BRILL-NOETHER GEOMETRY ON MODULI SPACES OF SPIN CURVES

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The aim of this paper is to initiate a study of geometric divisors of Brill-Noether type on the moduli space $\overline{\mathcal{S}}_g$ of spin curves of genus g. The moduli space $\overline{\mathcal{S}}_g$ is a compactification the parameter space $\mathcal{S}_g$ of pairs $[C, \eta]$, consisting of a smooth genus g curve C and a theta-characteristic $\eta \in \mathrm{Pic}^{g-1}(C)$, see [C]. The study of the birational properties of $\overline{\mathcal{S}}_g$ as well as other moduli spaces of curves with level structure has received an impetus in recent years, see [BV] [FL], [F2], [Lud], to mention only a few results. Using syzygy divisors, it has been proved in [FL] that the Prym moduli space $\overline{\mathcal{R}}_g := \overline{\mathcal{M}}_g(\mathcal{B}\mathbb{Z}_2)$ classifying curves of genus g together with a point of order 2 in the Jacobian variety, is a variety of general type for $g \geq 13$ and $g \neq 15$. The moduli space $\overline{\mathcal{S}}_g^+$ of even spin curves of genus g is known to be of general type for $g > 8$, uniruled for $g < 8$, see [F2], whereas the Kodaira dimension of $\overline{\mathcal{S}}_8^+$ is equal to zero, [FV]. This was the first example of a naturally defined moduli space of curves of genus $g \geq 2$, having intermediate Kodaira dimension. An application of the main construction of this paper, gives a new way of computing the class of the divisor $\overline{\Theta}_{\mathrm{null}}$ of vanishing theta-nulls on $\overline{\mathcal{S}}_g^+$, reproving thus the main result of [F2].

Virtually all attempts to show that a certain moduli space $\overline{\mathcal{M}}_{g,n}$ is of general type, rely on the calculation of certain effective divisors $D \subset \overline{\mathcal{M}}_{g,n}$ enjoying extremality properties in their effective cones $\mathrm{Eff}(\overline{\mathcal{M}}_{g,n})$, so that the canonical class $K_{\overline{\mathcal{M}}_{g,n}}$ lies in the cone spanned by $[D]$, boundary classes $\delta_{i:S}$, tautological classes $\lambda, \psi_1, \ldots, \psi_n$, and possible other effective geometric classes. Examples of such a program being carried out, can be found in [EH2], [HM]-for the case of *Brill-Noether divisors* on $\overline{\mathcal{M}}_g$ consisting of curves with a $\mathfrak{g}^r_d$ when $\rho(g, r, d) = -1$, [Log]-where pointed Brill-Noether divisors on $\overline{\mathcal{M}}_{g,n}$ are studied, and [F1]-for the case of *Koszul divisors* on $\overline{\mathcal{M}}_g$, which provide counterexamples to the Slope Conjecture on $\overline{\mathcal{M}}_g$. A natural question is what the analogous geometric divisors on the spin moduli space of curves $\overline{\mathcal{S}}_g$ should be?

In this paper we propose a construction for *spin Brill-Noether divisors* on both spaces $\overline{\mathcal{S}}_g^+$ and $\overline{\mathcal{S}}_g^-$, defined in terms of the relative position of theta-characteristics with respect to difference varieties on Jacobians. Precisely, we fix integers $r, s \geq 1$ such that $d := rs + r \equiv 0 \mod 2$, and then set $g := rs + s$. One can write $d = 2i$. By standard Brill-Noether theory, a general curve $[C] \in \mathcal{M}_g$ carries a finite number of (necessarily complete and base point free) linear series $\mathfrak{g}^r_d$. One considers the following loci of spin curves (both odd and even)

$$\mathcal{U}^r_{g,d} := \{[C, \eta] \in \mathcal{S}_g^{\mp} : \exists L \in W^r_d(C) \text{ such that } \eta \otimes L^\vee \in C_{g-i-1} - C_i\}.$$

Thus $\mathcal{U}^r_{g,d}$ consists of spin curves such that the embedded curve $C \xrightarrow{|\eta \otimes L|} \mathbf{P}^{d-1}$ admits an i-secant $(i-2)$-plane. We shall prove that for $s \geq 2$, the locus $\mathcal{U}^r_{g,d}$ is always a divisor on

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$\mathcal{S}_g^{\mp}$, and we find a formula for the class of its compactification in $\overline{\mathcal{S}}_g^{\mp}$. For simplicity, we display this formula in the introduction only in the case $r = 1$, when $g \equiv 2 \bmod 4$:

Theorem 0.1. *We fix an integer $a \geq 1$ and set $g := 4a + 2$. The locus*

$$\mathcal{U}^1_{4a+2,2a+2} := \{[C, \eta] \in \mathcal{S}^{\mp}_{4a+2} : \exists L \in W^1_{2a+2}(C) \text{ such that } \eta \otimes L^{\vee} \in C_{3a} - C_{a+1}\}$$

is an effective divisor and the class of its compactification in $\overline{\mathcal{S}}_g^{\mp}$ is given by

$$\overline{\mathcal{U}}^1_{4a+2,2a+2} \equiv \binom{4a}{a}\binom{4a+2}{2a}\frac{a+2}{8(2a+1)(4a+1)}\Big(\big(192a^3 + 736a^2 + 692a + 184\big)\lambda -$$

$$-\big(32a^3 + 104a^2 + 82a + 19\big)\alpha_0 - \big(64a^3 + 176a^2 + 148a + 36\big)\beta_0 - \cdots\Big) \in \mathrm{Pic}(\overline{\mathcal{S}}_g^{\mp}).$$

To specialize further, in Theorem 0.1 we set $a = 1$, and find the class of (the closure of) the locus of spin curves $[C, \eta] \in \mathcal{S}_6^{\mp}$, such that there exists a pencil $L \in W^1_4(C)$ for which the linear series $C \stackrel{|\eta \otimes L|}{\longrightarrow} \mathbf{P}^3$ is not very ample:

$$\overline{\mathcal{U}}^1_{6,4} \equiv 451\lambda - \frac{237}{4}\alpha_0 - 106\beta_0 - \cdots \in \mathrm{Pic}(\overline{\mathcal{S}}_6^{\mp}).$$

The case $s = 1$, when necessarily $L = K_C \in W^{g-1}_{2g-2}(C)$, produces a divisor only on $\overline{\mathcal{S}}_g^{+}$, and we recover in this way the main calculation from [F2], used to prove that $\overline{\mathcal{S}}_g^{+}$ is a variety of general type for $g > 8$. We recall that $\Theta_{\mathrm{null}} := \{[C, \eta] \in \mathcal{S}_g^{+} : H^0(C, \eta) \neq 0\}$ denotes the divisor of *vanishing theta-nulls*.

Theorem 0.2. *Let $\pi : \overline{\mathcal{S}}_g^{+} \to \overline{\mathcal{M}}_g$ be the ramified covering which forgets the spin structure. For $g \geq 3$, one has the following equality $\overline{\mathcal{U}}^{g-1}_{g,2g-2} = 2 \cdot \overline{\Theta}_{\mathrm{null}}$ of codimension 1-cycles on the open subvariety $\pi^{-1}(\mathcal{M}_g \cup \Delta_0)$ of $\overline{\mathcal{S}}_g^{+}$. Moreover, there is an equality of classes*

$$\overline{\mathcal{U}}^{g-1}_{g,2g-2} \equiv 2 \cdot \overline{\Theta}_{\mathrm{null}} \equiv \frac{1}{2}\lambda - \frac{1}{8}\alpha_0 - 0 \cdot \beta_0 - \cdots \in \mathrm{Pic}(\overline{\mathcal{S}}_g^{+}).$$

We remark once more, the low slope of the divisor $\overline{\Theta}_{\mathrm{null}}$. No similar divisor with such remarkable class is known to exist on $\overline{\mathcal{R}}_g$. In Section 4, we present a third way of calculating the class $[\overline{\Theta}_{\mathrm{null}}]$, by rephrasing the condition that a curve C have a vanishing theta-null η, if and only if, for a pencil A on C of minimal degree, the multiplication map of sections

$$H^0(C, A) \otimes H^0(C, A \otimes \eta) \to H^0(C, A^{\otimes 2} \otimes \eta)$$

is not an isomorphism. For $[C] \in \mathcal{M}_g$ sufficiently general, we note that

$$\dim H^0(C, A) \otimes H^0(C, A \otimes \eta) = \dim H^0(C, A^{\otimes 2} \otimes \eta).$$

In this way, $\overline{\Theta}_{\mathrm{null}}$ appears as the push-forward of a degeneracy locus of a morphism between vector bundles of the same rank defined over a Hurwitz stack of coverings. To compute the push-forward of tautological classes from a Hurwitz stack, we use the techniques developed in [F1] and [Kh].

In the last section of the paper, we study the divisor $\overline{\Theta}_{g,1}$ on the universal curve $\overline{\mathcal{M}}_{g,1}$, which consists of points in the support of odd theta-characteristics. This divisor, somewhat similar to the divisor $\overline{\mathcal{W}}_g$ of Weierstrass points on $\overline{\mathcal{M}}_{g,1}$, cf. [Cu], should be of some importance in the study of the birational geometry of $\overline{\mathcal{M}}_{g,1}$:

Theorem 0.3. *The class of the compactification in $\overline{\mathcal{M}}_{g,1}$ of the effective divisor*

$$\Theta_{g,1} := \{[C,q] \in \mathcal{M}_{g,1} : q \in \mathrm{supp}(\eta) \text{ for some } [C,\eta] \in \mathcal{S}_g^-\}$$

is given by the following formula:

$$\overline{\Theta}_{g,1} \equiv 2^{g-3}\Bigl((2^g-1)\bigl(\lambda+2\psi\bigr)-2^{g-3}\delta_{\mathrm{irr}}-(2^g-2)\delta_1-\sum_{i=1}^{g-1}(2^i+1)(2^{g-i}-1)\delta_i\Bigr) \in \mathrm{Pic}(\overline{\mathcal{M}}_{g,1}).$$

When $g=2$, the divisor Θ_2 specializes to the divisor of Weierstrass points:

$$\Theta_{2,1} = \mathcal{W}_2 := \{[C,q] \in \mathcal{M}_{2,1} : q \in C \text{ is a Weierstrass point}\}.$$

If we use Mumford's formula $\lambda = \delta_0/10 + \delta_1/5 \in \mathrm{Pic}(\overline{\mathcal{M}}_2)$, Theorem 0.3 reads

$$\overline{\Theta}_{2,1} \equiv \frac{3}{2}\lambda + 3\psi - \frac{1}{4}\delta_{\mathrm{irr}} - \frac{3}{2}\delta_1 = -\lambda + 3\psi - \delta_1 \in \mathrm{Pic}(\overline{\mathcal{M}}_{2,1}),$$

that is, we recover the formula for the class of the Weierstrass divisor on $\overline{\mathcal{M}}_{2,1}$, cf. [EH2]. When $g=3$, the condition $[C,q] \in \Theta_{3,1}$, states that the point $q \in C$ lies on one of the 28 bitangent lines of the canonically embedded curve $C \xrightarrow{|K_C|} \mathbf{P}^2$.

Corollary 0.4. *The class of the compactification in $\overline{\mathcal{M}}_{3,1}$ of the bitangent locus*

$$\Theta_{3,1} := \{[C,q] \in \mathcal{M}_{3,1} : q \text{ lies on a bitangent of } C\}$$

is equal to $\overline{\Theta}_{3,1} \equiv 7\lambda + 14\psi - \delta_{\mathrm{irr}} - 9\delta_1 - 5\delta_2 \in \mathrm{Pic}(\overline{\mathcal{M}}_{3,1})$.

If $p : \overline{\mathcal{M}}_{g,1} \to \overline{\mathcal{M}}_g$ is the map forgetting the marked point, we note the equality

$$\overline{\mathcal{D}}_3 \equiv p^*(\overline{\mathcal{M}}^1_{3,2}) + 2\cdot \overline{\mathcal{W}}_3 + 2\psi \in \mathrm{Pic}(\overline{\mathcal{M}}_{3,1}),$$

where $\overline{\mathcal{W}}_3 \equiv -\lambda + 6\psi - 3\delta_1 - \delta_2$ is the divisor of Weierstrass points on $\overline{\mathcal{M}}_{3,1}$. Since the class $\psi \in \mathrm{Pic}(\overline{\mathcal{M}}_{3,1})$ is big and nef, it follows that $\overline{\Theta}_{3,1}$ (unlike the divisor $\overline{\Theta}_{2,1} \in \mathrm{Pic}(\overline{\mathcal{M}}_{2,1})$), lies in the interior of the cone of effective divisors $\mathrm{Eff}(\overline{\mathcal{M}}_{3,1})$, or it other words, it is big. In particular, it cannot be contracted by a rational map $\overline{\mathcal{M}}_{3,1} \dashrightarrow X$ to any projective variety X. This phenomenon extends to all higher genera:

Corollary 0.5. *For every $g \geq 3$, the divisor $\overline{\Theta}_{g,1} \in \mathrm{Eff}(\overline{\mathcal{M}}_{g,1})$ is big.*

It is not known whether the Weierstrass divisor $\overline{\mathcal{W}}_g$ lies on the boundary of the effective cone $\mathrm{Eff}(\overline{\mathcal{M}}_{g,1})$ for g sufficiently large.

1. Generalities about $\overline{\mathcal{S}}_g$

As usual, we follow that the convention that if $\mathbf{M}$ is a Deligne-Mumford stack, then $\mathcal{M}$ denotes its associated coarse moduli space. We first recall basic facts about Cornalba's stack of stable spin curves $\pi : \overline{\mathbf{S}}_g \to \overline{\mathbf{M}}_g$, see [C], [F2], [Lud] for details and other basic properties. If X is a nodal curve, a smooth rational component $R \subset X$ is said to be *exceptional* if $\#(R \cap \overline{X-R}) = 2$. The curve X is said to be *quasi-stable* if $\#(R \cap \overline{X-R}) \geq 2$ for any smooth rational component $R \subset X$, and moreover, any two exceptional components of X are disjoint. A quasi-stable curve is obtained from a stable curve by possibly inserting a rational curve at each of its nodes. We denote by $[\mathrm{st}(X)] \in \overline{\mathcal{M}}_g$ the stable model of the quasi-stable curve X.

Definition 1.1. A *spin curve* of genus g consists of a triple (X,η,β), where X is a genus g quasi-stable curve, $\eta\in \mathrm{Pic}^{g-1}(X)$ is a line bundle of degree $g-1$ such that $\eta_R=\mathcal{O}_R(1)$ for every exceptional component $R\subset X$, and $\beta:\eta^{\otimes 2}\rightarrow \omega_X$ is a sheaf homomorphism which is generically non-zero along each non-exceptional component of X.

Stable spin curves of genus g form a smooth Deligne-Mumford stack $\overline{\mathbf{S}}_g$ which splits into two connected components $\overline{\mathbf{S}}_g^+$ and $\overline{\mathbf{S}}_g^-$, according to the parity of $h^0(X,\eta)$. Let $f:\mathcal{C}\rightarrow \overline{\mathbf{S}}_g$ be the universal family of spin curves of genus g. In particular, for every point $[X,\eta,\beta]\in \overline{\mathcal{S}}_g$, there is an isomorphism between $f^{-1}([X,\eta,\beta])$ and the quasi-stable curve X. There exists a (universal) spin line bundle $\mathcal{P}\in \mathrm{Pic}(\mathcal{C})$ of relative degree $g-1$, as well as a morphism of $\mathcal{O}_{\mathcal{C}}$-modules $B:\mathcal{P}^{\otimes 2}\rightarrow \omega_f$ having the property that $\mathcal{P}_{|f^{-1}([X,\eta,\beta])}=\eta$ and $B_{|f^{-1}([X,\eta,\beta])}=\beta:\eta^{\otimes 2}\rightarrow \omega_X$, for all spin curves $[X,\eta,\beta]\in \overline{\mathcal{S}}_g$. Throughout we use the canonical isomorphism $\mathrm{Pic}(\overline{\mathbf{S}}_{\mathbf{g}})_{\mathbb{Q}}\cong \mathrm{Pic}(\overline{\mathcal{S}}_{\mathbf{g}})_{\mathbb{Q}}$ and we make little distinction between line bundles on the stack and the corresponding moduli space.

1.1. The boundary divisors of $\overline{\mathcal{S}}_g$.

We discuss the structure of the boundary divisors of $\overline{\mathcal{S}}_g$ and concentrate on the case of $\overline{\mathcal{S}}_g^+$, the differences compared to the situation on $\overline{\mathcal{S}}_g^-$ being minor. We describe the pull-backs of the boundary divisors $\Delta_i\subset \overline{\mathcal{M}}_g$ under the map π. First we fix an integer $1\leq i\leq [g/2]$ and let $[X,\eta,\beta]\in \pi^{-1}([C\cup_y D])$, where $[C,y]\in \mathcal{M}_{i,1}$ and $[D,y]\in \mathcal{M}_{g-i,1}$. For degree reasons, then $X=C\cup_{y_1} R\cup_{y_2} D$, where R is an exceptional component such that $C\cap R=\{y_1\}$ and $D\cap R=\{y_2\}$. Furthermore $\eta=\big(\eta_C,\eta_D,\eta_R=\mathcal{O}_R(1)\big)\in \mathrm{Pic}^{g-1}(X)$, where $\eta_C^{\otimes 2}=K_C$ and $\eta_D^{\otimes 2}=K_D$. The theta-characteristics η_C and η_D have the same parity in the case of $\overline{\mathcal{S}}_g^+$ (and opposite parities for $\overline{\mathcal{S}}_g^-$). One denotes by $A_i\subset \overline{\mathcal{S}}_g^+$ the closure of the locus corresponding to pairs of pointed spin curves

$$\big([C,y,\eta_C],[D,y,\eta_D]\big)\in \mathcal{S}_{i,1}^+\times \mathcal{S}_{g-i,1}^+$$

and by $B_i\subset \overline{\mathcal{S}}_g^+$ the closure of the locus corresponding to pairs

$$\big([C,y,\eta_C],[D,y,\eta_D]\big)\in \mathcal{S}_{i,1}^-\times \mathcal{S}_{g-i,1}^-.$$

If $\alpha:=[A_i],\beta_i:=[B_i]\in \mathrm{Pic}(\overline{\mathbf{S}}_g^+)$, we have the relation $\pi^*(\delta_i)=\alpha_i+\beta_i$.

Next, we describe $\pi^*(\delta_0)$ and pick a stable spin curve $[X,\eta,\beta]$ such that $\mathrm{st}(X)=C_{yq}:=C/y\sim q$, with $[C,y,q]\in \mathcal{M}_{g-1,2}$. There are two possibilities depending on whether X possesses an exceptional component or not. If $X=C_{yq}$ and $\eta_C:=\nu^*(\eta)$ where $\nu:C\rightarrow X$ denotes the normalization map, then $\eta_C^{\otimes 2}=K_C(y+q)$. For each choice of $\eta_C\in \mathrm{Pic}^{g-1}(C)$ as above, there is precisely one choice of gluing the fibres $\eta_C(y)$ and $\eta_C(q)$ such that $h^0(X,\eta)\equiv 0 \bmod 2$. We denote by A_0 the closure in $\overline{\mathcal{S}}_g^+$ of the locus of points $[C_{yq},\ \eta_C\in \mathrm{Pic}^{g-1}(C),\ \eta_C^{\otimes 2}=K_C(y+q)]$ as above.

If $X=C\cup_{\{y,q\}} R$, where R is an exceptional component, then $\eta_C:=\eta\otimes \mathcal{O}_C$ is a theta-characteristic on C. Since $H^0(X,\omega)\cong H^0(C,\omega_C)$, it follows that $[C,\eta_C]\in \mathcal{S}_{g-1}^+$. We denote by $B_0\subset \overline{\mathcal{S}}_g^+$ the closure of the locus of points

$$\big[C\cup_{\{y,q\}} R,\ \eta_C\in \sqrt{K_C},\ \eta_R=\mathcal{O}_R(1)\big]\in \overline{\mathcal{S}}_g^+.$$

A local analysis carried out in [C], shows that B_0 is the branch locus of π and the ramification is simple. If $\alpha_0 = [A_0] \in \mathrm{Pic}(\overline{\mathcal{S}}_g^+)$ and $\beta_0 = [B_0] \in \mathrm{Pic}(\overline{\mathcal{S}}_g^+)$, we have the relation

$$\pi^*(\delta_0) = \alpha_0 + 2\beta_0. \tag{1}$$

2. Difference varieties and theta-characteristics

We describe a way of calculating the class of a series of effective divisors on both moduli spaces $\overline{\mathcal{S}}_g^-$ and $\overline{\mathcal{S}}_g^+$, defined in terms of the relative position of a theta-characteristic with respect to the divisorial difference varieties in the Jacobian of a curve. These loci, which should be thought of as divisors of Brill-Noether type on $\overline{\mathcal{S}}_g$, inherit a determinantal description over the entire moduli stack of spin curves, via the interpretation of difference varieties in $\mathrm{Pic}^{g-2i-1}(C)$ as Raynaud theta-divisors for exterior powers of Lazarsfeld bundles provided in [FMP]. The determinantal description is then extended over a partial compactification $\widetilde{\mathbf{S}}_g$ of $\mathbf{S_g}$, using the explicit description of stable spin curves. The formulas we obtain for the class of these divisors are identical over both $\overline{\mathbf{S}}_g^-$ and $\overline{\mathbf{S}}_g^+$, therefore we sometimes use the symbol $\overline{\mathbf{S}}_g^{\mp}$ (or even $\overline{\mathbf{S}}_g$), to denote one of the two spin moduli spaces.

We start with a curve $[C] \in \mathcal{M}_g$ and denote as usual by $Q_C := M_{K_C}^{\vee}$ the associated *Lazarsfeld bundle* [L] defined via the exact sequence on C

$$0 \to M_{K_C} \to H^0(C, K_C) \otimes \mathcal{O}_C \stackrel{\mathrm{ev}}{\to} K_C \to 0.$$

Note that Q_C is a semistable vector bundle on C (even stable, when the curve C is non-hyperelliptic), and $\mu(Q_C) = 2$. For integers $0 \leq i \leq g-1$, one defines the *divisorial difference variety* $C_{g-i-1} - C_i \subset \mathrm{Pic}^{g-2i-1}(C)$ as being the image of the difference map

$$\phi : C_{g-i-1} \times C_i \to \mathrm{Pic}^{g-2i-1}(C), \;\; \phi(D, E) := \mathcal{O}_C(D - E).$$

The main result from [FMP] provides a scheme-theoretic identification of divisors on the Jacobian variety

$$C_{g-i-1} - C_i = \Theta_{\wedge^i Q_C} \subset \mathrm{Pic}^{g-2i-1}(C), \tag{2}$$

where the right-hand-side denotes the *Raynaud locus* [R]

$$\Theta_{\wedge^i Q_C} := \{\eta \in \mathrm{Pic}^{g-2i-1}(C) : H^0(C, \wedge^i Q_C \otimes \eta) \neq 0\}.$$

The non-vanishing $H^0(C, \wedge^i Q_C \otimes \xi) \neq 0$ for all line bundles $\xi = \mathcal{O}_C(D - E)$, where $D \in C_{g-i-1}$ and $E \in C_i$, follows from [L]. The thrust of [FMP] is that the reverse inclusion $\Theta_{\wedge^i Q_C} \subset C_{g-i-1} - C_i$ also holds. Moreover, identification (2) shows that, somewhat similarly to Riemann's Singularity Theorem, the product $C_{g-i-1} \times C_i$ can be thought of as a canonical desingularization of the generalized theta-divisor $\Theta_{\wedge^i Q_C}$.

We fix integers $r, s > 0$ and set $d := rs + r, g := rs + s$, therefore the Brill-Noether number $\rho(g, r, d) = 0$. We assume moreover that $d \equiv 0 \bmod 2$, that is, either r is even or s is odd, and write $d = 2i$. We define the following locus in the spin moduli space $\mathcal{S}_g^{\mp}$:

$$\mathcal{U}_{g,d}^r := \{[C, \eta] \in \mathcal{S}_g^{\mp} : \exists L \in W_d^r(C) \text{ such that } \eta \otimes L^{\vee} \in C_{g-i-1} - C_i\}.$$

Using (2), the condition $[C, \eta] \in \mathcal{U}_{g,d}^r$ can be rewritten in a determinantal way as,

$$H^0(C, \wedge^i M_{K_C} \otimes \eta \otimes L) \neq 0.$$

Tensoring by $\eta \otimes L$ the exact sequence coming from the definition of M_{K_C}, namely

$$0 \longrightarrow \wedge^i M_{K_C} \longrightarrow \wedge^i H^0(C, K_C) \otimes \mathcal{O}_C \longrightarrow \wedge^{i-1} M_{K_C} \otimes K_C \longrightarrow 0,$$

then taking global sections and finally using that M_{K_C} (hence all of its exterior powers) are semi-stable vector bundles, we find that $[C, \eta] \in \mathcal{U}^r_{g,d}$ if and only if the map

(3) $\quad \phi(C, \eta, L) : \wedge^i H^0(C, K_C) \otimes H^0\big(C, \eta \otimes L) \to H^0(C, \wedge^{i-1} M_{K_C} \otimes K_C \otimes \eta \otimes L\big)$

is not an isomorphism for a certain $L \in W^r_d(C)$. Since $\mu(\wedge^{i-1} M_{K_C} \otimes K_C \otimes \eta \otimes L) \geq 2g-1$ and $\wedge^{i-1} M_{K_C}$ is a semi-stable vector bundle on C, it follows that

$$h^0(C, \wedge^{i-1} M_{K_C} \otimes K_C \otimes \eta \otimes L) = \chi(C, \wedge^{i-1} M_{K_C} \otimes K_C \otimes \eta \otimes L) = \binom{g}{i} d.$$

We assume that $h^1(C, \eta \otimes L) = 0$. This condition is satisfied outside a locus of $\mathcal{S}^{\mp}_g$ of codimension at least 2; if $H^1(C, \eta \otimes L) \neq 0$, then $H^1(C, K_C \otimes L^{\otimes(-2)}) \neq 0$, in particular the Petri map

$$\mu_0(C, L) : H^0(C, L) \otimes H^0(C, K_C \otimes L^{\vee}) \to H^0(C, K_C)$$

is not injective. Then $h^0(C, L \otimes \eta) = d$ and we note that $\phi(C, \eta, L)$ is a map between vector spaces of the same rank. This obviously suggests a determinantal presentation of $\mathcal{U}^r_{g,d}$ as the (push-forward of) a degeneracy locus between vector bundles of the same rank. In what follows we extend this presentation over a partial compactification of $\overline{\mathbf{S}}^{\mp}_g$. We refer to [FL] Section 2 for a similar calculation over the Prym moduli stack $\overline{\mathbf{R}}_g$.

We denote by $\mathbf{M}^0_g \subset \mathbf{M}_g$ the open substack classifying curves $[C] \in \mathcal{M}_g$ such that $W^r_{d-1}(C) = \emptyset$, $W^{r+1}_d(C) = \emptyset$ and moreover $H^1(C, L \otimes \eta) = 0$, for every $L \in W^r_d(C)$ and each odd-theta characteristic $\eta \in \mathrm{Pic}^{g-1}(C)$. From general Brill-Noether theory one knows that $\mathrm{codim}(\mathcal{M}_g - \mathcal{M}^0_g, \mathcal{M}_g) \geq 2$. Then we define $\widetilde{\Delta}_0 \subset \Delta_0$ to be the open substack consisting of 1-nodal stable curves $[C_{yq} := C/y \sim q]$, where $[C] \in \mathcal{M}_{g-1}$ is a curve satisfying the Brill-Noether theorem and $y, q \in C$. We then set $\overline{\mathbf{M}}^0_g := \mathbf{M}^0_g \cup \widetilde{\Delta}_0$, hence $\overline{\mathbf{M}}^0_g \subset \widetilde{\mathbf{M}}_g$ and then $\overline{\mathbf{S}}^0_g := \pi^{-1}(\overline{\mathbf{M}}^0_g) = (\overline{\mathbf{S}}^0_g)^+ \cup (\overline{\mathbf{S}}^0_g)^-$. Following [EH1], [F1], we consider the proper Deligne-Mumford stack

$$\sigma_0 : \mathfrak{G}^r_d \to \overline{\mathbf{M}}^0_g$$

classifying pairs $[C, L]$ with $[C] \in \overline{\mathcal{M}}^0_g$ and $L \in W^r_d(C)$. For any curve $[C] \in \overline{\mathcal{M}}^0_g$ and $L \in W^r_d(C)$, we have that $h^0(C, L) = r+1$, that is, $\mathfrak{G}^r_d$ parameterizes only complete linear series. For a point $[C_{yq} := C/y \sim q] \in \widetilde{\Delta}_0$, we have the identification

$$\sigma_0^{-1}\big[C_{yq}\big] = \{L \in W^r_d(C) : h^0(C, L \otimes \mathcal{O}_C(-y-q)) = r\},$$

that is, we view linear series on singular curves as linear series on the normalization such that the divisor of the nodes imposes only one condition. We denote by $f^r_d : \mathfrak{C}^r_{g,d} := \overline{\mathbf{M}}^0_{g,1} \times_{\overline{\mathbf{M}}^0_g} \mathfrak{G}^r_d \to \mathfrak{G}^r_d$ the pull-back of the universal curve $p : \overline{\mathbf{M}}^0_{g,1} \to \overline{\mathbf{M}}^0_g$ to $\mathfrak{G}^r_d$. Once we have chosen a Poincaré bundle $\mathcal{L}$ on $\mathfrak{C}^r_{g,d}$, we can form the three codimension 1 tautological classes in $A^1(\mathfrak{G}^r_d)$:

(4)

$$\mathfrak{a} := (f^r_d)_*\big(c_1(\mathcal{L})^2\big),\ \mathfrak{b} := (f^r_d)_*\big(c_1(\mathcal{L}) \cdot c_1(\omega_{f^r_d})\big),\ \mathfrak{c} := (f^r_d)_*\big(c_1(\omega_{f^r_d})^2\big) = (\sigma_0)^*\big((\kappa_1)_{\overline{\mathbf{M}}^0_g}\big).$$

The dependence on $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ on the choice of $\mathcal{L}$ is discussed in both [F2] and [FL]. We introduce the stack of $\mathfrak{g}^r_d$'s on spin curves

$$\sigma : \mathfrak{G}^r_d(\overline{\mathbf{S}}^0_g/\overline{\mathbf{M}}^0_g) := \overline{\mathbf{S}}^0_g \times_{\overline{\mathbf{M}}^0_g} \mathfrak{G}^r_d \to \overline{\mathbf{S}}^0_g$$

and then the corresponding universal spin curve over the $\mathfrak{g}^r_d$ parameter space

$$f' : \mathcal{C}^r_d := \mathcal{C} \times_{\overline{\mathbf{S}}^0_g} \mathfrak{G}^r_d(\overline{\mathbf{S}}^0_g/\overline{\mathbf{M}}^0_g) \to \mathfrak{G}^r_d(\overline{\mathbf{S}}^0_g/\overline{\mathbf{M}}^0_g).$$

We note that f' is a family of quasi-stable curves carrying at the same time a spin structure as well as a $\mathfrak{g}^r_d$. Just like in [FL], the boundary divisors of $\mathfrak{G}^r_d(\overline{\mathbf{S}}^0_g/\overline{\mathbf{M}}^0_g)$ are denoted by the same symbols, that is, one sets $A_0' := \sigma^*(A_0')$ and $B_0' := \sigma^*(B_0')$ and then

$$\alpha_0 := [A_0'], \;\; \beta_0 := [B_0'] \in A^1(\mathfrak{G}^r_d(\overline{\mathbf{S}}^0_g/\overline{\mathbf{M}}^0_g)).$$

We observe that two tautological line bundles live on $\mathcal{C}^r_d$, namely the pull-back of the universal spin bundle $\mathcal{P}^r_d \in \mathrm{Pic}(\mathcal{C}^r_d)$ and a Poincaré bundle $\mathcal{L} \in \mathrm{Pic}(\mathcal{C}^r_d)$ singling out the $\mathfrak{g}^r_d$'s, that is, $\mathcal{L}_{|f'^{-1}[X,\eta,\beta,L]} = L \in W^r_d(C)$, for each point $[X, \eta, \beta, L] \in \mathfrak{G}^r_d(\overline{\mathbf{S}}^0_g/\overline{\mathbf{M}}^0_g)$. Naturally, one also has the classes $\mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in A^1(\mathfrak{G}^r_d(\overline{\mathbf{S}}^0_g/\overline{\mathbf{M}}^0_g))$ defined by the formulas (4).

The following result is easy to prove and we skip details:

Proposition 2.1. *We denote by $f' : \mathcal{C}^r_d \to \mathfrak{G}^r_d(\overline{\mathbf{S}}^0_g/\overline{\mathbf{M}}^0_g)$ the universal quasi-stable spin curve and by $\mathcal{P}^r_d \in \mathrm{Pic}(\mathcal{C}^r_d)$ the universal spin bundle of relative degree $g-1$. One has the following formulas in $A^1(\mathfrak{G}^r_d(\overline{\mathbf{S}}^0_g/\overline{\mathbf{M}}^0_g))$:*

(i) $f'_*(c_1(\omega_{f'}) \cdot c_1(\mathcal{P}^r_d)) = \frac{1}{2}\mathfrak{c}$.
(ii) $f'_*(c_1(\mathcal{P}^r_d)^2) = \frac{1}{4}\mathfrak{c} - \frac{1}{2}\beta_0$.
(iii) $f'_*(c_1(\mathcal{L}) \cdot c_1(\mathcal{P}^r_d)) = \frac{1}{2}\mathfrak{b}$.

We determine the class of a compactification of $\mathcal{U}^r_{g,d}$ by pushing-forward a codimension 1 degeneracy locus via the map $\sigma : \mathfrak{G}^r_d(\overline{\mathbf{S}}^0_g/\overline{\mathbf{M}}^0_g) \to \overline{\mathbf{S}}^0_g$. To that end, we define a sequence of tautological vector bundles on $\mathfrak{G}^r_d(\overline{\mathbf{S}}^0_g/\overline{\mathbf{M}}^0_g)$: First, for $l \geq 0$ we set

$$\mathcal{A}_{0,l} := f'_*(\mathcal{L} \otimes \omega_{f'}^{\otimes l} \otimes \mathcal{P}^r_d).$$

It is easy to verify that $R^1 f'_*(\mathcal{L} \otimes \omega_{f'}^{\otimes l} \otimes \mathcal{P}^r_d) = 0$, hence $\mathcal{A}_{0,l}$ is locally free over $\mathfrak{G}^r_d(\overline{\mathbf{S}}^0_g/\overline{\mathbf{M}}^0_g)$ of rank equal to $h^0(X, L \otimes \omega_X^{\otimes l} \otimes \eta) = l(2g-2) + d$. Next we introduce the global Lazarsfeld vector bundle $\mathcal{M}$ over $\mathcal{C}^r_d$ by the exact sequence

$$0 \longrightarrow \mathcal{M} \longrightarrow (f')^*\big(f'_*\omega_{f'}\big) \longrightarrow \omega_{f'} \longrightarrow 0,$$

and then for all integers $a, j \geq 1$ we define the sheaf over $\mathfrak{G}^r_d(\overline{\mathbf{S}}^0_g/\overline{\mathbf{M}}^0_g)$

$$\mathcal{A}_{a,j} := f'_*(\wedge^a \mathcal{M} \otimes \omega_{f'}^{\otimes j} \otimes \mathcal{L} \otimes \mathcal{P}^r_d).$$

In a way similar to [FL] Proposition 2.5 one shows that $R^1 f'_*\big(\wedge^a \mathcal{M} \otimes \omega_{f'}^{\otimes(i-a)} \otimes \mathcal{L} \otimes \mathcal{P}^r_d\big) = 0$, therefore by Grauert's theorem $\mathcal{A}_{a,i-a}$ is a vector bundle over $\mathfrak{G}^r_d(\overline{\mathbf{S}}^0_g/\overline{\mathbf{M}}^0_g)$ of rank

$$\mathrm{rk}(\mathcal{A}_{a,i-a}) = \chi\big(X, \wedge^a M_{\omega_X} \otimes \omega_X^{\otimes(i-a)} \otimes L \otimes \eta\big) = 2(i-a)g\binom{g-1}{a}.$$

Furthermore, for all $1 \leq a \leq i-1$, the vector bundles $\mathcal{A}_{a,i-a}$ sit in exact sequences

$$0 \longrightarrow \mathcal{A}_{a,i-a} \longrightarrow \wedge^a f'_*(\omega_{f'}) \otimes \mathcal{A}_{0,i-a} \longrightarrow \mathcal{A}_{a-1,i-a+1} \longrightarrow 0, \tag{5}$$

where the right exactness boils down to showing that $H^1(X, \wedge^a M_{\omega_X} \otimes \omega_X^{\otimes(i-a)} \otimes \eta \otimes L) = 0$ for all $[X, \eta, \beta, L] \in \mathfrak{G}^r_d(\overline{\mathbf{S}}^0_g / \overline{\mathbf{M}}^0_g)$.

We denote as usual $\mathbb{E} := f'_*(\omega_{f'})$ the Hodge bundle over $\mathfrak{G}^r_d(\overline{\mathbf{S}}^0_g / \overline{\mathbf{M}}^0_g)$ and then note that there exists a vector bundle map

$$\phi : \wedge^i \mathbb{E} \otimes \mathcal{A}_{0,0} \to \mathcal{A}_{i-1,1} \tag{6}$$

between vector bundles of the same rank over $\mathfrak{G}^r_d(\overline{\mathbf{S}}^0_g / \overline{\mathbf{M}}^0_g)$. For $[C, \eta, L] \in \sigma^{-1}(\mathcal{M}^0_g)$ the fibre of this morphism is precisely the map $\phi(C, \eta, L)$ defined by (3).

Theorem 2.2. *The vector bundle morphism $\phi : \wedge^i \mathbb{E} \otimes \mathcal{A}_{0,0} \to \mathcal{A}_{i-1,1}$ is generically non-degenerate over $\mathfrak{G}^r_d(\overline{\mathbf{S}}^0_g / \overline{\mathbf{M}}^0_g)$. It follows that $\mathcal{U}^r_{g,d}$ is an effective divisor over $\mathcal{S}^+_g$ for all $s \geq 1$, and over $\mathcal{S}^-_g$ as well for $s \geq 2$.*

Proof. We specialize C to a hyperelliptic curve, and denote by $A \in W^1_2(C)$ the hyperelliptic involution. The Lazarsfeld bundle splits into a sum of line bundles $Q_C \cong A^{\oplus(g-1)}$, therefore the condition $H^0(C, \wedge^i M_{K_C} \otimes \eta \otimes L) = 0$ translates into $H^0(C, \eta \otimes A^{\otimes i} \otimes L^\vee) = 0$. Suppose that $h^0(C, \eta \otimes A^{\otimes i} \otimes L^\vee) \geq 1$ for any $L = A^{\otimes r} \otimes \mathcal{O}_C(x_1 + \cdots + x_{d-2r}) \in W^r_d(C)$, where the $x_1, \ldots, x_{d-2r} \in C$ are arbitrarily chosen points. This implies that $h^0(C, \eta \otimes A^{\otimes(i-r)}) \geq d - 2r + 1$. Any theta-characteristic on C is of the form

$$\eta = A^{\otimes m} \otimes \mathcal{O}_C(p_1 + \cdots + p_{g-2m-1}),$$

where $1 \leq m \leq (g-1)/2$ and $p_1, \ldots, p_{g-2m-1} \in C$ are Weierstrass points. Choosing a theta-characteristic on C for which $m \leq i - r - 1$ (which can be done in all cases except on $\mathcal{S}^-_g$ when $i = r$), we obtain that $h^0(C, \eta \otimes A^{\otimes(i-r)}) \leq d - 2r$, a contradiction. $\square$

Proof of Theorem 0.1. To compute the class of the degeneracy locus of ϕ we use repeatedly the exact sequence (5). We write the following identities in $A^1(\mathfrak{G}^r_d(\overline{\mathbf{S}}^0_g / \overline{\mathbf{M}}^0_g))$:

$$c_1\big(\mathcal{A}_{i-1,1} - \wedge^i \mathbb{E} \otimes \mathcal{A}_{0,0}\big) = \sum_{l=0}^{i} (-1)^{l-1} c_1(\wedge^{i-l} \mathbb{E} \otimes \mathcal{A}_{0,l}) =$$

$$= \sum_{l=0}^{i} (-1)^{l+1} \Big((2l(g-1) + d) \binom{g-1}{i-l-1} c_1(\mathbb{E}) + \binom{g}{i-l} c_1(\mathcal{A}_{0,l}) \Big).$$

Using Proposition 2.1 one can show via the Grothendieck-Riemann-Roch formula applied to $f' : \mathcal{C}^r_d \to \mathfrak{G}^r_d(\overline{\mathbf{S}}^0_g / \overline{\mathbf{M}}^0_g)$ that one has that

$$c_1(\mathcal{A}_{0,l}) = \lambda + \Big(\frac{l^2}{2} - \frac{1}{8}\Big)\mathfrak{c} + \frac{1}{2}\mathfrak{a} + l\mathfrak{b} - \frac{1}{4}\beta_0 \in A^1(\mathfrak{G}^r_d(\overline{\mathbf{S}}^0_g / \overline{\mathbf{M}}^0_g)).$$

To determine $\sigma_*\big(c_1(\mathcal{A}_{i-1,1} - \wedge^i \mathbb{E})\big) \in A^1(\overline{\mathbf{S}}_g)$ we use [F1], [Kh]: If

$$N := \deg(\sigma) = \#(W^r_d(C))$$

denotes the number of $\mathfrak{g}^r_d$'s on a general curve $[C] \in \mathcal{M}_g$, then there exists a precisely described choice of a Poincaré bundle on $\mathfrak{C}^r_{g,d}$ such that the push-forwards of the tautological classes on $\mathfrak{G}^r_d(\overline{\mathbf{S}}^0_g/\overline{\mathbf{M}}^0_g)$ are given as follows (cf. [F1], [Kh] and especially [FL] Section 2, for the similar argument in the Prym case):

$$\sigma_*(\mathfrak{a}) = \frac{dN}{(g-1)(g-2)}\Big((gd-2g^2+8d-8g+4)\lambda + \frac{1}{6}(2g^2-gd+3g-4d-2)(\alpha_0+2\beta_0)\Big)$$

and

$$\sigma_*(\mathfrak{b}) = \frac{dN}{2g-2}\Big(12\lambda - \alpha_0 - 2\beta_0\Big) \in A^1(\mathfrak{G}^r_d(\overline{\mathbf{S}}^0_g/\overline{\mathbf{M}}^0_g)).$$

One notes that $c_1(\mathcal{A}_{i-1,1} - \wedge^i \mathbb{E} \otimes \mathcal{A}_{0,0}) \in A^1(\mathfrak{G}^r_d(\overline{\mathbf{S}}^0_g/\overline{\mathbf{M}}^0_g))$ does not depend of the Poincaré bundle. Using the previous formulas, after some arithmetic, one computes the class of the partial compactification of $\mathcal{U}^r_{g,d}$ and finishes the proof. $\square$

When $s = 2a+1$, hence $g = (2a+1)(r+1)$ and $d = 2r(a+1)$, our calculation shows that

$$\overline{\mathcal{U}}^r_{g,d} \equiv c_{a,r}\big(\bar{\lambda}\,\lambda - \bar{\alpha_0}\,\alpha_0 - \bar{\beta}_0\,\beta_0 - \cdots\big) \in \mathrm{Pic}(\overline{\mathcal{S}}^{\mp}_g),$$

where $c_{a,r} \in \mathbb{Q}_{>0}$ is explicitly known and

$$\bar{\lambda} = 12r^3 - 12r^2 - 48a^2 + 96a^3 + 48r^4a + 2208r^3a^3 + 1968r^3a^2 + 3936r^2a^3 + 2208ra^3 + 552r^3a + 3984r^2a^2 + 1080r^2a + 2160ra^2 + 528ra + 192r^4a^4 + 384r^4a^3 + 768r^3a^4 + 960r^2a^4 + 240r^4a^2 + 384ra^4,$$

$$\bar{\alpha}_0 = 220ra^2 + 536r^2a^3 + 32r^4a^4 + 36ra + 24a^3 + 328r^3a^3 + 296ra^3 + 8r^4a + 64r^4a^3 + 3r^3 + 468r^2a^2 + 128r^3a^4 + 74r^3a + 40r^4a^2 + 160r^2a^4 + 64ra^4 + 268r^3a^2 + 110r^2a - 3r^2 - 12a^2$$

and

$$\bar{\beta}_0 = 96ra + 64r^4a^4 + 16r^4a + 416ra^2 + 928r^2a^3 + 448ra^3 + 208r^2a + 608r^3a^3 + 256r^3a^4 + 112r^3a + 80r^4a^2 + 320r^2a^4 + 128ra^4 + 464r^3a^2 + 128r^4a^3 + 816r^2a^2.$$

These formulas, though unwieldy, carry a great deal of information about $\overline{\mathcal{S}}_g$. In the simplest case, $s = 1$ (that is, $a = 0$) and $r = g-1$, then necessarily $L = K_C \in W^{g-1}_{2g-2}(C)$ and the condition $\eta - K_C \in -C_{g-1}$ is equivalent to $H^0(C, \eta) \neq 0$. In this way we recover the theta-null divisor $\overline{\Theta}_{\mathrm{null}}$ on $\overline{\mathcal{S}}^+_g$, or more precisely also taking into account multiplicities [F2],

$$\mathcal{U}^{g-1}_{g,2g-2} = 2 \cdot \Theta_{\mathrm{null}}.$$

At the same time, on $\mathcal{S}^+_g$ one does not get a divisor at all. In particular, we find that

$$\overline{\mathcal{U}}^{g-1}_{g,2g-2} \equiv 2 \cdot \overline{\Theta}_{\mathrm{null}} \equiv \frac{1}{2}\lambda - \frac{1}{8}\alpha_0 - 0 \cdot \beta_0 - \cdots \in \mathrm{Pic}(\overline{\mathcal{S}}^+_g).$$

Another interesting case is when $r = 2$, hence $g = 3s$, $L \in W^2_{2s+2}(C)$ and the condition $\eta \otimes L^\vee \in C_{2s-2} - C_{s+1}$ is equivalent to requiring that the embedded curve $C \stackrel{|\eta \otimes L|}{\longrightarrow} \mathbf{P}^{2s+1}$ has an $(s+1)$-secant $(s-1)$-plane:

Theorem 2.3. *For $g = 3s, d = 2s+2$, the class of the closure in $\overline{\mathbf{S}}_g^{\mp}$ of the effective divisor*

$$\mathcal{U}^2_{g,d} := \{[C,\eta] \in \mathcal{S}^{\mp}_{3s} : \exists L \in W^2_{2s+2}(C) \text{ such that } \eta \otimes L^{\vee} \in C_{2s-2} - C_{s+1}\}$$

is given by the formula in $\mathrm{Pic}(\overline{\mathcal{S}}^{\mp}_g)$*:*

$$\overline{\mathcal{U}}^2_{g,d} \equiv \binom{g}{s+2}\binom{g}{s,s,s}\frac{1}{24g(g-1)^2(g-2)(s+1)^2}\Big(4(216s^4+513s^3-348s^2-387s+18)\lambda-$$

$$-\big(144s^4+225s^3-268s^2-99s+10\big)\alpha_0-\big(288s^4+288s^3+320s^2+32\big)\beta_0-\cdots\Big).$$

For instance, for $g=9$, we obtain the class of the closure of the locus spin curves $[C,\eta] \in \mathcal{S}^{\mp}_9$, for which there exists a net $L \in W^2_8(C)$ such that $\eta \otimes L^{\vee} \in C_4 - C_4$:

$$\overline{\mathcal{U}}^2_{9,8} \equiv 235 \cdot 35\Big(\frac{36}{5}\lambda - \alpha_0 - \frac{428}{235}\beta_0 - \cdots\Big) \in \mathrm{Pic}(\overline{\mathcal{S}}^{\mp}_9).$$

3. The class of $\overline{\Theta}_{\mathrm{null}}$ on $\overline{\mathcal{S}}^{+}_g$: An alternative proof using the Hurwitz stack

We present an alternative way of computing the class of the divisor $[\overline{\Theta}_{\mathrm{null}}]$ (in even genus), as the push-forward of a determinantal cycle on a Hurwitz scheme of degree k coverings of genus g curves. We set

$$g = 2k-2, \ r=1, \ d=k,$$

hence $\rho(g,1,k)=0$, and use the notation from the previous section. In particular, we have the proper morphism $\sigma_0 : \mathfrak{G}^1_k \to \overline{\mathbf{M}}^0_g$ from the Hurwitz stack of $\mathfrak{g}^1_k$'s, and the universal spin curve over the Hurwitz stack

$$f' : \mathcal{C}^k_1 := \mathcal{C} \times_{\overline{\mathbf{S}}^0_g} \mathfrak{G}^1_k(\overline{\mathbf{S}}^0_g/\overline{\mathbf{M}}^0_g) \to \mathfrak{G}^1_k(\overline{\mathbf{S}}^0_g/\overline{\mathbf{M}}^0_g).$$

Once more, we introduce a number of vector bundles over $\mathfrak{G}^1_k(\overline{\mathbf{S}}^0_g/\overline{\mathbf{M}}^0_g)$: First, we set $\mathcal{H} := f'_*(\mathcal{L})$. By Grauert's theorem, $\mathcal{H}$ is a vector bundle of rank 2 over $\mathfrak{G}^1_k(\overline{\mathbf{S}}^0_g/\overline{\mathbf{M}}^0_g)$, having fibre $\mathcal{H}[X,\eta,\beta,L] = H^0(X,L)$, where $L \in W^1_k(X)$. Then for $j \geq 1$ we define

$$\mathcal{B}_j := f'_*(\mathcal{L}^{\otimes j} \otimes \mathcal{P}^1_k).$$

Since $R^1 f'_*(\mathcal{L}^{\otimes j} \otimes \mathcal{P}^1_k) = 0$, we find that $\mathcal{B}_j$ is a vector bundle over $\mathfrak{G}^1_k(\overline{\mathbf{S}}^0_g/\overline{\mathbf{M}}^0_g)$ of rank equal to $h^0(X, L^{\otimes j} \otimes \eta) = kj$.

Proposition 3.1. *If $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ are the codimension 1 tautological classes on $\mathfrak{G}^1_k(\overline{\mathbf{S}}^0_g/\overline{\mathbf{M}}^0_g)$ defined by (4), then for all $j \geq 1$ one has the following formula in $A^1(\mathfrak{G}^1_k(\overline{\mathbf{S}}^0_g/\overline{\mathbf{M}}^0_g))$:*

$$c_1(\mathcal{B}_j) = \lambda - \frac{1}{8}\mathfrak{c} + \frac{j^2}{2}\mathfrak{a} - \frac{j}{2}\mathfrak{b} - \frac{1}{4}\beta_0.$$

Proof. We apply Grothendieck-Riemann-Roch to the morphism $f' : \mathcal{C}^1_k \to \mathfrak{G}^1_k(\overline{\mathbf{S}}^0_g/\overline{\mathbf{M}}^0_g)$:

$$c_1(\mathcal{B}_j) = c_1\big(f'_!(\mathcal{L}^{\otimes j} \otimes \mathcal{P}^1_k)\big) =$$

$$= f'_*\Big[\Big(1 + c_1(\mathcal{L}^{\otimes j} \otimes \mathcal{P}^1_k) + \frac{c_1^2(\mathcal{L}^{\otimes j} \otimes \mathcal{P}^1_k)}{2}\Big)\Big(1 - \frac{c_1(\omega_{f'})}{2} + \frac{c_1^2(\omega_{f'}) + [\mathrm{Sing}(f')]}{12}\Big)\Big]_2,$$

where $\mathrm{Sing}(f') \subset \mathcal{X}_k^1$ denotes the codimension 2 singular locus of the morphism f', therefore $f'_*[\mathrm{Sing}(f')] = \alpha_0 + 2\beta_0$. We then use Mumford's formula [HM] pulled back from $\overline{\mathbf{M}}_g^0$ to $\mathfrak{G}_k^1(\overline{\mathbf{S}}_g^0/\overline{\mathbf{M}}_g^0)$, to write that

$$\kappa_1 = f'_*(c_1^2(\omega_{f'})) = 12\lambda - (\alpha_0 + 2\beta_0)$$

and then note that $f'_*(c_1(\mathcal{L}) \cdot c_1(\mathcal{P}_k^1)) = 0$ (the restriction of $\mathcal{L}$ to the exceptional divisor of $f' : \mathcal{C}_k^1 \to \mathfrak{G}_k^1(\overline{\mathbf{S}}_g^0/\overline{\mathbf{M}}_g^0)$ is trivial). Similarly, we note that $f'_*(c_1(\omega_{f'}) \cdot c_1(\mathcal{P}_k^1)) = \mathfrak{c}/2$. Finally, we write that $f'_*(c_1^2(\mathcal{P}_k^1)) = \mathfrak{c}/4 - \beta_0/2$. □

For $j \geq 1$ there are natural vector bundle morphisms over $\mathfrak{G}_k^1(\overline{\mathbf{S}}_g^0/\overline{\mathbf{M}}_g^0)$

$$\chi_j : \mathcal{H} \otimes \mathcal{B}_j \to \mathcal{B}_{j+1}.$$

Over a point $[C, \eta_C, L] \in \mathcal{S}_g^+ \times_{\mathcal{M}_g} \mathfrak{G}_k^1$ corresponding to an even theta-characteristic η_C and a pencil $L \in W_k^1(C)$, the morphism χ_j is given by multiplications of global sections

$$\chi_j[C, \eta, L] : H^0(C, L) \otimes H^0(C, L^{\otimes j} \otimes \eta_C) \to H^0(C, L^{\otimes(j+1)} \otimes \eta_C).$$

In particular, $\chi_1 : \mathcal{H} \otimes \mathcal{B}_1 \to \mathcal{B}_2$ is a morphism between vector bundles of the same rank. From the base point free pencil trick, the degeneration locus $Z_1(\chi_1)$ is (set-theoretically) equal to the inverse image $\sigma^{-1}\big(\overline{\Theta}_{\mathrm{null}} \cap (\overline{\mathcal{S}}_g^0)^+\big)$.

Theorem 3.2. *We fix $g = 2k-2$. The vector bundle morphism $\chi_1 : \mathcal{H} \otimes \mathcal{B}_1 \to \mathcal{B}_2$ defined over $\mathfrak{G}_k^1(\overline{\mathbf{S}}_g^0/\overline{\mathbf{M}}_g^0)$ is generically non-degenerate and we have the following formula for the class of its degeneracy locus:*

$$[Z_1(\chi_1)] = c_1(\mathcal{B}_2 - \mathcal{H} \otimes \mathcal{B}_1) = \frac{1}{2}\lambda - \frac{1}{8}\alpha_0 + \mathfrak{a} - kc_1(\mathcal{H}) \in A^1(\mathfrak{G}_k^1(\overline{\mathbf{S}}_g^0/\overline{\mathbf{M}}_g^0)).$$

The class of the push-forward $\sigma_*[Z_1(\chi_1)]$ to $\overline{\mathcal{S}}_g^+$ is given by the formula:

$$\sigma_*\big(c_1(\mathcal{B}_2 - \mathcal{H} \otimes \mathcal{B}_1)\big) \equiv \frac{(2k-2)!}{k!(k-1)!}\Big(\frac{1}{2}\lambda - \frac{1}{8}\alpha_0 - 0 \cdot \beta_0\Big) \equiv \frac{2(2k-2)!}{k!(k-1)!}\,\overline{\Theta}_{\mathrm{null}\ |\overline{\mathcal{S}}_g^+} \in \mathrm{Pic}(\overline{\mathcal{S}}_g^+).$$

Proof. The first part follows directly from Theorem 3.1. To determine the push-forward of codimension 1 tautological classes to $(\overline{\mathcal{S}}_g^0)^+$, we use again [F1], [Kh]: One writes the following relations in $A^1((\overline{\mathbf{S}}_g^0)^+) = A^1((\overline{\mathcal{S}}_g^0)^+)$:

$$\sigma_*(\mathfrak{a}) = \deg(\mathfrak{G}_k^1/\overline{\mathbf{M}}_g^0)\,\Big(-\frac{3k(k+1)}{2k-3}\,\lambda + \frac{k^2}{2(2k-3)}(\alpha_0 + 2\beta_0)\Big),$$

$$\sigma_*(\mathfrak{b}) = \deg(\mathfrak{G}_k^1/\overline{\mathbf{M}}_g^0)\,\Big(\frac{6k}{2k-3}\,\lambda - \frac{k}{2(2k-3)}(\alpha_0 + 2\beta_0)\Big),$$

and

$$\sigma_*\big(c_1(\mathcal{H})\big) = \deg(\mathfrak{G}_k^1/\overline{\mathbf{M}}_g^0)\,\Big(-3\frac{k+1}{2k-3}\,\lambda + \frac{k}{2(2k-3)}(\alpha_0 + 2\beta_0)\Big),$$

where

$$N := \deg(\mathfrak{G}_k^1/\overline{\mathbf{M}}_g^0) = \frac{(2k-2)!}{k!(k-1)!}$$

denotes the *Catalan number* of linear series $\mathfrak{g}^1_k$ on a general curve of genus $2k-2$. This yields yet another proof of the main result from [F2], in the sense that we compute the class of the divisor $\overline{\Theta}_{\mathrm{null}}$ of vanishing theta-nulls:

$$\sigma_*(Z_1(\chi_1)) = \deg(\mathfrak{G}^1_k/\overline{\mathbf{M}}^0_g)\big(\frac{1}{2}\lambda - \frac{1}{8}\alpha_0\big) \equiv 2\deg(\mathfrak{G}^1_k/\overline{\mathbf{M}}^0_g)\ [\overline{\Theta}_{\mathrm{null}\ |(\overline{\mathcal{S}}^0_g)^+}].$$

$\square$

Remark 3.3. The multiplicity 2 appearing in the expression of $\sigma_*(Z_1(\chi_1))$ is justified by the fact that $\dim \mathrm{Ker}(\chi_1(t)) = h^0(C,\eta)$ for every $[C,\eta,L] \in \sigma^{-1}((\mathcal{S}^0_g)^+)$. This of course is always an even number. Thus we have the equality cycles

$$Z_1(\chi_1) = Z_2(\chi_1) = \{t \in \mathfrak{G}^1_k(\overline{\mathbf{S}}^0_g/\overline{\mathbf{M}}^0_g) : \text{co-rank}(\phi_1(t)) \geq 2\},$$

that is χ_1 degenerates in codimension 1 with corank 2, and $Z_1(\chi_1)$ is an everywhere non-reduced scheme.

4. The divisor of points of odd theta-characteristics

In this section we compute the class of the divisor $\overline{\Theta}_{g,1}$. The study of geometric divisors on $\overline{\mathcal{M}}_{g,1}$ begins with [Cu], where the locus of Weierstrass points is determined:

$$\overline{\mathcal{W}_g} \equiv -\lambda + \binom{g+1}{2}\psi - \sum_{i=1}^{g-1}\binom{g-i+1}{2}\delta_{i:1} \in \mathrm{Pic}(\overline{\mathcal{M}}_{g,1}).$$

More generally, if $\bar{\alpha} : 0 \leq \alpha_0 \leq \ldots \leq \alpha_r \leq d-r$ is a *Schubert index* of type (r,d) such that $\rho(g,r,d) - \sum_{i=0}^r \alpha_i = -1$, one defines the *pointed Brill-Noether divisor* $\mathcal{M}^r_{g,d}(\bar{\alpha})$ as being the locus of pointed curves $[C,q] \in \mathcal{M}_{g,1}$ possessing a linear series $l \in G^r_d(C)$ with ramification sequence $\alpha^l(q) \geq \bar{\alpha}$. It follows from [EH3] that the cone spanned by the pointed Brill-Noether divisors on $\overline{\mathcal{M}}_{g,1}$ is 2-dimensional, with generators $[\overline{\mathcal{W}}_g]$ and the pull-back of the Brill-Noether class from $\overline{\mathcal{M}}_g$. Our aim is to analyze the divisor $\overline{\Theta}_{g,1}$, whose definition is arguably simpler than that of the divisors $\overline{\mathcal{M}}^r_{g,d}(\bar{\alpha})$, and which seems to have been overlooked until now. A consequence of the calculation is that (as expected) $[\overline{\Theta}_{g,1}]$ lies outside the Brill-Noether cone of $\overline{\mathcal{M}}_{g,1}$.

We begin by recalling basic facts about divisors on $\overline{\mathcal{M}}_{g,1}$. For $i = 1,\ldots,g-1$, the divisor Δ_i on $\overline{\mathcal{M}}_{g,1}$ is the closure of the locus of pointed curves $[C \cup D, q]$, where C and D are smooth curves of genus i and $g-i$ respectively, and $q \in C$. Similarly, Δ_{irr} denotes the closure in $\overline{\mathcal{M}}_{g,1}$ of the locus of irreducible 1-pointed stable curves. We set $\delta_i := [\Delta_i], \delta_{\mathrm{irr}} := [\Delta_{\mathrm{irr}}] \in \mathrm{Pic}(\overline{\mathcal{M}}_{g,1})$, and recall that $\psi \in \mathrm{Pic}(\overline{\mathcal{M}}_{g,1})$ is the universal cotangent class. Clearly, $p^*(\delta_{\mathrm{irr}}) = \delta_{\mathrm{irr}}$ and $p^*(\delta_i) = \delta_i + \delta_{g-i} \in \mathrm{Pic}(\overline{\mathcal{M}}_{g,1})$ for $1 \leq i \leq [g/2]$. For $g \geq 3$, the group $\mathrm{Pic}(\overline{\mathcal{M}}_{g,1})$ is freely generated by the classes $\lambda, \psi, \delta_{\mathrm{irr}}, \delta_1, \ldots, \delta_{g-1}$, cf. [AC1]. When $g = 2$, the same classes generate $\mathrm{Pic}(\overline{\mathcal{M}}_{2,1})$ subject to the *Mumford relation*

$$\lambda = \frac{1}{10}\delta_{\mathrm{irr}} + \frac{1}{5}\delta_1,$$

expressing that λ is a boundary class. We expand the class $[\overline{\Theta}_{g,1}]$ in this basis of $\mathrm{Pic}(\overline{\mathcal{M}}_{g,1})$,

$$\overline{\Theta}_{g,1} \equiv a\lambda + b\psi - b_{\mathrm{irr}}\delta_{\mathrm{irr}} - \sum_{i=1}^{g-1} b_i\delta_i \in \mathrm{Pic}(\overline{\mathcal{M}}_{g,1}),$$

and determine the coefficients in a classical way, by understanding the restriction of $\overline{\Theta}_{g,1}$ to sufficiently many geometric subvarieties of $\overline{\mathcal{M}}_{g,1}$. To ease calculations, we set

$$N_g^- := 2^{g-1}(2^g-1) \text{ and } N_g^+ := 2^{g-1}(2^g+1),$$

to be the number of odd (respectively even) theta-characteristic on a curve of genus g.

We define some test-curves in the boundary of $\overline{\mathcal{M}}_{g,1}$. For an integer $2 \le i \le g-1$, we choose general (pointed) curves $[C] \in \mathcal{M}_i$ and $[D,x,q] \in \mathcal{M}_{g-i,2}$. In particular, we may assume that $x, q \in D$ do not appear in the support of any odd theta-characteristic η_D^- on D, and that $h^0(D,\eta_D^+) = 0$, for any even theta-characteristic η_D^+. By joining C and D at a variable point $x \in C$, we obtain a family of 1-pointed stable curves

$$F_{g-i} := \{[C \cup_x D,\ q] : x \in C\} \subset \Delta_{g-i} \subset \overline{\mathcal{M}}_{g,1},$$

where the marked point $q \in D$ is fixed. It is clear that $F_{g-i} \cdot \delta_{g-i} = 2-2i$, $F_{g-i} \cdot \lambda = F_{g-i} \cdot \psi = 0$. Moreover, F_{g-i} is disjoint from all the other boundary divisors of $\overline{\mathcal{M}}_{g,1}$.

Proposition 4.1. *For each $2 \le i \le g-1$, one has that $b_{g-i} = N_i^- \cdot N_{g-i}^+/2$.*

Proof. We observe that the curve $F_{g-i} \times_{\overline{\mathcal{M}}_{g,1}} \overline{\mathcal{S}}_g^-$ splits into $N_i^+ \cdot N_{g-i}^- + N_i^- \cdot N_{g-i}^+$ irreducible components, each isomorphic to C, corresponding to a choice of a pair of theta-characteristics of opposite parities on C and D respectively. Let $t \in F_{g-i} \cdot \overline{\Theta}_{g,1}$ be an arbitrary point in the intersection, with underlying stable curve $C \cup_x D$, and spin curves $\big([C,\eta_C],[D,\eta_D]\big) \in \mathcal{S}_i \times \mathcal{S}_{g-i}$ on the two components.

Suppose first that $\eta_C = \eta_C^+$ and $\eta_D = \eta_D^-$, that is, t corresponds to an even theta-characteristic on C and an odd theta-characteristic on D. Then there exist non-zero sections $\sigma_C \in H^0\big(C, \eta_C^+ \otimes \mathcal{O}_C((g-i)x)\big)$ and $\sigma_D \in H^0\big(D, \eta_D^- \otimes \mathcal{O}_D(ix)\big)$ such that

$$\mathrm{ord}_x(\sigma_C) + \mathrm{ord}_x(\sigma_D) \ge g-1, \text{ and } \sigma_D(q) = 0. \tag{7}$$

In other words, σ_C and σ_D are the aspects of a limit $\mathfrak{g}^0_{g-1}$ on $C \cup_x D$ which vanishes at $q \in D$. Clearly, $\mathrm{ord}_x(\sigma_C) \le g-i-1$, hence $\mathrm{div}(\sigma_D) \ge ix + q$, that is, $q \in \mathrm{supp}(\eta_D^-)$. This contradicts the generality assumption on $q \in D$, so this situation does not occur.

Thus, we are left to consider the case $\eta_C = \eta_C^-$ and $\eta_D = \eta_D^+$. We denote again by $\sigma_C \in H^0(C, \eta_C^- \otimes \mathcal{O}_C((g-i)x))$ and $\sigma_D \in H^0(D, \eta_D^+ \otimes \mathcal{O}_D(ix))$ the sections satisfying the compatibility relations (7). The condition $h^0(D, \eta_D^+ \otimes \mathcal{O}_D(x-q)) \ge 1$ defines a correspondence on $D \times D$, cf. [DK], in particular, we can choose the points $x, q \in D$ general enough such that $H^0(D, \eta_D^+ \otimes \mathcal{O}_D(x-q)) = 0$. Then $\mathrm{ord}_x(\sigma_D) \le i-2$, thus $\mathrm{ord}_x(\sigma_C) \ge g-i+1$. It follows that we must have equality $\mathrm{ord}_x(\sigma_C) = g-i+1$, and then, $x \in \mathrm{supp}(\eta_C^-)$. An argument along the lines of [EH3] Lemma 3.4, shows that each of these intersection points has to be counted with multiplicity 1, thus $F_{g-i} \cdot \overline{\Theta}_{g,1} = \#\mathrm{supp}(\eta_C^-) \cdot N_i^- \cdot N_{g-i}^+$. We conclude by noting that $(2i-2)b_{g-i} = F_{g-i} \cdot \overline{\Theta}_{g,1}$. □

Proposition 4.2. *The relation $b = N_g^-/2$ holds.*

Proof. Having fixed a general curve $[C] \in \overline{\mathcal{M}}_g$, by considering the fibre $p^*([C])$ inside the universal curve, one writes the identity $(2g-2)b = p^*([C]) \cdot \overline{\Theta}_{g,1} = (g-1)N_g^-$. □

We compute the class of the restriction of the divisor $\Theta_{g,1}$ over $\mathcal{M}_{g,1}$:

Proposition 4.3. *One has the equivalence $\Theta_{g,1} \equiv N_g^-(\psi/2 + \lambda/4) \in \mathrm{Pic}(\mathcal{M}_{g,1})$.*

Proof. We consider the universal pointed spin curve $\mathrm{pr}: \mathbf{S}^-_{g,1} := \mathbf{S}^-_g \times_{\mathbf{M}_g} \mathbf{M}_{g,1} \to \mathbf{M}_{g,1}$. As usual, $\mathcal{P} \in \mathrm{Pic}(\mathbf{S}^-_{g,1})$ denotes the universal spin bundle, which over the stack $\mathbf{S}^-_{g,1}$, is a root of the dualizing sheaf ω_{pr}, that is, $2c_1(\mathcal{P}) = \mathrm{pr}^*(\psi)$. We introduce the divisor

$$\mathcal{Z} := \{[C, \eta, q] \in \mathcal{S}^-_{g,1} : q \in \mathrm{supp}(\eta)\} \subset \mathcal{S}^-_{g,1},$$

and clearly $\Theta_{g,1} := \mathrm{pr}_*(\mathcal{Z})$. We write $[\mathcal{Z}] = c_1(\mathcal{P}) - c_1\big(\mathrm{pr}^*(\mathrm{pr}_*(\mathcal{P}))\big)$, and take into account that $c_1(\mathrm{pr}_!(\mathcal{P})) = 2c_1(\mathrm{pr}_*(\mathcal{P})) = -\lambda/2$. The rest follows by applying the projection formula. $\Box$

In order to determine the remaining coefficients b_0, b_1, we study the pull-back of $\overline{\Theta}_{g,1}$ under the map $\nu : \overline{\mathcal{M}}_{1,2} \to \overline{\mathcal{M}}_{g,1}$, given by $\nu([E, x, q]) := [C \cup_x E, q] \in \overline{\mathcal{M}}_{g,1}$, where $[C, x] \in \mathcal{M}_{g-1,1}$ is a fixed general pointed curve.

On the surface $\overline{\mathcal{M}}_{1,2}$, if we denote a general element by $[E, x, q]$, one has the following relations between divisors classes, see [AC2]:

$$\psi_x = \psi_q, \ \lambda = \psi_x - \delta_{0:xq}, \ \delta_{\mathrm{irr}} = 12(\psi_x - \delta_{0:xq}).$$

We describe the pull-back map $\nu^* : \mathrm{Pic}(\overline{\mathcal{M}}_{g,1}) \to \mathrm{Pic}(\overline{\mathcal{M}}_{1,2})$ at the level of divisors:

$$\nu^*(\lambda) = \lambda, \ \nu^*(\psi) = \psi_q, \ \nu^*(\delta_{\mathrm{irr}}) = \delta_{\mathrm{irr}}, \ \nu^*(\delta_1) = -\psi_x, \ \nu^*(\delta_{g-1}) = \delta_{0:xq}.$$

By direct calculation, we write $\nu^*(\overline{\Theta}_{g,1}) \equiv (a + b - 12b_0 + b_1)\psi_x - (a + b_{g-1} - 12b_0)\delta_{0:xq}$. We compute b_0 and b_1 by describing $\nu^*(\overline{\Theta}_{g,1})$ viewed as an explicit divisor on $\overline{\mathcal{M}}_{1,2}$:

Proposition 4.4. *One has the relation* $\nu^*(\overline{\Theta}_{g,1}) \equiv N^-_{g-1} \cdot \overline{\mathfrak{T}}_2 \in \mathrm{Pic}(\overline{\mathcal{M}}_{2,1})$, *where*

$$\mathfrak{T}_2 := \{[E, x, q] \in \mathcal{M}_{1,2} : 2x \equiv 2q\}.$$

Proof. We fix an arbitrary point $t := [C \cup_x E, q] \in \nu^*(\overline{\Theta}_{g,1})$. Suppose first that E is a smooth elliptic curve, that is, $j(E) \neq \infty$ and $x \neq q$. Then there exist theta-characteristics of opposite parities η_C, η_E on C and E respectively, together with non-zero sections

$$\sigma_C \in H^0\big(C, \eta_C \otimes \mathcal{O}_C(x)\big) \ \text{ and } \ \sigma_E \in H^0\big(E, \eta_E \otimes \mathcal{O}_E((g-1)x)\big),$$

such that $\sigma_E(q) = 0$ and $\mathrm{ord}_x(\sigma_C) + \mathrm{ord}_x(\sigma_E) \geq g - 1$.

First we assume that $\eta_C = \eta^+_C$ and $\eta_E = \eta^-_E$, thus, $\eta_E = \mathcal{O}_E$. Since $H^0(C, \eta^+_C) = 0$, one obtains that $\mathrm{ord}_x(\sigma_C) = 0$, that is $\mathrm{ord}_x(\sigma_E) = g-1$, which is impossible, because σ_E must vanish at q as well. Thus, one is lead to study the remaining case, when $\eta_C = \eta^-_C$ and $\eta_E = \eta^+_E$. Since $x \notin \mathrm{supp}(\eta^-_C)$, we obtain $\mathrm{ord}_x(\sigma_C) \leq 1$, and then by compatibility, the last inequality becomes equality, while $\mathrm{ord}_x(\sigma_E) = g - 2$, hence $\eta^+_E = \mathcal{O}_E(x - q)$, or equivalently, $[E, x, q] \in \mathfrak{T}_2$. The multiplicity N^-_{g-1} in the expression of $\nu^*(\overline{\Theta}_{g,1})$ comes from the choices for the theta-characteristics η^-_C, responsible for the C-aspect of a limit $\mathfrak{g}^0_{g-1}$ on $C \cup_x E$. It is an easy moduli count to show that the cases when $j(E) = \infty$, or $[E, x, q] \in \delta_{0:xq}$ (corresponding to the situation when x and q coalesce on E), do not occur generically on a component of $\nu^*(\overline{\Theta}_{g,1})$. $\Box$

Proposition 4.5. $\overline{\mathfrak{T}}_2$ *is an irreducible divisor on* $\overline{\mathcal{M}}_{1,2}$ *of class* $\overline{\mathfrak{T}}_2 \equiv 3\psi_x \in \mathrm{Pic}(\overline{\mathcal{M}}_{1,2})$.

Proof. We write $\overline{\mathfrak{T}}_2 \equiv \alpha\psi_x - \beta\delta_{0:xq} \in \mathrm{Pic}(\overline{\mathcal{M}}_{1,2})$, and we need to understand the intersection of $\overline{\mathfrak{T}}_2$ with two test curves in $\overline{\mathcal{M}}_{1,2}$. First, we fix a general point $[E, q] \in \overline{\mathcal{M}}_{1,1}$ and consider the family $E_1 := \{[E, x, q] : x \in E\} \subset \overline{\mathcal{M}}_{1,2}$. Clearly, $E_1 \cdot \delta_{0:xq} = E_1 \cdot \psi_x = 1$. On the other hand $E_1 \cdot \overline{\mathfrak{T}}_2$ is a 0-cycle simply supported at the points $x \in E - \{q\}$ such that $x - q \in \mathrm{Pic}^0(E)[2]$, that is, $E_1 \cdot \overline{\mathfrak{T}}_2 = 3$. This yields the relation $\alpha - \beta = 3$.

As a second test curve, we denote by $[L, u, x, q] \in \overline{\mathcal{M}}_{0,3}$ the rational 3-pointed rational curve, and define the pencil $R := \{[L \cup_u E_\lambda, x, q] : \lambda \in \mathbf{P}^1\} \subset \overline{\mathcal{M}}_{1,2}$, where $\{E_\lambda\}_{\lambda \in \mathbf{P}^1}$ is a pencil of plane cubic curves. Then $R \cap \overline{\mathfrak{T}}_2 = \emptyset$. Since $R \cdot \lambda = 1$ and $R \cdot \delta_{\mathrm{irr}} = 12$, we obtain the additional relation $\beta = 0$, which completes the proof. $\square$

Putting together Propositions 4.1, 4.3 and 4.5, we obtain the system of equations

$$a + b_{g-1} - 12b_{\mathrm{irr}} = 0,\ a - 12b_{\mathrm{irr}} + b + b_1 = 3N_{g-1}^-,\ a = \frac{1}{4}N_g^-,\ b = \frac{1}{2}N_g^-,\ b_1 = \frac{3}{2}N_{g-1}^-.$$

Thus $b_{\mathrm{irr}} = 2^{2g-6}$ and $b_{g-1} = 2^{g-3}(2^{g-1}+1)$. This completes the proof of Theorem 0.3.

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