DIVISORS ON $\mathcal{M}_{g,g+1}$ AND THE MINIMAL RESOLUTION CONJECTURE FOR POINTS ON CANONICAL CURVES

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Introduction

The Minimal Resolution Conjecture for points in projective space has attracted considerable attention in recent years, starting with the original [Lo1], [Lo2] and continuing most notably with [Ga], [BG], [Wa], [HS], [EP], [EPSW]. The purpose of this paper is to explain how a completely analogous problem can be formulated for sets of points on arbitrary varieties embedded in projective space, and then study in detail the case of curves. Similarly to the well-known analysis of syzygies of curves carried out by Green and Lazarsfeld ([GL1], [GL2], [GL3]), we divide our work into a study of resolutions of points on canonical curves and on curves of large degree. The central result of the paper states that the Minimal Resolution Conjecture is true on any canonical curve. In contrast, it always fails for curves embedded with large degree, although a weaker result, called the Ideal Generation Conjecture, holds also in this case. These results turn out to have surprisingly deep connections with the geometry of difference varieties in Jacobians, special divisors on moduli spaces of curves with marked points, and moduli spaces of stable bundles.

Let X be a projective variety over an algebraically closed field, embedded by a (not necessarily complete) linear series. We begin by formulating a general version of the Minimal Resolution Conjecture (MRC), in analogy with the case of $\mathbf{P}^n$, predicting how the Betti numbers of a general subset of points of X in the given embedding are related to the Betti numbers of X itself. More precisely (cf. Theorem 1.2 below), for a large enough general set of points Γ on X, the Betti diagram consisting of the graded Betti numbers $b_{i,j}(\Gamma)$ is obtained from the Betti diagram of X by adding two more nontrivial rows, at places well determined by the length of Γ. Recalling that the Betti diagram has the Betti number $b_{i,j}$ in the (j,i)-th position, and assuming that the two extra rows are indexed by $i = r-1$ and $i = r$, for some integer r, the MRC predicts that

$$b_{i+1,r-1}(\Gamma) \cdot b_{i,r}(\Gamma) = 0,$$

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i.e. at least one of the two Betti numbers on any "diagonal" is zero. As the difference $b_{i+1,r-1} - b_{i,r}$ can be computed exactly, this implies a precise knowledge of the Betti numbers in these two rows. Summing up, knowing the Betti diagram of Γ would be the same as knowing the Betti diagram of X. A subtle question is however to understand how the shape of the Betti diagram of X influences whether MRC is satisfied for points on X. An example illustrating this is given at the end of Section 1.

The Minimal Resolution Conjecture has been extensively studied in the case $X = \mathbf{P}^n$. The conjecture holds for $n \leq 4$ by results of Gaeta, Ballico and Geramita, and Walter (see [Ga], [BG] and [Wa], respectively). Moreover, Hirschowitz and Simpson proved in [HS] that it holds if the number of points is large enough with respect to n. However, the conjecture does not hold in general: it fails for every $n \geq 6$, $n \neq 9$ for almost $\sqrt{n}/2$ values of the number of points, by a result of Eisenbud, Popescu, Schreyer and Walter (see [EPSW]). We refer to [EP] and [EPSW] for a nice introduction and an account of the present status of the problem in this case.

The main body of the paper is dedicated to a detailed study of MRC in the case of curves. We will simply say that a curve *satisfies MRC* in a given embedding if MRC is satisfied by a general set of points Γ of any sufficiently large degree (for the precise numerical statements see Section 1). We will also sometimes say that MRC holds for a line bundle L if it holds for C in the embedding given by L. Our main result says that MRC holds in the most significant case, namely the case of canonically embedded nonhyperelliptic curves.

Theorem. *If C is a canonical curve, then C satisfies MRC.*

In contrast, under very mild assumptions on the genus, the MRC always fails in the case of curves of large degree, at well-determined spots in the Betti diagram (cf. Section 2 for precise details). The statement $b_{2,r-1} \cdot b_{1,r} = 0$, i.e. the case $i = 1$, does hold though; this is precisely the Ideal Generation Conjecture, saying that the minimal number of generators of I_Γ / I_X is as small as possible.

Theorem. *(a) If L is a very ample line bundle of degree $d \geq 2g$, then IGC holds for L.*
(b) If $g \geq 4$ and L is a line bundle of degree $d \geq 2g+16$, then there exists a value of γ such that $C \subseteq \boldsymbol{P}H^0(L)$ does not satisfy MRC for $i = \lfloor \frac{g+1}{2} \rfloor$. The same holds if $g \geq 15$ and $d \geq 2g+5$.

It is interesting to note that by the "periodicity" property of Betti diagrams of general points on curves (see [Mu] §2), the theorem above implies that on curves of high degree, MRC fails for sets of points of arbitrarily large length. This provides a very different picture from the case of projective space (cf. [HS]), where asymptotically the situation is as nice as possible.

We explain the strategy involved in the proof of these results in some detail, as it appeals to some new geometric techniques in the study of syzygy related questions. For simplicity we assume here that C is a smooth curve embedded in projective space by means of a complete linear series corresponding to a very ample line bundle L (but see §2 for more general statements). A well-known geometric approach, developed by Green and Lazarsfeld in the study of syzygies of curves (see [Lz1] for a survey), is to find vector bundle statements equivalent to the algebraic ones, via Koszul cohomology. This program can be carried out completely in the case of MRC, and for curves we get a particularly clean statement. Assume that M_L is the kernel of the evaluation map $H^0(L) \otimes \mathcal{O}_C \to L$ and $Q_L := M_L^*$. Then (cf. Corollary 1.8 below) MRC holds for a collection of $\gamma \geq g$ general points on C if and only if the following is true:

$$h^0(\wedge^i M_L \otimes \xi) = 0, \text{ for all } i \text{ and } \xi \in \mathrm{Pic}^{g-1+\lfloor \frac{di}{n} \rfloor}(C) \text{ general.} \quad (*)$$

Condition $(*)$ above is essentially the condition studied by Raynaud [Ra], related to the existence of theta divisors for semistable vector bundles. In the particular situation of $\wedge^i M_L$, with L a line bundle of large degree, it has been considered in [Po] in order to produce base points for the determinant linear series on the moduli spaces $SU_C(r)$ of semistable bundles of rank r and trivial determinant. A similar approach shows here the failure of condition $(*)$ (and so of MRC) for $i = [\frac{g+1}{2}]$. On the other hand, the fact that IGC holds is a rather elementary application of the Base Point Free Pencil Trick [ACGH] III §3.

The case of canonical curves is substantially more involved, but in the end one is rewarded with a positive answer. As above, it turns out that MRC is equivalent to the vanishing:

$$h^0(\wedge^i Q \otimes \xi) = 0, \text{ for all } i \text{ and } \ \xi \in \mathrm{Pic}^{g-2i-1}(C) \text{ general,}$$

where Q is the dual of the bundle M defined by the evaluation sequence:

$$0 \longrightarrow M \longrightarrow H^0(\omega_C) \otimes \mathcal{O}_C \longrightarrow \omega_C \longrightarrow 0.$$

As the slope of $\wedge^i Q$ is $2i \in \mathbb{Z}$, this is in turn equivalent to the fact that $\wedge^i Q$ has a theta divisor $\Theta_{\wedge^i Q} \in \mathrm{Pic}^{g-2i-1}(C)$. On a fixed curve, if indeed a divisor, $\Theta_{\wedge^i Q}$ will be identified as being precisely the difference variety $C_{g-i-1} - C_i \subseteq \mathrm{Pic}^{g-2i-1}(C)$ (cf. [ACGH] Ch.V.D), where C_n is the n-th symmetric product of C. This is achieved via a filtration argument and a cohomology class calculation similar to the classical Poincaré theorem (cf. Proposition 3.6). A priori though, on an arbitrary curve the nonvanishing locus $\{\xi \mid h^0(\wedge^i Q \otimes \xi) \neq 0\}$ may be the whole $\mathrm{Pic}^{g-2i-1}(C)$, in which case this identification is meaningless. We overcome this problem by working with all curves at once, that is by setting up a similar universal construction on the moduli space of curves with marked points $\mathcal{M}_{g,g+1}$. Here we slightly oversimplify the exposition in order to present the main idea, but for the precise technical details see Section 3. We essentially consider the "universal nonvanishing locus" in $\mathcal{M}_{g,g+1}$:

$$\mathcal{Z} = \{(C, x_1, \ldots, x_{g-i}, y_1, \ldots, y_{i+1}) | h^0(\wedge^i Q_C \otimes \mathcal{O}(x_1 + \ldots + x_{g-i} - y_1 - \ldots - y_{i+1})) \neq 0\}.$$

The underlying idea is that the difference line bundles $\mathcal{O}_C(x_1+\ldots+x_{g-i}-y_1-\ldots-y_{i+1})$ in fact cover the whole $\mathrm{Pic}^{g-2i-1}(C)$ (i.e. $C_{g-i}-C_{i+1} = \mathrm{Pic}^{g-2i-1}(C)$), and so for any given curve C, $\mathcal{Z}|_C$ is precisely the nonvanishing locus described above. The advantage of writing it in this form is that we are led to performing a computation on $\mathcal{M}_{g,g+1}$ rather than on a universal Picard, where for example one does not have a canonical choice of generators for the Picard group. A "deformation to hyperelliptic curves" argument easily implies that MRC holds for general canonical curves, so $\mathcal{Z}$ is certainly a divisor. We then show that $\mathcal{Z}$ is the degeneracy locus of a morphism of vector bundles of the same rank and compute its class using a Grothendieck-Riemann-Roch argument (cf. Proposition 3.11).

On the other hand, one can define an (a priori different) divisor D in $\mathcal{M}_{g,g+1}$ which is a global analogue of the preimage of $C_{g-i-1} - C_i$ in $C^{g-i} \times C^{i+1}$ via the difference map. It is convenient to see D as the locus of curves with marked points $(C, x_1, \ldots, x_{g-i}, y_1, \ldots, y_{i+1})$ having a $\mathfrak{g}^1_g$ which contains $x_1, \ldots, x_{g-i}$ in a fiber and $y_1, \ldots, y_{i+1}$ in a different fiber. An equivalent formulation of the discussion above is that $D \subseteq \mathcal{Z}$, and in order for MRC to hold for all canonical curves one should have precisely $D = \mathcal{Z}$. As we show that D is reduced (cf. Proposition 4.2), it suffices then to prove that the class of D coincides with that of $\mathcal{Z}$. To this end we consider the closure of D in the compactification $\overline{\mathcal{M}}_{g,g+1}$, where the corresponding boundary condition is defined by means of limit linear series. The computation of the class of D via this closure is essentially independent of the rest of the paper. It relies on degeneration and enumerative techniques in the spirit of [HaMu] and [EH1].

The results of both this and the computation of the class of $\mathcal{Z}$ are summarized in the following theorem. For the statement, we recall that $\mathrm{Pic}(\mathcal{M}_{g,n})_{\mathbb{Q}}$ is generated by the class λ of the Hodge bundle and the classes ψ_j, $1 \leq j \leq n$, where $\psi_j := c_1(p_j^*\omega)$, with ω the relative dualizing sheaf on the universal curve $\mathcal{C}_g \to \mathcal{M}_g$ and $p_j : \mathcal{M}_{g,n} \to \mathcal{C}_g$ the projection onto the j-th factor.

Theorem. *The divisors $\mathcal{Z}$ and D defined above have the same class in* $\mathrm{Pic}(\mathcal{M}_{g,g+1})_{\mathbb{Q}}$, *namely*

$$-\Big(\binom{g-1}{i} - 10\binom{g-3}{i-1}\Big)\lambda + \binom{g-2}{i}\Psi_x + \binom{g-2}{i-1}\Psi_y,$$

where $\Psi_x = \sum_{j=1}^{g-i} \psi_j$ and $\Psi_y = \sum_{j=g-i+1}^{g+1} \psi_j$. In particular $D = \mathcal{Z}$.

As mentioned above, this implies that $\wedge^i Q$ always has a theta divisor, for all i, so equivalently that MRC holds for an arbitrary canonical curve. We record the more precise identification of this theta divisor, which now follows in general.

Corollary. *For any nonhyperelliptic curve C, $\Theta_{\wedge^i Q} = C_{g-i-1} - C_i$.*

In this particular form, our result answers positively a conjecture of R. Lazarsfeld. It is worth mentioning that it also answers negatively a question that was raised in connection with [Po], namely if $\wedge^i Q$ provide base points for determinant linear series on appropriate moduli spaces of vector bundles.

The paper is structured as follows. In Section 1 we give some equivalent formulations of the Minimal Resolution Conjecture and we describe the vector bundle setup used in the rest of the paper. In Section 2 we treat the case of curves embedded with large degree, proving IGC and showing that MRC fails. Section 3 is devoted to the main result, namely the proof of MRC for canonical curves, and here is where we look at the relationship with difference varieties and moduli spaces of curves with marked points. The divisor class computation in $\overline{\mathcal{M}}_{g,g+1}$, on which part of the proof relies, is carried out in Section 4 by means of limit linear series.

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1. **Several formulations of the Minimal Resolution Conjecture**

Notations and conventions. We work over an algebraically closed field k which, unless explicitly mentioned otherwise, has arbitrary characteristic. Let V be a vector space over k with $\dim_k V = n+1$ and $S = Sym(V) \simeq k[X_0, \dots, X_n]$ the homogeneous coordinate ring of the corresponding projective space $\mathbf{P}V \simeq \mathbf{P}^n$.

For a finitely generated graded S-module N, the *Betti numbers* $b_{i,j}(N)$ of N are defined from the minimal free resolution $F_\bullet$ of N by

$$F_i = \oplus_{j \in \mathbb{Z}} S(-i-j)^{b_{i,j}(N)}.$$

The *Betti diagram* of N has in the (j,i)-th position the Betti number $b_{i,j}(N)$. The *regularity* reg(N) of $N \neq 0$ can be defined as the index of the last nontrivial row in the Betti diagram of N (see [Ei], 20.5 for the connection with the cohomological definition).

We will use the computation of Betti numbers via Koszul cohomology: $b_{i,j}(N)$ is the dimension over k of the cohomology of the following piece of the Koszul complex:

$$\wedge^{i+1} V \otimes N_{j-1} \longrightarrow \wedge^i V \otimes N_j \longrightarrow \wedge^{i-1} V \otimes N_{j+1}$$

(see [Gr] for details).

For an arbitrary subscheme $Z \subseteq \mathbf{P}^n$, we denote by $I_Z \subseteq S$ its saturated ideal and let $S_Z = S/I_Z$. We denote by P_Z and H_Z the Hilbert polynomial and Hilbert function of Z, respectively. The regularity reg(Z) of Z is defined to be the regularity of I_Z, if $Z \neq \mathbf{P}^n$, and 1 otherwise. Notice that with this convention, in the Betti diagram of Z, which by definition is the Betti diagram of S_Z, the last nontrivial row is always indexed by reg(Z) $- 1$.

For a projective variety X, a line bundle L on X, and a linear series $V \subseteq H^0(L)$ which generates L, we denote by M_V the vector bundle which is the kernel of the evaluation map

$$0 \longrightarrow M_V \longrightarrow V \otimes \mathcal{O}_X \xrightarrow{ev} L \longrightarrow 0.$$

When $V = H^0(L)$ we use the notation $M_L := M_V$. If C is a smooth curve of genus $g \geq 1$, and ω_C is the canonical line bundle, then M_C denotes the vector bundle M_{ω_C}. The dual vector bundles will be denoted by Q_V, Q_L and Q_C, respectively. Whenever there is no risk of confusion, we will simply write M and Q, instead of M_C and Q_C.

The Minimal Resolution Conjecture for points on embedded varieties. In this section $X \subseteq \mathbf{P}V \simeq \mathbf{P}^n$ is a fixed irreducible projective variety of positive dimension. We study the Betti numbers of a general set of γ points $\Gamma \subseteq X$. Since the Betti numbers are upper semicontinuous functions, for every positive integer γ, there is an open subset U_γ of $X^\gamma \setminus \cup_{p \neq q}\{x \colon x_p = x_q\}$ such that for all i and j, $b_{i,j}(\Gamma)$ takes its minimum value for $\Gamma \in U_\gamma$. Notice that as the regularity is bounded in terms of γ, we are concerned with finitely many Betti numbers. From now on, Γ *general* means $\Gamma \in U_\gamma$.

It is easy to determine the Hilbert function of a general set of points Γ in terms of the Hilbert function of X (see [Mu]). We have the following:

Proposition 1.1. *If $\Gamma \subseteq X$ is a general set of γ points, then*

$$H_\Gamma(t) = \min\{H_X(t), \gamma\}.$$

To determine the Betti numbers of a general set of points Γ is a much more subtle problem. If γ is large enough, then the Betti diagram of Γ looks as follows: in the upper part we have the Betti diagram of X and there are two extra nontrivial rows at the bottom. Moreover, the formula in Proposition 1.1 gives an expression for the differences of the Betti numbers in these last two rows. We record the formal statement in the following theorem and for the proof we refer to [Mu].

Theorem 1.2. *Assume that $\Gamma \subseteq X$ is a general set of γ points, with $P_X(r-1) \leq \gamma < P_X(r)$ for some $r \geq m+1$, where $m = \operatorname{reg} X$.*

(i) *For every i and $j \leq r-2$, we have $b_{i,j}(\Gamma) = b_{i,j}(X)$.*

(ii) *$b_{i,j}(\Gamma) = 0$, for $j \geq r+1$ and there is an i such that $b_{i,r-1}(\Gamma) \neq 0$.*

(iii) *For every $j \geq m$, we have*

$$b_{i,j}(\Gamma) = b_{i-1,j+1}(I_\Gamma/I_X) = b_{i-1,j+1}(\oplus_{l\geq 0} H^0(\mathcal{I}_{\Gamma/X}(l))).$$

(iv) *If $d = \dim X$, then for every $i \geq 0$, we have $b_{i+1,r-1}(\Gamma) - b_{i,r}(\Gamma) = Q_{i,r}(\gamma)$, where*

$$Q_{i,r}(\gamma) = \sum_{l=0}^{d-1} (-1)^l \binom{n-l-1}{i-l} \Delta^{l+1} P_X(r+l) - \binom{n}{i}(\gamma - P_X(r-1)).$$

We will focus our attention on the Betti numbers in the bottom two rows in the Betti diagram of Γ. The equation in Theorem 1.2 (iv) gives lower bounds for these numbers, namely $b_{i+1,r-1}(\Gamma) \geq \max\{Q_{i,r}(\gamma), 0\}$ and $b_{i,r}(\Gamma) \geq \max\{-Q_{i,r}(\gamma), 0\}$.

Definition 1.3. In analogy with the case $X = \mathbf{P}^n$ (see [Lo1] and [Lo2]), we say that the *Minimal Resolution Conjecture* (to which we refer from now on as MRC) holds for a fixed value of γ as above if for every i and every general set Γ, $b_{i+1,r-1}(\Gamma) = \max\{Q_{i,r}(\gamma), 0\}$ and $b_{i,r}(\Gamma) = \max\{-Q_{i,r}(\gamma), 0\}$. Equivalently, it says that

$$b_{i+1,r-1}(\Gamma) \cdot b_{i,r}(\Gamma) = 0 \text{ for all } i.$$

This conjecture has been extensively studied in the case $X = \mathbf{P}^n$, $L = \mathcal{O}_{\mathbf{P}^n}(1)$. It is known to hold for small values of n ($n = 2$, 3 or 4) and for large values of γ, depending on n, but not in general. In fact, it has been shown that for every $n \geq 6$, $n \neq 9$, MRC fails for almost $\sqrt{n}/2$ values of γ (see [EPSW], where one can find also a detailed account of the problem).

Note that the assertion in MRC holds obviously for $i = 0$. The first nontrivial case $i = 1$ is equivalent by Theorem 1.2 to saying that the minimal number of generators of I_Γ/I_X is as small as possible. This suggests the following:

Definition 1.4. We say that the *Ideal Generation Conjecture* (IGC, for short) holds for γ as above if for a general set of points $\Gamma \subseteq X$ of cardinality γ, we have $b_{2,r-1}(\Gamma) \cdot b_{1,r}(\Gamma) = 0$.

Example 1.5. ([Mu]) MRC holds for every X when $\gamma = P_X(r-1)$, since in this case $b_{i,r}(\Gamma) = 0$ for every i. Similarly, MRC holds for every X when $\gamma = P_X(r) - 1$, since in this case $b_{1,r-1} = 1$ and $b_{i,r-1}(\gamma) = 0$ for $i \geq 2$.

We derive now a cohomological interpretation of MRC. From now on we assume that X is nondegenerate, so that we have $V \subseteq H^0(\mathcal{O}_X(1))$. Using a standard Koszul cohomology argument, we can express the Betti numbers in the last two rows of the Betti diagram of Γ as follows.

Proposition 1.6. *With the above notation, for every $i \geq 0$ we have*

$$b_{i+1,r-1}(\Gamma) = h^0(\wedge^i M_V \otimes \mathcal{I}_{\Gamma/X}(r)),$$

$$b_{i,r}(\Gamma) = h^1(\wedge^i M_V \otimes \mathcal{I}_{\Gamma/X}(r)).$$

Proof. We compute the Betti numbers via Koszul cohomology, using the formula in Theorem 1.2 (iii).

Consider the complex:

$$\wedge^i V \otimes H^0(\mathcal{I}_{\Gamma/X}(r)) \xrightarrow{f} \wedge^{i-1} V \otimes H^0(\mathcal{I}_{\Gamma/X}(r+1)) \xrightarrow{h} \wedge^{i-2} V \otimes H^0(\mathcal{I}_{\Gamma/X}(r+2))$$

Since $H^0(\mathcal{I}_{\Gamma/X}(r-1)) = 0$, it follows that $\dim_k(\mathrm{Ker}\,\mathrm{f}) = b_{i+1,r-1}(\Gamma)$ and $\dim_k(\mathrm{Ker}\,h/\mathrm{Im}\,f) = b_{i,r}(\Gamma)$. The exact sequence

$$0 \longrightarrow M_V \longrightarrow V \otimes \mathcal{O}_X \longrightarrow \mathcal{O}_X(1) \longrightarrow 0$$

induces long exact sequences

$$0 \longrightarrow \wedge^i M_V \longrightarrow \wedge^i V \otimes \mathcal{O}_X \longrightarrow \wedge^{i-1} M_V \otimes \mathcal{O}_X(1) \longrightarrow 0. \quad (*)$$

By tensoring with $\mathcal{I}_{\Gamma/X}(r)$ and taking global sections, we get the exact sequence

$$H^0(\wedge^i M_V \otimes \mathcal{I}_{\Gamma/X}(r)) \hookrightarrow \wedge^i V \otimes H^0(\mathcal{I}_{\Gamma/X}(r)) \xrightarrow{f} \wedge^{i-1} V \otimes H^0(\mathcal{I}_{\Gamma/X}(r+1)).$$

This proves the first assertion in the proposition.

We have a similar exact sequence:

$$H^0(\wedge^{i-1} M_V \otimes \mathcal{I}_{\Gamma/X}(r+1)) \hookrightarrow \wedge^{i-1} V \otimes H^0(\mathcal{I}_{\Gamma/X}(r+1)) \xrightarrow{h} \wedge^{i-2} V \otimes H^0(\mathcal{I}_{\Gamma/X}(r+2)).$$

Therefore $b_{i,r}(\Gamma)$ is the dimension over k of the cokernel of

$$g \,:\, \wedge^i V \otimes H^0(\mathcal{I}_{\Gamma/X}(r)) \longrightarrow H^0(\wedge^{i-1} M_V \otimes \mathcal{I}_{\Gamma/X}(r+1)).$$

Using again the exact sequence $(*)$, by tensoring with $\mathcal{I}_{\Gamma/X}(r)$ and taking a suitable part of the long exact sequence, we get:

$$\wedge^i V \otimes H^0(\mathcal{I}_{\Gamma/X}(r)) \longrightarrow H^0(\wedge^{i-1} M_V \otimes \mathcal{I}_{\Gamma/X}(r+1)) \longrightarrow$$
$$H^1(\wedge^i M_V \otimes \mathcal{I}_{\Gamma/X}(r)) \longrightarrow \wedge^i V \otimes H^1(\mathcal{I}_{\Gamma/X}(r)).$$

Since $\mathrm{reg}\,\Gamma \leq \mathrm{r}+1$, we have $\mathrm{reg}\,\mathcal{I}_{\Gamma/\mathrm{X}} \leq \mathrm{r}+1$ and therefore $H^1(\mathcal{I}_{\Gamma/X}(r)) = 0$. From the above exact sequence we see that $\mathrm{Coker}\,g \simeq H^1(\wedge^i M_V \otimes \mathcal{I}_{\Gamma/X}(r))$, which proves the second assertion of the proposition. □

Remark 1.7. The higher cohomology groups $H^p(\wedge^i M_V \otimes \mathcal{I}_{\Gamma/X}(r))$, $p \geq 2$, always vanish. Indeed, using the exact sequences in the proof of the proposition, we get

$$h^p(\wedge^i M_V \otimes \mathcal{I}_{\Gamma/X}(r)) = h^1(\wedge^{i-p+1} M_V \otimes \mathcal{I}_{\Gamma/X}(r+p-1)) = b_{i-p+1,r+p-1}(\Gamma) = 0.$$

Therefore we have $Q_{i,r}(\gamma) = \chi(\wedge^i M_V \otimes \mathcal{I}_{\Gamma/X}(r))$ and MRC can be interpreted as saying that for general Γ, the cohomology of $\wedge^i M_V \otimes \mathcal{I}_{\Gamma/X}(r)$ is supported in cohomological degree either zero or one.

In the case of a curve C, MRC can be reformulated using Proposition 1.6 in terms of general line bundles on C. We will denote by $\lfloor x \rfloor$ and $\lceil x \rceil$ the integers defined by $\lfloor x \rfloor \leq x < \lfloor x \rfloor + 1$ and $\lceil x \rceil - 1 < x \leq \lceil x \rceil$.

Corollary 1.8. *Suppose that $C \subseteq \boldsymbol{P}V$ is a nondegenerate, integral curve of arithmetic genus g and degree d. We consider the following two statements:*

(i) *For every i and for a general line bundle $\xi \in \mathrm{Pic}^j(C)$, where $j = g - 1 + \lceil \frac{di}{n} \rceil$, we have $H^1(\wedge^i M_V \otimes \xi) = 0$.*

(ii) *For every i and for a general line bundle $\xi \in \mathrm{Pic}^j(C)$, where $j = g - 1 + \lfloor \frac{di}{n} \rfloor$, we have $H^0(\wedge^i M_V \otimes \xi) = 0$.*

Then MRC holds for C for every $\gamma \geq \max\{g, P_C(\operatorname{reg} X)\}$ if and only if both (i) *and* (ii) *are true. Moreover, if C is locally Gorenstein, then* (i) *and* (ii) *are equivalent.*

Proof. If $\gamma \geq g$, then for a general set Γ of γ points, $\mathcal{I}_{\Gamma/C}$ is a general line bundle on C of degree $-\gamma$. Since in this case $\mathcal{I}_{\Gamma/C}(r)$ is a general line bundle of degree $j = dr - \gamma$ and $d(r-1) + 1 - g \leq \gamma \leq dr + 1 - g$, Proposition 1.6 says that MRC holds for every $\gamma \geq \max\{g, P_C(\operatorname{reg} C)\}$ if and only if for every j such that $g - 1 \leq j \leq d + g - 1$ and for a general line bundle $\xi' \in \mathrm{Pic}^j(C)$, either $H^0(\wedge^i M_V \otimes \xi') = 0$ or $H^1(\wedge^i M_V \otimes \xi') = 0$.

Since $\dim C = 1$, we have $b_{i+1,r-1}(\Gamma) - b_{i,r}(\Gamma) = d\binom{n-1}{i} - (\gamma - P_C(r-1))\binom{n}{i}$. It follows immediately that $b_{i+1,r-1}(\Gamma) - b_{i,r}(\Gamma) \geq 0$ if and only if $j \geq g - 1 + di/n$.

The first statement of the corollary follows now from the fact that if E is a vector bundle on a curve and P is a point, then $H^0(E) = 0$ implies $H^0(E \otimes \mathcal{O}(-P)) = 0$ and $H^1(E) = 0$ implies $H^1(E \otimes \mathcal{O}(P)) = 0$. The last statement follows from Serre duality and the isomorphism $\wedge^i Q_V \simeq \wedge^{n-i} M_V \otimes \mathcal{O}_C(1)$. □

Remark 1.9. The corresponding assertion for IGC says that X satisfies IGC for every $\gamma \geq \max\{g, P_C(\operatorname{reg} C)\}$ if and only if both (i) and (ii) are true for $i = 1$. Note that if X is locally Gorenstein, then by Serre duality condition (ii) for $i = 1$ is equivalent to condition (i) for $i = n - 1$.

Remark 1.10. If C is a locally Gorenstein integral curve such that $d/n \in \mathbb{Z}$, then in order to check MRC for all $\gamma \geq \max\{g, P_C(\operatorname{reg} C)\}$, it is enough to check condition (i) in Corollary 1.8 only for $i \leq n/2$. Indeed, using Serre duality and Riemann-Roch, we see that the conditions for i and $n - i$ are equivalent.

In light of Corollary 1.8, we make the following:

Definition 1.11. If $C \subseteq \mathbf{P}\,V$ is a nondegenerate integral curve of arithmetic genus g and regularity m, we say that C satisfies MRC if a general set of γ points on C satisfies MRC for every $\gamma \geq \max\{g, P_C(m)\}$. If L is a very ample line bundle on a curve C as before, we say that L satisfies MRC if $C \subset \mathbf{P}\,H^0(L)$ satisfies MRC. Analogous definitions are made for IGC.

Example 1.12. (**Rational quintics in $\mathbf{P}^3$.**) We illustrate the above discussion in the case of smooth rational quintic curves in $\mathbf{P}^3$. We consider two explicit examples, the first when the curve lies on a (smooth) quadric and the second when it does not. Let X be given parametrically by $(u,v)\in \mathbf{P}^1 \longrightarrow (u^5,u^4v,uv^4,v^5)\in \mathbf{P}^3$, so that it lies on the quadric $X_0X_3=X_1X_2$. The Betti diagram of X is

$$\begin{array}{c|cccc}
\hline
0 & 1 & - & - & - \\
1 & - & 1 & - & - \\
2 & - & - & - & - \\
3 & - & 4 & 6 & 2
\end{array}$$

and if $\Gamma\subset X$ is a set of 28 points, then the Betti diagram of Γ is

$$\begin{array}{c|cccc}
\hline
0 & 1 & - & - & - \\
1 & - & 1 & - & - \\
2 & - & - & - & - \\
3 & - & 4 & 6 & 2 \\
4 & - & - & - & - \\
5 & - & 3 & 4 & 1 \\
6 & - & - & 2 & 2
\end{array}$$

As $b_{3,5}(\Gamma)=1$ and $b_{2,6}(\Gamma)=2$, we see that MRC is not satisfied by X for this number of points.

Let now Y be the curve given parametrically by $(u,v)\in \mathbf{P}^1 \longrightarrow (u^5+u^3v^2, u^4v-u^2v^3, uv^4, v^5)\in \mathbf{P}^3$. In this case Y does not lie on a quadric, and in fact, its Betti diagram is given by

$$\begin{array}{c|cccc}
\hline
0 & 1 & - & - & - \\
1 & - & - & - & - \\
2 & - & 4 & 3 & - \\
3 & - & 1 & 2 & 1
\end{array}$$

If $\Gamma'\subset Y$ is a set of 28 points, then the Betti diagram of Γ' is

$$\begin{array}{c|cccc}
\hline
0 & 1 & - & - & - \\
1 & - & - & - & - \\
2 & - & 4 & 3 & - \\
3 & - & 1 & 2 & 1 \\
4 & - & - & - & - \\
5 & - & 3 & 4 & - \\
6 & - & - & 1 & 2
\end{array}$$

which shows that MRC is satisfied for Y and this number of points.

These two examples show the possible behavior with respect to the MRC for smooth rational quintics in $\mathbf{P}^3$. The geometric condition of lying on a quadric translates into a condition on the splitting type of $M_V = \Omega_{\mathbf{P}^3}(1)|_X$. More precisely, it is proved in [EV] that if $X \subset \mathbf{P}^3$ is a smooth rational quintic curve, then X lies on a quadric if and only if we have $\Omega_{\mathbf{P}^3}(1)|_X \simeq \mathcal{O}_{\mathbf{P}^1}(-3) \oplus \mathcal{O}_{\mathbf{P}^1}(-1)^{\oplus 2}$ (the other possibility, which is satisfied by a general such quintic, is that $\Omega_{\mathbf{P}^3}(1)|_X \simeq \mathcal{O}_{\mathbf{P}^1}(-1) \oplus \mathcal{O}_{\mathbf{P}^1}(-2)^{\oplus 2}$). Corollary 1.8 explains therefore the behaviour with respect to MRC in the above examples.

2. Curves of large degree and a counterexample to MRC

In this section we assume that C is a smooth projective curve of genus g and L is a very ample line bundle on C. Our aim is to investigate whether C satisfies MRC, or at least IGC, for every $\gamma \geq g$, in the embedding given by the complete linear series $|L|$. As before, m will denote the regularity of C.

Example 2.1. If $g = 0$ or 1, then C satisfies MRC for all $\gamma \geq P_C(m)$ in every embedding given by a complete linear series (see [Mu], Proposition 3.1).

In higher genus we will concentrate on the study of MRC for canonical curves and curves embedded with high degree, in direct analogy with the syzygy questions of Green-Lazarsfeld (cf. [GL1], [GL2], [GL3]). The main conclusion of this section will be that, while IGC is satisfied in both situations, the high-degree embeddings always fail to satisfy MRC at a well-specified spot in the Betti diagram. This is in contrast with our main result, proved in §3, that MRC always holds for canonical curves, and the arguments involved here provide an introduction to that section. The common theme of the proofs is the vector bundle interpretation of MRC described in §1.

Review of filtrations for Q_L and Q [Lz1]. Here we recall a basic property of the vector bundles Q_L which will be essential for our arguments. Let L be a very ample line bundle on C of degree d, and recall from §1 that Q_L is given by the defining sequence

$$0 \longrightarrow L^{-1} \longrightarrow H^0(L)^* \otimes \mathcal{O}_C \longrightarrow Q_L \longrightarrow 0.$$

Assume first that L is non-special and $x_1, \ldots, x_d$ are the points of a general hyperplane section of $C \subseteq \mathbf{P}H^0(L)$. One shows (see e.g. [Lz1] §1.4) that there exists an exact sequence

$$(1) \qquad 0 \longrightarrow \bigoplus_{i \in \{1, \ldots, d-g-1\}} \mathcal{O}_C(x_i) \longrightarrow Q_L \longrightarrow \mathcal{O}_C(x_{d-g} + \ldots + x_d) \longrightarrow 0.$$

On the other hand, assuming that C is nonhyperelliptic and $L=\omega_C$, if $x_1,\dots,x_{2g-2}$ are the points of a general hyperplane section, the analogous sequence reads:

$$0 \longrightarrow \bigoplus_{i\in\{1,\dots,g-2\}} \mathcal{O}_C(x_i) \longrightarrow Q \longrightarrow \mathcal{O}_C(x_{g-1}+\dots+x_{2g-2}) \longrightarrow 0. \tag{2}$$

We start by looking at the case of curves embedded with large degree. The main results are summarized in the following:

Theorem 2.2. (a) *If L is a very ample line bundle of degree $d\geq 2g$, then IGC holds for L.*
(b) *If $g\geq 4$ and L is a line bundle of degree $d\geq 2g+10$, then there exists a value of γ such that $C\subseteq \boldsymbol{P}H^0(L)$ does not satisfy MRC for $i=\lfloor\frac{g+1}{2}\rfloor$. The same holds if $g\geq 14$ and $d\geq 2g+5$.*

Proof. (a) Let L be a very ample line bundle of degree $d\geq 2g$. By Corollary 1.8 and Serre duality, it is easy to see that IGC holds for L if:
(i) $h^1(Q_L\otimes\eta)=0$ for $\eta\in \mathrm{Pic}^{g-2}(C)$ general
and
(ii) $h^0(Q_L\otimes\eta)=0$ for $\eta\in \mathrm{Pic}^{g-3}(C)$ general.

Condition (i) is a simple consequence of the filtration (1). More precisely, if $x_1,\dots,x_d$ are the points of a general hyperplane section of $C\subseteq \mathbf{P}H^0(L)$, from the exact sequence

$$0 \longrightarrow \bigoplus_{i\in\{1,\dots,d-g-1\}} \mathcal{O}_C(x_i) \longrightarrow Q_L \longrightarrow \mathcal{O}_C(x_{d-g}+\dots+x_d) \longrightarrow 0.$$

we conclude that it would be enough to prove:

$$h^1(\eta(x_i))=0 \text{ and } h^1(\eta(x_{d-g}+\dots+x_d))=0$$

for $\eta\in \mathrm{Pic}^{g-2}(C)$ general. Now for every $i\in\{1,\dots,d-g-1\}$, $\eta(x_i)$ is a general line bundle of degree $g-1$, so $h^1(\eta(x_i))=0$. On the other hand deg $\eta(x_{d-g}+\dots+x_d)\geq 2g-1$, so clearly $h^1(\eta(x_{d-g}+\dots+x_d))=0$.

For condition (ii) one needs a different argument. By twisting the defining sequence of Q_L:

$$0\longrightarrow L^{-1}\longrightarrow H^0(L)^*\otimes\mathcal{O}_C\longrightarrow Q_L\longrightarrow 0$$

by $\eta\in \mathrm{Pic}^{g-3}(C)$ general and taking cohomology, we see that (ii) holds if and only if the map

$$\alpha^*: H^1(L^{-1}\otimes\eta)\to H^0(L)^*\otimes H^1(\eta)$$

is injective, or dually if and only if the cup-product map

$$\alpha: H^0(L)\otimes H^0(\omega_C\otimes\eta^{-1})\to H^0(L\otimes\omega_C\otimes\eta^{-1})$$

is surjective. We make the following:

Claim. $|\omega_C\otimes\eta^{-1}|$ *is a base point free pencil.*

Assuming this for the time being, one can apply the Base Point Free Pencil Trick (see [ACGH] III §3) to conclude that

$$\text{Ker } \alpha = H^0(L \otimes \omega_C^{-1} \otimes \eta).$$

But $L \otimes \omega_C^{-1} \otimes \eta$ is a general line bundle of degree $d - g - 1 \geq g - 1$ and so $h^1(L \otimes \omega_C^{-1} \otimes \eta) = 0$. By Riemann-Roch this means $h^0(L \otimes \omega_C^{-1} \otimes \eta) = d - 2g$. On the other hand $h^0(L) = d - g + 1$, $h^0(\omega_C \otimes \eta^{-1}) = 2$ and $h^0(L \otimes \omega_C \otimes \eta^{-1}) = d + 2$, so α must be surjective.

We are only left with proving the claim. Since $\eta \in \text{Pic}^{g-3}(C)$ is general, $h^0(\eta) = 0$, and so we easily get:

$$h^0(\omega_C \otimes \eta^{-1}) = h^1(\eta) = g - 1 - (g - 3) = 2.$$

Also, for every $p \in C$, $\eta(p) \in \text{Pic}^{g-2}(C)$ is general, hence still noneffective. Thus:

$$h^0(\omega_C \otimes \eta^{-1}(-p)) = h^1(\eta(p)) = g - 1 - (g - 2) = 1.$$

This implies that $|\omega_C \otimes \eta^{-1}|$ is base point free.

(b) Here we follow an argument in [Po] leading to the required nonvanishing statement. First note that it is clear from (1) that for every i with $1 \leq i \leq d - g - 1$ there is an inclusion

$$\mathcal{O}_C(x_1 + \ldots + x_i) \hookrightarrow \wedge^i Q_L,$$

where $x_1, \ldots, x_i$ are general points on C. This immediately implies that

$$h^0(\wedge^i Q_L \otimes \mathcal{O}_C(E_i - D_i)) \neq 0,$$

where E_i and D_i are general effective divisors on C of degree i. On the other hand we use the fact (see e.g. [ACGH] Ex. V. D) that every line bundle $\xi \in \text{Pic}^0(C)$ can be written as a difference

$$\xi = \mathcal{O}_C(E_{\lfloor \frac{g+1}{2} \rfloor} - D_{\lfloor \frac{g+1}{2} \rfloor}),$$

which means that

$$h^0(\wedge^{\lfloor \frac{g+1}{2} \rfloor} Q_L \otimes \xi) \neq 0, \ \forall \xi \in \text{Pic}^0(C) \text{ general }.$$

Now by Serre duality:

$$H^0(\wedge^i Q_L \otimes \xi) \cong H^1(\wedge^i M_L \otimes \omega_C \otimes \xi^{-1})^*,$$

so that Corollary 1.8 easily implies that C does not satisfy MRC for $i = \lfloor \frac{g+1}{2} \rfloor$ as long as $2g - 2 \geq g - 1 + \frac{di}{d-g}$. A simple computation gives then the stated conclusion. □

Remark 2.3. Motivation for the argument in (b) above was quite surprisingly provided by the study [Po] of the base locus of the determinant linear series on the moduli space $SU_C(r)$ of semistable bundles of rank r and trivial determinant on a curve C. In fact this argument produces explicit base points for the determinant linear series under appropriate numerical conditions.

Remark 2.4. The technique in Theorem 2.2 (b) can be extended to produce examples of higher dimensional varieties for which appropriate choices of γ force the failure of MRC for general sets of γ points. More precisely, the varieties in question are projective bundles $\mathbf{P}E \to C$ over a curve C, associated to very ample vector bundles E on C of arbitrary rank and large degree, containing sub-line bundles of large degree. Using the interpretation given in Proposition 1.6, the problem is reduced to a cohomological question about the exterior powers $\wedge^i M_E$, where M_E is defined analogously as the kernel of the evaluation map

$$0 \longrightarrow M_E \longrightarrow H^0(E) \otimes \mathcal{O}_X \longrightarrow E \longrightarrow 0.$$

This question is then treated essentially as above, and we do not enter into details. Unfortunately once a bundle E of higher rank is fixed, this technique does not seem to produce couterexamples for arbitrarily large values of γ, as in the case of line bundles. Such examples would be very interesting, in light of the asymptotically nice behavior of general points in $\mathbf{P}^n$ (cf. [HS]).

Finally we turn to the case of canonical curves with the goal of providing an introduction to the main result in Section 3. Let C be a nonhyperelliptic curve of genus g, $V = H^0(\omega_C)$ and $C \hookrightarrow \mathbf{P}V \simeq \mathbf{P}^{g-1}$ the canonical embedding. We note here that an argument similar to Theorem 2.2 (a) immediately implies IGC for C. This will be later subsumed in the general Theorem 3.1.

Proposition 2.5. *IGC holds for the canonical curve C.*

Proof. The argument is similar (and in fact simpler) to the proof of (ii) in Theorem 2.2 (a). In this case, again by interpreting Proposition 1.6 (se Remark 1.9 IGC holds if and only if

$$H^0(Q \otimes \xi) = 0 \text{ for } \xi \in \mathrm{Pic}^{g-3}(C) \text{ general.}$$

This is in turn equivalent to the surjectivity of the multiplication map:

$$H^0(\omega_C) \otimes H^0(\omega_C \otimes \xi^{-1}) \to H^0(\omega_C^{\otimes 2} \otimes \xi^{-1}),$$

which is again a quick application of the Base Point Free Pencil Trick. □

The geometric picture in the present case of canonical curves can be described a little more precisely. In fact, for $\xi \in \mathrm{Pic}^{g-3}(C)$, we have

$$\mu(Q \otimes \xi) = g - 1,$$

where $\mu(E) := \deg(E)/\mathrm{rk}(E)$ denotes in general the *slope* of the vector bundle E. By standard determinantal results, the subset

$$\Theta_Q := \{\xi \mid h^0(Q \otimes \xi) \neq 0\} \subseteq \mathrm{Pic}^{g-3}(C)$$

is either a divisor or the whole variety. The statement of IGC is then equivalent to saying that Θ_Q is indeed a divisor in $\mathrm{Pic}^{g-3}(C)$ (one says that Q *has a theta* *divisor*). A simple filtration argument based on the sequence (3) above shows that in fact

$$\Theta_Q = C_{g-2} - C := \{\mathcal{O}_C(p_1 + \ldots + p_{g-2} - q) \mid p_1, \ldots, p_{g-2}, q \in C\},$$

which has already been observed by Paranjape and Ramanan in [PR] A generalization of this observation to the higher exterior powers $\wedge^i Q$ will be the starting point for our approach to proving MRC for canonical curves in what follows.

3. MRC for canonical curves

In this section C will be a canonical curve, i.e. a smooth curve of genus g embedded in $\mathbf{P}^{g-1}$ by the canonical linear series $|\omega_C|$ (in particular C is not hyperelliptic). Our goal is to prove the following:

Theorem 3.1. *If C is a canonical curve, then C satisfies MRC.*

Remark 3.2. In fact, since C is canonically embedded, its regularity is $m = 3$, and as $g \geq 3$ we always have $P_C(m) = 5(g-1) \geq g$. Thus the statement means that MRC holds for every $\gamma \geq P_C(m)$.

The general condition required for a curve to satisfy MRC which was stated in Corollary 1.8 (see also Remark 1.10) takes a particularly clean form in the case of canonical embeddings. We restate it for further use.

Lemma 3.3. *Let C be a canonical curve. Then C satisfies MRC if and only if, for all $1 \leq i \leq \frac{g-1}{2}$ we have*

$$h^0(\wedge^i M \otimes \eta) = h^1(\wedge^i M \otimes \eta) = 0, \text{ for } \eta \in \mathrm{Pic}^{g+2i-1}(C) \text{ general},$$

or equivalently

$$(*)\ h^0(\wedge^i Q \otimes \xi) = h^1(\wedge^i Q \otimes \xi) = 0, \text{ for } \xi \in \mathrm{Pic}^{g-2i-1}(C) \text{ general}.$$

Remark 3.4. Note that $\mu(Q) = 2$, so $\mu(\wedge^i Q) = 2i \in \mathbb{Z}$. This means that the condition $(*)$ in Lemma 3.3 is equivalent to saying that $\wedge^i Q$ has a theta divisor (in $\mathrm{Pic}^{g-2i-1}(C)$), which we denote $\Theta_{\wedge^i Q}$. In other words, the set defined by

$$\Theta_{\wedge^i Q} := \{\xi \in \mathrm{Pic}^{g-2i-1}(C) \mid h^0(\wedge^i Q \otimes \xi) \neq 0\}$$

with the scheme structure of a degeneracy locus of a map of vector bundles of the same rank is an actual divisor as expected (cf. [ACGH] II §4).

Hyperelliptic curves. Note that the statement $(*)$ in Lemma 3.3 makes sense even for hyperelliptic curves. Again Q is the dual of M, where M is the kernel of the evaluation map for the canonical line bundle. Therefore we will say slightly abusively that MRC is satisfied for some smooth curve of genus $g \geq 2$ if $(*)$ is satisfied for all i, $1 \leq i \leq (g-1)/2$. In fact, the hyperelliptic case is the only one for which we can give a direct argument.

Proposition 3.5. *MRC holds for hyperelliptic curves.*

Proof. We show that for every i, $h^0(\wedge^i Q \otimes \xi) = 0$, if $\xi \in \mathrm{Pic}^{g-2i-1}(C)$ is general. Since C is hyperelliptic, we have a degree two morphism $f : C \to \mathbf{P}^1$ and if $L = f^*(\mathcal{O}_{\mathbf{P}^1}(1))$, then $\omega_C = L^{g-1}$. Therefore the morphism $\widetilde{f} : C \longrightarrow \mathbf{P}^{g-1}$ defined by ω_C is the composition of the Veronese embedding $\mathbf{P}^1 \hookrightarrow \mathbf{P}^{g-1}$ with f. Note that we have $M = \widetilde{f}^*(\Omega_{\mathbf{P}^{g-1}}(1))$.

Since on $\mathbf{P}^1$ we have the exact sequence:

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}^1}(-1)^{\oplus(g-1)} \longrightarrow H^0(\mathcal{O}_{\mathbf{P}^1}(g-1)) \otimes \mathcal{O}_{\mathbf{P}^1} \longrightarrow \mathcal{O}_{\mathbf{P}^1}(g-1) \longrightarrow 0,$$

we get $M \simeq (L^{-1})^{\oplus(g-1)}$. Therefore for every i, we have

$$\wedge^i Q \simeq (L^i)^{\oplus\binom{g-1}{i}}.$$

Now if $\xi \in \mathrm{Pic}^{g-2i-1}(C)$ is general, then $\xi \otimes L^i$ is a general line bundle of degree $g-1$ and so $h^0(\wedge^i Q \otimes \xi) = 0$. $\square$

Theta divisors and difference varieties for a fixed curve. We noted above that MRC is satisfied for C if and only if $\Theta_{\wedge^i Q}$ is a divisor. We now identify precisely what the divisor should be, assuming that this happens. (At the end of the day this will hold for all canonical curves.) Recall that by general theory, whenever a divisor, $\Theta_{\wedge^i Q}$ belongs to the linear series $|\binom{g-1}{i}\Theta|$, where we slightly abusively denote by Θ a certain theta divisor on $\mathrm{Pic}^{g-2i-1}(C)$ (more precisely Θ_N, where N is a $\binom{g-1}{i}$-th root of $\det(\wedge^i Q)$).

From now on we always assume that we are in this situation. The Picard variety $\mathrm{Pic}^{g-2i-1}(C)$ contains a *difference subvariety* $C_{g-i-1} - C_i$ defined as the image of the difference map

$$\phi : C_{g-i-1} \times C_i \longrightarrow \mathrm{Pic}^{g-2i-1}(C)$$

$$(x_1 + \ldots + x_{g-i-1}, y_1 + \ldots + y_i) \to \mathcal{O}_C(x_1 + \ldots + x_{g-i-1} - y_1 - \ldots - y_i).$$

The geometry of the difference varieties has interesting links with the geometry of the curve [Lz2] and [ACGH] (see below). The key observation is that our theta divisor is nothing else but the difference variety above.

Proposition 3.6. *For every smooth curve C of genus g, we have*

$$C_{g-i-1} - C_i \subseteq \Theta_{\wedge^i Q}.$$

Moreover, if C is nonhyperelliptic and $\Theta_{\wedge^i Q}$ is a divisor, then the above inclusion is an equality.

We start with a few properties of the difference varieties, which for instance easily imply that $C_{g-i-1} - C_i$ is a divisor. More generally, we study the difference variety $C_a - C_b$, $a \geq b$, defined analogously. Note that this study is suggested in a series of exercises in [ACGH] Ch.V.D and Ch.VI.A in the case $a = b$, but the formula in V.D-3 there giving the cohomology class of $C_a - C_a$ is unfortunately incorrect, as we first learned from R. Lazarsfeld. The results we need are collected in the following:

Proposition 3.7. *(a) Assume that $1 \le b \le a \le \frac{g-1}{2}$. Then the difference map:*

$$\phi : C_a \times C_b \longrightarrow C_a - C_b \subseteq \mathrm{Pic}^{a-b}(C)$$

is birational onto its image if C is nonhyperelliptic. When C is hyperelliptic, ϕ has degree $\binom{a}{b}\, 2^b$ onto its image.
(b) If C is nonhyperelliptic, the cohomology class $c_{a,b}$ of $C_a - C_b$ in $\mathrm{Pic}^{a-b}(C)$ is given by

$$c_{a,b} = \binom{a+b}{a} \theta^{g-a-b},$$

where θ is the class of a theta divisor.

Assuming this, the particular case $a = g - i - 1$ and $b = i$ quickly implies the main result.

Proof. (of Proposition 3.6) From Proposition 3.7 (b) we see that if C is nonhyperelliptic, then the class of $C_{g-i-1} - C_i$ is given by:

$$c_{g-i-1,i} = \binom{g-1}{i} \theta.$$

On the other hand, as $\Theta_{\wedge^i Q}$ is associated to the vector bundle $\wedge^i Q$, if it is a divisor, then its cohomology class is $\binom{g-1}{i}\theta$ (recall that $\Theta_{\wedge^i Q}$ has the same class as $\binom{g-1}{i}\Theta$). As in this case both $\Theta_{\wedge^i Q}$ and $C_{g-i-1} - C_i$ are divisors, in order to finish the proof of the proposition it is enough to prove the first statement.

To this end, we follow almost verbatim the argument in Theorem 2.2 (b). Namely, the filtration (2) in §2 implies that for every $i \ge 1$ there is an inclusion:

$$\mathcal{O}_C(x_1 + \ldots + x_i) \hookrightarrow \wedge^i Q,$$

where $x_1, \ldots , x_i$ are general points on C. This means that

$$h^0(\wedge^i Q \otimes \mathcal{O}_C(E_{g-i-1} - D_i)) \neq 0$$

for all general effective divisors E_{g-i-1} of degree $g-i-1$ and D_i of degree i, which gives the desired inclusion. □

We are left with proving Proposition 3.7. This follows by more or less standard arguments in the study of Abel maps and Poincaré formulas for cohomology classes of images of symmetric products.

Proof. (of Proposition 3.7) (a) This is certainly well known (cf. [ACGH] Ch.V.D), and we do not reproduce the proof here.
(b) Assume now that C is nonhyperelliptic, so that

$$\phi : C_a \times C_b \longrightarrow C_a - C_b$$

is birational onto its image. For simplicity we will map everything to the Jacobian of C, so fix a point $p_0 \in C$ and consider the commutative diagram:

$$\begin{tikzcd} & C^{a+b} \arrow[dl, "\psi"'] \arrow[drr, "\alpha"] & & \\ C_a \times C_b \arrow[r, "\phi"] & C_a - C_b \arrow[rr, "-(a-b)p_0"] & & J(C) \end{tikzcd}$$

where C^{a+b} is the $(a+b)$-th cartesian product of the curve and the maps are either previously defined or obvious. We will in general denote by $[X]$ the fundamental class of the compact variety X. Since ψ clearly has degree $a! \cdot b!$, and since ϕ is birational by (a), we have:

$$\alpha_*[C^{a+b}] = a! \cdot b! \cdot c_{a,b}.$$

This means that it is in fact enough to prove that $\alpha_*[C^{a+b}] = (a+b)! \cdot \theta^{g-a-b}$, and note that $(a+b)! \cdot \theta^{g-a-b}$ is the same as the class $u_*[C^{a+b}]$, where u is the usual Abel map:

$$u : C^{a+b} \longrightarrow J(C)$$
$$(x_1, \ldots, x_{a+b}) \to \mathcal{O}_C(x_1 + \ldots + x_{a+b} - (a+b)p_0).$$

The last statement is known as Poincaré's formula (see e.g. [ACGH] I §5). We are now done by the following lemma, which essentially says that adding or subtracting points is the same when computing cohomology classes. □

Lemma 3.8. *Let $u, \alpha : C^{a+b} \longrightarrow J(C)$ defined by:*

$$u(x_1, \ldots, x_{a+b}) = \mathcal{O}_C(x_1 + \ldots + x_{a+b} - (a+b)p_0)$$

and

$$\alpha(x_1, \ldots, x_{a+b}) = \mathcal{O}_C(x_1 + \ldots + x_a - x_{a+1} - \ldots - x_{a+b} - (a-b)p_0).$$

Then $u_*[C^{a+b}] = \alpha_*[C^{a+b}] \in H^{2(g-a-b)}(J(C), \mathbb{Z})$.

Proof. For simplicity, in this proof only, we will use additive divisor notation, although we actually mean the associated line bundles. Consider the auxiliary maps:

$$u_0 : C^{a+b} \longrightarrow J(C)^{a+b}$$
$$(x_1, \ldots, x_{a+b}) \to (x_1 - p_0, \ldots, x_{a+b} - p_0),$$
$$\alpha_0 : C^{a+b} \longrightarrow J(C)^{a+b}$$
$$(x_1, \ldots, x_{a+b}) \to (x_1 - p_0, \ldots, x_a - p_0, p_0 - x_{a+1}, \ldots, p_0 - x_{a+b})$$

and the addition map:

$$a : J(C)^{a+b} \longrightarrow J(C)$$
$$(\xi_1, \ldots, \xi_{a+b}) \to \xi_1 + \ldots + \xi_{a+b}.$$

Then one has:

$$u = au_0 \text{ and } \alpha = a\alpha_0 = a\mu u_0,$$

where μ is the isomorphism

$$\mu : J(C)^{a+b} \longrightarrow J(C)^{a+b}$$

$$(\xi_1, \dots, \xi_{a+b}) \to (\xi_1, \dots, \xi_a, -\xi_{a+1}, \dots, -\xi_{a+b}).$$

Now the statement follows from the more general fact that μ induces an isomorphism on cohomology. This is in turn a simple consequence of the fact that the involution $x \to -x$ induces the identity on $H^1(J(C), \mathbb{Z})$. $\square$

Remark 3.9. The equality in Proposition 3.6 holds set-theoretically also for hyperelliptic curves. Indeed, we have the inclusion $C_{g-i-1} - C_i \subseteq \Theta_{\wedge^i Q}$, and we have seen in the proof of Proposition 3.5 that $\Theta_{\wedge^i Q}$ is irreducible (with the reduced structure, it is just a translate of the usual theta divisor).

General canonical curves. Since we have seen that MRC holds for hyperelliptic curves, a standard argument shows that it holds for general canonical curves. In fact, our previous result about the expected form of the theta divisors $\Theta_{\wedge^i Q}$ allows us to say something more precise about the set of curves in $\mathcal{M}_g$ which might not satisfy MRC.

Proposition 3.10. *For every i, the set of curves $\{[C] \in \mathcal{M}_g \mid \Theta_{\wedge^i Q_C} = \mathrm{Pic}^{g-2i-1}(C)\}$ is either empty or has pure codimension one. In particular, the same assertion is true for the set of curves in $\mathcal{M}_g$ which do not satisfy MRC.*

Proof. As the arguments involved are standard we will just sketch the proof.

Start by considering, for a given $d \geq 2g+1$, the Hilbert scheme $\mathcal{H}$ of curves in $\mathbf{P}^{d-g}$ with Hilbert polynomial $P(T) = dT + 1 - g$ and $\mathcal{U} \subseteq \mathcal{H}$ the open subset corresponding to smooth connected nondegenerate curves.

Let $f : \mathcal{Z} \longrightarrow \mathcal{U}$ be the universal family over $\mathcal{U}$, which is smooth of relative dimension 1, and $\omega_{\mathcal{Z}/\mathcal{U}} \in \mathrm{Pic}(\mathcal{Z})$ the relative cotangent bundle. By base change there is an exact sequence

$$0 \longrightarrow \mathcal{Q}^\vee \longrightarrow f^* f_* \omega_{\mathcal{Z}/\mathcal{U}} \longrightarrow \omega_{\mathcal{Z}/\mathcal{U}} \longrightarrow 0,$$

where $\mathcal{Q}$ is a vector bundle on $\mathcal{Z}$ such that if $u \in \mathcal{U}$ corresponds to a curve $C = \mathcal{Z}_u$ (in a suitable embedding), then $\mathcal{Q}|_{\mathcal{Z}_u} \simeq Q_C$.

The usual deformation theory arguments show that $\mathcal{H}$ is smooth and has dimension $(d-g+1)^2 + 4(g-1)$. Moreover, the universal family $\mathcal{Z}$ defines a surjective morphism $\pi : \mathcal{U} \longrightarrow \mathcal{M}_g$ whose fibers are irreducible and have dimension $(d-g+1)^2 + g - 1$. It is immediate to see from this that $\mathcal{U}$ is connected.

Fix now l such that $d = (g-2i-1) + l(2g-2) \geq 2g+1$. Consider $\mathcal{U}$ and $\mathcal{Z}$ as above and let $\mathcal{F} := \wedge^i \mathcal{Q} \otimes \omega_{\mathcal{Z}/\mathcal{U}}^{-l} \otimes p^* \mathcal{O}_{\mathbf{P}^{d-g}}(1)$, where p is the composition of the inclusion $\mathcal{Z} \hookrightarrow \mathcal{U} \times \mathbf{P}^{d-g}$ and the projection onto the second factor.

We consider also the closed subset of $\mathcal{U}$:

$$\mathcal{D}_1 = \{u \in \mathcal{U} \mid h^0(\mathcal{F}|_{\mathcal{Z}_u}) \geq 1\}.$$

It is clear by definition that $\pi^{-1}([C]) \subseteq \mathcal{D}_1$ if and only if $\Theta_{\wedge^i Q_C} = \mathrm{Pic}^{g-2i-1}(C)$. In particular, Proposition 3.5 implies that $\mathcal{D}_1 \neq \mathcal{U}$.

$\mathcal{D}_1$ is the degeneracy locus of a morphism between two vector bundles of the same rank. Indeed, if $H \subset \mathbf{P}^{d-g}$ is a hyperplane, $\widetilde{H} = p^{-1}H$, and $r \gg 0$, then $\mathcal{D}_1$ is the degeneracy locus of

$$f_*(\mathcal{F} \otimes \mathcal{O}_{\mathcal{Z}}(r\widetilde{H})) \longrightarrow f_*(\mathcal{F} \otimes \mathcal{O}_{r\widetilde{H}}(r\widetilde{H})).$$

Note that these are both vector bundles of rank $rd\binom{g-1}{i}$ (we use base change and the fact that by Corollary 3.5 in [PR], for every smooth curve C, the bundle Q_C is semistable). We therefore conclude that $\mathcal{D}_1$ is a divisor on $\mathcal{U}$.

On the other hand, it is easy to see that the set

$$\mathcal{D}_2 = \{u \in \mathcal{U} \mid \mathcal{O}_{\mathcal{Z}_u}(1) \otimes \omega_{\mathcal{Z}_u}^{-l} \in (\mathcal{Z}_u)_{g-i-1} - (\mathcal{Z}_u)_i\}$$

is closed. Moreover, Proposition 3.6 (see also Remark 3.9) shows that $\mathcal{D}_2 \subseteq \mathcal{D}_1$, and if $\pi^{-1}([C]) \not\subseteq \mathcal{D}_1$, then $\pi^{-1}([C]) \cap \mathcal{D}_1 = \pi^{-1}([C]) \cap \mathcal{D}_2$.

Let $\mathcal{S}$ be the set of irreducible components of $\mathcal{D}_1$ which are not included in $\mathcal{D}_2$. Using the fact that π has irreducible fibers, all of the same dimension, it is easy to see that if $Y \in \mathcal{S}$, then $\pi(Y)$ is closed in $\mathcal{M}_g$, that it is in fact a divisor, and $Y = \pi^{-1}(\pi(Y))$. Moreover, the locus of curves in $\mathcal{M}_g$ for which $\Theta_{\wedge^i Q}$ is not a divisor is $\cup_{Y \in \mathcal{S}} \pi(Y)$, which proves the proposition. $\square$

The class of the degeneracy locus on $\mathcal{M}_{g,g+1}$. We first fix the notation. We will denote by $\mathcal{M}_g^0$ the open subset of $\mathcal{M}_g$ which corresponds to curves with no nontrivial automorphisms. From now on we assume that $g \geq 4$, since for $g = 3$ MRC is equivalent to IGC, which is the content of Proposition 2.5. Thus $\mathcal{M}_g^0$ is nonempty and its complement has codimension $g - 2 \geq 2$ (see [HaMo] pg.37), so working with this subset will not affect the answers we get for divisor class computations on $\mathcal{M}_g$ or $\mathcal{M}_{g,n}$.

In this case we have a universal family over $\mathcal{M}_g^0$ denoted by $\mathcal{C}_g^0$ and for every $n \geq 1$, the open subset of $\mathcal{M}_{g,n}$ lying over $\mathcal{M}_g^0$ (which we denote by $\mathcal{M}_{g,n}^0$) is equal to the fiber product $(\times_{\mathcal{M}_g^0} \mathcal{C}_g^0)^n$ minus all the diagonals. We assume that $n \geq g+1$.

Consider the following cartezian diagram:

$$\begin{tikzcd} \mathcal{X} \arrow[r, "q"] \arrow[d, "f"] & \mathcal{M}_{g,n}^0 \arrow[d, "h"] \\ \mathcal{C}_g^0 \arrow[r, "p"] & \mathcal{M}_g^0 \end{tikzcd}$$

in which all the morphisms and varieties are smooth and p (hence also q) is proper.

Let $\omega \in \mathrm{Pic}(\mathcal{C}_g^0)$ be the relative canonical line bundle for p, $E = p_*(\omega)$ the Hodge vector bundle and $\mathcal{Q}$ the rank $g-1$ vector bundle on $\mathcal{C}_g^0$ such that $\mathcal{Q}^\vee$ is the kernel of the evaluation map $p^*E \longrightarrow \omega$. For every $[C] \in \mathcal{M}_g^0$, we have $\mathcal{Q}|_{p^{-1}([C])} \simeq Q_C$.

The projection on the j-th factor $p_j : \mathcal{M}_{g,n}^0 \longrightarrow \mathcal{C}_g^0$ induces a section $q_j : \mathcal{M}_{g,n}^0 \longrightarrow \mathcal{X}$ of q. If $E_j = \mathrm{Im}(q_j)$, then E_j is a relative divisor over $\mathcal{M}_{g,n}^0$. Consider the following vector bundle on $\mathcal{X}$:

$$\mathcal{E} = \wedge^i f^* \mathcal{Q}\left(\sum_{j=1}^{g-i} E_j - \sum_{j=g-i+1}^{g+1} E_j\right)$$

and let $\mathcal{Z} = \{u \in \mathcal{M}_{g,n}^0 | \ h^0(\mathcal{E}|_{\mathcal{X}_u}) \geq 1\}$.

The algebraic set $\mathcal{Z}$ comes equipped with a natural stucture of degeneracy locus. Suppose for example that Y is a sum of m divisors E_j (possibly with repetitions), where $m \gg 0$. In this case, $\mathcal{Z}$ is the degeneracy locus of the morphism

$$\mathcal{F} := q_*(\mathcal{E} \otimes \mathcal{O}_{\mathcal{X}}(Y)) \longrightarrow \mathcal{F}' := q_*(\mathcal{E} \otimes \mathcal{O}_{\mathcal{X}}(Y)|_Y).$$

This scheme structure does not depend on the divisor Y we have chosen. In fact, it is the universal subscheme over which the 0-th Fitting ideal of the first higher direct image of $\mathcal{E}$ becomes trivial (see e.g. [ACGH] Ch.IV §3 for the proof of an analogous property). Note that $\mathcal{Z}$ is a divisor and not the whole space, since by Proposition 3.10 we know that for a general curve C, there is $\xi \in \mathrm{Pic}^{g-2i-1}(C)$ such that $H^0(\wedge^i Q_C \otimes \xi) = 0$ (note also that the difference map $C^{g-i} \times C^{i+1} \longrightarrow \mathrm{Pic}^{g-2i-1}(C)$ is surjective, cf. [ACGH] Ch.V.D).

We will use the notation $\lambda = c_1(h^*(E))$, $\psi_j = c_1(p_j^*(\omega))$, and $\Psi_x = \sum_{j=1}^{g-i} \psi_j$ and $\Psi_y = \sum_{j=g-i+1}^{g+1} \psi_j$. It is well known that λ together with ψ_j, $1 \leq j \leq n$, form a basis for $\mathrm{Pic}(\mathcal{M}_{g,n}^0)_{\mathbb{Q}}$.

Proposition 3.11. *With the above notation, for every $n \geq g+1$, the class of $\mathcal{Z}$ in $\mathrm{Pic}(\mathcal{M}_{g,n}^0)_{\mathbb{Q}}$ is given by*

$$(3) \qquad [\mathcal{Z}] = -\left(\binom{g-1}{i} - 10\binom{g-3}{i-1}\right)\lambda + \binom{g-2}{i}\Psi_x + \binom{g-2}{i-1}\Psi_y.$$

Proof. Note that the pull-back of divisors induced by the projection to the first $(g+1)$ components induces injective homomorphisms $\mathrm{Pic}(\mathcal{M}_{g,g+1}^0)_{\mathbb{Q}} \hookrightarrow \mathrm{Pic}(\mathcal{M}_{g,n}^0)_{\mathbb{Q}}$. From the universality of the scheme structure on $\mathcal{Z}$ it follows that the computation of $c_1(\mathcal{Z})$ is independent of n. Therefore we may assume that n is large enough, so that in defining the scheme structure of $\mathcal{Z}$ as above, we may take $Y = \sum_{j=g+2}^n E_j$. We introduce also the notation $\Psi_z = \sum_{j=g+2}^n \psi_j$.

As a degeneracy locus, the class of $\mathcal{Z}$ is given by $c_1(\mathcal{F}') - c_1(\mathcal{F})$. It is clear that we have $E_j \cap E_l = \emptyset$ if $j \neq l$ and via $E_j \simeq \mathcal{M}_{g,n}^0$, we have $\mathcal{O}_{E_j}(-E_j) \simeq p_j^*(\omega)$.

Since $\mathcal{Q}^{\vee}$ is the kernel of the evaluation map for ω, we get

$$c_1(f^*Q) = f^*(c_1(\omega)) - q^*(\lambda). \tag{4}$$

Before starting the computation of $c_1(\mathcal{F})$ and $c_1(\mathcal{F}')$, we record the following well-known formulas for Chern classes.

Lemma 3.12. *Let R be a vector bundle of rank n on a variety X and $L \in \mathrm{Pic}(X)$.*

(i) $c_1(R \otimes L) = c_1(R) + nc_1(L)$.

(ii) $c_2(R \otimes L) = c_2(R) + (n-1)c_1(R)c_1(L) + \binom{n}{2}c_1(L)^2$.

(iii) $c_1(\wedge^i R) = \binom{n-1}{i-1}c_1(R)$, *if $n \geq 2$ and $1 \leq i \leq n$.*

(iv) $c_2(\wedge^i R) = \frac{1}{2}\binom{n-1}{i-1}\left(\binom{n-1}{i-1} - 1\right)c_1(R)^2 + \binom{n-2}{i-1}c_2(R)$, *if $n \geq 3$ and $1 \leq i \leq n$.*

From the previous discussion and Lemma 3.12 (i) and (iii), we get

$$c_1(q_*(\mathcal{E} \otimes \mathcal{O}_{E_j}(Y))) = -\binom{g-2}{i-1}\lambda - \binom{g-2}{i}\psi_j,$$

from which we deduce

$$c_1(\mathcal{F}') = -(n-g-1)\binom{g-2}{i-1}\lambda - \binom{g-2}{i}\Psi_z.$$

In order to compute $c_1(\mathcal{F})$, we apply the Grothendieck-Riemann-Roch formula for q and $\mathcal{E}(Y)$ (see [Fu] 15.2). Note that the varieties are smooth and q is smooth and proper. Moreover, since we assume $n \gg 0$, we have $R^j q_*(\mathcal{E}(Y)) = 0$, for $j \geq 1$. Therefore we get

$$ch(q_*(\mathcal{E}(Y))) = q_*(ch(\mathcal{E}(Y)) \cdot td(f^*\omega^{-1})).$$

From this we deduce

$$(*)\; c_1(\mathcal{F}) = q_*\left(\frac{1}{2}c_1(\mathcal{E}(Y))^2 - c_2(\mathcal{E}(Y)) - \frac{1}{2}f^*c_1(\omega)\cdot c_1(\mathcal{E}(Y)) + \frac{1}{12}\binom{g-1}{i}f^*c_1(\omega)^2\right).$$

We compute now each of the classes involved in the above equation. In order to do this we need to know how to make the push-forward of the elementary classes on $\mathcal{X}$. We list these rules in the following:

Lemma 3.13. *With the above notation, we have*

(i) $q_*(f^*c_1(\omega)^2) = 12\lambda$.

(ii) $q_*(q^*\lambda \cdot f^*c_1(\omega)) = (2g-2)\lambda$.

(iii) $q_*q^*(\lambda^2) = 0$.

(iv) $q_*(c_1(E_j) \cdot q^*\lambda) = \lambda$.

(v) $q_*(c_1(E_j) \cdot f^*c_1(\lambda)) = \psi_j$.

(vi) $q_*q^*c_2(h^*E) = 0$.

(vii) $q_*(c_1(E_j)^2) = -\psi_j$.

Proof of Lemma 3.13. The proof of (i) is analogous to that of the relation $p_*(c_1(\omega)^2) = 12\lambda$ (see [HaMo] §3E). The other formulas are straightforward. $\square$

Using Lemma 3.12 (i) and (iii) and the formula (4) for $c_1(f^*Q)$, we deduce that

$$c_1(\mathcal{E}(Y)) = \binom{g-2}{i-1}(f^*c_1(\omega) - q^*\lambda) + \binom{g-1}{i}\left(\sum_{j=1}^{g-i} c_1(E_j) - \sum_{j=g-i+1}^{g+1} c_1(E_j) + \sum_{j=g+2}^{n} c_1(E_j)\right).$$

Applying Lemma 3.13, we get

$$q_*(c_1(\mathcal{E}(Y))^2/2) = \left((8-2g)\binom{g-2}{i-1}^2 - (n-2i-2)\binom{g-2}{i-1}\binom{g-1}{i}\right)\lambda +$$
$$\binom{g-2}{i-1}\binom{g-1}{i}(\Psi_x - \Psi_y + \Psi_z) - \frac{1}{2}\binom{g-1}{i}^2(\Psi_x + \Psi_y + \Psi_z).$$

From the above formula for $c_1(\mathcal{E}(Y))$ and Lemma 3.13, we get

$$q_*\left(-\frac{1}{2}f^*c_1(\omega)\cdot c_1(\mathcal{E}(Y))\right) = (g-7)\binom{g-2}{i-1}\lambda - \frac{1}{2}\binom{g-1}{i}(\Psi_x - \Psi_y + \Psi_z).$$

Lemma 3.13 (i) gives

$$q_*\left(\frac{1}{12}\binom{g-1}{i}f^*c_1(\omega)^2\right) = \binom{g-1}{i}\lambda.$$

From the defining exact sequence of $\mathcal{Q}^\vee$ we compute

$$c_2(f^*\mathcal{Q}) = q^*c_2(h^*E) + f^*c_1(\omega)\cdot(f^*c_1(\omega) - q^*\lambda).$$

Using now Lemma 3.12 (ii) and (iv) and Lemma 3.13, we deduce

$$q_*c_2(\mathcal{E}(Y)) = \left((8-2g)\binom{g-2}{i-1}\left(\binom{g-2}{i-1} - 1\right) + (14-2g)\binom{g-3}{i-1}\right)\lambda -$$
$$(n-2i-2)\binom{g-2}{i-1}\left(\binom{g-1}{i} - 1\right)\lambda + \binom{g-2}{i-1}\left(\binom{g-1}{i} - 1\right)(\Psi_x - \Psi_y + \Psi_z) -$$
$$\frac{1}{2}\binom{g-1}{i}\left(\binom{g-1}{i} - 1\right)(\Psi_x + \Psi_y + \Psi_z).$$

Using these formulas and equation $(*)$, we finally obtain

$$c_1(\mathcal{F}) = \left((2g-14)\binom{g-3}{i-1} - (n+g-2i-3)\binom{g-2}{i-1} + \binom{g-1}{i}\right)\lambda -$$

$$\binom{g-2}{i}(\Psi_x+\Psi_z)-\binom{g-2}{i-1}\Psi_y.$$

Since the class of $\mathcal{Z}$ is equal with $c_1(\mathcal{F}')-c_1(\mathcal{F})$, we deduce the statement of the proposition. $\square$

Proof of the main result. We introduce next a divisor D on $\mathcal{M}_{g,g+1}$ which is a global analogue of the preimage of $C_{g-i-1}-C_i$ under the difference map $C^{g-i}\times C^{i+1}\to \mathrm{Pic}^{g-2i-1}(C)$. This is motivated by Proposition 3.6, and our goal is roughly speaking to prove a global version of that result.

Definition 3.14. For $g\geq 3$ and $1\leq i\leq \frac{g-1}{2}$ we define the divisor D on $\mathcal{M}_{g,g+1}$ to be the locus of smooth pointed curves $(C,x_1,\ldots,x_{g-i},y_1,\ldots y_{i+1})$ having a linear series $\mathfrak{g}^1_g$ containing $x_1+\cdots+x_{g-i}$ in a fiber and $y_1+\cdots+y_{i+1}$ in another fiber. Note that this means that we can in fact write the line bundle $\mathcal{O}_C(x_1+\ldots+x_{g-i}-y_1-\ldots-y_{i+1})$ as an element in $C_{g-i-1}-C_i$.

We consider in what follows the divisor $\overline{\mathcal{Z}}$, the closure in $\mathcal{M}_{g,g+1}$ of the divisor $\mathcal{Z}$ studied above (we take now $n=g+1$, but as we mentioned, this does not affect the formula for its class). In Section 4 we prove that D is reduced and that $D\equiv_{\mathbb{Q}-lin}\overline{\mathcal{Z}}$ (cf. Theorem 4.1). This being granted we are in a position to complete the proof of Theorem 3.1:

Proof of Theorem 3.1. Note that for $g=3$ our assertion is just the statement of Proposition 2.5. Thus we can assume $g\geq 4$. As mentioned above D is reduced, and from Proposition 3.6 we see that $\mathrm{supp}(D)\subseteq\mathrm{supp}(\overline{\mathcal{Z}})$. We get that $\overline{\mathcal{Z}}-D$ is effective, and in fact $\overline{\mathcal{Z}}-D=h^*(E)$, where E is an effective divisor on $\mathcal{M}_g$ and $h:\mathcal{M}_{g,g+1}\to\mathcal{M}_g$ is the projection. Moreover, the map $h^*:\mathrm{Pic}(\mathcal{M}_g)_{\mathbb{Q}}\to\mathrm{Pic}(\mathcal{M}_{g,g+1})_{\mathbb{Q}}$ is injective (cf. [AC2]), hence $E\equiv_{\mathbb{Q}-lin}0$. Since the Satake compactification of $\mathcal{M}_g$ has boundary of codimension 2 (see e.g [HaMo], pg.45) this implies $E=0$, that is, $\overline{\mathcal{Z}}=D$. Therefore $\Theta_{\wedge^i Q_C}$ is a divisor in $\mathrm{Pic}^{g-2i-1}(C)$ and the identification $\Theta_{\wedge^i Q_C}=C_{g-i-1}-C_i$ holds for *every* nonhyperelliptic curve C. $\square$

4. A divisor class computation on $\mathcal{M}_{g,g+1}$

In this section we compute the class of the divisor D on $\mathcal{M}_{g,g+1}$ defined in the previous section. We start by recalling a few facts about line bundles on $\overline{\mathcal{M}}_{g,n}$. Let us fix $g\geq 3$, $n\geq 0$ and a set N of n elements. Following [AC2], we identify $\overline{\mathcal{M}}_{g,n}$ with the moduli space $\overline{\mathcal{M}}_{g,N}$ of stable curves of genus g with marked points indexed by N. We denote by $\pi_q:\overline{\mathcal{M}}_{g,N\cup\{q\}}\to\overline{\mathcal{M}}_{g,N}$ the map forgetting the marked point indexed by q. For each $z\in N$ we define the cotangent line bundle ψ_z on $\overline{\mathcal{M}}_{g,N}$ as follows: the projection $\pi_q:\overline{\mathcal{M}}_{g,N\cup\{q\}}\to\overline{\mathcal{M}}_{g,N}$ has a section $\sigma_z:\overline{\mathcal{M}}_{g,N}\to\overline{\mathcal{M}}_{g,N\cup\{q\}}$

which associates to an n-pointed curve $(C, \{x_i\}_{i\in N})$ the curve $C \cup_{x_z\sim 0} \mathbf{P}^1$, where we label the points $1, \infty \in \mathbf{P}^1$ by q and z. We define $\psi_z := \sigma_z^*(c_1(\omega_{\pi_q}))$. Note that although we are using an apparently different definition, these ψ classes are the same as those which appear in the previous section.

For $0 \le i \le g$ and $S \subseteq N$, the boundary divisor $\Delta_{i:S}$ corresponds to the closure of the locus of nodal curves $C_1 \cup C_2$, with C_1 smooth of genus i, C_2 smooth of genus $g-i$, and such that the marked points sitting on C_1 are precisely those labelled by S. Of course $\Delta_{i:S} = \Delta_{g-i:S^c}$ and we set $\Delta_{0:S} := 0$ when $|S| \le 1$. We also consider the divisor Δ_{irr} consisting of irreducible pointed curves with one node. We denote by $\delta_{i:S} \in \mathrm{Pic}(\overline{\mathcal{M}}_{g,n})_{\mathbb{Q}}$ the class of $\Delta_{i:S}$ and by δ_{irr} that of Δ_{irr}. It is well known that the Hodge class λ, δ_{irr}, the ψ_z's and the $\delta_{i:S}$'s freely generate $\mathrm{Pic}(\overline{\mathcal{M}}_{g,n})_{\mathbb{Q}}$ (cf. [AC2]).

Recall that for $g \ge 3$ and $0 \le i \le \frac{g-1}{2}$ we have defined the divisor D on $\mathcal{M}_{g,g+1}$ to be the locus of curves $(C, x_1, \dots, x_{g-i}, y_1, \dots, y_{i+1})$ having a linear series $\mathfrak{g}^1_g$ containing $x_1 + \dots + x_{g-i}$ in a fiber and $y_1 + \dots + y_{i+1}$ in another fiber. We denote by $\overline{D}$ the closure of D in $\overline{\mathcal{M}}_{g,g+1}$.

The divisor D comes equipped with a scheme structure induced by the forgetful map $\mathcal{H} \to \mathcal{M}_{g,g+1}$, where $\mathcal{H}$ is the Hurwitz scheme parametrizing objects $(C, f, \vec{x}, \vec{y})$, where $\vec{x} = (x_1, \dots, x_{g-i})$, $\vec{y} = (y_1, \dots, y_{i+1})$ and $[C, \vec{x}, \vec{y}] \in \mathcal{M}_{g,g+1}$, while $f : C \to \mathbf{P}^1$ is a g-sheeted cover mapping all the x_j's to a point and all the y_j's to another point. The main result of the section is the following:

Theorem 4.1. *The divisor D is reduced and its class in $\mathrm{Pic}(\overline{\mathcal{M}}_{g,g+1})_{\mathbb{Q}}$ is*

$$[D] = -\left(\binom{g-1}{i} - 10\binom{g-3}{i-1}\right)\lambda + \binom{g-2}{i}\Psi_x + \binom{g-2}{i-1}\Psi_y,$$

where $\Psi_x = \sum_{j=0}^{g-i} \psi_{x_j}$ and $\Psi_y = \sum_{j=0}^{i+1} \psi_{y_j}$.

We begin by proving the first part of Theorem 4.1:

Proposition 4.2. *The divisor D is reduced.*

Proof. We denote by $\pi : \mathcal{H} \to \mathcal{M}_{g,g+1}$ the map forgetting the covering. Since the Hurwitz scheme $\mathcal{H}$ is smooth of pure dimension $4g-3$ (standard deformation theory), it suffices to show that $\mathrm{codim}_{\mathcal{M}_{g,g+1}}(A) \ge 2$, where

$$A := \{[C, \vec{x}, \vec{y}] \in \mathcal{M}_{g,g+1} \mid |\pi^{-1}(C, \vec{x}, \vec{y})| \ge 2\}.$$

Indeed, suppose that $[C, \vec{x}, \vec{y}] \in \mathcal{M}_{g,g+1}$ is such that there are different g-sheeted maps $f_1, f_2 : C \to \mathbf{P}^1$ both containing $\vec{x}$ and $\vec{y}$ in different fibers. Then the product map $f = (f_1, f_2) : C \to \mathbf{P}^1 \times \mathbf{P}^1$ is birational onto its image and the curve $f(C)$ will have points of multiplicity at least $g-i$ and $i+1$ at $a = f(\vec{x})$ and $b = f(\vec{y})$ respectively. If we denote by $S = \mathrm{Bl}_{\{a,b\}}(\mathbf{P}^1 \times \mathbf{P}^1)$, we get in this way a dominant map from the moduli space of maps $\mathcal{M}_g(S, \gamma)$ to $A/(S_{g-i} \times S_{i+1})$, where $S_{g-i} \times S_{i+1}$ acts on A by permuting the x_j's and the y_j's separately, while $\gamma = gl + gm - (g-i)E_a - (i+1)E_b \in \mathrm{Pic}(\mathrm{S})$. Here l and m are pullbacks of the rulings on $\mathbf{P}^1 \times \mathbf{P}^1$, and E_a, E_b are the exceptional divisors.

An argument coming from [AC1], Proposition 2.4, shows that if M is an irreducible component of $\mathcal{M}_g(S,\gamma)$ whose general point corresponds to a generically injective map and $\dim(M) \geq g+1$, then $\dim(M) = g - 1 - \gamma \cdot K_S$. Indeed, if $[f : C \to S] \in M$ is a general point, let us denote by N_f the normal sheaf of f. We use the tangent space identification $T_{[f]}\mathcal{M}_g(S,\gamma) = H^0(C, N_f)$ and assume $h^0(C, N_f) \geq g+1$. By [AC1] Lemma 1.4, we can also assume that N_f is locally free. Clifford's Theorem now implies $H^1(C, N_f) = 0$, hence $\mathcal{M}_g(S,\gamma)$ is smooth at $[f]$ and $\dim_{[f]}\mathcal{M}_g(S,\gamma) = h^0(C, N_f) = g - 1 - \gamma \cdot K_S$. Therefore $\dim(A) \leq \dim(M) - \dim \mathrm{Aut}(S) = 4g - 4$. $\square$

We will prove the second part of Theorem 4.1 using degeneration techniques and enumerative geometry.

Recap on limit linear series (cf. [EH1]**).** We recall that for a smooth curve C, a point $p \in C$ and a linear series $l = (L, V)$ with $L \in \mathrm{Pic}^d(C)$ and $V \in G(r+1, H^0(L))$, the *vanishing sequence* of l at p is obtained by ordering the set $\{\mathrm{ord}_p(\sigma)\}_{\sigma \in V}$, and it is denoted by

$$a^l(p) : 0 \leq a_0^l(p) < \ldots < a_r^l(p) \leq d.$$

The *weight* of l at p is defined as $w^l(p) := \sum_{i=0}^r (a_i^l(p) - i)$.

Given a curve C of compact type, a *limit* $\mathfrak{g}_d^r$ on C is a collection of honest linear series $l_Y = (L_Y, V_Y) \in G_d^r(Y)$ for each component Y of C, satisfying the compatibility condition that if Y and Z are components of C meeting at p then

$$a_i^{l_Y}(p) + a_{r-i}^{l_Z}(p) \geq d \quad \text{for } i = 0, \ldots, r.$$

We note that limit linear series appear as limits of ordinary linear series in 1-dimensional families of curves and there is a useful sufficient criterion for a limit $\mathfrak{g}_d^r$ to be *smoothable* (cf. [EH1], Theorem 3.4).

We will need the following enumerative result (cf. [Ha], Theorem 2.1):

Proposition 4.3. *Let C be a general curve of genus g, $d \geq \frac{g+2}{2}$ and $p \in C$ a general point.*

- *The number of $\mathfrak{g}_d^1$'s on C containing $(2d-g)q$ in a fiber, where $q \in C$ is an unspecified point, is*

$$b(d,g) = (2d-g-1)(2d-g)(2d-g+1)\frac{g!}{d!(g-d)!} \ .$$

- *If $\beta \geq 1$, $\gamma \geq 1$ are integers such that $\beta + \gamma = 2d - g$, the number of $\mathfrak{g}^1_d$'s on C containing $\beta p + \gamma q$ in a fiber for some point $q \in C$ is*
$$c(d, g, \gamma) = (\gamma^2(2d - g) - \gamma)\frac{g!}{d!\ (g-d)!}\ .$$

Remark 4.4. The original proof of Proposition 4.3 used some involved calculations in $H^*(J(C), \mathbb{Z})$. A quicker way of computing these numbers is via degeneration. For instance, in order to compute $b(d, g)$ we let C specialize to the curve of compact type $C_0 = \mathbf{P}^1 \cup E_1 \cup \ldots \cup E_g$, where E_i are general elliptic curves, $\{p_i\} = E_i \cap \mathbf{P}^1$ and $p_1, \ldots, p_g \in \mathbf{P}^1$ are general points.

One has to count limit $\mathfrak{g}^1_d$'s on C_0 having vanishing $\geq 2d - g$ at some point $q \in C_0$. Let l be such a limit linear series. Using the additivity of the Brill-Noether numbers (cf. [EH1], Proposition 4.6) it follows that q has to sit on one of the elliptic tails, say $q \in E_1$. Moreover $a^{l_{\mathbf{P}^1}}(p_1) = a^{l_{E_1}}(q) = (0, 2d - g)$, hence $q - p_1$ is a $(2d - g)$-torsion in $\mathrm{Pic}^0(E_1)$. Also, since $a^{l_{E_i}}(p_i) \leq (d - 2, d)$ we get $a^{l_{\mathbf{P}^1}}(p_i) = (0, 2)$ for $2 \leq i \leq g$.

Clearly $q \in E_1$ can be chosen in $(2d - g)^2 - 1$ ways and there are g tails on which q can sit. Since the points $p_1, \ldots, p_g \in \mathbf{P}^1$ are general, the variety of $\mathfrak{g}^1_d$'s on $\mathbf{P}^1$ having vanishing $2d - g$ at p_1 and simple ramification at $p_2, \ldots, p_g$ is reduced, 0-dimensional, and its number of points is given by the product of Schubert cycles $\sigma^{g-1}_{(0,1)}\sigma_{(0,2d-g-1)}\big(\in H^{top}(\mathbb{G}(1, d), \mathbb{Z})\big)$. Thus we obtain that

$$b(d, g) = g\big((2d - g)^2 - 1\big)\sigma^{g-1}_{(0,1)}\sigma_{(0,2d-g-1)}$$

and the conclusion follows.

The following simple observation will be used repeatedly:

Proposition 4.5. *Fix $y, z \in N$ and denote by $\pi_z : \overline{\mathcal{M}}_{g,N} \to \overline{\mathcal{M}}_{g,N-\{z\}}$ the map forgetting the marked point labelled by z. If E is any divisor class on $\overline{\mathcal{M}}_{g,N}$, then the λ and the ψ_x coefficients of E are the same as those of $(\pi_z)_*(E \cdot \delta_{0:yz})$ for all $x \in N - \{y, z\}$.*

Proof. We use that $(\pi_z)_*(\lambda \cdot \delta_{0:yz}) = \lambda$, $(\pi_z)_*(\psi_x \cdot \delta_{0:yz}) = \psi_x$ for $x \in N - \{y, z\}$ and that $(\pi_z)_*(\delta_{i:S} \cdot \delta_{0:yz})$ is boundary in all cases except $(\pi_z)_*(\delta^2_{0:yz}) = -\psi_y$ (cf. [AC2], Lemma 1.2 and [Log], Theorem 2.3). Note also that $(\pi_z)_*(\psi_x \cdot \delta_{0:yz}) = 0$ for $x \in \{y, z\}$. □

By a succession of push-forwards, we will reduce the problem of computing the class of D to two divisor class computations in $\overline{\mathcal{M}}_{g,3}$.

We define the following sequence of divisors: starting with $\overline{D} = D_{y_{i+1}}$, for $1 \leq j \leq i$ we define inductively the divisors D_{y_j} on $\overline{\mathcal{M}}_{g,g-i+j}$ by
$$D_{y_j} := (\pi_{y_{j+1}})_*(\Delta_{0:y_j y_{j+1}} \cdot D_{y_{j+1}}).$$

Loosely speaking, D_{y_j} is obtained from $D_{y_{j+1}}$ by letting the marked points y_j and y_{j+1} coalesce. Then we define $D_{x_{g-i}} := D_{y_1}$ and we let the marked points $x_2, \ldots, x_{g-i}$ coalesce: for $2 \leq j \leq g-i-1$ we define inductively the divisors D_{x_j} on $\overline{\mathcal{M}}_{g,j+1}$ by

$$D_{x_j} := (\pi_{x_{j+1}})_*(\Delta_{0:x_j x_{j+1}} \cdot D_{x_{j+1}}).$$

Proposition 4.5 ensures that the ψ_{x_1} and the λ coefficients of $[\overline{D}]$ are the same as those of $[D_{x_2}]$.

Proposition 4.6. *The divisor D_{x_2} is reduced and it is the closure in $\overline{\mathcal{M}}_{g,3}$ of the locus of those smooth pointed curves (C, x_1, x_2, y) for which there exists a $\mathfrak{g}^1_g$ with $(i+1)y$ in a fiber and $x_1 + (g-i-1)x_2$ in another fiber.*

Proof. For simplicity we will only prove that D_{y_i} is reduced and that it is the closure of the locus of those smooth pointed curves $(C, x_1, \ldots, x_{g-i}, y_1, \ldots, y_i)$ for which $x_1 + \cdots + x_{g-i}$ and $y_1 + \cdots + y_{i-1} + 2y_i$ are in different fibers of the same $\mathfrak{g}^1_g$. Then by iteration we will get a similar statement for D_{x_2}.

Let $(X = C \cup_q \mathbf{P}^1, x_1, \ldots, x_{g-i}, y_1, \ldots, y_{i+1})$ with $y_i, y_{i+1} \in \mathbf{P}^1$ be a general point in a component of $D_{y_{i+1}} \cap \Delta_{0:y_i y_{i+1}}$. A standard dimension count shows that C must be smooth. There exists a limit $\mathfrak{g}^1_g$ on X, say $l = (l_C, l_{\mathbf{P}^1})$, together with sections $\sigma_{\mathbf{P}^1} \in V_{\mathbf{P}^1}$ and $\sigma_C, \tau_C \in V_C$, such that $\mathrm{div}(\tau_C) \geq x_1 + \cdots + x_{g-i}$, $\mathrm{div}(\sigma_C) \geq y_1 + \cdots + y_{i-1}$, $\mathrm{div}(\sigma_{\mathbf{P}^1}) \geq y_i + y_{i+1}$ and moreover $\mathrm{ord}_q(\sigma_{\mathbf{P}^1}) + \mathrm{ord}_q(\sigma_C) \geq g$ (apply [EH1], Proposition 2.2).

Clearly $\mathrm{ord}_q(\sigma_{\mathbf{P}^1}) \leq g-2$, hence $\mathrm{div}(\sigma_C) \geq 2q + y_1 + \cdots + y_{i-1}$. The contraction map $\pi_{y_{i+1}}$ collapses $\mathbf{P}^1$ and identifies q and y_i, so the second part of the claim follows.

To conclude that D_{y_i} is also reduced we use that both $D_{y_{i+1}}$ and $\Delta_{0:y_i y_{i+1}}$ are reduced and that they meet transversally. This is because the limit $\mathfrak{g}^1_g$ we found on X is smoothable in such a way that all ramification is kept away from the nodes (cf. [EH1], Proposition 3.1). $\square$

In a similar way, by letting first all x_j with $1 \leq j \leq g-i$ and then all y_j with $2 \leq j \leq i+1$ coalesce, we obtain a reduced divisor D_{y_2} on $\overline{\mathcal{M}}_{g,3}$ which is the closure of the locus of smooth curves (C, x, y_1, y_2) having a $\mathfrak{g}^1_g$ with $(g-i)x$ and $y_1 + iy_2$ in different fibers. Moreover, the λ and the ψ_{y_1} coefficients of $[\overline{D}]$ coincide with those of $[D_{y_2}]$. Applying once more Proposition 4.5 it follows that the λ and the ψ_{y_1} coefficients of $[D_{y_2}]$ are the same as those of $(\pi_x)_*([D_{y_2}] \cdot \delta_{0:xy_2})$. Similarly, the ψ_{x_1} coefficient of $[D_{x_2}]$ is the same as that of $(\pi_y)_*([D_{x_2}] \cdot \delta_{0:x_2 y})$.

Proposition 4.7. *We have that*

$$(\pi_x)_*(D_{y_2} \cdot \Delta_{0:xy_2}) = \sum_{j=0}^{i} Y_j,$$

where for $j < i$ the reduced divisor Y_j is the closure in $\overline{\mathcal{M}}_{g,2}$ of the locus of curves (C, y_1, y_2) having a $\mathfrak{g}^1_{g-j}$ with $(g-2j-1)y_2 + y_1$ in a fiber, while the reduced divisor Y_i consists of curves (C, y_1, y_2) with a $\mathfrak{g}^1_{g-i}$ having $(g-2i)y_2$ in a fiber (and no condition on y_1).

Proof. Once again, let $(X = C \cup_q \mathbf{P}^1, x, y_1, y_2)$ be a point in $D_{y_2} \cap \Delta_{0:xy_2}$, with $y_1 \in C$ and $x, y_2 \in \mathbf{P}^1$. Then there exists a limit $\mathfrak{g}^1_g$, say $l = (l_C, l_{\mathbf{P}^1})$ on X together with sections $\sigma_{\mathbf{P}^1}, \tau_{\mathbf{P}^1} \in V_{\mathbf{P}^1}$ and $\sigma_C \in V_C$ such that $\mathrm{div}(\sigma_{\mathbf{P}^1}) \geq iy_2, \mathrm{div}(\tau_{\mathbf{P}^1}) \geq (g-i)x, \mathrm{div}(\sigma_C) \geq y_1$ and moreover $\mathrm{ord}_q(\sigma_{\mathbf{P}^1}) + \mathrm{ord}_q(\sigma_C) \geq g$.

The Hurwitz formula on $\mathbf{P}^1$ and the condition defining a limit linear series give that $w^{l_C}(q) \geq w^{l_{\mathbf{P}^1}}(x) + w^{l_{\mathbf{P}^1}}(y_2) \geq g-2$. On the other hand, since (X, x, y_1, y_2) moves in a family of dimension $\geq 3g-2$ it follows that (C, q) also moves in a family of dimension $\geq 3g-3$ in $\mathcal{M}_{g,1}$ (i.e. codimension ≤ 1). Applying [EH2], Theorem 1.2, we get $w^{l_C}(q) \leq g-1$. Moreover, if $w^{l_C}(q) = g-1$, then there will be no separate condition on y_1. There are two possibilities:

i) $w^{l_C}(q) = g-2$. Let us denote $j = a_0^{l_C}(q)$, hence $a_1^{l_C}(q) = g-1-j$ and $a_k^{l_C}(q) + a_{1-k}^{l_{\mathbf{P}^1}}(q) = g$ for $k = 0, 1$. Therefore $j+1 = a_0^{l_{\mathbf{P}^1}}(q) \leq \mathrm{ord}_q(\tau_{\mathbf{P}^1}) \leq i$. Moreover, since $\mathrm{ord}_q(\sigma_{\mathbf{P}^1}) \leq g-i \leq g-j-1$, we obtain that $\mathrm{ord}_q(\sigma_C) \geq j+1$, hence $\mathrm{div}(\sigma_C) \geq y_1 + (g-1-j)y_2$, that is, $l_C(-jq)$ is a $\mathfrak{g}^1_{g-j}$ on C with $(g-2j-1)q + y_1$ in a fiber, or $[C, y_1, q] \in Y_j$.

To see that conversely $\bigcup_{j=0}^{i-1} Y_j \subseteq (\pi_x)_*(D_{y_2} \cdot \Delta_{0:xy_2})$ we pick a general pointed curve (C, y_1, q) having a $\mathfrak{g}^1_{g-j}$ with $(g-2j-1)q + y_1$ in a fiber and we construct a Harris-Mumford admissible covering $f : X' \to B$ of degree g, where X' is a curve semistably equivalent to X defined as above, and $B = (\mathbf{P}^1)_1 \cup_t (\mathbf{P}^1)_2$ is the transversal union of two lines (see Fig. 1): we take $f_{|C} : C \to (\mathbf{P}^1)_1$ to be the degree $g-j$ covering such that $(g-2i-1)q + y_1 \subseteq f^*_{|C}(t)$, while $f_{|\mathbf{P}^1} : \mathbf{P}^1 \to (\mathbf{P}^1)_2$ is the degree $g-j-1$ map containing $(g-i)x$ and iy_2 in different fibers and with $(g-2j-1)q$ in the fiber over t. It is clear that there is a unique such $\mathfrak{g}^1_{g-j-1}$ on $\mathbf{P}^1$. Furthermore, at y_1 we insert a rational curve R mapping isomorphically onto $(\mathbf{P}^1)_2$ and at the remaining j points in $f^{-1}_{|C}(t) - \{y_1, q\}$ we insert rational curves mapping with degree 1 onto $(\mathbf{P}^1)_2$ while at the $g-j$ points in $f^{-1}_{|\mathbf{P}^1}(t) - \{q\}$ we insert copies of $\mathbf{P}^1$ mapping isomorphically onto $(\mathbf{P}^1)_1$. We denote the resulting curve by X'. If $y_1' = f^{-1}_{|R}(f(y_2))$, then (X', x, y_1', y_2) is stably equivalent to (X, x, y_1, y_2), hence $[X, x, y_1, y_2] \in D_{y_2} \cap \Delta_{0:xy_2}$. (The result of this construction is represented in the following figure.)

FIGURE 1

ii) $w^{l_C}(q) = g-1$. We denote $a^{l_C}(q) = a^{l_{\mathbf{P}^1}}(q) = (j, g-j)$. Since $\mathrm{ord}_q(\tau_{\mathbf{P}^1}) \leq i$ we get that $j \leq i$. Now $w^{l_C}(q) = g-1$ is already a codimension 1 condition on $\mathcal{M}_{g,1}$, so it follows that $\mathrm{ord}_q(\sigma_C) = j$, hence $\mathrm{div}(\sigma_{\mathbf{P}^1}) \geq (g-j)q + iy_2$. This yields $i = j$ and $\mathrm{div}(\sigma_{\mathbf{P}^1}) = (g-i)q + iy_2$. We thus get that $[C, y_1, q] \in Y_i$.

Conversely, given $(C, y_1, q) \in \mathcal{M}_{g,2}$ together with a $\mathfrak{g}^1_{g-i}$ on C with $(g-2i)q$ in a fiber, we construct a degree g admissible covering $f : X' \to (\mathbf{P}^1)_1 \cup_t (\mathbf{P}^1)_2$, which will prove that $[C, q, y_2] \in (\pi_x)_*(D_{y_2} \cdot \Delta_{0:xy_2})$: we first take $f_{|C} : C \to (\mathbf{P}^1)_1$ of degree $g-i$ with $(g-2i)q \subseteq f^*_{|C}(t)$. Then $f_{|\mathbf{P}^1} : \mathbf{P}^1 \to (\mathbf{P}^1)_2$ is of degree $g-i$, completely ramified at x and with $f^{-1}_{|\mathbf{P}^1}(t) = (g-2i)q + iy_2$. At $y_2 \in \mathbf{P}^1$ we insert a rational curve R which we map $i:1$ to $(\mathbf{P}^1)_1$ such that we have total ramification both at y_2 and at the point $y_2' \in R$ characterized by $f_{|C}(y_1) = f_{|R}(y_2')$. Finally, at each of the points in $f^{-1}_{|C}(t) - \{q\}$ we insert a $\mathbf{P}^1$ which we map isomorphically onto $(\mathbf{P}^1)_2$.

Thus we have proved that $\mathrm{supp}(\pi_x)_*(D_{y_2} \cdot \Delta_{0:xy_2}) = \cup_{j=0}^{i} \mathrm{supp}(Y_j)$. The conclusion now follows if we notice that D_{y_2} is reduced and all admissible coverings we constructed are smoothable, hence $D_{y_2} \cdot \Delta_{0:xy_2}$ is reduced too. $\square$

Proposition 4.8. *For $0 \leq j \leq i$ we have the following relations in $\mathrm{Pic}(\mathcal{M}_{g,2})_{\mathbb{Q}}$:*

$$Y_j \equiv_{lin} a_j\lambda + b_{1j}\psi_{y_1} + b_{2j}\psi_{y_2}, \quad \textit{where}$$

$$a_j = -\frac{g-2j}{g}\binom{g}{j} + \frac{10(g-2j)}{g-2}\binom{g-2}{j-1} \quad \textit{for all } 0 \leq j \leq i,$$

$$b_{1j} = \frac{g-2j-1}{g-1}\binom{g-1}{j} \textit{ when } j \leq i-1, b_{1i} = 0, b_{2i} = \frac{(g-2i)^3 - (g-2i)}{2g-2}\binom{g}{i}$$

$$b_{2j} = \frac{(g-2j-1)(g^3-g^2-4g^2j+4j^2g+2jg-2j)(g-2)!}{2j!(g-1)!} \textit{ for } j \leq i-1.$$

Proof. We will compute the class of Y_j when $j \leq i-1$. The class of Y_i is computed similarly. Let us write the following relation in $\mathrm{Pic}(\overline{\mathcal{M}}_{g,2})_{\mathbb{Q}}$:

$$Y_j \equiv_{lin} a_j\lambda + b_{1j}\psi_{y_1} + b_{2j}\psi_{y_2} - c_j\delta_{0:y_1y_2} + (\text{ other boundary terms }).$$

Then by Proposition 4.5 we have that

$$Z_j := (\pi_{y_2})_*(Y_j \cdot \Delta_{0:y_1y_2}) \equiv_{lin} a_j\lambda + c_j\psi_{y_1} + (\text{ boundary }). \tag{5}$$

Using the same reasoning as in Proposition 4.6, we obtain that Z_j is the closure in $\overline{\mathcal{M}}_{g,1}$ of the locus of curves (C, y_1) carrying a $\mathfrak{g}^1_{g-j}$ with $(g-2j)y_1$ in a fiber.

In order to determine the coefficient c_j we intersect both sides of (5) with a general fiber F of the map $\overline{\mathcal{M}}_{g,1} \to \overline{\mathcal{M}}_g$: we get that $c_j = Z_j \cdot F/\psi_{y_1} \cdot F = b(g-j,g)/(2g-2)$ (cf. Proposition 4.3).

To determine b_{1j} and b_{2j} we use two test curves in $\overline{\mathcal{M}}_{g,2}$: first, we fix a general curve C of genus g and we obtain a family $C_{[1]} = \{(C, y_1, y_2)\}_{y_1 \in C}$, by fixing a general point $y_2 \in C$ and letting y_1 vary on C. From (5), clearly $C_{[1]} \cdot Z_j = (2g-1)b_{1j} + b_{2j} - c_j$. On the other hand, according to Proposition 4.3 $C_{[1]} \cdot Z_j = c(g-j, g, 1)$.

For a new relation between b_{1j} and b_{2j} we use the test curve $C_{[2]} = \{(C, y_1, y_2)\}_{y_2 \in C}$ in $\overline{\mathcal{M}}_{g,2}$, where this time y_1 is a fixed general point while y_2 varies on C. We have the equation $(2g-1)b_{2j} + b_{1j} - c_j = C_{[2]} \cdot Z_j = c(g-j, g, g-2j-1)$, and since c_j is already known we get in this way both b_{1j} and b_{2j}.

We are only left with the computation of a_j. From [EH2], Theorem 4.1 we know that the class of Z_j is a linear combination of the Brill-Noether class and of the class of the divisor of Weierstrass points, that is, $Z_j \equiv_{lin} \mu BN + \nu\mathcal{W}$, where

$$BN := (g+3)\lambda - \frac{g+1}{6}\delta_{irr} - \sum_{i=1}^{g-1} i(g-i)\delta_{i:y_1} \text{ and}$$

$$\mathcal{W} := -\lambda + \frac{g(g+1)}{2}\psi_{y_1} - \sum_{i=1}^{g-1}\binom{g-i+1}{2}\delta_{i:y_1}.$$

We already know that $\nu = 2c_j/(g(g+1))$. To determine μ we use the following test curve in $\overline{\mathcal{M}}_{g,1}$: we take a general curve B of genus $g-1$ and a general 2-pointed elliptic curve $(E, 0, y_1)$. We consider the family $\overline{B} = \{X_q = B \cup_{q\sim 0} E, y_1\}_{q\in B}$ obtained by identifying the variable point $q \in B$ with the fixed point $0 \in E$. We easily get $\overline{B} \cdot \psi_{y_1} = \overline{B} \cdot \lambda = 0$, $\overline{B} \cdot \delta_{1:y_1} = -\mathrm{deg}K_B = 4-2g$, while $\overline{B}$ vanishes on all the other boundaries. On the other hand $\overline{B} \cdot Z_j$ is the number of limit $\mathfrak{g}^1_{g-j}$'s on the curves X_q having vanishing $g-2j$ at the fixed point $y_1 \in E$. If $l = (l_B, l_E)$ is such a linear series, then using again the additivity of the Brill-Noether numbers (cf.

[EH1], Proposition 4.6) and the assumption that $y_1 - 0 \in \mathrm{Pic}^0(E)$ is not torsion, we obtain that $w^{l_B}(q) = g-2j$, so either $a^{l_B}(q) = (1, g-2j)$ or $a^{l_B}(q) = (0, g-2j+1)$. Thus $\overline{B} \cdot Z_j = b(g-j-1, g-1) + b(g-j, g-1)$ and we can write a new relation enabling us to compute a_j. $\square$

We can now complete the proof of Theorem 4.1:

Proof of Theorem 4.1. Let us write $D \equiv_{lin} A\lambda + B_1\Psi_x + B_2\Psi_y$, where $\Psi_x := \sum_{j=1}^{g-i} \psi_{x_j}$ and $\Psi_y := \sum_{j=1}^{i+1} \psi_{y_j}$. As noticed before, the $\{\lambda, \Psi_y\}$-part of $[D]$ and the $\{\lambda, \psi_{y_1}\}$-part of $\sum_{j=0}^{i} [Y_j]$ coincide, hence using Proposition 4.8

$$A = \sum_{j=0}^{i} a_j = -\binom{g-1}{i} + 10\binom{g-3}{i-1} \text{ and } B_2 = \sum_{j=0}^{i} b_{1j} = \binom{g-2}{i-1}.$$

Finally, to determine B_1 one has to compute the ψ_{x_1} coefficient of the divisor D_{x_2} on $\overline{\mathcal{M}}_{g,3}$. Arguing in a way that is entirely similar to Proposition 4.7 we obtain that $B_1 = \binom{g-2}{i}$. $\square$

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