, K)$ is obtained as the cokernel of the map

$$(21) \quad \delta_{2,q} : K \otimes \text{Sym}^q V \longrightarrow \text{Im}(\delta_{2,q})$$

and in particular we have that $W_q(V, K) \neq 0$ when $\dim(\text{Im}(\delta_{2,q})) > \dim(K \otimes \text{Sym}^q V)$. We say that the *divisorial case* occurs when the map (21) is given by a square matrix, i.e.

$$\dim(K \otimes \text{Sym}^q V) = \dim(\text{Im}(\delta_{2,q})).$$

Proposition 3.5. *Let $K \subseteq \bigwedge^2 V$ be an arbitrary subspace of dimension $m = 2n - 3$. For any $0 \leq q \leq n - 4$ we have $W_q(V, K) \neq 0$, and $q = n - 3$ is a divisorial case.*

Proof. Using the fact that $\dim \text{Sym}^k V = \binom{n+k-1}{k}$ and $\dim(K) = 2n - 3$ we obtain

$$\dim(K \otimes \text{Sym}^q V) = (2n - 3) \binom{n+q-1}{q}.$$

The exactness of the Koszul complex implies that $\delta_{1,q+1}$ is surjective and that we have the equality $\text{Im}(\delta_{2,q}) = \text{Ker}(\delta_{1,q+1})$, so

$$\dim(\text{Im}(\delta_{2,q})) = \dim(V \otimes \text{Sym}^{q+1} V) - \dim(\text{Sym}^{q+2} V) = n \cdot \binom{n+q}{q+1} - \binom{n+q+1}{q+2},$$

A direct computation leads for $q \geq 0$ to the formula

$$(22) \quad \dim(\text{Im}(\delta_{2,q})) - \dim(K \otimes \text{Sym}^q V) = \binom{n+q-1}{q} \frac{(n-2)(n-q-3)}{q+2},$$

which is positive for $0 \leq q \leq n-4$ and vanishes for $q = n-3$, proving that $W_q(V, K) \neq 0$ for $0 \leq q \leq n-4$, and that the case $q = n-3$ is divisorial. \square

We next analyze in more detail the divisorial case, where the vanishing of $W_{n-3}(V, K)$ can be rephrased into a transversality condition as follows.

Lemma 3.6. *Let $K \subseteq \bigwedge^2 V$ be a subspace of dimension $2n - 3$. The following are equivalent:*

- (a) $W_{n-3}(V, K) = 0$.
- (b) $(K \otimes \text{Sym}^{n-3} V) \cap \text{Ker}(\delta_{2,n-3}) = 0$.
- (c) $(K \otimes \text{Sym}^q V) \cap \text{Ker}(\delta_{2,q}) = 0$ for all $q \leq n-3$.

Proof. It is clear that (c) implies (b), while for the reverse implication it suffices to observe that multiplication by a linear form in V yields for each $q \geq 0$ an injective map from $(K \otimes \text{Sym}^q V) \cap \text{Ker}(\delta_{2,q})$ to $(K \otimes \text{Sym}^{q+1} V) \cap \text{Ker}(\delta_{2,q+1})$.

Using the isomorphism $\text{Im}(\delta_{2,n-3}) = (\bigwedge^2 V \otimes \text{Sym}^{n-3} V) / \text{Ker}(\delta_{2,n-3})$ and (7) we get

$$(23) \quad W_{n-3}(V, K) = \left(\bigwedge^2 V \otimes \text{Sym}^{n-3} V \right) / \left(K \otimes \text{Sym}^{n-3} V + \text{Ker}(\delta_{2,n-3}) \right).$$

Since we are in the divisorial case we have $\dim(\text{Im}(\delta_{2,n-3})) = \dim(K \otimes \text{Sym}^{n-3} V)$, thus

$$(24) \quad \dim\left(\bigwedge^2 V \otimes \text{Sym}^{n-3} V \right) = \dim(K \otimes \text{Sym}^{n-3} V) + \dim(\text{Ker}(\delta_{2,n-3})).$$

Combining (23) with (24) we conclude that $W_{n-3}(V, K) = 0$ if and only if the sum $K \otimes \text{Sym}^{n-3} V + \text{Ker}(\delta_{2,n-3})$ is direct, proving the equivalence of (a) and (b). \square

Corollary 3.7. *If $K \subseteq \bigwedge^2 V$ has dimension $2n - 3$ and $\mathcal{R}(V, K) = \{0\}$, then the map (21) is of maximal rank for all q .*

Proof. Using (22), it suffices to prove that (21) is injective for $q \leq n - 3$ and surjective for $q \geq n - 3$. Since $\mathcal{R}(V, K) = \{0\}$, we know from Theorem 3.1 that $W_{n-3}(V, K) = 0$, hence the assertion in Lemma 3.6(c) holds, proving the injectivity of (21) in the range $q \leq n - 3$. The surjectivity of (21) is equivalent to the vanishing of $W_q(V, K)$, which is true for $q \geq n - 3$ because $W_{n-3}(V, K) = 0$ and $W(V, K)$ is generated in degree 0. \square

We now have all the tools necessary to prove Theorem 3.2.

Proof of Theorem 3.2. We write $m = \dim(K)$ as usual, and suppose first that $m = 2n - 3$. By Corollary 3.7 we know that the maps (21) have maximal rank for all q . Since their cokernels are the graded components of $W(V, K)$, it follows from (22) that

$$\dim W_q(V, K) = \binom{n+q-1}{q} \frac{(n-2)(n-q-3)}{q+2} \text{ for all } q = 0, \dots, n-4.$$

We next consider the case $m > 2n - 3$. If we set $\mathbf{G} := \text{Gr}_2(V^\vee) \subseteq \mathbb{P}(\bigwedge^2 V^\vee)$ then (10) implies that $\mathbb{P}(K^\perp) \cap \mathbf{G} = \emptyset$. We consider the linear projection away from $\mathbb{P}(K^\perp)$

$$\pi : \mathbb{P}\left(\bigwedge^2 V^\vee\right) \dashrightarrow \mathbb{P}^{m-1}$$

Since $\dim(\mathbf{G}) = 2n - 4$ it follows that $\pi(\mathbf{G})$ is a subvariety of \mathbb{P}^{m-1} of codimension at least $m - 1 - (2n - 4) = m - 2n + 3$, hence a general linear subspace $\mathbb{P}^{m-2n+2} \subset \mathbb{P}^{m-1}$ is disjoint from $\pi(\mathbf{G})$, i.e. it satisfies $\pi^{-1}(\mathbb{P}^{m-2n+2}) \cap \mathbf{G} = \emptyset$. Since $\pi^{-1}(\mathbb{P}^{m-2n+2})$ is a linear subspace of $\mathbb{P}(\bigwedge^2 V^\vee)$ of codimension $2n - 3$ containing $\mathbb{P}(K^\perp)$, we can write $\pi^{-1}(\mathbb{P}^{m-2n+2}) = \mathbb{P}(K'^\perp)$ for some $K' \subseteq K$ with $\dim(K') = 2n - 3$. It follows from (8) (with the roles of K and K' exchanged) that $\dim W_q(V, K) \leq \dim W_q(V, K')$ for all q , which combined with the case $m = 2n - 3$ proves the desired inequality and completes our proof. \square

Remark 3.8. A special case of Theorem 3.2 is obtained by taking $V = \text{Sym}^{n-1} \mathbb{C}^2$ and $K = \text{Sym}^{2n-4} \mathbb{C}^2$, in which case $W(V, K)$ is the *Weyman module* $W(n-1)$ (see [39, Section 5] and [18, Section 3.I.B]). It follows that Theorem 3.2 computes the Hilbert series of the said module, answering a problem left open in [39, Section 1.2]. For $n \leq 8$, Alex Suciu used a computer search to construct subspaces $K_n \subseteq \bigwedge^2 \mathbb{C}^n$ with $m = 2n - 3$ and $\mathcal{R}(\mathbb{C}^n, K_n) = \{0\}$. He also computed the Hilbert series of the Koszul modules $W(\mathbb{C}^n, K_n)$. He conjectured the existence of a family of subspaces $K_n \subseteq \bigwedge^2 \mathbb{C}^n$ with $m = 2n - 3$ and $\mathcal{R}(\mathbb{C}^n, K_n) = \{0\}$, for all values of n , with Hilbert series given by the formula from Theorem 3.2. Our results prove that the expectations were correct.

4. TOPOLOGICAL INVARIANTS OF GROUPS

In this section we explain the topological motivation for investigating Koszul modules, and derive a number of consequences of our results to the study of invariants of fundamental groups. Some of our main applications concern groups G that are 1-formal (in the sense of Sullivan [44]), where a direct relationship between certain invariants of G and those of the corresponding Koszul modules can be derived. The class

of 1-formal groups contains important examples such as the fundamental groups of compact Kähler manifolds, hyperplane arrangement groups, or the Torelli group of the mapping class group of a genus g surface.

4.1. The Koszul module and the resonance variety of a group G . We recall from Example 2.3 the construction of the *Koszul module* $W(G)$ associated to a finitely generated group G . The module $W(G) := W(V, K)$ is obtained by setting $V := H_1(G, \mathbb{C})$, and letting $K^\perp \subseteq \bigwedge^2 V^\vee$ be the kernel of the cup-product map

$$\cup_G : \bigwedge^2 H^1(G, \mathbb{C}) \rightarrow H^2(G, \mathbb{C}).$$

To the Koszul module $W(G)$ we can associate its *resonance variety* $\mathcal{R}(G)$ as in Definition 2.4. Alternatively, we can define $\mathcal{R}(G)$ without reference to Koszul modules as follows. For an element $a \in H^1(G, \mathbb{C})$, we consider the complex of abelian groups

$$(G, a) : 0 \longrightarrow H^0(G, \mathbb{C}) \xrightarrow{\cup_a} H^1(G, \mathbb{C}) \xrightarrow{\cup_a} H^2(G, \mathbb{C}) \xrightarrow{\cup_a} \dots$$

The resonance variety $\mathcal{R}(G)$ is then obtained as

$$\mathcal{R}(G) = \{a \in H^1(G, \mathbb{C}) : H^1(G, a) \neq 0\}.$$

In the 1-formal context, $\mathcal{R}(G)$ has yet another interpretation given as follows.

Remark 4.1. For a character $\rho \in \widehat{G} := \text{Hom}(G, \mathbb{C}^*)$, we denote by \mathbb{C}_ρ the $\mathbb{C}[G]$ -module \mathbb{C} with multiplication given by $g \cdot z := \rho(g)z$. The resonance variety $\mathcal{R}(G)$ turns out to be an infinitesimal analogue of the much studied *characteristic variety*

$$\mathcal{V}(G) := \{\rho \in \widehat{G} : H^1(G, \mathbb{C}_\rho) \neq 0\},$$

which already appeared in the work of Green–Lazarsfeld [20], Beauville [3], or Libgober [30]. It is shown in [17, Theorems A and C] that when G is 1-formal, $\mathcal{R}(G)$ is isomorphic to the tangent cone at 1 of the characteristic variety $\mathcal{V}(G)$.

Reinterpreted in the context of groups, our Theorems 3.1 and 3.2 imply the following. We write $b_1(G) := \dim H_1(G, \mathbb{C})$ for the *first Betti number* of the group G .

Theorem 4.2. *Let G be a finitely generated group and let $n = b_1(G)$. If $n \geq 3$ then we have the equivalence*

$$\mathcal{R}(G) = \{0\} \iff W_{n-3}(G) = 0.$$

Moreover, if the equivalent statements above hold then

$$\dim W_q(G) \leq \binom{n+q-1}{q} \frac{(n-2)(n-q-3)}{q+2} \quad \text{for } 0 \leq q \leq n-4.$$

This theorem can be used to obtain upper bounds on the Chen ranks of a group G with vanishing resonance, once we establish a relationship between the said ranks and the Hilbert function of $W(G)$. This is achieved via the study of the Alexander invariant of G in the next section.

4.2. Alexander invariants. Alexander-type invariants have a long history in topology, starting with the definition of the Alexander polynomial of knots and links [1]. For a connected CW-complex X with fundamental group $G = \pi_1(X)$, each surjective group homomorphism $\alpha : G \rightarrow H$ gives rise to a sequence of Alexander-type invariants of X , namely the twisted homology groups $H_i(X, L)$, where L is the local system $\mathbb{C}[H]$. By construction, these invariants are $\mathbb{C}[G]$ -modules. When $i = 1$ the construction depends solely on G , is purely algebraic and can be carried out for any group. The classical Alexander invariant of a knot $Y \subseteq S^3$ corresponds to the case when $X := S^3 \setminus Y$ is the knot complement, $i = 1$ and $\alpha : G \rightarrow \mathbb{Z}$ is the abelianization map.

Definition 4.3. If G is a finitely generated group, its *Alexander invariant* is defined as $B(G) := H_1(G', \mathbb{Z}) = G'/G''$, viewed as a module over the group ring $\mathbb{Z}[G_{\text{ab}}]$, where $G_{\text{ab}} := G/G'$, the action being given by conjugation. We write $B(G)_{\mathbb{C}} := B(G) \otimes_{\mathbb{Z}} \mathbb{C}$.

Geometrically, if X is a CW-complex with $\pi_1(X) \cong G$ and $X_{\text{ab}} \rightarrow X$ is the maximal abelian cover corresponding to the group $H_1(X, \mathbb{Z})$, then $B(G)$ can be identified with $H^1(X', \mathbb{Z})$, viewed as a module over $G_{\text{ab}} \cong H_1(X, \mathbb{Z})$.

The Alexander invariant is equipped with the I -adic filtration, where $I := \text{Ker}(\epsilon)$ is the kernel of the augmentation map $\epsilon : \mathbb{Z}[G/G'] \rightarrow \mathbb{Z}$. A key result of Massey [33] asserts that for each $q \geq 0$, one has an isomorphism

$$(25) \quad \text{gr}_q B(G) := \text{gr}_q^I B(G) \cong \text{gr}_{q+2}(G/G'').$$

In particular, the generating series of the Chen ranks of G is given as a Hilbert series

$$\sum_{q=0}^{\infty} \theta_{q+2}(G) t^q = \text{Hilb}(\text{gr } B(G)_{\mathbb{C}}, t).$$

Remark 4.4. The Chen ranks are usually more easily computable than the largely inaccessible lower central ranks of a group G . An instance of this is the case of hyperplane arrangement groups associated to an arrangement $\mathcal{A} = \{H_1, \dots, H_n\}$ of hyperplanes in \mathbb{C}^m . Whereas the fundamental group $G(\mathcal{A}) := \pi_1(\mathbb{C}^m - \cup_{H \in \mathcal{A}} H)$ of the arrangement is *not* determined by the intersection lattice of \mathcal{A} , the lower central series and the corresponding ranks $\phi_q(G(\mathcal{A}))$ and $\theta_q(G(\mathcal{A}))$ are, and in particular they are combinatorial objects. An explicit formula for $\phi_q(G(\mathcal{A}))$ remains elusive, but the Chen ranks are subject to a conjecture of Suciu, proven in many cases in [40] and in full generality in [12]. It asserts that for $q \gg 0$, one has the formula

$$\theta_q(G(\mathcal{A})) = \sum_{r \geq 1} h_r \theta_q(F_{r+1}),$$

where h_r is the number of r -dimensional components of the resonance variety $\mathcal{R}(G(\mathcal{A}))$ and the Chen ranks of the free group are given by (see [8])

$$\theta_1(F_{r+1}) = r + 1 \quad \text{and} \quad \theta_q(F_{r+1}) = (q-1) \binom{q+r-1}{q} \quad \text{for } q \geq 2.$$

The Alexander invariants of G exhibit the same type of behaviour as their cousins, the topological jump loci $\mathcal{V}(X) := \mathcal{V}(\pi_1(X))$ from Remark 4.1. They are complicated

objects encoding useful information, with complexity bounded by invariants depending on the cohomology ring. Work from [16, 33, 37, 38, 43] implies the existence of a natural surjection

$$W(G) \rightarrow \text{gr } B(G)_{\mathbb{C}}.$$

In particular, for all $q \geq 0$,

$$(26) \quad \dim \text{gr}_q B(G)_{\mathbb{C}} \leq \dim W_q(G),$$

with equality when the group G is 1-formal [39, Corollary 6.1].

One should regard the relation between $\text{gr } B(G)_{\mathbb{C}}$ and $W(G)$ as a commutative analogue of the relationship between the Lie algebra $\text{gr}(G)_{\mathbb{C}}$ and the holonomy Lie algebra of G , whose definition we recall next. Denoting by $\partial_G : \bigwedge^2 H_1(G, \mathbb{C}) \rightarrow H_2(G, \mathbb{C})$ the dual of the cup product map \cup_G , one defines following Chen [8] the *holonomy Lie algebra*

$$\mathfrak{h}(G) := \mathbb{L}(H_1(G, \mathbb{C})) / \text{Ideal}(\text{Im}(\partial_G)),$$

where $\mathbb{L}(H_1(G, \mathbb{C}))$ is the free complex Lie algebra generated by $H_1(G, \mathbb{C})$. Work of Sullivan [44] implies that for a 1-formal group G one has an isomorphism of Lie algebras

$$\text{gr}(G)_{\mathbb{C}} \cong \mathfrak{h}(G),$$

that is, the entire lower central series of G is determined by the first cup-product map.

In the following applications we relate the I -adic filtrations of the Alexander invariant $B(G)$ of G to the cohomology ring of G in low degrees in a more precise way.

Theorem 4.5. *Let G be a finitely generated group with $b_1(G) \geq 3$. If $\mathcal{R}(G) = \{0\}$, then*

$$(I^q \cdot B(G))_{\mathbb{C}} = (I^{q+1} \cdot B(G))_{\mathbb{C}}$$

for all $q \geq b_1(G) - 3$.

Proof. Combine Theorem 4.2 with (26). □

We recall that a module M over a group ring is nilpotent if $I^q \cdot M = 0$ for some q , where I is the augmentation ideal.

Corollary 4.6. *If $\mathcal{R}(G) = \{0\}$ and $B(G)$ is nilpotent, then $(I^{b_1(G)-3} \cdot B(G))_{\mathbb{C}} = 0$. If moreover G is 1-formal, then $\dim B(G)_{\mathbb{C}} = \dim W(G) < \infty$.*

In general, the Alexander invariant of a group may not be nilpotent as a module, but this is the case when the group itself is nilpotent. Recall that a group G is said to be nilpotent if there exists $c \geq 1$ with $\Gamma_c(G) = \{1\}$. The largest c such that $\Gamma_c(G) \neq \{1\}$ is called the *nilpotency class* of G , and is denoted $\text{nc}(G)$. A group G is *virtually nilpotent* if there exists a finite index nilpotent subgroup $H \leq G$. We define the *virtual nilpotency class* $\text{vnc}(G)$ of G to be the following quantity

$$(27) \quad \text{vnc}(G) := \min\{\text{nc}(H) : H \leq G \text{ is a nilpotent finite index subgroup}\}.$$

Since every virtually nilpotent group contains a torsion free finite index subgroup, and since the nilpotency class of a torsion free nilpotent group equals that of any of its finite index subgroups, it follows that for a virtually nilpotent group G we have

$$(28) \quad \text{vnc}(G) = \text{nc}(H) \text{ for any torsion free nilpotent subgroup } H \leq G \text{ of finite index.}$$

One useful bound for the virtual nilpotency class of G is obtained as follows.

Lemma 4.7. *If G contains a torsion-free finite index subgroup H and $\Gamma_c(G)$ is finite, then $\text{vnc}(G) = \text{nc}(H) \leq c - 1$.*

Proof. Since $\Gamma_c(H) \leq \Gamma_c(G)$ and $\Gamma_c(H)$ is torsion-free, it follows that $\Gamma_c(H) = \{1\}$ and therefore $\text{nc}(H) \leq c - 1$. The equality $\text{vnc}(G) = \text{nc}(H)$ follows from (28). \square

Corollary 4.8. *If G is nilpotent and $\mathcal{R}(G) = \{0\}$, then $(I^{b_1(G)-3} \cdot B(G))_{\mathbb{C}} = 0$. It follows that $\text{vnc}(G/G'') \leq b_1(G) - 2$.*

Proof. Since G is nilpotent, the Alexander invariant is also nilpotent so the conclusion $(I^{b_1(G)-3} \cdot B(G))_{\mathbb{C}} = 0$ follows from Corollary 4.6. It follows moreover from (25) and (26) that $\theta_q(G) = 0$ for $q \geq b_1(G) - 1$. Since G/G'' is nilpotent, we must have that $\theta_q(G)$ is finite for $q \geq b_1(G) - 1$, so $\text{vnc}(G/G'') \leq b_1(G) - 2$ by Lemma 4.7. \square

The conclusion of Corollary 4.8 *does* require the vanishing resonance assumption.

Example 4.9. Indeed, let F_n be the free group on $n \geq 2$ generators, and consider the nilpotent group $G := F_n/\Gamma_k(F_n)$, where $k \geq 3$. As explained for instance in [35, Remark 2.4], the resonance variety of a group depends only on its third nilpotent quotient. Since $G/\Gamma_3(G) \cong F_n/\Gamma_3(F_n)$, it follows that $\mathcal{R}(G) \cong \mathcal{R}(F_n) \cong \mathbb{C}^n$. Note also that $b_1(G) = n$. On the other hand, as already pointed out, the Hilbert series $\sum_{q \geq 1} \theta_q(F_n)t^q$ was computed by Chen [8] and *all* of its coefficients are strictly positive. Since $\theta_q(F_n) = \theta_q(G)$ for $q < k$, it follows from (25) that $I^q \cdot B(G) \neq 0$ for $q < k - 2$, thus the nilpotency class cannot be bounded solely in terms of $n = b_1(G)$.

In fact the resonance variety of a nilpotent group of nilpotency class 2 can be arbitrarily complicated, cf. [35, Remark 2.4]. However, this phenomenon does not occur if we assume 1-formality. The resonance variety of a finitely generated, nilpotent, 1-formal group G must vanish, see [5, Lemma 2.4] and the proof of Theorem 4.10 below. Theorem 1.1 in the Introduction is the special case when G is a Kähler group of the following.

Theorem 4.10. *Let G be a finitely generated 1-formal group, and suppose its first Betti number is $n = b_1(G) \geq 3$. If G/G'' is nilpotent, then $\theta_q(G) = 0$ for $q \geq n - 1$ and*

$$\theta_q(G) \leq \binom{n+q-3}{n-1} \cdot \frac{(n-2)(n-1-q)}{q} \quad \text{for } q = 2, \dots, n-2.$$

Proof. The condition that G/G'' be nilpotent implies that $\theta_q(G) = \phi_q(G/G'') = 0$ for $q \gg 0$. Since G is 1-formal, we have using (25) and (26) that $\dim W_q(G) = \theta_{q+2}(G)$ for all $q \geq 0$. In particular $W(G)$ is a finite dimensional module, hence $\mathcal{R}(G) = \{0\}$ and Theorem 4.2 applies to give the desired conclusions about $\theta_q(G) = \dim W_{q-2}(G)$. \square

We proved in Theorem 3.1 that $W_{n-3}(V, K) = 0$ when $\mathcal{R}(V, K) = \{0\}$, and therefore $W_q(V, K) = 0$ for all $q \geq n - 3$. In the borderline case $m = 2n - 3$, we infer from Proposition 3.5 that this is the best possible vanishing result for Koszul modules. In the general case, the following examples coming from nilpotent groups show that the bound $q(n, m) = n - 3$ may be improved, at least in some cases.

Example 4.11. For $k \geq 1$, we denote by H_k the fundamental group of the *Heisenberg nilmanifold* of dimension $2k + 1$. It is well-known that H_k is finitely generated and

$\Gamma_3(H_k) = \{1\}$. We set $V := H_1(H_k, \mathbb{C})$ and denote by $K^\perp \subseteq \bigwedge^2 V^\vee$ the kernel of the cup-product map as before. If $k \geq 2$, we know from [34] that H_k is 1-formal and $\mathcal{R}(H_k) = \{0\}$. In this case, $b_1(H_k) = 2k$ and $K \subseteq \bigwedge^2 V$ has codimension 1. The fact that $\Gamma_3(H_k) = \{1\}$ and (26) together imply that $W_1(H_k) = 0$, but note that $1 < b_1(H_k) - 3$ as soon as $k > 2$.

4.3. A bound on the degree of growth of a group. The goal of this section is to provide a proof of Theorem 1.2. Suppose G is a finitely generated group and let $S \subseteq G$ be a finite set of generators. For an integer $m \geq 0$, we denote by $\ell_G(m) = \ell_{G,S}(m)$ the number of elements $g \in G$ expressible as a product of m elements from $S \cup S^{-1}$. Bass [2] and Guivarc'h [23] showed that a finitely generated nilpotent group G has polynomial growth. More precisely, the inequality $\ell_G(m) \leq Cm^{d(G)}$ holds for $m \gg 0$, where $d(G)$ is the *degree of polynomial growth* of the group, which in the case of nilpotent groups is given by the formula

$$(29) \quad d(N) = \sum_{q \geq 1} q \cdot \phi_q(N).$$

We also recall a celebrated theorem of Gromov [22] which asserts that conversely, the virtually nilpotent groups are precisely the ones that have polynomial growth. If G is merely virtually nilpotent, the Bass-Guivarc'h formula (29) fails as seen for instance by taking G to be the infinite dihedral group, where $d(G) = 1$, whereas $\phi_q(G) = 0$ for all q . For a quantitative version of Gromov's result, we refer to [31, Theorem 4].

Before stating the following result, we recall that the *Hirsch index* of a finitely generated nilpotent (or more generally polycyclic) group G is defined to be the quantity $h(G) := \sum_{q \geq 1} \phi_q(G)$, see [42, Chapter 1]. It is well-known that for a normal subgroup $N \trianglelefteq G$ one has the additivity property $h(G) = h(N) + h(G/N)$.

Lemma 4.12. *Let G be a finitely generated group, and H be a finite index subgroup of G . For every $k \geq 1$ we have that*

$$(30) \quad \sum_{q=1}^k \phi_q(H) \geq \sum_{q=1}^k \phi_q(G).$$

Moreover, if $[H \cap \Gamma_{k+1}(G) : \Gamma_{k+1}(H)] < \infty$, then the above inequality is in fact an equality.

Proof. The quotient $\overline{H} := H/\Gamma_{k+1}(H)$ is nilpotent and its Hirsch number is given by

$$h(\overline{H}) = \sum_{q=1}^k \phi_q(H).$$

A similar formula holds for $\overline{G} := G/\Gamma_{k+1}(G)$. Since $H \leq G$, we must also have that $\Gamma_{k+1}(H) \leq \Gamma_{k+1}(G)$, and therefore we obtain an induced group homomorphism

$$\psi : \overline{H} \longrightarrow \overline{G}.$$

Since $[G : H] < \infty$, the image of ψ has finite index in \overline{G} , therefore $h(\text{Im } \psi) = h(\overline{G})$. Since $\text{Im } \psi$ is a quotient of \overline{H} , it follows that $h(\overline{H}) = h(\text{Ker } \psi) + h(\text{Im } \psi) \geq h(\text{Im } \psi)$, which

proves that $h(\overline{H}) \geq h(\overline{G})$. If we assume that $\Gamma_{k+1}(H)$ has finite index in $H \cap \Gamma_{k+1}(G)$ then $\text{Ker } \psi$ is a finite group and therefore $h(\text{Ker } \psi) = 0$, which implies $h(\overline{H}) = h(\overline{G})$. \square

In the next proof, recall that $M := G/G''$ denotes the maximal metabelian quotient of a 1-formal finitely generated group G .

Proof of Theorem 1.2. Since $\Gamma_q(M)$ is finite for $q \gg 0$, it follows that $\theta_q(G) = \phi_q(M) = 0$ for $q \gg 0$. Using (25) and the fact that (26) is an equality when G is 1-formal, we obtain

$$\dim W_q(G) = \phi_{q+2}(M) = 0 \text{ for } q \gg 0.$$

It follows that $W(G)$ is finite dimensional, which by (10) and Theorem 3.1 implies that $W_q(G) = 0$ for $q \geq n - 3$ and thus $\phi_q(M) = 0$ for $q \geq n - 1$. Combining this with the fact that $\Gamma_q(M)$ is finite for $q \gg 0$, we conclude that in fact $\Gamma_{n-1}(M)$ is finite proving conclusion (1) of Theorem 1.2.

We note that if N is a finite normal subgroup of $M = G/G''$ then the groups M and M/N have the same growth rate. Taking $N = \Gamma_q(M)$ where q is such that $\Gamma_q(M)$ is finite, we get that M has the same growth rate as the nilpotent group M/N , and in particular M is virtually nilpotent by Gromov's theorem. We can then apply Lemma 4.7 to deduce part (2) of Theorem 1.2.

Consider now H to be a nilpotent finite index subgroup of M . Using the last part of Lemma 4.12 and the fact that $\Gamma_{n-1}(M)$ is finite we get that

$$(31) \quad \sum_{q=1}^{n-2} \phi_q(H) = \sum_{q=1}^{n-2} \phi_q(M).$$

Applying the Bass–Guivarc'h Formula 1.2 to H yields

$$\begin{aligned} d(M) &= d(H) = \sum_{q \geq 1} q \cdot \phi_q(H) = (n-2) \cdot \left(\sum_{q=1}^{n-2} \phi_q(H) \right) - \sum_{k=1}^{n-3} \left(\sum_{q=1}^k \phi_q(H) \right) \\ &\stackrel{(30),(31)}{\leq} (n-2) \cdot \left(\sum_{q=1}^{n-2} \phi_q(M) \right) - \sum_{k=1}^{n-3} \left(\sum_{q=1}^k \phi_q(M) \right) = \sum_{q=1}^{n-2} q \cdot \phi_q(M) \\ &= \phi_1(M) + \sum_{q=0}^{n-4} (q+2) \cdot \dim W_q(G). \end{aligned}$$

Using the fact that $\phi_1(M) = b_1(G) = n$ and the inequality in Theorem 3.2, it follows that

$$d(M) \leq n + \sum_{q=0}^{n-4} \binom{n+q-1}{q} \cdot (n-2) \cdot (n-q-3) = n + (n-2) \cdot \binom{2n-3}{n-4}. \quad \square$$

Remark 4.13. Just as in the case of the nilpotency class, in the absence of 1-formality there can be no uniform bound in terms of $b_1(G)$ for the degree of polynomial growth of $M = G/G''$. Indeed, if we take $G = F_n/\Gamma_k(F_n)$ as in Example 4.9, then M is nilpotent. Combining (29) with the fact that $\phi_q(M) = \theta_q(G) = \theta_q(F_n)$ is strictly positive for $q < k$, we obtain that $d(M)$ can be made arbitrarily large if we let $k \rightarrow \infty$.

Remark 4.14. The only place in the proof of Theorem 1.2 that required 1-formality is the assertion that the vanishing of the Chen ranks implies the vanishing of $W_q(G)$. If we replace the 1-formality assumption with the hypothesis that $\mathcal{R}(G) = \{0\}$ then the vanishing of $W_q(G)$ follows from Theorem 4.2, and the proof of Theorem 1.2 carries through without any further modifications.

4.4. Kähler groups. Our results can be applied to analyze the nilpotency class of fundamental groups of compact Kähler manifolds. Suppose X is a compact Kähler manifold and $G := \pi_1(X)$, hence $H_1(G, \mathbb{Z}) \cong H_1(X, \mathbb{Z})$. The first thing we observe is that $\mathcal{R}(G) \neq 0$ if and only if the $(2, 0)$ -part of the cup product map

$$\psi_X = \cup_X^{2,0} : \bigwedge^2 H^0(X, \Omega_X^1) \rightarrow H^0(X, \Omega_X^2)$$

is zero on a degenerate element $0 \neq \omega_1 \wedge \omega_2 \in \bigwedge^2 H^0(X, \Omega_X^1)$. This follows because $\cup_X^{0,2}$ is the conjugate of $\cup_X^{2,0}$, whereas the $(1, 1)$ -part of the cup-product in the Hodge decomposition $\cup_X^{1,1} : H^{1,0}(X) \otimes H^{0,1}(X) \rightarrow H^{1,1}(X)$ cannot vanish on decomposable tensors. Following Catanese's generalization of the Castelnuovo-de Franchis Theorem [6], it follows that $\mathcal{R}(X) \neq 0$ if and only if X is fibred onto a curve C of genus at least 2. In this case, we have a surjection $\pi_1(X) \twoheadrightarrow \pi_1(C)$, thus $\pi_1(X)$ is not nilpotent. We say that in this case X is *fibred*. Since $\pi_1(X)$ is 1-formal, we obtain as a special case of Theorem 1.2 the following.

Theorem 4.15. *Suppose that X is a non-fibred compact Kähler manifold. If $\pi_1(X)/\pi_1(X)''$ is nilpotent, then its virtual nilpotency class is at most $2q(X) - 2$.*

Constructing explicit Kähler manifolds with nilpotent fundamental group has proven to be a challenging task. Examples of Kähler groups with a non-trivial nilpotent filtration (in fact, of nilpotency class 2) have been produced first by Campana [4]. On the other hand, a Kähler group is either fibred or each of its solvable quotients is virtually nilpotent [14]. In particular, solvable Kähler groups are virtually nilpotent.

A recent example is provided by the Schoen surfaces discovered in [41] and studied further in [10]. They are minimal algebraic surfaces X of general type having invariants

$$q(X) = 4, p_g(X) = 5, K_X^2 = 16, h^{1,1}(X) = 12.$$

Remarkably, $p_g(X) = 2q(X) - 3$, that is, we are in the divisorial case when $\text{Ker}(\psi_X)$ may not contain decomposable elements. That is the case, for Schoen surfaces are not fibred. In fact $\text{Ker}(\psi_X)$ is of dimension one, implying $\dim \text{Ker}(\cup_X) \in \{6, 7\}$. It is unknown whether the fundamental group of such an X is nilpotent. Applying Theorem 4.15, we obtain that the virtual nilpotency class of $\pi_1(X)/\pi_1(X)''$ is at most $6 = b_1(X) - 2$.

4.5. The Torelli group of the mapping class group. A particularly interesting class of applications of our results is provided by the *Torelli groups* T_g of genus $g \geq 2$. We fix a closed oriented surface Σ_g of genus g and denote by $\pi_g := \pi_1(\Sigma_g)$ its fundamental group and by $H := H_1(\Sigma_g, \mathbb{Z})$ its first homology. Let Mod_g be the mapping class group of isotopy classes of orientation-preserving homeomorphisms of Σ_g . If \mathcal{X}_g denotes the Teichmüller space of genus g , then Mod_g is the orbifold fundamental group of the moduli space \mathcal{M}_g of smooth curves of genus g , which can be realized as the quotient

$\mathcal{M}_g \cong \mathcal{X}_g/\text{Mod}_g$. Nielsen theory offers an alternative description $\text{Mod}_g \cong \text{Out}^+(\pi_g)$ of the mapping class group as the group of orientation preserving outer isomorphisms of π_g . The Torelli group consists of those outer automorphisms of π_g that act trivially on H . The *Torelli space*

$$\mathcal{T}_g := \mathcal{X}_g/T_g$$

can then be thought of as the moduli space of pairs $[C, \alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g]$, consisting of a smooth curve C of genus g and a symplectic basis of $H_1(C, \mathbb{Z})$.

Johnson [27] defined the surjective homomorphism

$$\tau : T_g \rightarrow \bigwedge^3 H/H,$$

where the $\text{Sp}_{2g}(\mathbb{Z})$ -equivariant injective map $H \hookrightarrow \bigwedge^3 H$ is given by $z \mapsto z \wedge \omega$, where $\omega := \alpha_1 \wedge \beta_1 + \dots + \alpha_g \wedge \beta_g \in \bigwedge^2 H$, with $(\alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g)$ being a symplectic basis of H . Then he proved that T_g is finitely generated and that the *Johnson kernel* defined by the exact sequence

$$1 \longrightarrow K_g \longrightarrow T_g \longrightarrow \bigwedge^3 H/H \longrightarrow 1,$$

is precisely the subgroup of Mod_g generated by Dehn twists along separated simple closed curves in Σ_g . Finally, Johnson [28] showed that the abelianization of T_g is given (up to 2-torsion) by the map τ , that is one has an isomorphism

$$\tau_* : H_1(T_g, \mathbb{Q}) \cong \bigwedge^3 H_{\mathbb{Q}}/H_{\mathbb{Q}},$$

which in particular implies that

$$b_1(T_g) = \binom{2g}{3} - 2g.$$

For $g \geq 4$, the Torelli group is 1-formal [24, 25] and $\mathcal{R}(T_g) = \{0\}$, see [15, Theorem 4.4]. Furthermore, Dimca, Hain and Papadima [16, Corollary 3.5] proved that $B(T_g)_{\mathbb{C}} = H_1(K_g, \mathbb{C})$ is a nilpotent module over $\mathbb{C}[T_g/T'_g]$. We deduce from Corollary 4.6 the following new estimate of nilpotency of the Alexander invariant of T_g .

Theorem 4.16. *If $g \geq 4$ and $I \subseteq \mathbb{C}[T_g/T'_g]$ denotes the augmentation ideal, then*

$$I^q \cdot B(T_g)_{\mathbb{C}} = 0 \text{ for } q \geq \binom{2g}{3} - 2g - 3.$$

More recently, it was shown in [19, Corollary 1.5] that the metabelian quotient T_g/T''_g is nilpotent for $g \geq 12$, and in particular it follows that Theorem 1.2 applies for the Torelli groups. It follows that $\text{vnc}(T_g/T''_g) \leq b_1(T_g) - 2$, which is the statement of Theorem 1.3 in the Introduction. Of course, the groups T_g and Mod_g are very far from being nilpotent. In fact, if $G \leq \text{Mod}_g$ is a subgroup, either G is virtually abelian, or it contains a non-abelian free subgroup, see [36].

4.6. The Torelli group of a free group. More generally, for an arbitrary finitely generated group G , we define its *Torelli group* $T(G) \subseteq \text{Out}(G)$ to be the subgroup of outer automorphisms of G consisting of automorphisms which induce the identity on $H_1(G, \mathbb{Z})$. The class of right-angled Artin groups, associated to finite simple graphs, is of considerable interest in geometric group theory. The groups G in this class interpolate between free abelian groups (corresponding to full graphs) and free groups (corresponding to discrete graphs). Of particular interest is the case when $G = F_g$ is a free group on g generators x_1, \dots, x_g . The corresponding Torelli group $T(F_g)$ denoted by OA_g , is thus defined by the following exact sequence

$$1 \longrightarrow \text{OA}_g \longrightarrow \text{Out}(F_g) \longrightarrow \text{GL}_g(\mathbb{Z}) \longrightarrow 1.$$

The Culler-Vogtmann *outer space* X_g of marked metric graphs of rank g can be thought of as the classifying space for OA_g , see [13]. The moduli space of graphs X_g / OA_g is (essentially) the moduli space $\mathcal{M}_g^{\text{tr}}$ of tropical curves of genus g , see [7] and references therein. In analogy with OA_g , one defines the subgroup IA_g of the group $\text{Aut}(F_g)$ of all automorphisms of the free group which induce the identity on $H_1(F_g, \mathbb{Z})$. Both IA_g and OA_g are finitely generated for $g \geq 3$ by [32], and there exist natural quotient maps $\text{IA}_g \twoheadrightarrow \text{OA}_g$.

It is known that for $g \geq 4$ one has $\mathcal{R}(\text{OA}_g) = \{0\}$, see [38, Theorem 9.7]. Moreover, the first Betti number of OA_g has been computed by Kawazumi [29], see also [38, Theorem 9.1] and the references therein and is given by

$$b_1(\text{OA}_g) = \frac{g(g+1)(g-2)}{2}.$$

Note [32] that IA_g is generated by the following set of *Magnus generators* of $\text{Aut}(F_g)$

$$\alpha_{ij} : x_i \mapsto x_j x_i x_j^{-1}, \quad x_\ell \mapsto x_\ell \text{ for } \ell \neq i, \text{ and}$$

$$\alpha_{ijk} : x_i \mapsto x_i \cdot [x_j, x_k], \quad x_\ell \mapsto x_\ell \text{ for } \ell \neq i,$$

where the indices are subject to the conditions $1 \leq i \neq j \leq g$ in the first case, respectively $1 \leq j < k \leq g$ and $i \neq j, k$, in the second case. One has $\frac{g^2(g-1)}{2} = b_1(\text{IA}_g)$ generators, see again [29]. When passing to OA_g , one notices the relation $\alpha_{1,\ell} \cdots \alpha_{g,\ell} = 1$ for $\ell = 1, \dots, g$ and one is left with $b_1(\text{IA}_g) - g = \frac{g(g+1)(g-2)}{2}$ generators, which explains the formula for $b_1(\text{OA}_g)$.

It is shown in [19, Corollary 1.5] that the metabelian quotients of IA_g are nilpotent for $g \geq 4$, so the same is true for the metabelian quotients of OA_g . Applying Theorem 1.4 with $G = \text{OA}_g$ we obtain the following result.

Theorem 4.17. *If $g \geq 4$ then the metabelian quotient $\text{OA}_g / \text{OA}_g''$ has virtual nilpotency class at most $\frac{g(g+1)(g-2)}{2} - 2$.*

Using Corollary 4.6 we also obtain an analogue of Theorem 4.16 for the Alexander invariant of OA_g .

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