THE NON-ABELIAN BRILL-NOETHER DIVISOR ON $\mathcal{M}_{13}$ AND THE KODAIRA DIMENSION OF $\mathcal{R}_{13}$

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ABSTRACT. We compute the class of the non-abelian Brill-Noether divisor on $\mathcal{M}_{13}$ of curves that have a stable rank 2 vector bundle with many sections. This provides the first example of an effective divisor on $\mathcal{M}_g$ with slope less than $6 + \frac{10}{g}$. Earlier work on the Slope Conjecture suggested that such divisors may not exist. The main geometric application of our result is a proof that the Prym moduli space $\mathcal{R}_{13}$ is of general type. Among other things, we also prove the Bertram-Feinberg-Mukai and the Strong Maximal Rank Conjectures on $\mathcal{M}_{13}$.

1. INTRODUCTION

One of the defining achievements of modern moduli theory is the result of Harris, Mumford and Eisenbud that $\mathcal{M}_g$ is of general type for $g \geq 24$ [HM, EH2]. An essential step in their proof is the calculation of the class of the Brill-Noether divisor $\mathcal{D}_{g,r,d}$ consisting of those curves $X$ of genus $g$ such that $G^r_d(X) \neq \emptyset$ in the case $\rho(g, r, d) := g - (r + 1)(g - d + r) = -1$. Recall that the slope of an effective divisor $D$ on $\mathcal{M}_g$ not containing any of the boundary divisors $\Delta_i$ is defined as the quantity $s(D) := a/\min_i b_i$, where $[D] = a\lambda_0 - b_0\delta_0 - \cdots - b_\lfloor \frac{g}{2} \rfloor \delta_{\lfloor \frac{g}{2} \rfloor} \in CH^1(\mathcal{M}_g)$. Eisenbud and Harris showed that the slope of $\mathcal{D}_{g,r,d}$ is $s(\mathcal{D}_{g,r,d}) = 6 + \frac{10}{g}$. After these seminal results from the 1980s, an important question arose whether one can construct effective divisors $D$ on $\mathcal{M}_g$ of slope $s(D) < 6 + \frac{10}{g}$ by using conditions defined in terms of higher rank vector bundles on curves.

Each effective divisor $D$ of slope $s(D)$ on $\mathcal{M}_g$ contains the locus $\mathcal{K}_g \subseteq \mathcal{M}_g$ of curves lying on a $K3$ surface [FP]. Since curves on $K3$ surfaces possess stable rank 2 vector bundles with canonical determinant and unexpectedly many sections [Laz, Mu2, Vo], it is then natural to focus on conditions defined in terms of rank 2 vector bundles with canonical determinant.

For a smooth curve $X$ of genus $g$, let $SU_X(2, \omega)$ be the moduli space of semistable rank 2 vector bundles $E$ on $X$ with $\det(E) \cong \omega_X$. For $k \geq 0$, Bertram-Feinberg [BF1, Conjecture, p.2] and Mukai [Mu2, Problem 4.8] conjectured that for a general curve $X$ of genus $g$,

$$SU_X(2, \omega, k) := \{ E \in SU_X(2, \omega) : h^0(X, E) \geq k \}$$

is of general type. Among other things, we also prove the Bertram-Feinberg-Mukai conjecture and the Strong Maximal Rank Conjecture on $\mathcal{M}_{13}$.
A general curve \( X \) of genus 13 carries precisely three stable vector bundles \( E \in SU_X(2, \omega, 8) \). The closure in \( \mathcal{M}_{13} \) of the non-abelian Brill-Noether divisor on \( \mathcal{M}_{13} \)

\[
\mathcal{M}_{13} := \left\{ [X] \in \mathcal{M}_{13} : \exists E \in SU_X(2, \omega, 8) \text{ with } \mu_E : \text{Sym}^2 H^0(E) \not\rightarrow H^0(\text{Sym}^2(E)) \right\}
\]

has slope equal to

\[
s([\mathcal{M}_{13}]) = \frac{4109}{610} = 6.735... < 6 + \frac{10}{13} = 6 + \frac{10}{g+1}.
\]

To explain the significance of this result, we recall that several infinite series of examples of divisors on \( \mathcal{M}_g \) for \( g \geq 10 \) with slope less than \( 6 + \frac{10}{g+1} \) have been constructed in \([FP, F, Kh, FJP]\), using syzygies on curves. Quite remarkably, the slopes \( s(D) \) of all these divisors on \( \mathcal{M}_g \) are supported on the loci, in \( \mathcal{M}_{g,r} \), of curves failing the Petri Theorem.

The slope \( 6 + \frac{10}{g+1} \) appeared as both the slope of the Brill-Noether divisors \( \mathcal{M}_{g,r} \), and as the slope of a Lefschetz pencil of curves of genus \( g \). Similarly, \( 6 + \frac{10}{g} \) is the slope of the family of curves \{\( X_{\ell} \}_{\ell \in \mathbb{P}^1}\) of curves of genus \( g - 1 \) on a \( K3 \) surface consisting of curves corresponding to base points of the pencil. The natural question was then: Is this slope achieved in a genuine way by curves of genus \( g \)?

Slope questions of this type can be reduced to the question posed in \([CFM]\). To explain the significance of this result, we recall that several infinite series of examples of divisors on \( \mathcal{M}_g \) for \( g \geq 10 \) with slope less than \( 6 + \frac{10}{g+1} \) have been constructed in \([FP, F, Kh, FJP]\), using syzygies on curves. Quite remarkably, the slopes \( s(D) \) of all these divisors on \( \mathcal{M}_g \) are supported on the loci, in \( \mathcal{M}_{g,r} \), of curves failing the Petri Theorem.

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1. The divisorial part of the Prym moduli space $\mathcal{R}_{13}$. One application of our work on the non-abelian Brill-Noether divisor class in Theorem 1.1 and hence for the proof of Theorem 1.2.

The Kodaira dimension of the Prym moduli space $\mathcal{R}_{13}$ is of general type for $g \geq 14$. In particular, the universal theta divisor $\Theta_{13}$ inside $\mathcal{M}_{13}$ is uniruled for $g \geq 14$ (see [B]). The Deligne-Mumford compactification $\overline{\mathcal{R}}_{13}$ is uniruled for $g \geq 14$ (see [FV]).

Hence, the push-forward of this locus is expected to be a divisor on the moduli space $\mathcal{M}_{13}$.

One application of Theorem 1.1 concerns the birational geometry of the moduli space of Prym curves $\mathcal{P}_{16}$ (see [AF] on indirect and proceed through a study of the failure locus of the Strong Maximal Rank Conjecture and computes the class $[\Theta_{13}]$ of the universal the Prym moduli space $\mathcal{P}_{16}$).

The next result verifies this case of the Strong Maximal Rank Conjecture and computes the class $[\Theta_{13}]$ of the Prym moduli space $\mathcal{P}_{16}$.

Theorem 1.3. The locus of curves $[X] \in \mathcal{M}_{13}$ of genus $g \geq 14$ whose closure in $\mathcal{M}_{13}$ is of codimension one (see [AF]) is a proper subvariety of $\mathcal{M}_{13}$.

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That the degree $g - d + 1$ of the linear system $(\mathcal{L} \otimes 2)$ is surjective is the main ingredient of Logan's result that guarantees that the virtual classes in that article are effective divisors on the moduli spaces of curves of genus $g$.

We use the method of tropical independence on chains of loops, as introduced in [JP1, JP2]. Our construction of the required tropical independence is similar to the one used in our proof that the Strong Maximal Rank Conjecture of [AF]. We prove this case, along with a stronger result that guarantees that the degeneracy locus $\mathcal{D}_{13}$ is uniruled for $g, n \geq 1$.

To resolve this difficulty, see Remarks 4.3 and 4.9. This improves on Logan's result that the divisor $\mathcal{D}_{13}$ is uniruled for $g, n \geq 1$.

We will show that the map from $\mathcal{SU}_{13}(2,\omega,8)$ to $\mathcal{M}_{13}$, $[X] \mapsto \mathcal{D}_{13}$, is surjective. In particular, $P(\text{Ker}(d)) \subseteq P(\Lambda^2 H^0(X, E)) \cong P^{27}$ is a 14-dimensional linear space. Since $h^0(X, \omega_X) = 2h^0(X, E) - 3$, it follows that the set of pairs $[X, E]$ satisfying the condition $P(\text{Ker}(d)) \cap G(2, H^0(X, E)) \neq \emptyset$

is an effective divisor in $\mathcal{M}_{13}$.

$$\mathcal{Res}_{13} = \mathcal{D}_{13} + 3 \cdot \mathcal{M}_{13,7}^{11}$$
2. The failure locus of the strong maximal rank conjecture on \( \overline{M}_{13} \)

We denote by \( \overline{M}_g \) the moduli stack of stable curves of genus \( g \geq 2 \) and by \( \overline{M}_g \) the associated coarse moduli space. We work throughout over an algebraically closed field \( K \) of characteristic 0 and the Chow groups that we consider are with rational coefficients. Via the isomorphism \( CH^*(\overline{M}_g) \cong CH^*(\overline{M}_g) \), we routinely identify cycle classes on \( \overline{M}_g \) with their push forward to \( \overline{M}_g \). Recall that for \( g \geq 3 \) the group \( CH^1(\overline{M}_g) \) is freely generated by the Hodge class \( \lambda \) and by the classes of the boundary divisors \( \delta_i = [\Delta_i] \), for \( i = 0, \ldots, [\frac{g}{2}] \).

In this section, we realize the virtual divisor class \( [D_{13}]^{vir} \) as the virtual class of a codimension 2 determinantal locus inside the moduli space \( \overline{M}_{13} \) of curves of type \( g_{13}^r \) over an open substack \( \mathfrak{M}_{13} \) of \( \overline{M}_{13} \), that differs from \( \mathfrak{M}_{13} \cup \mathfrak{M}_{13} \). We will use standard terminology from the theory of limit linear series \( \text{[EH1]} \), and begin by recalling a few of the basics.

**Definition 2.1.** Let \( X \) be a smooth curve of genus \( g \) with \( \ell \) a limit linear series. The *ramification sequence* of \( \ell \) at a point \( q \in X \) is denoted

\[
\alpha^{\ell}(q) : \alpha_0^{\ell}(q) \leq \cdots \leq \alpha_r^{\ell}(q).
\]

This is obtained from the *vanishing sequence* \( a^{\ell}(q) : a_0^{\ell}(q) < \cdots < a_r^{\ell}(q) \leq d \) of \( \ell \) at \( q \), by setting \( \alpha_i^{\ell}(q) := a_i^{\ell}(q) - i \), for \( i = 0, \ldots, r \). The *ramification weight of \( q \) with respect to \( \ell \) is* \( wt^{\ell}(q) := \sum_{i=0}^{r} \alpha_i^{\ell}(q) \).

We define \( \rho(\ell, q) := \rho(g, r, d) - wt^{\ell}(q) \).

A *generalized limit linear series* on a tree-like curve \( X \) of genus \( g \) consists of a collection

\[
\ell = \{(L_C, V_C) : C \text{ is a component of } X\},
\]

where \( L_C \) is an algebraic fiber space of type \( g_{13}^r \) over \( C \) and \( V_C \subseteq H^0(C, L_C) \) is a (\( r+1 \))-dimensional subspace satisfying compatibility conditions on the vanishing sequences at the nodes of \( X \), \( G_{d}^r(X) \) be the variety of generalized limit linear series of type \( g_{13}^r \) on \( X \).
In this section we set $g = 13$.

Among the constructions are set up for an arbitrary elliptic curve, it is easier to refer to such a curve by $E$, rather than by the codim($M^2_{13,5}$). We denote the locus of curves $X$ of genus $g$ which are unions of rational curves, and we denote the induced universal curve by $\Delta$. The fibre of $\pi: E(y_q) \to \Delta$ is an arbitrary point, together with their degenerations $\Delta^0 \cap \Delta^0$. We define the following open subset of $\mathcal{M}_g$:

$$\mathcal{M}_g := \mathcal{M}_g \setminus (\mathcal{M}_d, \pi_0 \cup \pi_1).$$

We define $\Delta_0 := \tilde{\mathcal{M}}_g \cap \Delta_0 \subseteq \Delta^0$ and $\Delta_1 := \tilde{\mathcal{M}}_g \cap \Delta_1 \subseteq \Delta^0$. Note that $\tilde{\mathcal{M}}_g$ and $\mathcal{M}_g \cup \Delta_0 \cup \Delta_1$ agree away from a set of codimension 2 in each. We identify $CH^1(\mathcal{M}_g) \cong \mathbb{Q}(\lambda, \delta_0, \delta_1)$, where $\lambda$ is the Hodge class, $\delta_0 := [\Delta_0]$ and $\delta_1 := [\Delta_1]$.

2.1. Stacks of limit linear series. Let $\tilde{G}_d^r$ be the stack of pairs $[X, \ell]$, where $[X] \in \tilde{\mathcal{M}}_g$ and $\ell$ is a (generalized) limit linear series of type $g_d^s$ on the tree-like curve $X$. We consider the proper projection $\sigma: \tilde{G}_d^r \to \tilde{\mathcal{M}}_g$.

Over a curve $[X \cup \ell \in E] \in \Delta_1$, we identify $\sigma^{-1}([X \cup E])$ with the variety of (generalized) limit linear series $\ell = (\ell_X, \ell_E) \in \mathcal{G}^d(X \cup E)$. The fibre $\sigma^{-1}(X)$ over an irreducible curve $[X]$ is canonically identified with the variety $W_d^r(X)$ of rank $r$ torsion free sheaves $L$ on $X$ having degree $d$.

Let $\mathcal{E}_g \to \tilde{\mathcal{M}}_g$ be the universal curve, and let $p_2: \mathcal{C}_g \times \mathcal{M}_g \to \tilde{G}_d^r$ be the projection map. We denote by $\mathcal{Z} \subseteq \mathcal{C}_g \times \tilde{\mathcal{M}}_g$ the codimension 2 substack consisting of pairs $[X, \ell, z]$, where $[X]$ is in $\Delta_0$, the point $z$ is the node of $X$, and $L \in W_d^r(X)$ is a non-locally free torsion free sheaf. Let $\epsilon: \mathcal{C}_g := B\lambda(\mathcal{C}_g \times \tilde{\mathcal{M}}_g) \to \mathcal{C}_g \times \tilde{\mathcal{M}}_g$ be the blow-up of this locus, and we denote the induced universal curve by $\varphi := p_2 \circ \epsilon: \mathcal{C}_g \to \tilde{G}_d^r$.

The fibre of $\varphi$ over a point $[X, \ell, L] \in \Delta_0$, where $L \in W_d^r(X) \setminus W_d^r(X)$, is the semistable curve $X \cup \ell$ of genus $g$, where $R$ is a smooth rational curve meeting $X$ transversally at $y$ and $q$. 
2.2. A degeneracy locus inside $\mathfrak{D}_{13}^\circ$. In order to define the degeneracy locus on $\mathfrak{D}_{13}^\circ$ whose push-forward produces $[\mathfrak{D}_{13}]^{\text{virt}}$, we first choose a Poincaré line bundle $L$ over the universal curve $\mathbb{C}_g$ with the following properties:

(i) If $[X \cup_y E] \in \Delta_1$ and $\ell = (\ell_X, \ell_E) \in \mathcal{T}_d^0(X \cup E)$ is a limit linear series, then

$$L_{[X \cup_y E, \ell]} \in \text{Pic}^d(X) \times \text{Pic}^0(E).$$

(ii) For a point $t = [X_{yq}, L]$, where $[X_{yq}] \in \Delta_0$ and $L \in W^d_d(X_{yq}) \setminus W^d_d(X_{yq})$, thus $L = \nu_*(A)$ for some $A \in W^d_d(X)$, we have $L_{[X]} \cong A$ and $L_{[X]} \cong \mathcal{O}_X(1)$. Here, $\nu^{-1}(t) = X \cup R$, whereas $\nu : X \to X_{yq}$ is the normalization map.

We now introduce the following two sheaves over $\mathfrak{D}_{13}^\circ$:

$$\mathcal{E} := \nu_*(L) \quad \text{and} \quad \mathcal{F} := \mathcal{F}_{\nu}(L).$$

Both $\mathcal{E}$ and $\mathcal{F}$ are locally free; the proof by local arguments (cf. [3.6]) goes through essentially without change.

There is a sheaf morphism over $\mathfrak{D}_{13}^\circ$ globalizing the Hilbert function

$$\phi : \text{Sym}^2(\mathcal{E}) \to \mathcal{F},$$

where $\mathcal{E} := \nu_*(L)$ and $\mathcal{F} := \mathcal{F}_{\nu}(L)$. Due to its determinantal nature, $\mathfrak{D}_{13}$ carries a virtual class in the expected codimension 2.

**Definition 2.2.** We define the virtual divisor class $[\mathfrak{D}_{13}]^{\text{virt}} := \sigma_*([\mathfrak{U}]^{\text{virt}})$ as

$$[\mathfrak{D}_{13}]^{\text{virt}} := \sigma_*\left(\ell_2(\text{Sym}^2(\mathcal{E})^\vee - \mathcal{F}^\vee)\right) \in CH^1(\mathfrak{M}_{13}).$$

If $\mathfrak{U}$ has pure codimension 2, then $\mathfrak{D}_{13}$ is a divisor on $\mathfrak{M}_{13}$ and $[\mathfrak{D}_{13}]^{\text{virt}} = [\mathfrak{D}_{13}]$. The following corollary provides a local description of the morphism $\phi$.

**Corollary 2.3.** The morphism $\phi : \text{Sym}^2(\mathcal{E}) \to \mathcal{F}$ has the following description on fibres:

(i) For $[X, L] \in \mathfrak{D}_{13}^\circ$, with $[X] \in \mathfrak{M}_g \setminus \mathfrak{M}_{g,d-1}$ smooth, the fibres are

$$\mathcal{E}_t(X, L) = H^0(X, L) \quad \text{and} \quad \mathcal{F}_t(X, L) = H^0(X, L^\otimes 2),$$

and $\phi(X, L) : \text{Sym}^2 H^0(X, L) \to H^0(X, L^\otimes 2)$ is the usual multiplication map of global sections.

(ii) Suppose $t = (X \cup_y E, \ell_X, \ell_E) \in \sigma^{-1}(\Delta_1)$, where $X$ is a curve of genus $g - 1$, $E$ is an elliptic curve and $\ell_X = |L_X|$ is the $X$-aspect of the corresponding limit linear series with $L_X \in W^d_d(X)$ such that $h^0(X, L_X(-2y)) \geq r$. If $L_X$ has no base point at $y$, then

$$\mathcal{E}_t = H^0(X, L_X) \cong H^0(X, L_X(-2y)) \oplus K \cdot u \quad \text{and} \quad \mathcal{F}_t = H^0(X, L_X^\otimes 2(-2y)) \oplus K \cdot u^2,$$

where $u \in H^0(X, L_X)$ is any section such that $\text{ord}_u(u) = 0$.

If $L_X$ has a base point at $y$, then $\mathcal{E}_t = H^0(X, L_X) \cong H^0(X, L_X(-y))$ and the image of $\mathcal{F}_t \to H^0(X, L_X^\otimes 2)$ is the subspace $H^0(X, L_X^\otimes 2(-2y)) \subseteq H^0(X, L_X^\otimes 2)$.

(iii) Let $t = [X_{yq}, L] \in \sigma^{-1}(\Delta_0)$ be a point with $q, y \in X$ and let $L \in W^d_d(X_{yq})$ be a locally free sheaf of rank 1, such that $h^0(X, \nu^*L(-y - q)) \geq r$, where $\nu : X \to X_{yq}$ is the normalization. Then one has the following description of the fibres:

$$\mathcal{E}_t = H^0(X, \nu^*L) \quad \text{and} \quad \mathcal{F}_t = H^0(X, \nu^*L^\otimes 2(-y - q)) \oplus K \cdot u^2,$$

where $u \in H^0(X, \nu^*L)$ is any section not vanishing at both points $y$ and $q$.

(iv) Let $t = [X_{yq}, \nu_*(A)]$, where $A \in W^d_d(X_{yq})$ be any section not vanishing at both points $y$ and $q$. Let $R$ be the fibre $\nu^{-1}(t)$. Then $\mathcal{E}_t = H^0(X \cup R, L_{X \cup R}) \cong H^0(X, A)$ and $\mathcal{F}_t = H^0(X, \nu^*(L_{X \cup R})) \oplus K \cdot u^2$, where $u$ is any section of $H^0(X, \nu^*(L_{X \cup R}))$. Furthermore, $\phi(t)$ is the multiplication map on $X \cup R$.

**Proof.** The proof is essentially identical to [18]; we omit the details. \(\square\)
Furthermore, we define the curve
\[ F_0 := \{ X_{yq} := X \cap E : y \in X \} \subseteq \Delta^0 \subseteq \overline{M}_g, \]
and
\[ F_1 := \{ X \cup_y E : y \in X \} \subseteq \Delta^1 \subseteq \overline{M}_g. \]

Furthermore, we define the curve
\[ F_{\text{ell}} := \{ [X \cup_q E_t] : t \in \mathbb{P}^1 \} \subseteq \Delta_1 \subseteq \overline{M}_g, \]
where \([E_t, q]_{t \in \mathbb{P}^1}\) denotes a pencil of plane cubics and \(q\) is a fixed point of the pencil. We record the intersection of these test curves with the generators of \(CH^1(\overline{M}_g)\):
\[
F_0 \cdot \lambda = 0, \quad F_0 \cdot \delta_0 = 2 - 2g, \quad F_0 \cdot \delta_1 = 1 \quad \text{and} \quad F_0 \cdot \delta_j = 0 \quad \text{for} \quad j = 2, \ldots, \left\lfloor \frac{g}{2} \right\rfloor, \\
F_{\text{ell}} \cdot \lambda = 1, \quad F_{\text{ell}} \cdot \delta_0 = 12, \quad F_{\text{ell}} \cdot \delta_1 = -1 \quad \text{and} \quad F_{\text{ell}} \cdot \delta_j = 0 \quad \text{for} \quad j = 2, \ldots, \left\lfloor \frac{g}{2} \right\rfloor. 
\]

Note also that \(F_1 \cdot \lambda = 0, \quad F_1 \cdot \delta_i = 4 - 2g, \quad \text{and} \quad F_1 \cdot \delta_j = 0 \quad \text{for} \quad j \neq 1.\)

We now describe the pull back \(\sigma^*(F_0) \subseteq \mathfrak{E}_{15}^3\). Having fixed a general point \((y, q) \in \overline{M}_{12,1}\), we introduce the variety
\[
Y := \left\{ (y, L) \in X \times W^5_{16}(X) : h^0(X, L) = 0 \right\},
\]
together with the projection \(\pi_1 : Y \to X\). Arguing in a way similar to [FJP, Proposition 3.10], we conclude that \(Y\) has pure dimension 2, that is, its actual dimension is 2 as a degeneracy locus. We consider two curves inside \(Y\), namely
\[
\Gamma_1 := \left\{ (y, A(y)) : y \in X, \quad A \in W^5_{16}(X) \right\}, \\
\Gamma_2 := \left\{ (y, A(q)) : y \in X, \quad A \in W^5_{15}(X) \right\},
\]
intersecting transversely along finitely many points. We then introduce the variety \(\tilde{Y}\) obtained from \(Y\) by identifying for each \((y, A) \in X \times W^5_{15}(X)\), the points \((y, A(y)) \in \Gamma_1\) and \((y, A(q)) \in \Gamma_2\). Let \(\vartheta : Y \to \tilde{Y}\) the projection map.

**Proposition 2.4.** With notation as above, there is a birational morphism
\[ f : \sigma^*(F_0) \to \tilde{Y}, \]
which is an isomorphism outside the degeneracy loci. \(f^{-1}(\vartheta^{-1}(q))\) forgets the aspect of each limit linear series on the elliptic curve \(E\). Furthermore, \(F_{\text{ell}}(F_0)\) and \(F_{\text{ell}}(F_0)\) are pull backs under \(f\) of vector bundles on \(\tilde{Y}\).

**Proof.** The proof is identical to [FJP, Proposition 3.11]. \(\square\)

We now describe the pull back \(\sigma^*(F_1)\) and define the locus
\[
Z := \left\{ L \in W^5_{15}(X) : h^0(X, L(-2y)) \geq 5 \right\}
\]
Because \(X\) is general, we find that \(Z\) is pure of dimension 2. Next we observe that in order to estimate the intersection of \([\overline{\mathfrak{E}}_{13}\] with the surface \(\sigma^*(F_1)\), it suffices to restrict ourselves to \(Z\):

**Proposition 2.5.** The variety \(Z\) is an irreducible component of \(\sigma^*(F_1)\) and
\[
c_2 \left( \text{Sym}^2(\mathfrak{E})^\vee - F^\vee \right) |_{\sigma^*(F_1)} = c_2 \left( \text{Sym}^2(\mathfrak{E})^\vee - F^\vee \right) |_{Z}.
\]
Proof. Let \((\ell_X, \ell_E) \in \sigma^{-1}([X \cup_v E])\) be a limit linear series. Observe that that \(\rho(13, 5, 16) = 1 \geq \rho(\ell_X, y) + \rho(\ell_E, y)\). Since \(\rho(\ell_E, y) \geq 0\), it follows that \(\rho(\ell_X, y) \in \{0, 1\}\). If \(\rho(\ell_X, y) = 0\), then \(\ell_X = 10y + |O_E(6y)|\) and the aspect \(\ell_X \in G^1_{10}(X)\) is a complete linear series with a cusp at the point \(y \in X\). Therefore \((y, \ell_X) \in \mathcal{Z}\), and in particular \(\mathcal{Z} \times \{\ell_E\} \cong \mathcal{Z}\) is a union of irreducible components of \(\sigma^*(F_1)\).

The remaining cases \(\rho(\ell_X, y) = 1\), are indexed by Schubert indices \(\alpha_0 \leq \cdots \leq \alpha_5 \leq 11 = 16 - 5\), such that \(\alpha \geq 1\) are indexed lexicographically, and \(\alpha_0 + \cdots + \alpha_5 \in \{6, 7\}\), for \(\rho(\ell_X, y) \geq -1\), for any point \(y \in X\). For a Schubert index \(\alpha\) satisfying these conditions, we let \(\alpha^c \defeq (11 - \alpha_0 - \cdots - \alpha_5)\) be an elementary Schubert index, and define

\[
\mathcal{Z}_\alpha \defeq \{ X(y) \geq \alpha \} \quad \text{and} \quad \mathcal{W}_\alpha \defeq \{ \ell_E \in G^1_{10}(E) : \alpha^{\ell_E}(y) \geq \alpha^c \}.
\]

Then the following relations hold for certain natural coefficients \(m_\alpha\).

\[
\sigma^*(F_1) = \mathcal{Z} + \sum_{\alpha \geq (0, 1, 1, 1, 1, 1)} m_\alpha (\mathcal{Z}_\alpha \times \mathcal{W}_\alpha).
\]

We now finish the proof by invoking the pointed Brill-Noether theorem on \(\mathcal{Z}\), which gives \(\dim \mathcal{Z} = 1 + \rho(13, 5, 16) - \alpha_0 - \cdots - \alpha_5 \leq 1\). In the context of \(\mathcal{Z}\), the point of attachment of \(\ell_E\) is the restriction of both \(\ell_X\) and \(\ell_E\) to \(Z_\alpha\) and one obtains \(\mathcal{Z} \times \{\ell_E\} \cong \mathcal{Z}\) for dimension reasons. \(\square\)

2.4. Top Cycles. We use various facts about intersection theory of Jacobians, for which we refer to [II-VIII]. We start with a general curve \(X\) of genus \(g\), fix a Poincaré line bundle \(\mathcal{P}_X\), and denote by \(\pi_1: X \times \Pic^d(X) \to \Pic^d(X)\) the two projection maps, \(\pi_1: X \times \Pic^d(X), \mathcal{Z}\), where \(x_0 \in X\) is a fixed point. We choose a symplectic basis \(\delta_1, \ldots, \delta_{2g} \in H^1(X, \mathcal{Z}) \cong H^1(\Pic^d(X), \mathcal{Z})\), and then consider the class \(\pi_1^*(\delta_1) - \pi_1^*(\delta_{g+a})\).

One has nice relations \(\gamma^1 = 0\, \gamma^2 = 0\, \eta^2 = 0\, \eta^2 = -2\eta\pi_2^*(\theta)\), for which we refer to [II-VIII]. Assuming \(W^d_{1+1}(X) = 0\) (which is what happens in the case of interest), the smooth variety \(W^d_X\) admits a rank \(r + 1\) vector bundle \(\mathcal{M} = (\pi_2)_* \left( \mathcal{P}_{X \times W^d_X} \right)\).

The Chern numbers of \(\mathcal{M}\) are computed via the Harris-Tu formula, formally

\[
\sum_{i=0}^{r} c_i(\mathcal{M}^r) = (1 + \rho(\ell_X, y) - k) \cdot (1 - k)!
\]

for \(c_i(\mathcal{M}^r) \in \mathcal{Z}\), the Chern numbers of \(H^{top}(W^d_X, \mathcal{Z})\), the Chern numbers of \(H^{top}(W^d_X, \mathcal{Z})\) can be computed by using repeatedly the following formula

\[
x_{r+1} \cdot \rho(g, r, d) = \sum_{i_1, \ldots, i_r} \frac{1}{(i_1 + \cdots + i_r + k)!} \cdot \frac{1}{(i_1! \cdots i_r!)} \cdot \frac{1}{(k)!}.
\]

A detailed discussion of how to read and apply the Harris-Tu formula in this context.
We now specialize to the case when \( X \) is a general curve of genus 12, thus \( W^6_{16}(X) \) is a smooth 6-fold. By Grauert’s Theorem, \( N := \langle R^1\pi_2 \rangle_* \left( P_{X \times W^6_{16}(X)} \right) \) is locally free of rank one. Set \( y_1 := c_1(N) \).

Proposition 2.6. For a general curve \( X \) of genus 12 set \( c_i := c_i(M^\vee) \), for \( i = 1, \ldots , 6 \), and \( y_1 := c_1(N) \). Then the following relations hold in \( H^k(W^6_{16}(X), \mathbb{Z}) \):

\[
c_i = \frac{\theta^i}{i!} - \sum_{j=1}^{i-1} \frac{\theta^j}{(i-1)!} y_1, \quad \text{for} \quad i = 1, \ldots , 6.
\]

Proof. For an effective divisor \( D \) of sufficiently large degree on \( X \), we have \( W \) the first degeneracy locus of \( \pi \), by Grauert’s Theorem, we have \( W \triangleleft \mathcal{M} \). Proposition 2.6, any Chern number on \( X \), with this preparation in place, we now compute the classes of the loci \( Z \) and \( Y \).

Recall that \( N \) is the vector bundle on the right in the exact sequence above. \( H \) as claimed. □

We now specialize to the case when \( X \) is a general curve of genus 12, thus \( W^6_{16}(X) \) is a smooth 6-fold. By Grauert’s Theorem, \( N := \langle R^1\pi_2 \rangle_* \left( P_{X \times W^6_{16}(X)} \right) \) is locally free of rank one. Set \( y_1 := c_1(N) \).

Proposition 2.7. Let \( [X, q] \) be a smooth projective curve of genus 12, let \( \mathcal{M} \) denote the tautological rank 6 vector bundle over \( W^6_{16}(X) \), and set \( c_i := c_i(M^\vee) \), as before. The following formulas hold:

(i) \( [Z] = \pi_2^*(c_5) - 6\eta\theta\pi_2^*c_2 \) and \( [Y] = \pi_2^*(c_6) - 2\eta\theta\pi_2^*c_3 \) in \( H^{10}(X \times W^6_{16}(X), \mathbb{Z}) \).

Proof. The locus \( Z \) has been given as the first degeneracy locus of a vector bundle morphism over the 7-dimensional smooth variety \( W^6_{16}(X) \). We observe again that \( W^6_{16}(X) = 0 \). For each \( (y, L) \in X \times W^6_{16}(X) \), there is a vector bundle morphism \( \mathcal{F} \), and the Porteous formula, \( [Z] = c_5 (\pi_2^*(M)^\vee - J_1(P)^\vee) \).

These maps viewed together induce a morphism \( \mathcal{F}_1(P)^\vee \rightarrow \pi_2^*(M)^\vee \) of vector bundles. Then \( Z \) is the first degeneracy locus of \( \mathcal{F} \) and applying the Porteous formula,

\[
[Z] = c_5 (\pi_2^*(M)^\vee - J_1(P)^\vee).
\]

The Chern classes of the jet bundle \( J_1(P) \) are computed using the standard exact sequence

\[
0 \rightarrow \pi_1^*(\omega_X) \otimes \mathcal{P} \rightarrow J_1(P) \rightarrow \mathcal{P} \rightarrow 0.
\]

We compute the total Chern class of the formal inverse of the jet bundle as follows:

\[
c_{\text{tot}}(J_1(P)^\vee)^{-1} = \left( \sum_{j \geq 0} (d(L)\eta + \gamma)^j \right) \left( \sum_{j \geq 0} ((2g(X) - 2 + d(L))\eta + \gamma)^j \right),
\]

\[
= (1 + 16\eta + \gamma + \gamma^2 + \cdots) \cdot (1 + 38\eta + \gamma^2 + \cdots),
\]

\[
= 1 + 54\eta + 2\gamma - 6\eta\theta.
\]

Multiplying this with the total class of \( \pi_2^*(M)^\vee \), one finds the claimed formula for \([Z]\).
To compute the class of $Y$ in $\mathcal{M}_{13}$ over the projections $\pi_1, \pi_2: \mathcal{M}_{13} \to X \times Pic^{16}(X)$, and let $\Delta \subseteq X \times X \times Pic^{16}(X)$ be the graph of the diagonal $\Delta := \{ (x, x, \pi_1^*\mathcal{M}) \} \subset X \times X \times Pic^{16}(X)$ and consider the vector bundle $\mathcal{B} := \mu_* (\nu^*(\mathcal{P}) \otimes \mathcal{O}_{X \times X \times Pic^{16}(X)}(\Delta))$ and apply Grothendieck-Riemann-Roch to the projection map $\pi_1: \mathcal{M}_{13} \to X$. By Grauert’s Theorem, we apply Grothendieck-Riemann-Roch to the projection map $\pi_1: \mathcal{M}_{13} \to X$. We introduce two further vector bundles which appear in many of our calculation. Their Chern classes are computed via Grothendieck-Riemann-Roch.

**Proposition 2.8.** Let $[X, q]$ be a general pointed curve of genus 12 and consider the vector bundles $\mathcal{A}_2$ and $\mathcal{B}_2$ on $X \times Pic^{16}(X)$ having fibres

$$\mathcal{A}_{2,(y, L)} = H^0(X, L^\otimes 2(-2y)) \quad \text{and} \quad \mathcal{B}_{2,(y, L)} = H^0(X, L^\otimes 2(-y - q)),$$

respectively. One then has the following formulas for their Chern classes:

$$c_1(\mathcal{A}_2) = -4\theta - 4\gamma - 86\eta, \quad c_1(\mathcal{B}_2) = -4\theta - 2\gamma - 31\eta,$$

$$c_2(\mathcal{A}_2) = 8\theta^2 + 320\eta\theta + 16\gamma \theta, \quad c_2(\mathcal{B}_2) = 8\theta^2 + 116\eta\theta + 8\theta \gamma.$$

**Proof.** We apply Grothendieck-Riemann-Roch to the projection map $\nu: X \times X \times Pic^{16}(X) \to X \times Pic^{16}(X)$. Via Grauert’s theorem, $\mathcal{A}_2$ can be realized as a push forward under the map $\nu$, precisely

$$\mathcal{A}_2 = \nu \left( \mu^*(\mathcal{P}^\otimes 2 \otimes \mathcal{O}_{X \times X \times Pic^{16}(X)}(\Delta)) \right) = \nu_* \left( \mu^*(\mathcal{P}^\otimes 2 \otimes \mathcal{O}_{X \times X \times Pic^{16}(X)}(-\Delta)) \right).$$

Applying Grothendieck-Riemann-Roch to $\nu$, we find $\text{ch}_2(\mathcal{A}_2) = 8\eta\theta$, and $\nu_*(c_1(\mathcal{P})^2) = -2\theta$. One then obtains $c_1(\mathcal{A}_2) = -4\theta - 4\gamma - (4d(L) + 2q(C) - 2)\eta$, which yields the formula for $c_2(\mathcal{A}_2)$. To determine the Chern classes of $\mathcal{B}_2$, we observe $c_1(\mathcal{B}_2) = -4\theta - 2\gamma - (2d - 1)\eta$ and $\text{ch}_2(\mathcal{B}_2) = 4\eta\theta$. \(\square\)

### 3. The Class of the Virtual Divisor $\tilde{\mathcal{D}}_{13}$

In this section we determine the virtual class $[\tilde{\mathcal{D}}_{13}]^{\text{virt}} := \sigma_* (c_2(\text{Sym}^2(\mathcal{E}))^\vee - \mathcal{F}^\vee)$ on $\overline{\mathcal{M}}_{13}$. We begin by recording the following formulas for a vector bundle $\mathcal{V}$ of rank $r + 1$ on a surface

$$c_1(\text{Sym}^2(\mathcal{V})) = (r + 2)c_1(\mathcal{V}), \quad c_2(\text{Sym}^2(\mathcal{V})) = \frac{r(r + 3)}{2} c_1^2(\mathcal{V}) + (r + 3)c_2(\mathcal{V}).$$

We apply these formulas for the first degeneracy locus of $\phi^\vee: \mathcal{F}^\vee \to \text{Sym}^2(\mathcal{E})^\vee$. By definition its class $[\mathcal{U}]^{\text{virt}}$ is given by

$$c_2(\text{Sym}^2(\mathcal{E})^\vee - \mathcal{F}^\vee) = c_2(\text{Sym}^2(\mathcal{E})^\vee) - c_1(\text{Sym}^2(\mathcal{E})^\vee) \cdot c_1(\mathcal{F}^\vee) + c_1^2(\mathcal{F}^\vee) - 2c_1^{\vee}(\mathcal{E}) + 8c_2(\mathcal{E}) - 7c_1(\mathcal{E}) \cdot c_1(\mathcal{F}) + c_1^2(\mathcal{F}) - c_2(\mathcal{F}).$$

In what follows we expand the virtual class in $CH^1(\overline{\mathcal{M}}_{13})$ as

$$[\mathcal{D}_{13}]^{\text{virt}} = a\lambda - b_0\delta_0 - b_1\delta_1.$$

We compute the coefficients $a, b_0$ and $b_1$, by intersecting both sides of this expression with the test curves $F_0, F_1$ and $F_{\text{ell}}$. We start with the coefficient $b_1$. 

Theorem 3.1. Let $X$ be a general curve of genus 12. The coefficient

$$b_1 = \frac{1}{2g(X) - 2} \sigma^*(F_1) \cdot c_2(S\text{ym}^2 \mathcal{E})$$

in (13) is:

$$b_1 = \frac{1}{2g(X) - 2} \sigma^*(F_1) \cdot c_2(S\text{ym}^2 \mathcal{E}) = c_2(S\text{ym}^2 \mathcal{E} - F')|_Z.$$ 

By Proposition 2.7, we find that

$$\phi: \Gamma \to \mathcal{F}_{\mathbb{A}}(16) \otimes \mathcal{O}_{\mathbb{A}}(-2y)$$

constructed a morphism $\zeta: J_1(P)^\vee \to \pi_2^*(\mathcal{M})^\vee$ of vector bundles on $Z$, where $\mathcal{M} = \mathcal{F}_{\mathbb{A}}(16)^\vee \to H^0(X, L)^\vee$. The kernel sheaf Ker$(\zeta)$ is locally free of rank 1.

Over a general point $(y, L)$, we intersect the degeneracy locus of the map $\phi$:

$$H^0(X, L(-2y)) \hookrightarrow H^0(X, L \otimes \mathcal{O}_{\mathbb{A}}(2y))$$

Proceeding as in the proof of Proposition 2.7, one computes the Chern classes of $\mathcal{M}$ with fibre $U$ over a point $(y, L)$.

We write

$$\pi_1^*(c_4) + (54\eta + 2\gamma)\pi_2^*(c_4) + (54\eta + 2\gamma)\pi_2^*(c_3) \cdot \xi|_Z.$$

Similarly, one has the intersection on the surface $Z$:

$$c_2^1(\text{Ker}(\zeta)) = \frac{1}{2g(X) - 2} \sigma^*(F_1) \cdot c_2(S\text{ym}^2 \mathcal{E}) - \xi|_Z.$$

In particular, we have

$$\gamma + 54\eta + c_1(\text{Ker}(\zeta)).$$

The Harris-Tu formula [HT]:

$$H^0(X, L) \otimes \mathcal{O}_{\mathbb{A}}(2y)$$

We also observe that

$$\mathcal{A}_3 = \text{Pic}^{16}(X).$$

Let $A_3$ denote the line bundles constructed as a push forward of a line bundle on $X \times X \times \text{Pic}^{16}(X)$. Then the line bundle $U^{\otimes 2}$ can be embedded in $A_3/A_2$. We consider the quotient

$$\mathcal{G} := A_3/A_2 \otimes U.$$
Recalling that $\mathcal{E}|_Z = \pi_2^*(\mathcal{M})$, we see that
\[
\sigma^*(F_1) \cdot c_2((\text{Sym}^2 \mathcal{E})^\vee - \mathcal{F}^\vee)
\]
is equal to:
\[
\eta \bar{\pi}^* \left( -602c_1 c_5 + 432c_2 c_4 - 120c_4^2 c_3 \theta + 168c_1 c_3 \theta^2 
- 4c_1^{12} - 4c_1^{10} c_3 - 1080c_1^2 c_3 \theta - 48c_2^2 c_3 \theta - 384c_4^2 \theta^2 \right) + 4(2\gamma + 14\eta) \cdot c_1(A_{2|Z}) + 14(2\gamma + 54\eta) \cdot \pi_2^*(c_1).
\]

This polynomial gets multiplied by the class $[Z]$, which is expressed in
\[
5 \text{ polynomial in } \theta, \eta \text{ and } \bar{\pi}_2^*(c_1).
\]
We obtain a homogeneous polynomial of degree
\[
13 \text{ in } \eta, \theta, \bar{\pi}_2^*(c_1)
\]
which is given by the following formula:
\[
\sigma^*(F_1) \cdot c_2((\text{Sym}^2 \mathcal{E})^\vee - \mathcal{F}^\vee) = \eta \bar{\pi}^* \left(-602c_1 c_5 + 432c_2 c_4 - 120c_4^2 c_3 \theta + 168c_1 c_3 \theta^2 
- 4c_1^{12} - 4c_1^{10} c_3 - 1080c_1^2 c_3 \theta - 48c_2^2 c_3 \theta - 384c_4^2 \theta^2 \right) + 4(2\gamma + 14\eta) \cdot c_1(A_{2|Z}) + 14(2\gamma + 54\eta) \cdot \pi_2^*(c_1).
\]

We suppress $\eta$ and the remaining expression is equal to
\[
\sigma^*(F_1) \cdot c_2((\text{Sym}^2 \mathcal{E})^\vee - \mathcal{F}^\vee) = \eta \bar{\pi}^* \left(-602c_1 c_5 + 432c_2 c_4 - 120c_4^2 c_3 \theta + 168c_1 c_3 \theta^2 
- 4c_1^{12} - 4c_1^{10} c_3 - 1080c_1^2 c_3 \theta - 48c_2^2 c_3 \theta - 384c_4^2 \theta^2 \right) + 4(2\gamma + 14\eta) \cdot c_1(A_{2|Z}) + 14(2\gamma + 54\eta) \cdot \pi_2^*(c_1)
\]

where for the last step we used the fact that
\[
b_1 = 2247.
\]
as required.

**Theorem 3.2.** Let $[X, q]$ be a general pencil in $\mathcal{C}_r$. Let $F_0 \subset \Delta_0 \subset \Delta_{13}$ be the associated test curve. Then the coefficient of $\delta_0$ in the homogenous polynomial of degree $13$ is equal to
\[
b_0 = \sigma^*(F_1) = \eta \bar{\pi}^* \left(-602c_1 c_5 + 432c_2 c_4 - 120c_4^2 c_3 \theta + 168c_1 c_3 \theta^2 
- 4c_1^{12} - 4c_1^{10} c_3 - 1080c_1^2 c_3 \theta - 48c_2^2 c_3 \theta - 384c_4^2 \theta^2 \right) + 4(2\gamma + 14\eta) \cdot c_1(A_{2|Z}) + 14(2\gamma + 54\eta) \pi_2^*(c_1) = \sqrt{2247}.
\]
This quadratic polynomial gets multiplied with the class \( \sigma(F_0) \). In particular, we have
\[
\sigma(F_0) \cdot c_2(\text{Sym}^2(E)^\vee - F^\vee)|_Y = c_2(\text{Sym}^2(E)^\vee - F^\vee)|_Y.
\]

We shall evaluate \( c_2(\text{Sym}^2(E)^\vee - F^\vee)|_Y \) via the line bundle \( V \) on \( Y \) with fibre
\[
V(y,L) = \mathcal{H}^0(X,L) \\
\mathcal{H}^0(X,L(-y-q)) \to \mathcal{H}^0(X,L \otimes O_{y+q})
\]
over a point \((y,L) \in Y\). We write the following exact sequence over \( Y \)
\[
0 \to V \to B \to (\text{Ker}(\chi))^\vee \to 0,
\]
where the morphism \( \chi \) is defined in the final part of the proof of
Theorem 3.1:
\[\text{In particular, we have } c_1(B) = c_1(\text{Ker}(\chi)) + c_1(\text{Ker}(\chi)).\]

The effect on multiplication on \( \text{Ker}(\chi) \) classified by \( \xi \in H^2(X \times W_{16}^5(X), \mathbb{Z}) \)
applying once more the following formula holds:
\[
c_1(\text{Ker}(\chi)) = 2\pi_2^*(c_4) + (15\eta + \gamma)\pi_2^*(c_5) + \xi|_Y,
\]
where we recall that \( \pi_2^*(c_4) \in H^{2i}(W_{16}^5(X), \mathbb{Z}) \). Similarly, for the self-
intersection on \( Y \) the following expression is valid:
\[
c_1(\text{Ker}(\chi)) = \pi_2^*(c_4) + (15\eta + \gamma)\pi^*_2(c_5) + \xi|_Y,
\]
where \( \pi_2^*(c_4) \in H^{2i}(W_{16}^5(X), \mathbb{Z}) \).

We have expanded this expression, computed in the final part of the proof of Proposition 2.7. Next, we
perform a local calculation along the lines of the one in
Proposition 2.8 the vector bundle \( B_2 \) on \( X \times \text{Pic}^{16}(X) \) with fibres
\( B_2(y,L) = H^1(X,L(-y-q)) \). Using the above calculation along the lines of the one in
the proof of Proposition 2.7, we can determine a decomposition on \( Y \), which can then be used to
determine the following formula holds:
\[
c_1(\mathcal{F}|_Y) = c_1(B_2|_Y) + c_2(B_2|_Y) + 2c_1(B_2|_Y)c_1(V).
\]

This exact sequence shows that for a general point \((y,L) \in Y\) one has a decomposition \( \mathcal{F}(y,L) = \mathcal{H}^0(X,L(-y-q)) \), \( y \in H^0(X,L) \) is a section not vanishing at \( y \) and \( q \). We thus obtain the formulas:
\[
c_1(\mathcal{F}|_Y) = c_1(B_2|_Y) + c_2(B_2|_Y) + 2c_1(B_2|_Y)c_1(V).
\]

To estimate \( c_2(\text{Sym}^2(E)^\vee - F^\vee)|_Y \), we have
\[
\sigma^*(F_0) \cdot c_2(\text{Sym}^2(E)^\vee - F^\vee)|_Y = \pi_2^*(c_4) + 7\pi_2^*(c_5) \cdot c_2\text{(B}_2|_Y \cdot c_1(V).
\]

We expand this expression, collecting the terms that contain \( c_1(\text{Ker}(\chi)) \)
resulting in the following:
\[
20\pi_2^*(c_1^2) - 7\eta \pi_2^*(c_1) - 28\theta \cdot \pi_2^*(c_1) + 4\theta \eta + 8\theta^2 + 8\pi_2^*(c_1).
\]

This quadratic polynomial gets multiplied with the class \( \xi \) computed in
Proposition 2.8, we collect the terms in \( \sigma^*(F_0) \cdot c_2(\text{Sym}^2(E)^\vee - F^\vee)|_Y \)
resulting in the following:
\[
\sigma^*(F_0) \cdot c_2(\text{Sym}^2(E)^\vee - F^\vee)|_Y = \eta \pi_2^*(c_1^2) + 4\theta \eta + 8\theta^2 + 8\pi_2^*(c_1).
\]

This quadratic polynomial gets multiplied with the class \( \xi \) computed in
Proposition 2.8, we collect the terms in \( \sigma^*(F_0) \cdot c_2(\text{Sym}^2(E)^\vee - F^\vee)|_Y \)
resulting in the following:
\[
\sigma^*(F_0) \cdot c_2(\text{Sym}^2(E)^\vee - F^\vee)|_Y = \eta \pi_2^*(c_1^2) + 4\theta \eta + 8\theta^2 + 8\pi_2^*(c_1).
\]
For the notation later, the vertices of \( \Gamma \) are labeled \( w_{13-g}, \ldots, w_{13}, v_{14-g}, \ldots, v_{14} \), as shown. In particular, we will consider positive dimensional fibres onto a divisor in \( \overline{\mathcal{M}_{13}} \). As in \[\text{[FJP]}\], this additional step is necessary to show that the push-forward of the virtual class \( [\overline{\mathcal{D}}_{13}]^{\text{virt}} \) is surjective. This is one case of the Strong Maximal Rank Conjecture in genus 13. As in \[\text{[FJP]}\], to which we refer the reader for details and further references, the motivation and technical foundations for this approach are detailed in \[\S\S\] 2.4-2.5, and \section{1.4-1.5}, to which we refer the reader for further details.

Let \( X \) be a smooth projective curve of genus 13. As in \[\text{[FJP]}\], this additional step is necessary to show that the push-forward of the virtual class \( [\overline{\mathcal{D}}_{13}]^{\text{virt}} \) is surjective. This is one case of the Strong Maximal Rank Conjecture in genus 13. As in \[\text{[FJP]}\], to which we refer the reader for details and further references, the motivation and technical foundations for this approach are detailed in \[\S\S\] 2.4-2.5, and \section{1.4-1.5}, to which we refer the reader for further details.

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Let \( X \) be a smooth projective curve of genus 13. As in \[\text{[FJP]}\], this additional step is necessary to show that the push-forward of the virtual class \( [\overline{\mathcal{D}}_{13}]^{\text{virt}} \) is surjective. This is one case of the Strong Maximal Rank Conjecture in genus 13. As in \[\text{[FJP]}\], to which we refer the reader for details and further references, the motivation and technical foundations for this approach are detailed in \[\S\S\] 2.4-2.5, and \section{1.4-1.5}, to which we refer the reader for further details.

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We write $\gamma_k$ for the $k$th loop in the chain of loops $\Gamma$, where the edges of length $\ell_k$ and $m_k$ between $v_k$ and $w_k$, for $14 - g \leq k \leq 13$, respectively. Let $w_{k-1}$ and $v_k$, for $0 \leq k \leq g$, which has length $n_k$. By Theorem 4.1, we assume that these edge lengths satisfy

\[ \ell_k \ll m_k \ll n_k \quad \text{for all } k. \]

These conditions are precisely as in [FJP, §8.2].

Recall that tropical independence is a sufficient condition for linear independence; if $\{\psi_0, \ldots, \psi_n\}$ on $X$ is a piecewise linear function with integer slopes on $\Gamma$, then it is tropically independent.

We will use the following generalization of Theorem 4.1 in our proof that $X$ is Brill-Noether general [CDPR].

**Theorem 4.1.** Let $X$ be a curve of genus 13 with a virtual class, and let $V$ be a linear series of degree 16 and dimension 5 on $X$, and let $\Sigma = \text{trop} \ V$. Then there is an independence $\theta$ among 20 pairwise sums of functions in $\Sigma$ which we highlight when they arise.

**Theorem 4.2.** Let $X$ be a curve of genus $g \geq 12$, and let $\Gamma$ be a loop of length $\ell_k$ between $v_k$ and $w_k$. Let $V$ be a linear series of degree 16 and dimension 5 on $X$. Assume that

(i) if $g = 12$, then $a_V^1(p) \geq 2$, and

(ii) if $g = 11$, then either $a_V^1(p) \geq 3$, or $a_V^0(p) + a_V^2(p) \geq 3$.

Then there is an independence $\theta$ among 20 pairwise sums of functions in $\Sigma$. The remainder of this section is devoted to the proof of Theorem 4.2. In all three cases, the adjusted Brill-Noether number $\rho(V, p)$ is equal to 1. In particular, this means that there is at most one unramified and vertex avoiding. Our approach to constructing the independence is similar to the proof that $X$ is Brill-Noether general. The situation is closely parallel to that in [FJP, §8.2], with a few important differences which we highlight when they arise.
Rem. 4.3. The graph is vertex avoiding. Even in the unramified and vertex avoiding case, we obtain an independence among only 19 functions if we break the graph into blocks in such a way that the lingering loop is the last loop in its block and has exactly two permissible functions. This allows us to assign a total number of functions in the independence to 20. See Remark 4.3.

Let us consider how the slope vector \( s_k \) changes as we go left to right across the graph. When crossing a loop other than the lingering loop \( \gamma \), the other 5 remain the same. So, after the first non-lingering loop, the slopes are \( s_k \) and after the second non-lingering loop, the slopes are \( s_k' \) or \( s_k \), respectively. Since \( \dim V = 6 \), the functions in \( \Sigma \) have exactly 6 distinct slopes along each tangent vector in \( \Gamma \).

\[ \begin{array}{c}
\gamma_k \\
\vdots \\
v_k \\
\vdots \\
w_k \\
\vdots \\
s_k \\
\vdots \\
s_k' \\
\end{array} \]

**Figure 2.** The slopes \( s_k \) and \( s_k' \).

**Definition 4.4.** Let \( s_k[0] < \cdots < s_k[5] \) and \( s_k'[0] < \cdots < s_k'[5] \) denote the 6 distinct rightward slopes that occur as \( s_k(\psi) \) and \( s_k'(\psi) \), for \( \psi \in \Sigma \).

Since \( D \) is vertex avoiding, there is a function \( \varphi_0 \in \Sigma \) such that

\[ s_k(\varphi_0) = s_k[i] \quad \text{and} \quad s_k'(\varphi_0) = s_k'[i] \]

for all \( k \), and it is an unramified divisor. Since \( \Sigma \) is also unramified, there is a unique lingering loop \( \gamma_\ell \), i.e., a unique loop \( \gamma_\ell \) such that \( s_k[5] = s_k'[5] = s_k[i] \) for all \( i \). Moreover, there is no function \( \varphi \in \Sigma \) with the property \( \dim V \geq s_k'[i] + 1 \). This last condition means that \( \gamma_\ell \) is not a switching loop, in the sense of [15, §6.19].

Our assumption that \( \Sigma \) is unramified implies that the break divisor \( D \) satisfies \( \deg w_0 D = 3 \), and the rightward slopes along \( \gamma_\ell \) at the points \( v_i \) are

\[ s_0[0], \ldots, s_0[5] = (-2, -1, 0, 1, 2, 3). \]

Let us consider how the slope \( s_k[i] \) changes as we go left to right across the graph. When crossing a loop other than the lingering loop \( \gamma_\ell \), one of these slopes increases by 1, and the other 5 remain the same. So, after the first non-lingering loop, the slopes are \( (-2, -1, 0, 1, 2, 4) \), and after the second non-lingering loop, the slopes are either \( (-2, -1, 0, 1, 2, 5) \) or \( (-2, -1, 0, 1, 3, 4) \). The data of these slopes is recorded by a standard Young tableau on a rectangle with 2 rows and 6 columns, filled with the symbols 1 through 13, excluding \( \ell \). If the symbol \( k \) appears in column \( i \), then it is the \( (5-i) \)th slope that increases on the loop \( \gamma_k \), i.e., \( s_k[i] = s_k[5-i] + 1 \). Note, in particular, that each slope increases exactly twice, so \( s_{13} = (0, 1, 2, 3, 4, 5) \) and no slope is ever greater than 5.

We first divide the graph into three blocks. Within each block, the slope of \( \theta \) will be nearly constant on each bridge, equal to 4 on bridges in the first block, 3 on bridges in the second block, and 2 on
bridges in the third block. Let

\[ z_1 = \min\{6, \ell\} \]
\[ z_2 = \max\{7, \ell\}. \]

We will construct our independence \( \theta \) so that its incoming slope at the loop \( \gamma_k \) is:

\[
s_k(\theta) = \begin{cases} 
4 & \text{if } k \leq z_1, \\
3 & \text{if } z_1 < k \leq z_2, \\
2 & \text{if } z_2 < k \leq 13.
\end{cases}
\]

In other words, \( \gamma_{z_1} \) and \( \gamma_{z_2} \) are the last loops of the first and second blocks, respectively. Note that either \( z_1 \) or \( z_2 \) is equal to \( \ell \), so the lingering loop \( \gamma_\ell \) is always the last loop in its block.

The following definition gives a natural necessary condition for a function in \( \Sigma \) to achieve the minimum at some point of a given loop.

**Definition 4.5.** Let \( \psi \in \text{PL}(\Gamma) \) be a function. We say that \( \psi \) is permissible on \( \gamma_k \) if

\[ s_k(\psi) \leq s_k(\theta) \quad \text{and} \quad s'_k(\psi) \geq s_k(\theta). \]

We say that \( \psi \) is permissible on a block if it is permissible on some loop in that block. A permissible function \( \psi \) is new if \( s_k(\psi) \leq s_k(\theta) - 1 \), and it is departing if \( s'_k(\psi) \geq s_k(\theta) + 1 \).

To understand this definition, recall that \( \theta \) has nearly constant slopes along bridges, and the bridges adjacent to a loop are much longer than the bridge itself. If \( s_k(\psi) \geq s_k(\theta) + 1 \), then the value of \( \psi \) at \( v_k \) exceeds the value of \( \theta \) at \( v_k \) by at least the length of the bridge \( \beta_k \) (or half this length, if \( \beta_k \) is the bridge between two blocks). Since this bridge is much longer than the loop \( \gamma_k \), it follows that \( \psi \) cannot achieve the minimum at any point of \( \gamma_k \). A similar argument shows that if \( s'_k(\psi) \leq s_k(\theta) - 1 \), then \( \psi \) cannot achieve the minimum at any point of \( \gamma_k \). Our choice of \( z_1 \) and \( z_2 \) completely determines which loops have new permissible functions.

**Proposition 4.6.** There is no new permissible function on \( \gamma_k \) if and only if \( k = \ell \) or

(i) \( \ell > 6 \) and \( k = 6 \); (iii) \( \ell < 9 \) and \( k = 9 \); or
(ii) \( \ell > 7 \) and \( k = 7 \); (iv) \( \ell \leq 7, s'_k[5] = 4 \), and \( k = 8 \).

**Proof.** There is no new permissible function on the lingering loop \( \gamma_\ell \). Suppose \( k \neq \ell \). Let \( j \) be the unique integer satisfying \( s'_k[j] = s_k[j] + 1 \). There is a new permissible function on \( \gamma_k \) if and only if either the function \( \varphi_{jj} \) is both new and departing, or there is an integer \( i \) such that \( s'_k(\varphi_{jj}) = s_k(\theta) \).

We now examine when such an \( i \) exists.

The values \( s'_k[i] \) are 6 distinct integers between \(-2\) and \(5\). Let \( a \) and \( b \) be the two integers in this range that are not equal to \( s'_k[i] \) for any \( i \). On the \( h \)th non-lingering loop, one has

\[
h = \sum_{i=0}^{5} (s'_k[i] + 2 - i) = 9 - (a + b).
\]

Since \( s'_k[j] = s_k[j] + 1 \), we must have that \( s'_k[j] \) is equal to either \( a + 1 \) or \( b + 1 \). Without loss of generality, assume that it is equal to \( a + 1 \). There does not exist \( i \) such that \( s'_k[i] + s'_k[j] = s'_k(\theta) \) if and only if \( s'_k(\theta) - (a + 1) \) is greater than \( 5 \), smaller than \(-2\), or equal to either \( a \) or \( b \). If it is equal to \( a \), then the function \( \varphi_{jj} \) is both new and departing. Since \( s'_k(\theta) \leq 4 \) and \( a + 1 \geq -1 \), we see that \( s'_k(\theta) - (a + 1) \) cannot be greater than \( 5 \), and \( s'_k(\theta) - (a + 1) \) is smaller than \(-2 \) if and only if \( s'_k(\theta) = 2 \) and \( a = 4 \). By the above calculation, \( b = s'_k(\theta) - (a + 1) \) if and only if \( h = 10 - s'_k(\theta) \).

The 6th non-lingering loop is contained in the first block if and only if \( \ell > 6 \). The 7th non-lingering loop is contained in the second block if and only if \( \ell > 7 \). The 8th non-lingering loop is contained in the third block if and only if \( \ell < 9 \). Finally, if \( a = 4 \), then \( \gamma_k \) is one of the first 7 non-lingering loops. If \( \gamma_k \) is in the third block, then since \( z_2 \geq 7 \), we have \( \ell \leq 7 \), and \( \gamma_k \) is the first loop in the third block. \( \square \)
The functions that appear in the tropical independence $\theta$ are as follows. Let
$$B = \{\varphi_i + \varphi_j : 0 \leq i \leq j \leq 5\}.$$ 
Note that $B$ has 21 elements. The tropical independence that we construct uses only 20 elements, which form a subset $B' \subseteq B$. We describe how to choose this subset $B'$. We make our choices so that the number of permissible functions on each block is 1 more than the number of loops in that block.

If $\ell \leq 7$, then we let $\psi \in B$ be a function that is permissible on the second block and $B' = B \setminus \{\psi\}$. If $\ell > 7$, then we let $\psi \in B$ be a function that is permissible on the third block and $B' = B \setminus \{\psi\}$.

**Remark 4.7.** Note that there are several ways to choose the permissible on the second or third block; it does not matter which of these we choose to omit from the set $B'$.

**Lemma 4.8.** On each block, the number of permissible functions in $B'$ is one more than the number of loops.

**Proof.** This follows directly from the assumption that $\ell > 7$. Since $z_1 = \min\{6, \ell\}$, there is a new permissible function in $B$ on each pair of bridges except for the last one. Since there are precisely two pairs $(i, j)$ such that $s_{ij} \leq 7$, the number of permissible functions on the first block is 1 more than the number of loops, and if $z_2 \leq 7$, then the number of permissible functions in $B$ on the second block is also one more. But when $z_2 > 7$, one of these functions is not in $B'$.

Finally, consider the middle block. We count the number of pairs $(i, j)$ such that $s_{ij} = 3$. Since 3 is odd, if $z_2 \geq 7$, then there are precisely two such pairs, and if $z_2 < 7$, there are three such pairs. By Remark 4.9, there is a new permissible function on every loop of the middle block. If the first block contains only one loop, and since this loop is not lingering, there are no new permissible functions on it. In both of these cases, the number of permissible functions in $B'$ on the second block is one more than the number of loops, but one of these functions is not in $B'$. Therefore, the number of permissible functions on $\gamma_7$ or $\gamma_\ell$, so the number of permissible functions in $B'$ on each block is one more than the number of loops.

□

The algorithm for constructing the tropical independence is identical to that presented in the vertex avoiding case. Specifically, we do not skip the lingering loops $\gamma_\ell$. Instead, since $\gamma_\ell$ is the last loop in the block, there are precisely two unassigned permissible functions on $\gamma_\ell$. These two functions have identical restrictions to $\gamma_\ell$. Thus, if we adjust their coefficients upward so that one of them will obtain the minimum uniquely at some point of the loop $\gamma_\ell$. We adjust $\varphi_{ij}$ to $\varphi_{ij} + \epsilon$, small enough so that $\varphi_{ij}$ does not achieve the minimum uniquely at some point of $\gamma_\ell$. The other will then obtain the minimum uniquely at $w_\ell$, and we assign this function to the bridge $\beta_{\ell+1}$.

We now verify that this algorithm produces a tropical independence.

**Lemma 4.10.** Suppose that $\varphi_{ij}$ is assigned to the loop $\gamma_k$, and $\varphi_{ij}$ does not achieve the minimum at any point to the right of $w_{k+1}$.

**Proof.** If $\gamma_k$ is a non-lingering loop, then the assumption that $\varphi_{ij}$ does not achieve the minimum is that $\psi \notin B'$. On the other hand, if $\gamma_k$ is the lingering loop, then the assumption that $\varphi_{ij}$ does not achieve the minimum is that $\psi \notin B'$. This follows directly from the fact that $B'$ contains precisely two permissible functions on $\gamma_k$. □

This completes the proof.

**Remark 4.11.** For future reference, we might want to state the assumptions. For instance, we might want to state the additional assumptions that each bridge is much longer than the next, and that the bridges are much longer than the loops. The assumption that each bridge is much longer than the next, and that the bridges are much longer than the loops, is only used when there are decreasing bridges, decreasing loops, or switching loops.
Remark 4.12. If $\Gamma'$ is the subgraph of $\Gamma$ to the right of $w_1$, then $\Gamma'$ is a chain of 12 loops whose edge lengths satisfy the required conditions, and the restriction of $\Sigma$ to $\Gamma'$ satisfies the ramification condition of Theorem 4.2. If the first loop is non-lingering, then the restriction of $\Sigma$ to this subgraph satisfies the ramification condition with equality. To produce an independence in these cases, assign each function in $B'$ with slope greater than 4 to the first bridge, and then proceed as above. There are precisely $15-g$ such functions, and they have distinct slopes along the first bridge. We pick these coefficients so that each one obtains the minimum uniquely at some point on the first bridge. Thus the unramified vertex avoiding cases of Theorem 4.2 for $g = 11$ and 12 follow from essentially the same argument as for $g = 13$. Our choice to index the vertices starting at $w_{13-g}$ reflects the idea that these linear series with ramification on a chain of $g = 11$ or 12 loops behave like linear series on a chain of 13 loops restricted to the subgraph to the right of $w_{13-g}$.

Example 4.13. We illustrate the construction with an example. Let $[D]$ be a vertex avoiding class of degree 16 and rank 5 associated to the tableau in Figure 3.

[Figure 3. The tableau corresponding to the divisor $D$.]

The independence $\theta = \min_{ij}(\phi_{ij} + c_{ij})$ that we construct is depicted schematically in Figure 4. The graph should be read from left to right and top to bottom, so the first 6 loops in the first row, with $\gamma_1$ on the left and $\gamma_6$ on the right, and $\gamma_{13}$ is the last loop in the third row, correspond to the three blocks. The 31 dots indicate the support of the divisor $D' = 2D + \text{div}(\theta)$, and $\text{deg}(D') = 32$; the point on the bridge $\beta_1$ appears with multiplicity 2, as marked. There is a function that is permissible on the second block in $B$ but not $B'$. The functions in $B$ that are permissible on the second block are precisely $\phi_{05}$, $\phi_{14}$, and $\phi_{23}$; we have chosen (arbitrarily) to omit $\phi_{23}$ from $B'$. Each of the 20 functions $\phi_{ij}$ in $B'$ achieves the minimum uniquely on the connected component of the complement of $\text{Supp}(D')$ labeled $ij$.

[Figure 4. The divisor $D' = 2D + \text{div}(\theta)$. The function $\phi_{ij}$ achieves the minimum uniquely on the region labeled $ij$ in $\Gamma \setminus \text{Supp}(D')$.]

4.2. No switching loops. Recall that a loop $\gamma_\ell$ is a switching loop for $\Sigma$ if there is some $\varphi \in \Sigma$ and some $h$ such that $s_\ell(\varphi) \leq s_\ell(h)$ and $s'_\ell(\varphi) \geq s'_\ell(h+1)$. It is a lingering loop if it is not a switching loop and $s_\ell(i) = s'_\ell(i)$ for all $i$. Recall also that $\gamma_\ell$ is a decreasing loop if $s_\ell(h) > s'_\ell(h)$. Similarly $\beta_\ell$ is a decreasing bridge if $s'_{\ell-1}(h) > s_\ell(h)$. 


Considering cases where the adjusted Brill-Noether number is 1, we may assume that there is either a lingering loop, a positive ramification weight, or no switching loop. The cases with a switching loop are discussed in Proposition 4.14. If \( \gamma \) is the last loop in its block, in which case the \( k \)-th loop is a decreasing loop, then by (iv) there is no function in \( B' \) that is permissible on more than one block, or no function of the form \( \phi_{ij} \) is permissible on \( \gamma_k \) and \( \gamma_k \) is a bridge between blocks, or no function of the form \( \phi_{ij} \) is permissible on \( \gamma_k \) and \( \gamma_k \) is a decreasing bridge and \( j \) is the unique value such that \( s_k'[^j] < s_k[j] \) is the unique value such that \( s_k s > s_k'[^j] \) is the unique value such that \( s_k s > s_k'[^j] \) is the unique value such that \( s_k s > s_k'[^j] \). In this case, we consider all cases where there is a unique function with slope equal to that of \( \theta \) at a point to the right of \( v \), and \( \phi_{ij} \) cannot achieve the minimum at any point to the right of \( v \), and \( \phi_{ij} \) cannot achieve the minimum at any point to the right of \( v \), and \( \phi_{ij} \) cannot achieve the minimum at any point to the right of \( v \). We show that \( s_k(\phi_i) = s_k[i] \) and \( s_k'(\phi_i) \) is locally nonlinear.

We again let \( B = \{ \phi_i + \phi_j : 0 \leq i, j \leq 5 \} \). As in the unramified vertex avoiding case, we choose a subset \( B' \subseteq B \) of 20 functions, and we choose integers \( z_1 \) and \( z_2 \) in order to divide the graph \( \Gamma \) into 3 blocks. We make our choices to satisfy the following conditions:

1. no two functions in \( B' \) that are permissible on \( \gamma_k \) differ by a constant on \( \gamma_k \),
2. no function in \( B' \) that is permissible on each block is at most one more than the number of loops in that block,
3. if \( \gamma_k \) is a lingering loop, then it is the last loop in its block,
4. if \( \gamma_k \) is a decreasing loop and \( j \) is the unique value such that \( s_k'[^j] < s_k[j] \), then no function of the form \( \phi_{ij} \) is permissible on \( \gamma_k \), and
5. if \( \beta_k \) is a decreasing bridge and \( j \) is the unique value such that \( s_k[j] < s_k'[^j] \), then either \( \gamma_k \) is a bridge between blocks, or no function of the form \( \phi_{ij} \) is permissible on \( \gamma_k \).

**Proposition 4.14.** If \( B' \) satisfies conditions (i)-(v), then the functions in \( B' \) are independent.

**Proof.** The algorithm for constructing the tropical independence is identical to that presented in [FJP, §7], with the following exceptions. First, as in Remark 4.12, we assign every function with slope greater than 4 to the first bridge. Second, we do not skip lingering loops and instead treat them as permissible on \( \gamma_k \). We initialize the coefficients of the new permissible functions on the next block so that they are equal to \( \theta \) at a point to the right of \( v \), and \( \phi_{ij} \) cannot achieve the minimum at any point to the right of \( v \), and \( \phi_{ij} \) cannot achieve the minimum at any point to the right of \( v \). We may therefore assume that \( \gamma_k \) is the last loop in its block. If \( \gamma_k \) is a decreasing loop, then by (v) there is no function in \( B' \) that is permissible on \( \gamma_k \) and contains the decreasing function as a summand, so the result holds again by Proposition 4.14. If \( \gamma_k \) has positive multiplicity, then by (vi) the argument is identical to the vertex avoiding case above. Otherwise, \( \beta_k \) is a decreasing bridge. By (iii) the function \( \phi_{ij} \) is only permissible on one block. Since \( \gamma_{k+1} \) is the last loop in its block, \( \phi_{ij} \) cannot achieve the minimum at any point to the right of \( \gamma_{k+1} \). □
For the rest of this section, we explain how to choose \( z_1, z_2 \), and the set \( B' \) in order to satisfy conditions (i)-(vi). This is done by a careful case analysis, depending on combinatorial properties of the tropical linear series \( \Sigma \).

**Case 1: There are no loops or bridges of positive multiplicity.** This guarantees that the linear series is ramified either at \( w_{13-g} \) or \( v_{14} \) (but not both, since \( \rho = 1 \)). In this case, we choose \( z_1 \) and \( z_2 \) so that \( \gamma_{z_1} \) is the first loop in the first block with no new function, and \( \gamma_{z_2+1} \) is the last loop in the last block with no departing function. These loops are guaranteed to exist by a counting argument, but we can in fact be more explicit.

If \( \Sigma \) is ramified at \( v_{14} \), let \( k \) be the smallest positive integer such that \( s_k[5] = 6 \), and define

\[
(21) \quad z_1 = \begin{cases} 
6 & \text{if } k \geq 7; \\
7 & \text{if } k \leq 6;
\end{cases} \quad \text{and} \quad z_2 = \max\{k-1, 7\}.
\]

If \( \Sigma \) is ramified at \( w_{13-g} \), let \( k \) be the largest positive integer such that \( s_k[0] = -3 \), and define

\[
(22) \quad z_1 = \min\{k, 6\}, \quad \text{and} \quad z_2 = \begin{cases} 
6 & \text{if } k \geq 8; \\
7 & \text{if } k \leq 7.
\end{cases}
\]

Let \( \psi \in B \) be a function that is permissible on the second block, and let \( B' = B \setminus \{\psi\} \). (In the case \( \psi \in B \), we construct an independence loop.) In the case where there is only one such loop or bridge of positive multiplicity 1.

**Case 2:** There is a switching bridge \( \beta \) of multiplicity 1, such that \( \Sigma \) ramifies at \( \beta \) and \( s_\ell[0] = -3 \), then define \( z_1 \) and \( z_2 \) as in (21). Otherwise, let \( \ell \) be the unique integer such that \( s_\ell[5] = 6 \) or \( \ell \leq 7 \) and \( s_\ell[0] = -3 \), then define \( z_1 \) and \( z_2 \) as in (22). Otherwise, let \( \ell \) be the unique integer such that \( s_\ell[4] < s_{\ell-1}[4] \). If \( \ell = 5, 6 \), then we will see in Lemma 4.15 that there is only one such loop or bridge of multiplicity 1.

**Case 3:** There is a decreasing loop \( \gamma \) of multiplicity 1, such that \( s_\ell[0] = -3 \), then define \( z_1 \) and \( z_2 \) as in (21). Otherwise, let \( \ell \) be the unique integer such that \( s_\ell[5] = 6 \) or \( \ell \leq 7 \) and \( s_\ell[0] = -3 \), then define \( z_1 \) and \( z_2 \) as in (22). Otherwise, let \( \ell \) be the unique integer such that \( s_\ell[4] < s_{\ell-1}[4] \). If \( \ell = 5, 6 \), then we will see in Lemma 4.15 that there is only one such loop or bridge of multiplicity 1.
In the cases above, we asserted that some permissible functions exist with specified slopes. To prove this, we need to generalize Proposition 4.6. We first define the following function.

Note that, if there is a loop of pecidity one, then the \( k \)th loop of multiplicity zero, then \( k = \tau(\ell) \).

For a fixed \( k \), suppose that \( -2 \leq s_k[i] \leq 5 \) for all \( i \). Let \( j \) be an integer such that \( s_k[j] - 1 \) is not equal to \(-3\) or \( s_k[i] \) for any \( i \). For \( s \) in the range \( 2 \leq s < 5 \), there does not exist \( i \) such that \( s_k[i] = s \) if and only if one of the following holds:

(i) \( \tau(k) = 10 - s \);
(ii) \( s = 5, j = 0, \) and \( s_k[0] = -1; \)
(iii) \( s = 3, j = 1, \) and \( s_k[1] = 5 \).

Proof. The argument is identical to that of Proposition 4.6. \( \square \)

There are additional relevant cases, when \( \tau(\ell) \) is a bridge of multiplicity one, and let \( \theta \) be the unique integer such that \( s_k[0] = \theta \).

If \( \phi \) is permissible, then we may assume that \( \phi \) is a decreasing loop. Let \( h \) be the unique integer such that \( s_k[h] = s_k[h] + 1 \), and let \( h' \) be the unique integer such that \( s_k[h'] = s_k[h'] + 1 \). Let \( \theta \) be the unique integer such that \( s_k[0] = \theta \).

Similarly, if \( \phi \) is permissible, then by Lemma 4.15, such an \( \phi \) exists if and only if one of the following holds:

(i) \( \tau(k) = 10 - s \);
(ii) \( s = 5, j = 0, \) and \( s_k[0] = -1; \)
(iii) \( s = 3, j = 1, \) and \( s_k[1] = 5 \).

Proof. Since \( \phi \) is permissible, then we may assume that \( \phi \) is a decreasing loop. Let \( h \) be the unique integer such that \( s_k[h] = s_k[h] + 1 \), and let \( h' \) be the unique integer such that \( s_k[h'] = s_k[h'] + 1 \).

Similarly, if \( \phi \) is permissible, then by Lemma 4.15, such an \( \phi \) exists if and only if one of the following holds:

(i) \( \tau(k) = 10 - s \);
(ii) \( s = 5, j = 0, \) and \( s_k[0] = -1; \)
(iii) \( s = 3, j = 1, \) and \( s_k[1] = 5 \).

Proof. The argument is identical to that of Proposition 4.6. \( \square \)

There are additional relevant cases, when \( \tau(\ell) \) is a bridge of multiplicity one, and let \( \theta \) be the unique integer such that \( s_k[0] = \theta \).

If \( \phi \) is permissible, then we may assume that \( \phi \) is a decreasing loop. Let \( h \) be the unique integer such that \( s_k[h] = s_k[h] + 1 \), and let \( h' \) be the unique integer such that \( s_k[h'] = s_k[h'] + 1 \). Let \( \theta \) be the unique integer such that \( s_k[0] = \theta \).

Similarly, if \( \phi \) is permissible, then by Lemma 4.15, such an \( \phi \) exists if and only if one of the following holds:

(i) \( \tau(k) = 10 - s \);
(ii) \( s = 5, j = 0, \) and \( s_k[0] = -1; \)
(iii) \( s = 3, j = 1, \) and \( s_k[1] = 5 \).

Proof. Since \( \phi \) is permissible, then we may assume that \( \phi \) is a decreasing loop. Let \( h \) be the unique integer such that \( s_k[h] = s_k[h] + 1 \), and let \( h' \) be the unique integer such that \( s_k[h'] = s_k[h'] + 1 \).

Similarly, if \( \phi \) is permissible, then by Lemma 4.15, such an \( \phi \) exists if and only if one of the following holds:

(i) \( \tau(k) = 10 - s \);
(ii) \( s = 5, j = 0, \) and \( s_k[0] = -1; \)
(iii) \( s = 3, j = 1, \) and \( s_k[1] = 5 \).

Proof. The argument is identical to that of Proposition 4.6. \( \square \)
an $i$ exists if and only if $\ell \geq 8$, and if $s_\ell[h] = 5$, then such an $i$ exists if and only if $\ell \leq 7$. In both cases, we have $\varphi_{hi} \notin B'$.

Next, consider the case where $\gamma_\ell$ is a decreasing loop. By construction, $\gamma_\ell$ is either the first or last loop in its block. Let $h$ be the unique integer such that $s'_\ell[h] = s_\ell[h] - 1$. If $\gamma_\ell$ is the last loop in its block and $\varphi_{ij}$ is permissible on both the block containing $\gamma_\ell$ and the next block, then $j = h$ and either $s_\ell[h] + s_i[i] = s_i(\theta)$, or $i = h$ and $2s_\ell[h] = s_i(\theta) + 1$. But then $\varphi_{ij} \notin B'$. Similarly, if $\gamma_\ell$ is the first loop in its block, and $\varphi_{ij}$ is permissible on both the block containing $\gamma_\ell$ and the preceding block, then $j = h$ and either $s_\ell[h] + s_i[i] = s_{i-1}(\theta)$, or $2s_\ell[h] = s_{i-1}(\theta) + 1$. If $\ell \neq 7$, then again $\varphi_{ij} \notin B'$. Finally, note that if $\gamma_\ell$ is both the first and last loop in its block, then $\ell = 7$, and the only functions $\varphi_{ij}$ that are permissible on $\gamma_7$ satisfy $s'_\ell[i] + s'_\ell[j] = 3$. The result follows.

**Condition (iv):** If $\gamma_\ell$ is a lingering loop, then we follow the construction of the vertex avoiding case of the previous subsection, in which $\gamma_\ell$ is the last loop in its block.

**Condition (v):** Let $\gamma_k$ be a decreasing loop, let $h$ be the unique integer such that $s'_k[h] = s_k[h] + 1$, and let $h'$ be the unique integer such that $\varphi_{hi} \notin B'$, as shown in the proof of condition (i).

**Condition (vi):** Let $\beta_k$ be a decreasing bridge and let $\gamma_k$ be a decreasing loop, let $k$ be the unique integer such that $\varphi_{hi} \notin B'$, as shown in the proof of condition (i).

This completes the proof.

### 4.3. Switching loops.

We now describe a switching loop $\gamma_\ell$, which switches slope $h$. This means that $s'_\ell[h] + 1 = s'_\ell[h + 1]$.

Let us now define $z_1$ and $z_2$ as follows:

$$z_1 = \begin{cases} 
\ell & \text{if } \ell < 6, \\
5 & \text{if } \ell = 6, \\
6 & \text{if } \ell > 6;
\end{cases} \quad \text{and} \quad z_2 = \begin{cases} 
7 & \text{if } \ell < 6, \\
\ell & \text{if } \ell \geq 6.
\end{cases}$$

From [9.18], there is a pencil $W \subseteq V$ with $\varphi_A$, $\varphi_B$, and $\varphi_C$ in $\text{trop}(W)$ such that:

- $\varphi_A < \ell$;
- $\varphi_B < \ell$;
- $\varphi_C < \ell$;
- all $k \leq \ell$; and $s'_k(\varphi_C) = s_k[h]$ for all $k \geq \ell$;
- $\varphi_A(1)$ and $s'_k(\varphi_A) \in \{s'_k[h], s'_k[h + 1]\}$ for all $k$.

Moreover, from [9.18], there are functions $\varphi^0_h$, $\varphi^0_{h+1}$, and $\varphi^\infty$ in $R(D)$ such that:

- $\varphi^0_h = s'_k[h]$ for all $k$;
- $\varphi^0_{h+1} = s'_k[h + 1]$ for all $k$;
- $\varphi^\infty = s'_k[h]$ for all $k \leq \ell$;
- $\varphi^\infty = s'_k[h + 1]$ for all $k \geq \ell$.

(4) The function $\varphi_A$ is a tropical linear combination of the functions $\varphi^0_h$ and $\varphi^\infty$, where the two functions simultaneously achieve the minimum at a point to the right of $\gamma_\ell$.

(5) The function $\varphi_B$ is a tropical linear combination of the functions $\varphi^0_{h+1}$ and $\varphi^\infty$, where the two functions simultaneously achieve the minimum at a point to the left of $\gamma_\ell$.

(6) The function $\varphi_C$ is a tropical linear combination of the functions $\varphi^0_h$ and $\varphi^0_{h+1}$, where the two functions simultaneously achieve the minimum on the loop $\gamma_\ell$ where they agree.

Note that $\varphi^0_h$, $\varphi^0_{h+1}$, and $\varphi^\infty$ are in $R(D)$ but not necessarily in $\Sigma$. Let

$$A = \{ \varphi_i : i \neq h, h + 1 \} \cup \{ \varphi^0_h, \varphi^0_{h+1}, \varphi^\infty \},$$
and let $B$ be the set of pairwise sums of elements of $A$. We first notice that if $B' = \emptyset$, and then construct a tropical linear combination $\vartheta$ of the elements of $B''$ achieving the best approximation of $\vartheta$ by certain pairwise sums of elements of $B''$. If there exists a $b_j$ such that $s_j' + s_j'' = s_{j+1}' = s_{j+1}''$. By construction, it suffices to consider the case where $\varphi$ is a function that is permissible on the second block, and let $B$ be the set of pairwise sums of elements of $\Sigma$ in independence among pairwise sums of elements of $B''$. This completes the proof of Theorem 4.2 in all cases where there is a switching loop for $\Sigma$.

Lemma 4.18. For each $\varphi \in A$, one of the following holds:

(i) $\varphi + \varphi_0' \notin B''$, 
(ii) $\varphi + \varphi_0'' \notin B''$, 
(iii) $\varphi + \varphi_0' + \varphi_0'' \notin B''$, or 
(iv) $\varphi + \varphi_0' + \varphi_0'' \in B''$.

Proof. By considering the case where $\varphi + \varphi_0'$ is not assigned to a loop $\gamma_k$ with $k < \ell$, but $\varphi + \varphi_0''$ is.

If $\ell < 6$, the output of this algorithm is an optimal function on $\gamma_k$ for all $k < \ell$. If $\ell$ is equal to 6 or 7, then it is the first block.

$$s_\ell(\varphi) + s_\ell(\varphi_0') = s_\ell(\varphi + \varphi_0').$$

Since $\varphi + \varphi_0'$ is permissible on the second block, and if both $\varphi + \varphi_0'$ and $\varphi + \varphi_0''$ are in $B''$, then we also assign $\varphi + \varphi_0''$ to $\gamma_k$. Similarly, if $\varphi + \varphi_0' + \varphi_0''$ is assigned to $\gamma_k$.

By construction, in this case $\varphi + \varphi_0'' \notin B''$.

If $\ell > 7$, then we could instead have

$$s_\ell(\varphi) + s_\ell(\varphi_0') = s_\ell(\varphi).$$

But in this case, $\varphi + \varphi_0'' \notin B''$. \hfill \square

We have the following.

Lemma 4.19. The best approximation of $\vartheta$ by $\varphi_C + \varphi_j$ achieves equality on the region where either $\varphi_j + \varphi_j$ or $\varphi_j + \varphi_j$ achieves the minimum.

\begin{itemize}
  \item If $\varphi_j + \varphi_j$ and $\varphi_j + \varphi_j$ then this is immediate. If not, it does not contain $\varphi_j + \varphi_j$. Then $s_\ell(\varphi_j) = s_\ell(\varphi_0'') + 1$. In this case, $\varphi_C + \varphi_j$ has slope greater than $s_\ell(\varphi_j)$ on $\beta_k$, so we replace $B_j = B'' \cap \{\varphi_j + \varphi_j, \varphi_j + \varphi_j, \varphi_j + \varphi_j\}$ with $\{\varphi_0' + \varphi_j, \varphi_j + \varphi_j\}$. 
  \item If $\varphi_j + \varphi_j$ achieves the minimum, then $s_\ell(\varphi_j) + 1 = s_\ell(\varphi_j)$, and we replace $B_j = B'' \cap \{\varphi_j + \varphi_j, \varphi_j + \varphi_j\}$ with $\{\varphi_j + \varphi_j\}$. 
  \item If $\varphi_j + \varphi_j \notin B''$, then we replace only with $\{\varphi_C + \varphi_j\}$. 
\end{itemize}

If $\Sigma$ contains all pairwise sums of elements of $\{\varphi_j + \varphi_j\}$, then the proof of independence among pairwise sums of elements of $\Sigma$. This completes the proof of the following two cases where there is a switching loop for $\Sigma$.
5. Effectivity of the virtual class

Recall that $\overline{\mathcal{M}}_{g,n}$ is the moduli stack of stable curves, and $\overline{\mathcal{G}}_d$ is a stack of generalized bundles $\phi: \mathcal{V} \rightarrow \overline{\mathcal{M}}_g$. Let $\sigma: \overline{\mathcal{G}}_d \rightarrow \overline{\mathcal{M}}_{g,n}$ be a morphism of vector bundles $\phi: \mathcal{V} \rightarrow \overline{\mathcal{M}}_g$.

The case of Theorem 4.2 where $g = 13$ is covered by Proposition 5.1 shows that $\sigma_*[\mathcal{U}]^{virt}$ under the proper forgetful map $\mathcal{U} \rightarrow \overline{\mathcal{M}}_g$ is effective, not just a rigid virtual class. We will use the additional cases where $g = 12$ or $g = 11$. We begin with the following:

**Theorem 5.1.** Let $\mathcal{U} \subseteq H^0(X, L)^{\otimes 2}$ be a linear series of rank $r > 0$ and degree $d = 16$. Assume that $\mathcal{U}$ is not surjective.

Then the multiplication map $\phi_{V}: \text{Sym}^2 V \rightarrow H^0(X, L)^{\otimes 2}$ is generically finite over each component of $\mathcal{U}$.

We now prove that $\mathcal{U}$ is generically finite over each component of $\mathcal{U}$, which implies that $\mathcal{U} \rightarrow \overline{\mathcal{M}}_{g,n}$ is effective. Our argument follows that of Lemma 5.2, except in many cases the arguments are identical, and we omit the proofs. We prove the following:

(i) $Z$ is a general point.
(ii) If $g = 12$, then $a_1^V(p) \geq 2$, and $a_0^V(p) \geq 3$, or $a_0^V(p) = 2$.
(iii) If $g = 11$, then either $a_1^V(p) \geq 3$, or $a_0^V(p) = 2$.

The image of the degeneracy locus $\mathcal{V}$ is not surjective.

The case of Theorem 4.2 where $g = 12$ is covered by Proposition 5.1 shows that $\sigma_*[\mathcal{U}]^{virt}$ under the proper forgetful map $\mathcal{U} \rightarrow \overline{\mathcal{M}}_g$ is effective, not just a rigid virtual class. We will use the additional cases where $g = 11$ or $g = 10$. We begin with the following:

**Theorem 5.1.** Let $\mathcal{U} \subseteq H^0(X, L)^{\otimes 2}$ be a linear series of rank $r > 0$ and degree $d = 16$. Assume that $\mathcal{U}$ is not surjective.

Then the multiplication map $\phi_{V}: \text{Sym}^2 V \rightarrow H^0(X, L)^{\otimes 2}$ is generically finite over each component of $\mathcal{U}$.

We now prove that $\mathcal{U}$ is generically finite over each component of $\mathcal{U}$, which implies that $\mathcal{U} \rightarrow \overline{\mathcal{M}}_{g,n}$ is effective. Our argument follows that of Lemma 5.2, except in many cases the arguments are identical, and we omit the proofs. We prove the following:

(i) $Z$ is the closure of a divisor in $\mathcal{U}_{13}$.
(ii) $j_2(Z) = 0$.
(iii) $Z$ does not contain any codimension 2 strata.

The only irreducible divisors in $\mathcal{U}_{13}$ are $\Delta_0$ and $\Delta_1$. Therefore, item (1), that $Z$ is the closure of a divisor in $\mathcal{U}_{13}$, is a consequence of the following.

**Proposition 5.2.** The image of the degeneracy locus $\mathcal{V}$ is not surjective.

**Proof.** The proof is identical to [FJP, §10.3].

The proofs of (2) and (3) use the following.

**Lemma 5.3.** If $X$ is a smooth curve of genus $13$, and $\mathcal{V} \subseteq H^0(X, L)^{\otimes 2}$ is a linear series $V \in G_{16}^5(X)$ that is ramified at $p$ such that $\phi_{V}$ is not surjective.

**Proof.** The proof is identical to [FJP, §10.3].

5.1. Pulling back the Weierstrass divisor $\overline{\mathcal{W}}_2$.

We consider the preimage of $Z$ under the map $j_2$.

**Lemma 5.4.** The preimage of $\overline{\mathcal{W}}_2$ under the Weierstrass divisor $\overline{\mathcal{W}}_2$ in $\mathcal{M}_{2,1}$.

**Proof.** The proof is identical to [FJP, §10.3].

To prove that $j_2^*\overline{\mathcal{W}}_2$ is generically finite, we use the following construction. Let $\Gamma$ be a chain of 13 loops with the following restrictions:

(i) $0 = m_0 := \sum_{k=1}^{12} n_k$.
(ii) $m_k \leq n_k$ for all $k \neq 2$.

The last condition says that, subject to the constraints of conditions (i) and (ii), the edge lengths other than the one that connect a smooth curve to a smooth curve away from its skeleton is $\Gamma$. We first note the following.

**Lemma 5.5.** $Z_{\Gamma} = 0$.

**Proof.** The proof is identical to the first part of Lemma 5.1 [FJP, §10.6].
Proposition 5.6.

Proof. Let \( X \) be a point specializing to \( v_1 \). Just as \([X] \notin Z\), we divide \( \Gamma \) into two subgraphs \( \Gamma' \) and \( \Gamma'' \) to the left and right, respectively, of the long bridge \( \beta_3 \). Let \( q \) be a point specializing to \( v_1 \), \( v_1' \), \( v_1'' \), which is ramified at \( q \). We may assume \( X \) is in \( \Gamma'' \) and \( \beta_3 \) goes through \( \Gamma' \). \( \Gamma \) has codimension \( 4 \). Next, we construct an independence among \( 15 \) pairwise sums of functions in \( \Sigma \) restricted to \( \Gamma'' \), with the property that \( s_3[i] \) is strictly positive. Since \( X \) is a break divisor, and consider \( \Sigma = \text{trop}(V) \). We will show that \( \Sigma \) is tropical independent pairwise sums of functions in \( \Sigma \) using a variant of the argument in \( \text{§} \), following the methods of \( \text{§} \) \( \text{[FJP]} \).

It remains to verify (3), that \( s_3[5] = 5 \). In this case, even though the restriction of \( \Sigma \) to \( \Gamma'' \) to the left and right, respectively, of the midpoint of the long bridge \( \beta_3 \). First, suppose that \( \ell \) is ramified at \( q \), and we also have \( s_{14}[5] \geq 5 \). Moreover, since the divisor \( D_{\Gamma''} - s_3[5]w_2 \) has degree \( 5 \) and no divisor of degree 1 on \( \Gamma'' \) has positive rank, \( s_3[5] \) must be exactly 3. Since the canonical class is the only divisor class of degree 2 and rank 1 on \( \Gamma'' \), we see that \( D_{\Gamma''} - s_3[5]w_2 \sim K_{\Gamma''} + 3w_2 \). This yields an upper bound on each of the slopes \( s_3[i] \), and these bounds determine the slopes for \( i \geq 2 \):

\[
\begin{align*}
\end{align*}
\]

Moreover, we also have \( s_3[i] \geq 2 \) for \( 2 \leq i \leq 5 \). Since \( \ell \) is ramified at \( q \), we also have \( s_{14}[5] \geq 6 \). These conditions together imply that the sum of the multiplicities of all loops and bridges on \( \Gamma'' \) is at most 1.

To construct an independence among \( 15 \) pairwise sums of functions in \( \Sigma \) restricted to \( \Gamma'' \), first we construct an independence among \( 5 \) functions in \( \Sigma \) restricted to \( \Gamma'' \), with the property that each of the functions \( \phi_{i} \) satisfies \( s_3[i] > 0 \). If each of the functions \( \phi_{i} \) below \( X \) is ramified at \( q \), we can obtain the minimum on \( \Gamma'' \) satisfies \( s_3[5] \geq 5 \). Since the bridge \( \beta_3 \) is very long, it follows that \( \Gamma'' \) satisfies \( s_3[5] \geq 5 \). The rest of the argument is exactly the same as that of \( \text{[FJP]} \).

5.2. Higher codimension boundary strata

It remains to construct an independence among \( 15 \) pairwise sums of functions in \( \Sigma \) restricted to \( \Gamma'' \), with the property that \( s_3[i] \) is strictly positive. Since \( X \) is a break divisor, the minimum uniquely at some point of \( \Gamma'' \). First, suppose that \( \beta_3 \) goes through \( \Gamma'' \). Following this construction, there will be 2 along the last bridge \( \beta_{14} \). We eliminate \( \psi_{14} \), and to the left and right, respectively, of the midpoint of the long bridge \( \beta_3 \). To do this, we run the algorithm \( \text{§} \), \( \text{[FJP]} \). Let \( 1 \) be a smooth curve of genus 2 over \( K \) whose skeleton \( \Gamma_1 \) is a chain of
2 loops with bridges, and let \( p \in Y \) be a third point specializing to the right endpoint of \( \Gamma_1 \). Similarly, let \( Y_2 \) and \( Y_3 \) the same, with \( p \) specializing to the right endpoint of \( \Gamma_2 \). Then, since the skeletons \( \Gamma_2 \) and \( \Gamma_3 \), are chains of \( \ell \) loops and \( \ell \) bridges, we can further that the edges in the final loop of \( \Gamma_2 \) are labeled \( \beta_1, \beta_2, \ldots, \beta_\ell \) in which case \( \beta \) obtain the minimum on a loop or bridge where it is permissible. In particular, for each \( \beta \) these properties guarantee that, even though the bridge \( \beta \) is not possible, using the tropical independence construction from \( \text{trop}(V) \). We have these functions \( s_k(\varphi_i) \geq 6 \). Also, since \( V \) is smooth and \( s_k(\varphi_i) \geq 6 \). These conditions imply that the multiplicity of every loop and bridge under \( \phi \) is at most 6. For each \( i \) there is a function \( \varphi_i \) satisfying

\[
s_k(\varphi_i) = s_{k-1}(\varphi_i) = s_k[i] = s_k(\varphi_i).
\]

These functions have constant slope along bridges, and the slopes \( s_k(\varphi_i) \) are nondecreasing in \( k \). These properties guarantee that, even though the bridge \( \beta \) is very long, a function \( \varphi_i \) can only obtain the minimum on a loop or bridge where it is permissible.

Even though the result relies on the tropicalization of a linear series on a curve of genus 11 with prescribed ramification at points specializing to the left and right endpoints of \( \Gamma \), it satisfies all of the combinatorics of the tropicalization of such a linear series, and we may apply the algorithm. Note that we are in a situation where the relative lengths of the bridges do not matter (Remark 4.11) the construction yields an independence among 20 pairwise sums of functions in \( \Sigma \), with bridges labeled \( \beta \).

6. THE BERTRAM-FEINBERG-MUKAI CONJECTURE IN GENUS 13

The aim of this section is to prove a basic part of the Bertram-Feinberg-Mukai conjecture on \( \overline{M}_{13} \). For a smooth curve \( X \) of genus \( g \), we denote by \( SU_X(2,\omega) \) the moduli space of \( S \)-equivalence classes of semistable rank 2 vector bundles \( E \) with \( \det(E) \equiv \omega_X \). For an integer \( k \geq 0 \), the Brill-Noether locus

\[
SU_X(2,\omega; k) = \{ X \to \mathbb{P}^1, E \in SU_X(2,\omega), h^0(X, E) \geq k \}
\]

has the structure of a Lagrangian subvariety of the component of \( SU_X(2,\omega) \). Moreover, there is a component of \( SU_X(2,\omega; k) \) at least \( \beta(2, g, k) = 3g - 3 \). The Brill-Noether locus \( SU_X(2,\omega; k) \) is smooth at a point \( [E] \) corresponding to a semistable bundle if and only if the Mukai-Petri map

\[
\beta(2, g, k) \to \omega_X \oplus \omega_X
\]

is injective. Of particular interest is the case \( \beta(2, g, k) = 2 \).

in which case \( \beta(2, 13, 8) = 0 \). First, using linkages methods, we show that a general curve \( \Gamma \) carries a stable vector bundle \( E \in SU_X(2,\omega; 8) \). Then using a Hecke correspondence, we compute the fundamental class of \( SU_X(2,\omega; 8) \).

**Theorem 6.1.** A general curve \( X \) of genus 13 carries a stable vector bundle \( E \) of rank 2 with \( \det(E) \equiv \omega_X \) and \( h^0(X, E) = 8 \).

As a first step towards proving this, we determine the extension type of the vector bundles in question.
**Proposition 6.2.** For a general curve $X$ of genus 13, every vector bundle $E \in SU_X(2, \omega, 8)$ can be represented as an extension

\[
0 \to \mathcal{O}_X(D) \to E \to \omega_X(-D) \to 0,
\]

where $D$ is an effective divisor such that $L := \omega_X(-D) \in W^6_{18}(X)$ is very ample and the map $\phi_L : \text{Sym}^2 H^0(L) \to \text{Ext}^1(L, \mathcal{O}_X(D))$ is surjective. Conversely, a very ample $L \in W^6_{18}(X)$ with $\phi_L$ not surjective induces a stable vector bundle $E \in SU_X(2, \omega, 8)$.

**Proof.** Using a result of [CGZ, 7.2], every semistable vector bundle $E$ on $X$ of rank 2 and canonical degree 8 and canonical determinant carries a line subbundle $\mathcal{O}_X \subset E$ with $\deg(D) \geq \frac{2g-2}{2}$. Therefore, in our case $\deg(D) \geq 6$.

If $h^0(X, \mathcal{O}_X(D)) = 8$ and the multiplication map $\phi_{\omega_X(-D)} : \text{Sym}^2 H^0(L) \to \text{Ext}^1(L, \mathcal{O}_X(D))$ is not surjective, which contradicts the general assumption $h^0(L) = 8$. Setting $L := \omega_X(-D) \in W^6_{18}(X)$, an extension (23) exists if and only if the multiplication map

\[
\phi_L : \text{Sym}^2 H^0(L) \to \text{Ext}^1(L, \mathcal{O}_X(D))
\]

is not surjective, which contradicts the general assumption $h^0(L) = 8$ and the multiplication map $\phi_{\omega_X(-D)}$ is not surjective, which contradicts $h^0(L) = 8$.

Since $X$ is general, by Theorem 1.5 the multiplication map $\phi_{\omega_X(-D)}$ is surjective, implying $h^0(L, \mathcal{O}_X(D)) \geq \frac{2g-2}{2} \subseteq \text{Im}(\phi_L)$. We deduce that $[E]$ lies in the kernel of the map

\[
\text{Ext}^1(L, D) \to \text{Ext}^1(L(-x-y), D)
\]

That is, the vector bundle $E$ can also be represented as an extension

\[
0 \to \mathcal{O}_X(D) \to E \to \omega_X(-D) \to 0,
\]

thus contradicting the semistability of $E$. Therefore, $h^0(L, \mathcal{O}_X(D)) \geq \frac{2g-2}{2}$.

Conversely, if $h^0(L, \mathcal{O}_X(D)) \geq \frac{2g-2}{2}$, then $\phi_L$ is surjective, which implies a stable vector bundle $E \in SU_X(2, \omega, 8)$.

By applying Riemann-Roch, one can then write

\[
h^0(X, \mathcal{O}_X(M)) + h^1(X, \mathcal{O}_X(-M)) = h^0(X, L) + h^1(X, L) - 2 \dim \frac{H^0(X, L)}{H^0(X, L(-M))} + \deg(M).
\]

Since

\[
h^0(X, L) + h^1(X, L) = h^0(X, L) \leq h^0(X, \mathcal{O}_X(D)) + h^1(X, \mathcal{O}_X(-D)) + h^1(X, \mathcal{O}_X(-M))
\]

it follows that

\[
\deg(M) \geq 2 \dim \frac{H^0(L)}{H^0(L(-M))}.
\]

Since $L$ is very ample, we find $\deg(M) \in \{4, 5, 6\}$. In each case, the Brill-Noether number of $L(-M)$ is negative, contradicting the generality of $X$. Therefore $E$ is stable. \qed
Picard variety parametrizing pairs

As pointed out in [Ve, Theorem 1.2], it follows from Mukai’s work [Mu1] that

The forgetful map

We introduce the variety:

Setting

where

and observe that \( \text{Ker}(\phi) \) has dimension at least \( 24 = \dim \text{Sym}^2(V) - h^0(L)^2 \). Choose a general 5-dimensional system of quadrics \( W \in G(5, H^0(\mathbb{P}^6, \mathcal{I}_{C,7}(2))) \). We then expect

\[
(24) \quad B_5 |W| = C + X \subseteq \mathbb{P}^6
\]
to be a nodal curve, and the curve \( X \) linked to \( C \) to be a smooth curve of degree 18 and genus 13.

Setting

we have that the line class \( \mathcal{L} \) is very ample and the embedded curve \( X \) on at least one component is a smooth curve of degree 18 and genus 13.

To check some transversality assumptions, we consider the universal family \( \mathcal{P}ic_{14}^4 \) over \( \mathcal{C}/\mathbb{P}^6 \), where \( C \) is a smooth curve of degree 18 and genus 13. As pointed out in [7, 2], it follows from Mukai’s theory that \( \mathcal{P}ic_{14}^4 \) is unirational. We introduce the base locus of this system of quadrics as \( B_5 |W| \)

The following lemmata give the structure of an iterated locally trivial projective bundle over \( \mathcal{P}ic_{14}^4 \), therefore \( \mathcal{Y} \) is unirational as well. Moreover,

One has a natural map

\[
(2, \omega, 8), \quad [C, L_C, V, W] \mapsto [X, L, E],
\]
where \( X \) is an embedded smooth curve of degree 18 in \( \mathbb{P}^6 \) and \( E \) is a rank 2 vector bundle over \( X \) via the extension

To show that \( \mathcal{I}_{C,7} \) is unirational, it suffices to produce one example of a point in \( \mathcal{Y} \) for which all these assumptions are realized. To that end, we consider 11 general points \( p_1, \ldots, p_5 \) and \( q_1, \ldots, q_6 \) respectively.

\[
H \equiv 6h - 2(E_{p_1} + \cdots + E_{p_5}) - (E_{q_1} + \cdots + E_{q_6})
\]
on the blow-up \( S = \text{Bl}_{11}(\mathbb{P}^2) \) at these points. Here \( h \) denotes the pullback of the line class from \( \mathbb{P}^2 \).

Via Macaulay2 one checks that \( S \xrightarrow{[H]} \mathbb{P}^6 \) is an embedding and the graded Betti diagram of \( S \) is the following:

\[
\begin{array}{ccccccc}
1 & - & - & - & - \\
- & 5 & - & - & - \\
- & - & 15 & 16 & 15
\end{array}
\]
Corollary 6.3. Proof. This follows from the proof of Theorem 6.1 and from the unirationality of \( \text{Pic} \) via \( \rho \) where \( \text{Pic} \).

Next we consider a general curve \( C \subseteq S \) in the linear system
\[
C \cong 10h - 4(E_{p_1} + E_{p_2} + E_{p_3} + E_{p_4}) - 3E_{p_5} - 2(E_{q_1} + E_{q_2}) - (E_{q_3} + E_{q_4} + E_{q_5} + E_{q_6}).
\]

Via \text{Macaulay2}, we verify that \( C \) is smooth, \( g(C) = 7 \) and \( \text{deg}(C) = 14 \). Furthermore, using that \( H^1(\mathbb{P}^6, \mathcal{I}_{S/\mathbb{P}^6}(2)) = 0 \), we have an exact sequence
\[
0 \to H^0(\mathbb{P}^6, \mathcal{I}_{S/\mathbb{P}^6}(2)) \to H^0(\mathbb{P}^6, \mathcal{I}_{C/\mathbb{P}^6}(2)) \to H^0(S, \mathcal{O}_S(2H - C)) \to 0.
\]

Note that \( \text{Pic} \) lies with \text{Macaulay2} that \( C \subseteq \mathbb{P}^6 \) is scheme-theoretically cut out by quadrics. Using \( [2.2] \), \( C \) lies on a smooth surface \( Y \subseteq \mathbb{P}^6 \) which is a complete intersection of 4 quadrics. Furthermore, the linear system \( |\mathcal{O}_Y(2H - C)| \) is base point free, so a general element \( C \) is a smooth curve of genus 13 meeting \( C \) transversally. Finally, a standard computation of the exact sequence \( 0 \to \mathcal{O}_X(H - X) \to \mathcal{O}_X(H) \to \mathcal{O}_X(H) \to 0 \) shows that since \( C \) is a smooth curve \( X \), \( \mathcal{O}_X(H) \to \mathcal{O}_X(1) = 1 \). This implies that the map \( \chi: \mathcal{Y} \to SU_{13}(2, \omega, 8) \) is well-defined and surjective. \( \square \)

Corollary 6.3. The parameter space \( \mathcal{Y} \) has a well-defined fundamental class. It is essential for our calculations to choose a point \( p \in X \). Since the moduli space \( SU_X(2, \omega) \) is singular, we define the fundamental class of the non-abelian Brill-Noether locus \( SU_X(2, \omega, k) \), and use instead the Hecke correspondence relating \( SU_X(2, \omega) \) to the moduli space. Hence there is a universal rank 2 vector bundle \( F \) on \( X \) with \( \text{det}(F) \cong \omega_X \).

6.1. The fundamental class of \( SU_X(2, \omega) \). 

endowed with the projection \( \pi_1: \mathbb{P} \to SU_X(2, \omega(p)) \). The points of \( \mathbb{P} \) are exact sequences
\[
0 \to E \to F \to K(p) \to 0,
\]
where \( F \in SU_X(2, \omega(p)) \), and therefore \( \text{det}(E) \cong \omega_X \). One has a diagram
\[
\begin{array}{ccc}
\mathbb{P} & \xrightarrow{\rho} & SU_X(2, \omega) \\
& \searrow & \downarrow \\
& & \pi_1 \end{array}
\]

where \( \rho \) assigns to \( \rho(1) = \rho^*c_1(\mathcal{L}_{ev}) \),

where \( \mathcal{L}_{ev} \) is the divisor associated to \( SU_X(2, \omega) \), associated to the effective divisor
\[
(2, \omega) : H^0(X, E) \neq 0 \}
\]

Set \( \alpha := c_1(\mathcal{L}_{\text{odd}}) \in H^2(SU_X(2, \omega(p)), \mathbb{Z}) \), where \( \mathcal{L}_{\text{odd}} \) is the ample generator of \( \text{Pic}(SU_X(2, \omega(p))) \). Note that \( \text{Pic}(\mathbb{P}) \) is generated by \( h \) and by \( \pi_1^*(\alpha) \).

For each \( k \in \mathbb{N} \), the non-abelian Brill-Noether locus
\[
B_p(k) := \left\{ [0 \to E \to F \to K(p) \to 0] \in \mathbb{P} : h^0(X, E) \geq k \right\}
\]
the virtual fundamental class $\mathcal{F}$, using that $c_2(\mathcal{F}) = \chi + \psi + g\alpha \otimes \varphi$, where $\chi$ is the fundamental class of the curve, $\chi \in H^4(SU_X(2, \omega(p)), \mathbb{Q})$ and $\psi$ is in $H^2(SU_X(2, \omega(p)), \mathbb{Q}) \otimes H^2(X, \mathbb{Q})$. Finally, we define the class $\gamma \in H^4(SU_X(2, \omega(p)), \mathbb{Q})$

by the formula $\psi^2 = \gamma \otimes \varphi$. One has the relation

$$h^2 = \alpha h - \frac{\alpha^2 - \beta}{4} \in H^4(\mathcal{P}, \mathbb{Q}),$$

from which we can recursively determine all powers of $h$. We summarize as follows:

**Proposition 6.4.** For each $n \geq 2$, the following relation holds:

$$h^n = \frac{h(-2\alpha + 2h)\sqrt{\beta} + 2\alpha h - \beta}{\sqrt{\beta(\alpha^2 - \beta)}} \left(\frac{\alpha + \sqrt{\beta}}{2}\right)^n.$$  

In this formula $\sqrt{\beta}$ is a formal root of the class $\beta$. One endows $B_{\mathcal{P}}(k)$ with the structure of a Lagrangian degeneracy locus bundle on $X \times \mathcal{P}$ defined by the following exact sequence:

$$0 \to E \to (id \times \pi_1)^\ast(\mathcal{F}) \to \mathcal{F},$$

where $p_2: X \times \mathcal{P} \to \mathcal{P}$ is the projection. Choose an effective divisor $D$ of large degree on $X$ and also denote by $D$ its pull-back under $X \times \mathcal{P} \to X$. Then $(p_2)_{\ast}\left(E/\mathcal{F}(-D)\right)$ and $(p_2)_{\ast}(E(D))$ are Lagrangian subbundles of $(p_2)_{\ast}(\mathcal{F}(D)/\mathcal{F}(-D))$. For each point $[0] \rightarrow \mathcal{F} \rightarrow E \rightarrow K(p) \to 0 \in \mathcal{P}$, one has $\chi = h^n \in H^4(\mathcal{F}, \mathbb{Q})$ to be 1-dimensional. Applying the formalism for Lagrangian degeneracy loci [M2, Proposition 1.11], we find the following determinantal formula for its virtual fundamental class $[B_{\mathcal{P}}(8)]^{\text{virt}}$:

$$[B_{\mathcal{P}}(8)]^{\text{virt}} = \left| \begin{array}{cccccccc} c_8 & c_{15} & c_{12} & c_{13} & c_9 & c_{14} \hline c_0 & c_3 & c_4 & c_5 & c_6 & c_7 \hline 0 & 0 & c_0 & c_1 & c_2 & c_3 \hline 0 & 0 & 0 & 0 & c_0 & c_1 \hline 0 & 0 & 0 & 0 & 0 & c_0 \end{array} \right|,$$

where $c_i \in H^{2i}(\mathcal{P}, \mathbb{Q})$ are defined recursively by the following formulas [2]:

$$c_1 = h, \quad c_2 = \frac{h^2}{2}, \quad c_3 = \frac{1}{3} \left(\frac{h^3}{2} + \frac{\beta h}{4} - \frac{\gamma}{2}\right), \quad c_4 = \frac{1}{4} \left(\frac{h^4}{2} + \frac{\beta h^2}{4} + \frac{\gamma h}{4}\right) \quad \text{and for each } n \geq 1,$$  

$$(n + 4)c_{n+4} - \frac{n + 2}{2} \beta c_{n+2} + \left(\frac{\beta}{4}\right)^2 n c_n = h c_{n+3} - \left(\frac{\beta h}{4} + \frac{\gamma}{4}\right)c_{n+1}.$$
In order to evaluate the determinant giving $[B_P(8)]^\text{virt}$, we shall use the formula obtained by Intersecting both sides of (30) with (29) and obtaining all top intersection numbers from $SU_X(2, \omega(p))$. Since $g = 3g - 3$, one has

$$(-1)^{g-p} \frac{g!m!}{(g-p)!q!} 2^{2g-2+p}$$

where $q = m + p + 1 - g$ and $B_q$ denotes the Bernoulli number; those appearing in our calculation are:

\[
\begin{align*}
B_2 &= \frac{1}{6}, & B_4 &= -\frac{1}{30}, & B_6 &= \frac{1}{42}, & B_8 &= -\frac{1}{30}, & B_{10} &= \frac{5}{66}, & B_{12} &= \frac{5}{2730}, & B_{14} &= \frac{7}{6}, \\
B_{16} &= -\frac{3617}{510}, & B_{18} &= \frac{43867}{798}, & B_{20} &= \frac{174611}{330}, & B_{22} &= \frac{854513}{138}, & B_{24} &= -\frac{23634091}{2730}.
\end{align*}
\]

**Theorem 6.5.** For a general curve $X$ of genus $13$, the locus $SU_X(2, \omega, \rho)$ consists of three reduced points corresponding to stable vector bundles.

*Proof.* As explained, the Lagrangian degeneracy locus $B_P(8)$ is expected to be a curve and we write

$$[B_P(8)]^\text{virt} = f(\alpha, \beta, \gamma) + h \cdot u(\alpha, \beta, \gamma),$$

where $f(\alpha, \beta, \gamma)$ and $u(\alpha, \beta, \gamma)$ are homogeneous polynomials of degree $26 = 2p + 2$ and $35 = 3g - 4$, respectively.

Observe that if $E \in SU_X(2, \omega, \rho)$ then necessarily $E$ is strictly semistable, in which case $E = B \oplus (\omega_X \otimes B')$, where $B$ is a $\mathbb{P}^1$-fibration on $X$. Since $\rho$ is a $\mathbb{P}^1$-fibration over the locus of stable vector bundles, it follows that $B_P(8)$ is a $\mathbb{P}^1$-fibration over $SU_X(2, \omega, \rho)$. Furthermore, the pull-back of the Petri map $\mu_E$ is an isomorphism for each vector bundle $E \in SU_X(2, \omega, \rho)$. We denote by $a$ its length, thus

$$[B_P(8)] = [B_P(8)]^\text{virt} = a \rho^*([E_0])$$

where $[E_0] \in SU_X(2, \omega)$ is general. Intersecting by $\omega$, we finally obtain

$$h \cdot f(\alpha, \beta, \gamma) = -\alpha \cdot \rho^*([E_0]),$$

Next observe (29), and rewrite, since $p + 1 = 12$, $\omega$ cuts the open locus of stable bundles and $\rho$ is zero. This follows the recursion

$$f(\alpha \cdot \rho^*([E_0])) = \omega \cdot \rho^*([E_0]) = -\alpha \cdot \rho^*([E_0]).$$

Intersecting both sides of this equation we find $2g = h \cdot \rho(\alpha, \beta, \gamma) = -h \cdot f(\alpha, \beta, \gamma)$, so

$$f(\alpha, \beta, \gamma) = \frac{2g}{h} = \frac{2 \cdot 13}{h}.$$
Theorem 1.1. The results in this section also lay the groundwork for the proof that the corresponding vector bundle on the universal non-abelian Brill-Noether locus.

Proposition 7.2. The map \( \vartheta : \mathcal{SU}_{13}^2(2, \omega, 8) \to \mathcal{M}_{13}^2 \) is proper. Furthermore, for each \([X, E] \in \mathcal{SU}_{13}^2(2, \omega, 8)\) the corresponding vector bundle \( E \) is globally generated.

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Let us consider the universal genus 13 curve
\[ \varphi: C_{13}^2 \to SU_{13}^2(2, \omega, 8), \]
then let \( \mathcal{E} \) be the universal rank two bundle over \( SU_{13}^2(2, \omega, 8) \). Note that we can normalize \( \mathcal{E} \) is such a way that \( \det(\mathcal{E}) \cong \omega_\varphi \).

**Definition 7.3.** We define the tautological class \( \gamma := \varphi_* \left( c_2(\mathcal{E}) \right) \in CH^1 \left( SU_{13}^2(2, \omega, 8) \right) \).

We aim to determine the push-forward to \( M_{13}^r \) of the class \( \gamma \) in terms of \( \lambda \) and \( \delta_0 \). To that end, we begin with the following:

**Proposition 7.4.** The push-forward \( \varphi_* (\mathcal{E}) \) is isomorphic to \( c_1(\mathcal{E}) = \varphi_* \left( c_1(\mathcal{E}) \right) \).

**Proof.** The fact that \( \varphi_* (\mathcal{E}) \) is locally free for \( \varphi: C_{13}^2 \to SU_{13}^2(2, \omega, 8) \) follows from Serre duality. Using \( \text{[HM, page 49]} \), observe that
\[
\text{ch}(\varphi_* (\mathcal{E})) = \varphi_* \left[ \left( 2 + c_1(\mathcal{E}) + \frac{c_1^2(\mathcal{E})}{2} \right) \right].
\]
We consider the degree one terms in this equality. Observe that
\[
c_1(\Omega^1_\varphi) = c_1(\omega_\varphi) \quad \text{and} \quad \text{R}^1 \varphi_* (\mathcal{E}) \cong \varphi_* (\mathcal{E}).
\]
By Serre duality, observe that \( \text{R}^1 \varphi_* (\mathcal{E}) \cong \varphi_* (\mathcal{E}) \).

\[
2c_1(\varphi_* (\mathcal{E})) - c_1(\varphi_* (\mathcal{E})) = 2c_1(\varphi_* (\mathcal{E})) - c_1(\varphi_* (\mathcal{E})) = 2\delta^* (\lambda) - \frac{1}{2} c_1(\gamma) \quad \text{and} \quad 2c_1(\mathcal{E}) - c_1(\mathcal{E}) = 2\delta^* (\lambda) - \frac{1}{2} c_1(\gamma).
\]
which leads to the claimed formula. \( \square \)

In view of our future applications to \( \mathcal{R}_{13} \), we introduce the rank 6 vector bundle
\[ \mathcal{M}_\varphi := \text{Ker} \left\{ \varphi^* (\varphi_* (\mathcal{E})) \to \mathbb{E} \right\}. \]
The fibre \( M_E := \mathcal{M}_\varphi | X, E \) over a point \( [X, E] \in SU_{13}^2(2, \omega, 8) \) sits in an exact sequence
\[
0 \to M_E \to H^0(X, E) \otimes \mathcal{O}_X \xrightarrow{ev} E \to 0.
\]

**Proposition 7.5.** The following formulas hold:
\[
c_1(\mathcal{M}_\varphi) = \varphi^* \left( \delta^* (\lambda) - \frac{3}{2} \right) - c_1(\mathcal{E}) \\
c_2(\mathcal{M}_\varphi) = \varphi^* c_2(\mathcal{E}) - c_2(\mathcal{E}) - c_1(\mathcal{E}) \cdot \varphi^* \left( \delta^* (\lambda) - \frac{3}{2} \right).
\]

**Proof.** This follows from the splitting principle applied to \( \mathcal{M}_\varphi \), coupled with

### 7.2. The resonance divisor in genus 13.

A general curve \( X \) of genus 13 has a subpencil defines a divisorial condition on the moduli space \( SU_{13}^2(2, \omega, 8) \). For a subpencil defines a divisorial condition on the moduli space \( SU_{13}^2(2, \omega, 8) \), we denote its determinant map by
\[
d: \bigwedge^2 H^0(X, E) \to H^0(X, \omega_X).
\]

**Definition 7.6.** The resonance divisor \( \mathcal{R}_{13}^r \) is the locus of curves \( [X] \in M_{13}^r \) for which
\[
G \left( 2, H^0(X, E) \right) \cap \mathbb{P} (\text{Ker}(d)) \neq 0,
\]
for some vector bundle \( E \in SU_{13}^2(2, \omega, 8) \). In other words, \( \mathcal{R}_{13}^r \) is the locus of \([X]\) for which there exists an element \( 0 \neq a \wedge b \in \bigwedge^2 H^0(X, E) \) such that \( d(a \wedge b) = 0 \).
We set $\mathfrak{Res}_{13} := \mathfrak{Res}_{13}^2 \cap \mathcal{M}_{13}$. Note that $\mathfrak{Res}_{13}^2$ comes with an induced scheme structure under the proper map $\phi: SU^2_{13}(2, \omega, 8) \to \mathcal{M}_{13}^2$. The points in $\mathfrak{Res}_{13}^2$ correspond to those curves $X$ for which a vector bundle $E \in SU_X(2, \omega, 8)$ carries a subpencil (which is generated by the sections $a, b, \ldots$). The class $\mathfrak{Res}_{13}^2$ can be computed in terms of certain tautological classes. On the other hand, we have a geometric characterization of points in $\mathfrak{Res}_{13}$: A resonance divisor coincides with $D_{13}$ away from the heptagonal locus $\mathcal{M}_{13}^2$.

Proof of Theorem 1.7. Let $\mathfrak{Res}_{13} = D_{13} + 3 \cdot M^1_{13,7}$
onumber

on $[X] \in \mathfrak{Res}_{13} \setminus M^1_{13,7}$ and let $E \in SU_X(2, \omega, 8)$ be the vector bundle which can be written as an extension

(32) \[ 0 \to A \to E \to \omega_X \otimes A^\vee \to 0, \]

where $h^0(X, A) \geq 2$. Since $\text{gon}(X) = 8$, and since $8 \leq h^0(X, E) \leq h^0(X, A) + h^0(X, \omega_X \otimes A^\vee)$, it follows that $A \in W^8_0(X)$ and $L := \omega_X \otimes A^\vee \in W^8_2(X)$. If such an extension exists, then the map $\phi_L$ is not surjective, therefore $[X] \in D_{13}$.

Conversely, if $[X] \in D_{13}$, there is some $L \in W^8_0(X)$ such that the multiplication map $\phi_L$ is not surjective. For $[X]$ a general point of an irreducible component of $D_{13}$, we may assume that the multiplication map $\phi_L$ has corank 1, for else $\varphi_L: X \to \mathbb{P}^5$ lies on a $(2, 2, 2)$ complete intersection in $\mathbb{P}^5$, which is an ordinary (non-degenerate) $K3$ surface. But the locus of curves $[X] \in \mathcal{M}_{13}$ lying on a $K3$ surface cannot exceed $g + 19 = 32 < 3g - 4$, a contradiction. We let

\[ E \in \mathbb{P}(\text{Ext}^1(L, \omega_X \otimes L^\vee)) \]

be a general divisor with $h^0(X, E) = h^0(X, L) + h^0(X, \omega_X \otimes L^\vee) = 8$. The argument of $E$ is stable, otherwise there would exist an effective divisor $M$ of degree $h^0(X, E) \geq 3$, hence this situation does not occur along a stable curve, away from the divisor $M^1_{13,7}$, the divisors $\mathfrak{Res}_{13}$ and $D_{13}$ coincide.

Each vector bundle $E \in SU_X(2, \omega, 8)$ that has a subpencil appears as an extension

(33) \[ 0 \to A \to E \to L \to 0. \]

In this case $h^0(X, E) = h^0(X, A) + h^0(X, L) - 1$. That is, $V := \text{Im}\{H^0(E) \to H^0(L)\}$ is 6-dimensional. Furthermore, the multiplication map $\mu_V: V \otimes H^0(X, L) \to H^0(X, L^\otimes 2)$ is not surjective. Conversely, each 6-dimensional subspace $V \subseteq H^0(X, L)$ such that $\mu_V$ is not surjective leads to a vector bundle $E \in \mathbb{P}(\text{Ext}^1(L, A))$ with $h^0(X, E) = 8$. The corresponding bundle $E$ is stable unless $V$ is of the form $H^0(X, L(-p))$ for a point $p \in X$, in which case $E$ can also be realized as an extension

\[ 0 \to L(-p) \to E \to A(p) \to 0. \]

To determine the number of such subspaces $V \subseteq H^0(X, L)$, we consider the projective space $\mathbb{P}^6 := \mathbb{P}(H^0(X, L)^\vee)$ and consider the vector bundle $A$ on $\mathbb{P}^6$ with fibre

\[ A(V) = \frac{V \otimes H^0(X, L)}{\wedge^2 V} \]

over a point $[V] \in \mathbb{P}^6$. There exists a bundle morphism $\mu: A \to H^0(X, L^\otimes 2) \otimes \mathcal{O}_{\mathbb{P}^6}$ given by multiplication and the subspaces $[V] \in \mathbb{P}^6$ for which $\mu_V$ is not surjective (or, equivalently, $\mu^\vee$ is not injective)
are precisely those lying in the degeneracy locus of $\mu$, that is, for which $\text{rk}(\mu(V)) = 21$. Applying the Porteous formula we find

$$[Z_{21}(\mu)] = c_6 \left( H^0(X, L^{\otimes 2})^\vee \otimes O_{\mathbb{P}^6} - A^\vee \right) = c_6(-A).$$

To compute the Chern classes of $A$, we recall that via the Euler sequence the rank 6 vector bundle $M_{\mathbb{P}^6}$ on $\mathbb{P}^6$ with $M_{\mathbb{P}^6}(V) = V \subseteq H^0(X, L)$ can be identified with $\Omega_{\mathbb{P}^6}(1)$. Then $A$ is isomorphic to $M_{\mathbb{P}^6} \otimes H^0(X, L) / \bigwedge^2 M_{\mathbb{P}^6}$. From the exact sequence

$$0 \rightarrow \bigwedge^2 M_{\mathbb{P}^6} \rightarrow H^0(X, L) \otimes O_{\mathbb{P}^6} \rightarrow M_{\mathbb{P}^6}(1) \rightarrow 0,$$

recalling that $c_{\text{tot}}(M_{\mathbb{P}^6}) = \frac{1}{1 + h}$, where $h = c_1(O_{\mathbb{P}^6}(1))$, we find $c_{\text{tot}}(\bigwedge^2 M_{\mathbb{P}^6}) = \frac{1 + 2h}{1 + h^2}$, therefore

$$[Z_{21}(\mu)] = \left[ \frac{1}{1 + h} \right] \cdot \left[ \frac{(1 + h)^7}{1 + 2h} \right]_6 = \left[ \frac{1}{1 + 2h} \right]_6 = 2^6 \cdot h^6 = 64.$$

From this, we subtract the excess contribution corresponding to the locus $X \hookrightarrow \mathbb{P}^6$, parametrizing the subspaces $V = H^0(X, L(-p))$ corresponding to unstable bundles. This locus appears in the class $[Z_{21}(\mu)]$ with a contribution of

$$c_1 \left( \text{Ker}(\mu^\vee) \otimes \text{Coker}(\mu^\vee) - N_{X/\mathbb{P}^6} \right) = -5c_1(\text{Ker}(\mu^\vee)) + c_1(A_X^\vee) - c_1(N_{X/\mathbb{P}^6}).$$

The restriction to $X \subseteq \mathbb{P}^6$ of the kernel bundle of $\mu^\vee$ can be identified with $L^\vee$, whereas $c_1(A_X^\vee) = -2c_1(M_{\mathbb{P}^6}|_X) = 2 \deg(L)$. Furthermore $c_1(N_{X/\mathbb{P}^6}) = 7\deg(L) + 2g(X) - 2$. All in all, the excess contribution to $[Z_{21}(\mu)]$ coming from $X$ equals

$$10 \deg(L) + 2 \deg(L) - 7 \deg(L) - (2 + 7) = 61.$$

Therefore, for a general curve $[X] \in M_{13}^4$, where $X$ is a curve, we parametrize bundles $E \in SU_X(2, \omega, 8)$ having $A$ as a subpencil, which finishes the proof. □

**Proposition 7.7.** One has $\vartheta_\ast(\gamma) = \frac{11288}{143} \lambda - \frac{1582}{143} \delta_0$.

**Proof.** The divisor $\text{Res}_{x}^4$ is defined as the locus $x \hookrightarrow M^4$ of the locus

$$\begin{array}{c}
\text{Res}_{x}^4 \rightarrow \vartheta_\ast(\omega_x) \\
\vartheta_\ast(\omega_x) \rightarrow e^{\vartheta_\ast(\omega_x)} \\
(\vartheta_\ast(e^{\vartheta_\ast(\omega_x)})) = 132 \left( -\frac{9}{4} \vartheta_\ast(\lambda) + \frac{13}{8} \gamma \right)
\end{array}$$

for the class of the heptagonal locus, while the degree of $e^{\vartheta_\ast(\omega_x)}$ is 3, we then find

$$\frac{1}{2} \left( 48 \lambda - 7 \delta_0 \right) = \frac{1128}{143} \lambda - \frac{1582}{143} \delta_0.$$ □

The non-Abelian Brill-Noether divisor on $\mathfrak{F}_{1,13}$ and the Kodaira dimension of $\mathfrak{F}_{1,13}$
7.3. The class of the non-abelian Brill-Noether divisor on $\mathcal{M}_{13}$.

In the introduction, we defined the non-abelian Brill-Noether divisor $\mathcal{M}^\sharp_{13}$ as the locus of curves $[X] \in \mathcal{M}_{13}$ for which there exists $E \in SU_X(2, \omega, 8)$ such that the map

$$\varphi_E : \text{Sym}^2 H^0(X, E) \to H^0(X, \text{Sym}^2 E)$$

is non-trivial equivariantly, the scheme $SU_X(2, \omega, 8)$ is not reduced. We now compute the class of this divisor.

Proposition 8. The non-abelian Brill-Noether divisor $\mathcal{M}^\sharp_{13}$ is the push-forward under the proper map $\varphi$ of the degeneracy locus of the natural map of vector bundles over $SU^\sharp_{13}(2, \omega, 8)$:

$$\text{Sym}^2 \varphi_*(\mathcal{E}) \to \varphi_*(\text{Sym}^2 \mathcal{E}).$$

Using again that $\text{Pic}^2 \mathcal{E} = \mathcal{O}_\mathcal{E}$ and $\mathcal{E}$ is not reduced. We now compute

$$c_1(\varphi_*(\text{Sym}^2 \mathcal{E})) = \varphi_* \left( 3 + 3c_1(\mathcal{E}) + \frac{5c_2^2(\mathcal{E}) - 8c_2(\mathcal{E})}{2} \right) \cdot \left( 1 - \frac{c_1(\Omega^1_{\mathcal{E}})}{2} + \frac{c_2(\Omega^1_{\mathcal{E}}) + c_2(\Omega^2_{\mathcal{E}})}{12} \right).$$

Using that $\varphi_* \left( 3 + \frac{5c_2^2(\mathcal{E}) - 8c_2(\mathcal{E})}{2} \right) = \theta^*(\lambda)$, we conclude that

$$c_1(\varphi_*(\text{Sym}^2 \mathcal{E})) = 3\theta^*(\lambda) + \varphi_* \left( \frac{c_2^2(\mathcal{E})}{2} \right) - 4\gamma = \theta^*(15\lambda - \delta_0 - 4\gamma).$$

Via the obvious isomorphisms $\mathcal{E} = 9c_1(\mathcal{E})$ and $\theta^*(\lambda) = \frac{6\lambda - \delta_0}{2}$, yielding

$$\left( 3\theta^*(\lambda) + \varphi_* \left( \frac{c_2^2(\mathcal{E})}{2} \right) - 4\gamma \right) = 3(6\lambda - \delta_0) + \frac{6\lambda - \delta_0}{2}.$$

Substituting into (8.1), we have $\mathcal{M}^\sharp_{13} = \frac{1}{133}(8218\lambda - 1220\delta_0)$. We turn our attention to showing that the Prym moduli space $\mathcal{M}_{13}$ is not an isomorphism, or equivalently, the scheme $SU_X(2, \omega, 8)$ is not reduced. We now compute

$$\text{Pic}^2 \mathcal{E} = \mathcal{O}_\mathcal{E}$$

8.1. The boundary divisors of $\mathcal{R}_g$. The geometry of the boundary of $\mathcal{R}_g$ is determined in the introduction. We recall some facts. If $[X_y] \in \mathcal{M}_{13}$ is such that $[X, y, q] \in \nu : X \to X_y$ the normalization map, there are three types of Prym curves in $\mathcal{M}_g$:

- First, one can choose a non-trivial 2-torsion point $\eta \in \text{Pic}^2(X_y)$. If $\nu^*(\eta) \neq \eta_X$, choosing a 2-torsion point $\eta_X \in \text{Pic}^2(X)[2] \setminus \{ \mathcal{O}_X \}$ together with an identification $\eta_Y(q) = \eta_X(q)$ at the points $y$ and $q$ respectively. As we vary $[X, y, q]$, points of this type fill-up the boundary divisor $\Delta_0$ in $\mathcal{R}_g$. The Prym curves corresponding to this situation $\nu^*(\eta) \cong \mathcal{O}_X$ fill-up the boundary divisor $\Delta_0$. Finally, choosing a line bundle $\eta_X$ on $X$ with $\eta_X^{\otimes 2} \cong \mathcal{O}_X(-y - q)$ leads to a Prym curve $[Y := X \cup_{y, q} \mathbb{P}^1, \eta, \beta]$, where $R$ is a smooth rational curve meeting $X$ at $y$ and $q$ and $\eta \in \text{Pic}^2(Y)$ is a line bundle such that $\eta_X = \eta_X$ and $\eta_R = \mathcal{O}_R(1)$. Points of this type fill-up the boundary divisor $\Delta_0^{\text{ram}}$ of $\mathcal{R}_g$. 
Proof. It follows that \( \Theta \) is the divisor over the fibre product and realize it as the push-forward of the degeneracy locus of a map of vector bundles of the same divisor.

Definition 8.1. The universal theta divisor \( \Theta_{13} \) on \( \mathcal{R}_{13} \) is defined as the locus of \( \delta \in \mathcal{R}_{13} \) for which there exists a vector bundle \( E \in SU_{13}(2, \omega, 8) \) such that \( H^0(X, E) = 0 \).

We first show that, as expected, this definition gives rise to a divisor on \( \mathcal{R}_{13} \).

Proposition 8.2. For a general Prym curve \([X, \eta] \in \mathcal{R}_{13}\) one has \( H^0(X, E \otimes \eta) = 0 \) for all \( E \in SU_{13}(2, \omega, 8) \).

We consider the open substack \( \mathcal{R}_{13}^2 := \pi^{-1}(\mathcal{M}_{13}^2) \) of \( \mathcal{R}_{13} \) and identify \( CH^1(\mathcal{R}_{13}^2) \) with the space \( \mathbb{Q}(\lambda, \delta_0, \delta_0', \delta_0^\text{ram}) \). In what follows we extend the structure on the universal theta divisor \( \Theta_{13} \) to \( \mathcal{R}_{13}^2 \) and realize it as the push-forward of the degeneracy map of the fiber bundles of the same rank over the fibre product.
is not injective. Since clearly \( H^0(X, M_E \otimes \eta) = 0 \), it follows that
\[
h^1(X, M_E \otimes \eta) = -\deg(M_E) + 6(g - 1) = 96 = 8 \cdot 12 = h^0(X, E) \cdot h^0(X, \omega_X \otimes \eta).
\]
That is, \( \nu \) is a map between vector space of the same dimension.

By slightly abusing notation, we still denote by
\[
\varphi: \mathcal{R} \mathcal{S} \mathcal{U}^4_{13} \rightarrow \mathcal{R} \mathcal{S} \mathcal{U}^4_{13}(2, \omega, 8)
\]
the universal curve of genus 13 over \( \mathcal{R} \mathcal{S} \mathcal{U}^4_{13}(2, \omega, 8) \). It comes equipped with a vector bundle \( \mathcal{E} \) such that \( \bigwedge^2 \mathcal{E} \cong \omega_\nu \) and \( \varphi_*(\mathcal{E}) \) is locally free of rank 8 (cf. Theorem 7.5). By slightly abusing notation, we still denote by \( \lambda = \varphi_*(\omega_\nu) \).

We consider the rank 6 vector bundle \( \mathcal{M}_E \) on \( \mathcal{R} \mathcal{S} \mathcal{U}^4_{13} \) defined by the push-forward to \( \mathcal{R} \mathcal{S} \mathcal{U}^4_{13} \) of the degeneracy locus of the morphism \( \varphi: \mathcal{R} \mathcal{S} \mathcal{U}^4_{13} \rightarrow \mathcal{R} \mathcal{S} \mathcal{U}^4_{13}(2, \omega, 8) \).

Theorem 8.3. For any point \( \varphi_!(\nu) \) on \( \mathcal{R} \mathcal{S} \mathcal{U}^4_{13} \) is given by
\[
1582 (\delta_0^0 + \delta_0^8) - \frac{5899 \delta_0^{am}}{2} \in CH^1(\mathcal{R} \mathcal{S} \mathcal{U}^4_{13}).
\]

To determine \( c_1(A) \) we apply Grothendieck-Ramanujan-Kac to the morphism \( \varphi_! \)
\[
\text{ch}(\varphi_!(\mathcal{M}_E \otimes \mathcal{L})) = \varphi_! \left( 6 + c_1(\mathcal{M}_E \otimes \mathcal{L}) + \frac{c_2^2(\mathcal{M}_E \otimes \mathcal{L}) - c_2(\mathcal{M}_E \otimes \mathcal{L})}{2} + \cdots \right)
\]
\[
\cdot \left( 1 - \frac{c_1(\Omega_\nu)}{2} + \frac{c_1^2(\Omega_\nu)}{12} + \cdots \right).
\]

Observe by direct calculation that the following formulas hold:
\[
c_1(\mathcal{M}_E \otimes \mathcal{L}) = c_1(\mathcal{M}_E) + 6c_1(\mathcal{L}), \quad c_2(\mathcal{M}_E \otimes \mathcal{L}) = c_2(\mathcal{M}_E) + 5c_1(\mathcal{M}_E) \cdot c_1(\mathcal{L}) + 15c_1^2(\mathcal{L}),
\]
then
\[
\varphi_*(\frac{c_2^2(\mathcal{M}_E \otimes \mathcal{L}) - c_2(\mathcal{M}_E \otimes \mathcal{L})}{2})
\]
where in the last formula we have used \( \varphi_*(\omega_\nu) = \omega_\nu \) and \( 2\varphi_*(\omega_\nu) = -\nu \).
Substituting in the expression for $\theta_{\text{ram}}$, we obtain
\[
\vartheta_\lambda = \vartheta_\lambda(x + \frac{\delta^{\text{ram}}}{2}) = 2\vartheta_\lambda(x) - 6\vartheta_\lambda + 2\delta^{\text{ram}}.
\]

Putting everything together we find $\vartheta_\lambda(x) = 1582(\delta_0 + \delta_0' + 2\delta^{\text{ram}})$ and the conclusion follows. \hfill \square

We use the symbol $\vartheta$ for general type.

Proposition 8.3.1. [FL, Theorem 6.1] that any $g$ pluricanonical forms defined on $\mathcal{M}_g$ are of general type if and only if $\mathcal{M}_g$ is of general type.

It follows from [Log] that up to a positive rational constant, the closure of $D_{13,2}$ inside $\mathcal{M}_{13}$ is $\mathcal{M}_{13,9}$. Then $\mathcal{M}_{13,9}$ is a general type.

By a simple argument using pencils of degree $\geq 3$ we show that each of the coefficients $a_1, \ldots, a_{12}$ or $a_{1,12}, \ldots, a_{6,7}$ is at least equal to $25/6$. See [Log]. Since $4362/3707 = 12.432... < 13$, comparing the class of $D$ to the one of $\mathcal{M}_{13,9}$, we conclude that $K_{\mathcal{M}_{13}}$ can be written as a positive combination of $[D]$ and a multiple of $\overline{\mathcal{M}}_{13}$. \hfill \square

8.3.2. The proof. In this subsection we established the result above.

Proposition 8.3.2. Let $\mathcal{M}_{13,9}$ be the closure of $D_{13,2}$ inside $\mathcal{M}_{13,9}$. Then $\mathcal{M}_{13,9}$ is of general type if and only if $\mathcal{M}_g$ is of general type.

The class of the closure in $\mathcal{M}_{13,9}$ of $D_{13,2}$ is
\[
\vartheta_\lambda(x) = -\lambda + 17 \sum_{i=1}^{9} \psi_i - 25 \delta_{0,2} - \cdots \in CH^1(\mathcal{M}_{13,9}).
\]

Observe that the class of the non-abelian Brill-Noether divisor $[\overline{\mathcal{M}}_{13}]$ the result follows. \hfill \square
References


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